(NORMAL) EXTREMALS VS LENGTH - MINIMIZERS

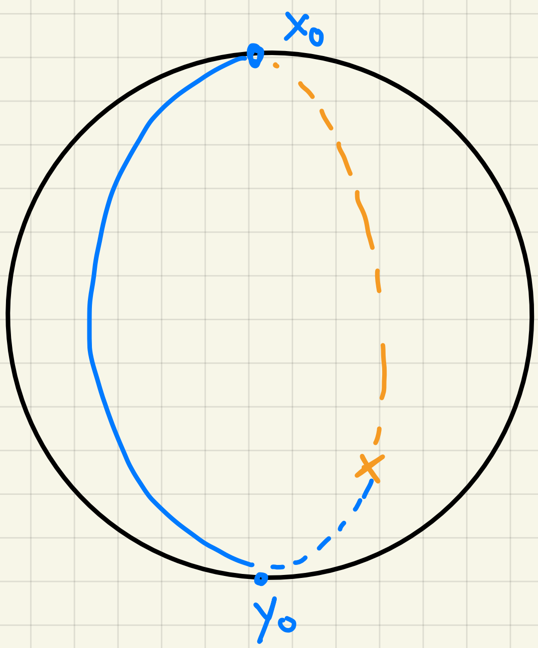
(and the Heisenberg group)

length-minimizer $\Longrightarrow$ condition (N) normal
condition (A) abnormal

necessary condition : extremals

A curve satisfying the necessary condition is a length minimizer ?

In general no (as in Riemannian geom)



S^2

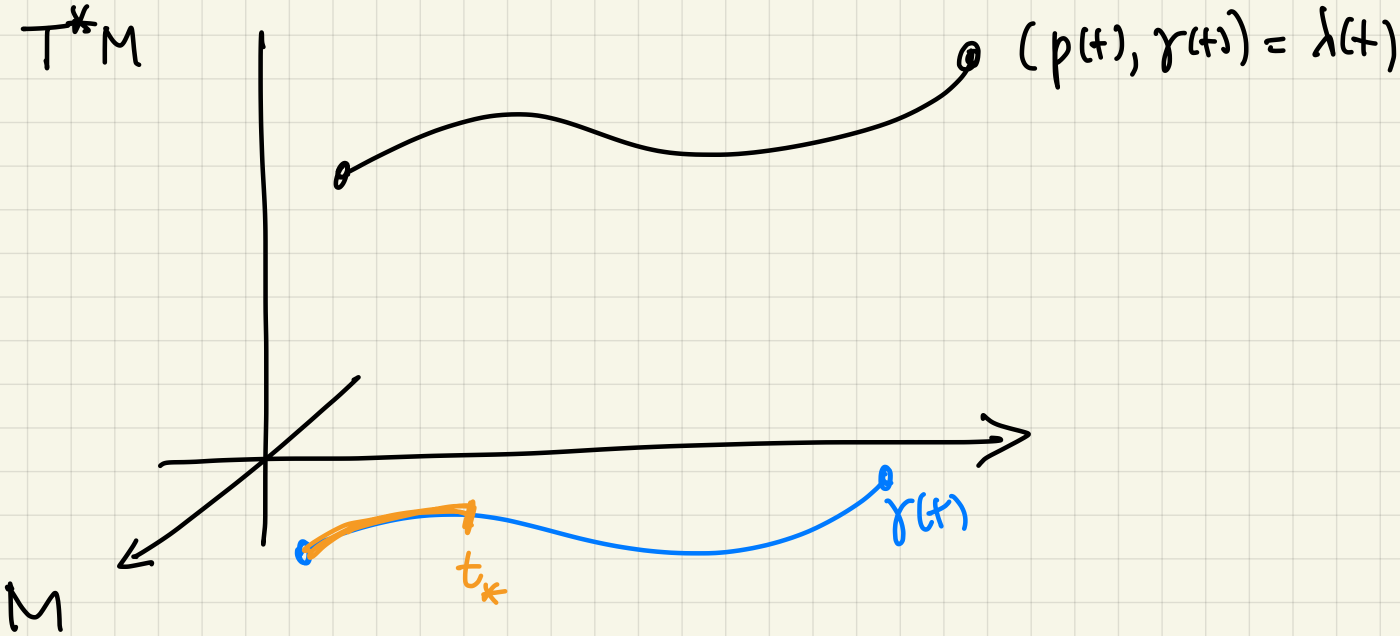
In Riem. geom. short arcs of geodesics (those satisfying (N)) are length-min.

