TOPICS IN SR GEOMETRY : PART 2

(by A. Agrachev)

notes by
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M smooth manifold (or $\mathbb{R}^n$) connected

Δ vector distribution

smooth subbundle of the tangent bundle

$$\Delta = \bigcup_{q \in M} \Delta_q \ , \quad \Delta_q \subset T_q M$$

$$\Delta_q = \text{span}\{ X_1(q), \dots, X_k(q) \} \cdot \quad \dim \Delta_q = k$$

we assume also in this 2nd part that the vector fields X_i $i = 1 \dots k$ are lin ind everywhere

notation $\text{Vec}(M)$ = set