Theorem let $\lambda(t) = (\gamma(t), p(t))$ for $t \in [0,T]$ be a solution of the Hamilton equation

$$\dot{\lambda}(t) = \vec{H}(\lambda(t)) \qquad H = \frac{1}{2}\sum_{i=1}^{k} h_{X_i}^2$$

sub-Riemannian hamiltonian

Then there exists $t_* > 0$ such that $\gamma|_{[0,t_*]}$ is a length minimizer among all horizontal curves with same endpoints.



$$\boxed{\text{Short arcs of normal extremals are length-min}}$$

satisfy the nec. cond.

Determining

$t_c = \sup\{t_* > 0 \mid \gamma|_{[0,t_*]} \text{ length-min}\}$ very difficult problem

CUT LOCUS.

HEISENBERG GROUP

We consider $\mathbb{R}^3$ with the SR structure defined.

$$X = \partial_x - \frac{y}{2}\partial_z \qquad Y = \partial_y + \frac{x}{2}\partial_z$$

$D = \text{span}\{X, Y\}$ and g def st X, Y o.n frame.

We have $[X,Y] = \partial_z$ hence $D + [D,D] = TM$ everywhere

$$\dim \text{span}\{X, Y, [X,Y]\}\Big|_g = 3 \qquad \forall g = (x,y,z)$$

→ No abnormal length min joining distinct pts.
$\Rightarrow$ all length min comes from the Ham. eq.
$\dot\lambda = \vec{H}(\lambda)$.

Lie group structure Introduce the group law

$$(x,y,z)\cdot(x',y',z') = \left(x+x',\ y+y',\ z+z' + \frac{1}{2}(xy'-x'y)\right)$$

$L_g(g') = g \cdot g'$ $\qquad (L_g)_*$ is the jacobian of this wrt (x',y',z') comp at $(0,0,0)$

this gives to $\mathbb{R}^3$ a Lie group structure

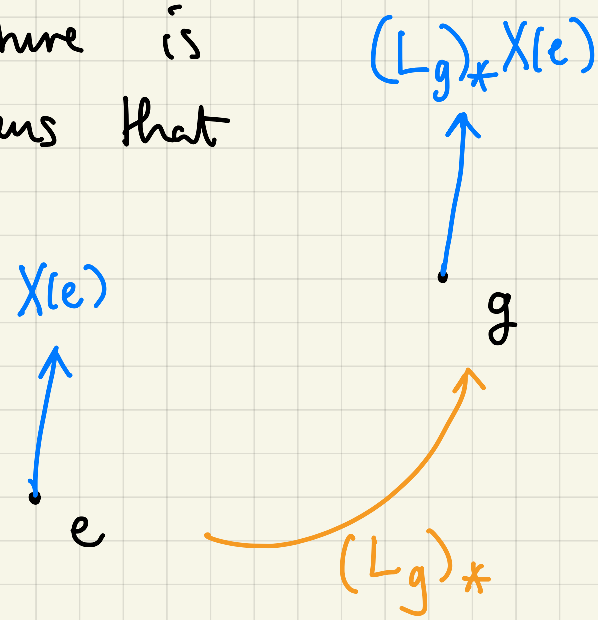
Notice that the identity of the group is $e = (0,0,0)$
and $(x,y,z)^{-1} = (-x,-y,-z)$

Denote the left translation $L_g(g') = g\cdot g'$

The sub-Riemannian structure is <u>left-invariant</u>. It means that

$$\boxed{X(g) = (L_g)_* X(e)}$$

$$Y(g) = (L_g)_* Y(e)$$



Indeed one can check that

$$(L_g)_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 1 \end{pmatrix}$$

$$\rightsquigarrow \quad X(g) = (L_g)_* X(e) \leftarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So we have that D and g are left invariant.

→ left translations of horizontal curves are horizontal

→ the length is invariant by left translations

$$l_{SR}(\gamma) = l_{SR}(L_g(\gamma))$$

→ the sub-Riemannian distance is left invariant

$$d_{SR}(L_h g, L_h g') = d_{SR}(g, g')$$

$$d_{SR}(g, g') = d_{SR}(0, g^{-1} g')$$

From this identity we also get

$$B_{SR}(g,r) = L_g(B_{SR}(0,r))$$

Notice that left translations are isometries

Corollary the SR structure $(\mathbb{H}, d_{SR})$ is complete.

indeed $\exists\, \bar{\varepsilon}>0$: $\overline{B_{SR}(g,\varepsilon)}$ is compact $\forall g \in \mathbb{H}$
$\forall\, \varepsilon \leq \varepsilon_0$.

completeness $\Rightarrow$ existence of length minimizers
$D+[D,D]=TM \Rightarrow$ all length min are normal.

Due to left inv we are reduced to find length min starting from the origin $(0,0,0)$.

Recall that horizontal curves satisfy

$$\dot{\gamma}(t) = u_1(t)\, X(\gamma(t)) + u_2(t)\, Y(\gamma(t))$$

$$\Updownarrow$$

$\pi : \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{z} = \frac{1}{2}(x u_2 - y u_1) \end{cases}$$

eq. hor. curves.

$$\ell_{SR}(\gamma) = \int_0^T u_1^2 + u_2^2\, dt = \ell_{Eu}(\pi(\gamma))$$

Compute normal extremals

the sub-Riemannian Hamiltonian

$$H = \frac{1}{2}(h_1^2 + h_2^2) \quad \text{where} \quad \begin{matrix} g = (x, y, z) \\ p = (p_x, p_y, p_z) \end{matrix}$$

$$h_1 = h_X(p, g) = p \cdot X(g) = p_x - \frac{y}{2} p_z$$

$$h_2 = h_Y(p, g) = p \cdot Y(g) = p_y + \frac{x}{2} p_z$$

$$H = \frac{1}{2}\left(\left(p_x - \frac{y}{2} p_z\right)^2 + \left(p_y + \frac{x}{2} p_z\right)^2 \right) \quad \begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}$$

It is a good idea to write down the equations in coord (x, y, z, h_1, h_2, h_0)

$$h_0 = h_{[X,Y]}(p, g) = p \cdot [X, Y](g) = p_z$$

the Hamilton equation is

$$\begin{cases} \dot{x} = h_1 \\ \dot{y} = h_2 \\ \dot{z} = \frac{1}{2}(x h_2 - y h_1) \end{cases} \qquad \begin{cases} \dot{h}_1 = -h_0 h_2 \\ \dot{h}_2 = h_0 h_1 \\ \dot{h}_0 = 0 \end{cases}$$

condition (N) $\quad h_1(\lambda(t)) = u_1(t)$

$\quad h_2(\lambda(t)) = u_2(t).$

$$\ddot{h}_1 = -h_0^2 h_1$$
$$\ddot{h}_2 = -h_0^2 h_2$$

Remember that H is constant along solutions so that we can fix a level set $\{H = \frac{1}{2}\} \iff \{h_1^2 + h_2^2 = 1\}$ corresponds to curves length-parametrized.

Solve the "h" part with

$$(h_1(0), h_2(0), h_0(0)) = (\cos\theta_0, \sin\theta_0, h_0)$$

$$\begin{cases} h_1(t) = \cos(\theta_0 + h_0 t) \\ h_2(t) = \sin(\theta_0 + h_0 t) \\ h_0(t) = h_0 \end{cases}$$

two controls.

initial condit. $(0,0,0)$.

One can solve the "x,y,z" part

$$(*) \quad \left.\begin{cases} x(t) = \dfrac{1}{h_0}\left(\sin(\theta_0 + h_0 t) - \sin(\theta_0)\right) \\ y(t) = -\dfrac{1}{h_0}\left(\cos(\theta_0 + h_0 t) - \cos(\theta_0)\right) \\ z(t) = \dfrac{1}{2h_0^2}\left(h_0 t - \sin(h_0 t)\right) \end{cases}\right] \quad h_0 \neq 0$$

$$\left.\begin{cases} x(t) = \cos(\theta_0)\, t \\ y(t) = \sin(\theta_0)\, t \\ z(t) = 0 \end{cases}\right] \quad h_0 = 0$$

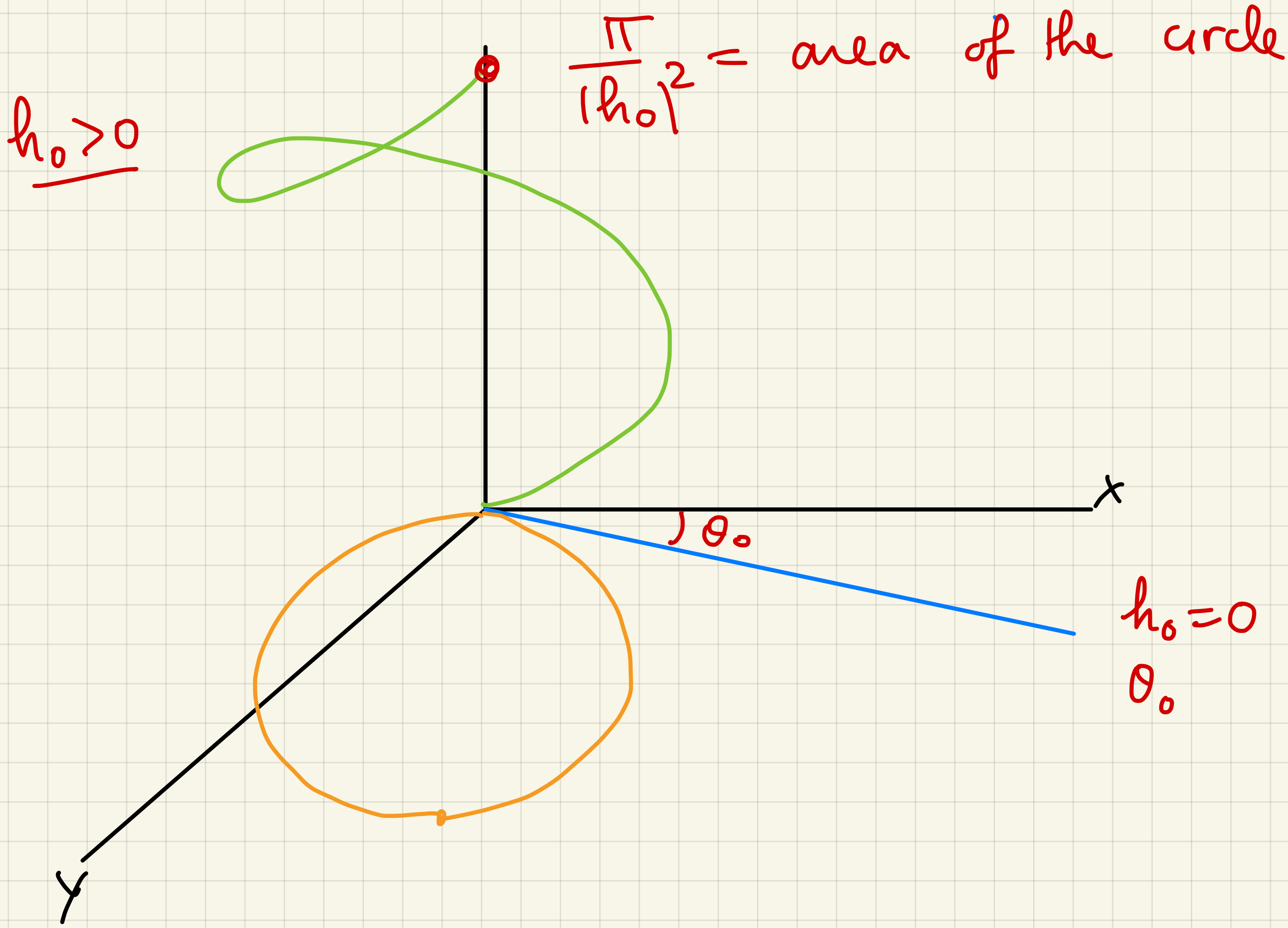
straight line in $\{z=0\}$

<u>RK</u> For all such curves $\begin{cases} \dot{x}(0) = \cos\theta_0 \\ \dot{y}(0) = \sin\theta_0 \\ \dot{z}(0) = 0 \end{cases}$ <u>in Δ</u>

Exercise: Show that the (x,y) part of the $(*)$ equations describe a circle

with center $C = \frac{1}{h_0}(-\sin\theta_0, \cos\theta_0)$

radius $\rho = \frac{1}{|h_0|}$



One can show that straigth lines are length min for all times

Curves that have $h_0 \neq 0$ are length min up to $t_* = \frac{2\pi}{|h_0|}$

After time $T = \frac{2\pi}{h_0}$ we reach the point

$$\left(0, 0, \frac{\pi}{h_0^2}\right)$$

these observation gives the following

$$\boxed{d_{SR}^2\left((0,0,0),(x,y,0)\right) = x^2 + y^2}$$

horiz.
X, Y

$$d_{SR}^2\left((0,0,0),\left(0,0,\frac{\pi}{h_0^2}\right)\right) = \frac{4\pi^2}{h_0^2}$$

$$\boxed{d_{SR}^2\left((0,0,0),(0,0,z)\right) = 4\pi|z| \qquad z \in \mathbb{R}}$$

vertical.
$[X, Y]$.

Ball - Box estimate

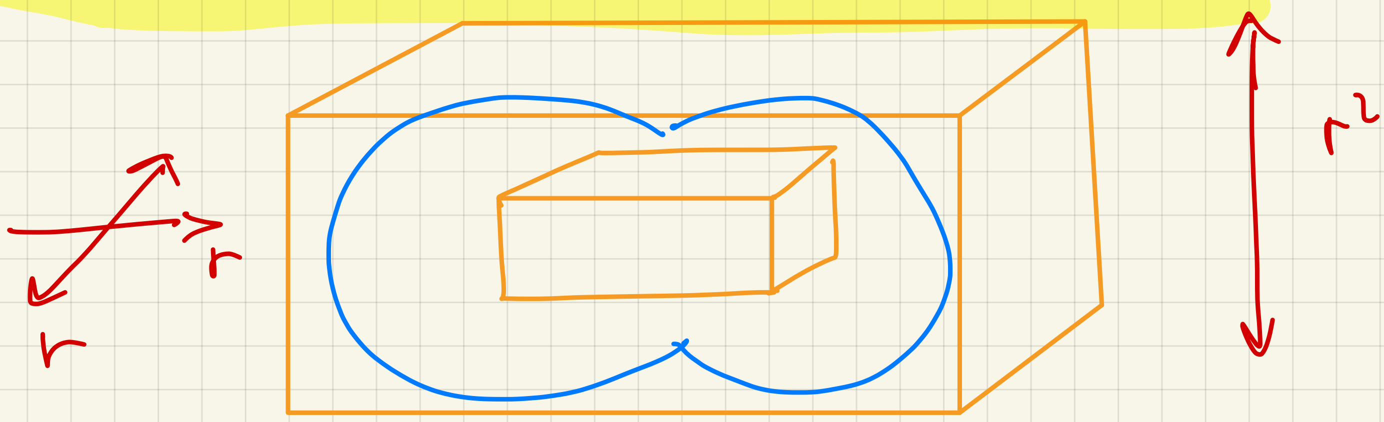
$$Box(r) = [-r, r] \times [-r, r] \times [-r^2, r^2]$$

then we have

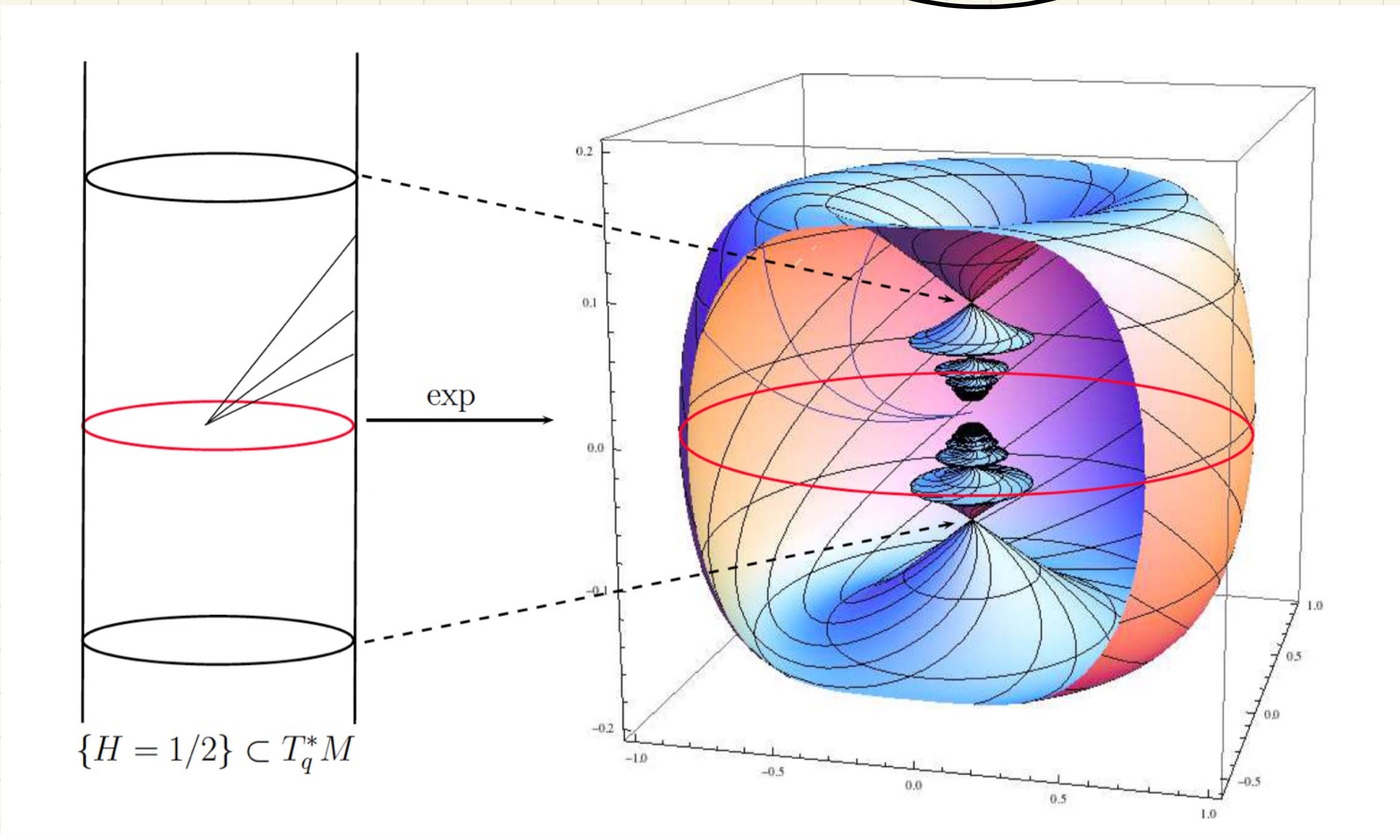
true if $D + [D, D] = TM$ in dim 3.

$$c_1 \, Box(r) \subseteq B_{SR}(x, r) \subseteq c_2 \, Box(r)$$

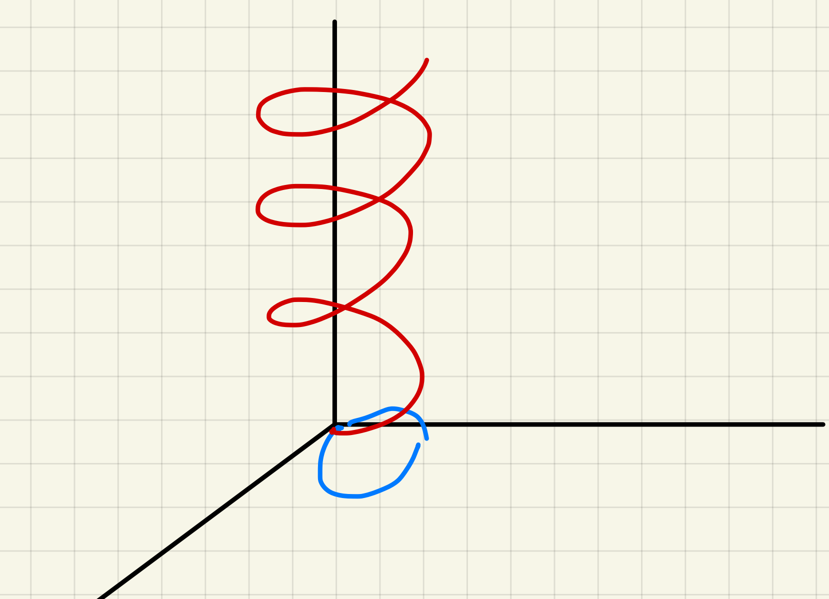
for $r > 0$ small enough.



$\boxed{T=1}$

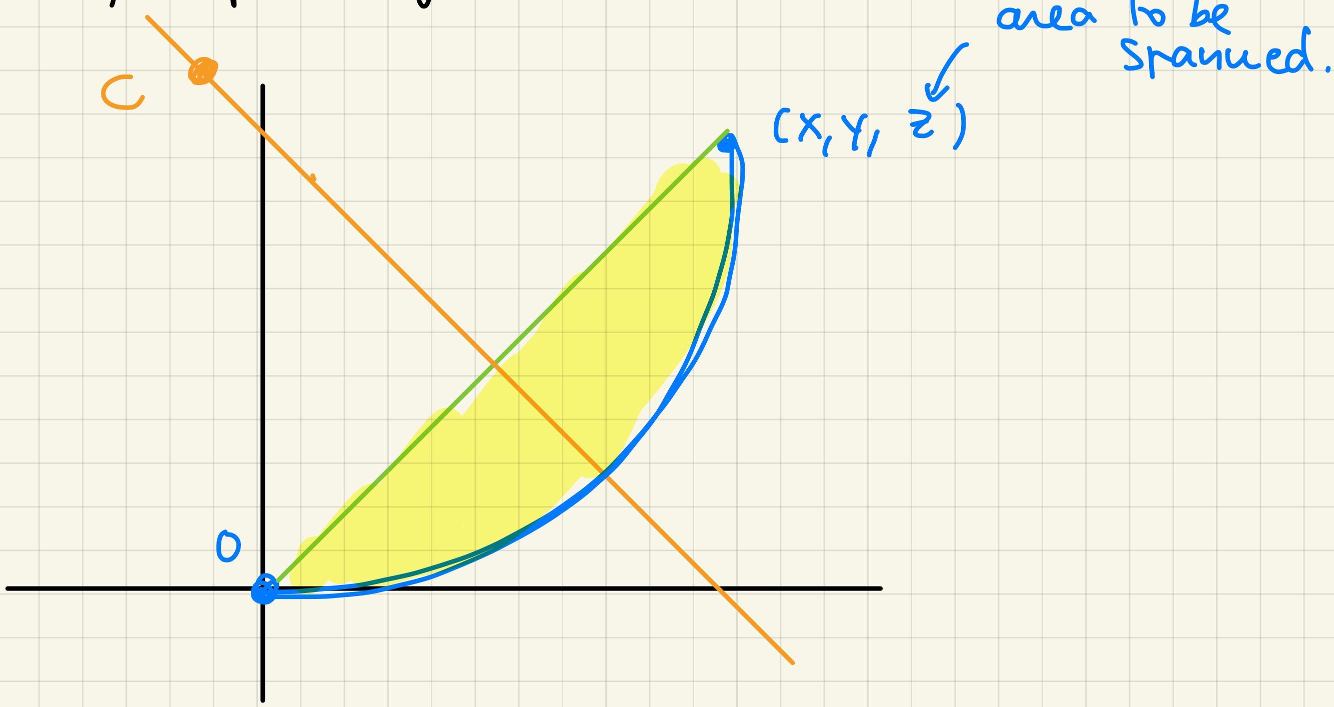


$\{H = \frac{1}{2}\} \simeq \{ h_1^2 + h_2^2 = 1 \}$ in the coord (h_1, h_2, h_0) , $h_0 \in \mathbb{R}$



Geometrically: how to find length minimizers

If I want to join $(0,0,0)$ with (x,y,z) this means join $(0,0)$ in $\mathbb{R}^2$ with (x,y) by spanning an area $= z$



Metric dimension of the Heisenberg group

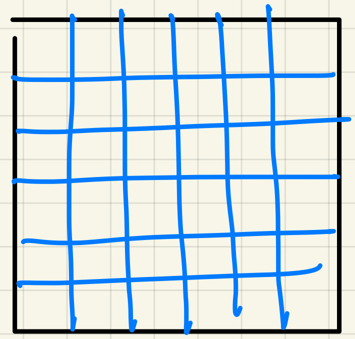
Given a metric space one can define a notion of dimension (Hausdorff dim)

Minkovski dimension. of (X,d)

fix $\varepsilon>0$ cover X by $N(\varepsilon)$ balls of radius $\varepsilon>0$

If $\boxed{N(\varepsilon) \sim \varepsilon^{-D}}$ $\Rightarrow$ $D = \dim X$

$$D = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$



$N(\varepsilon) = \varepsilon^{-2}$ squares of size ε.

<u>Prop</u> As a metric space the Heisenberg group has dimension 4.

$$B(x,r) \cong [-r,r] \times [-r,r] \times [-r^2, r^2]$$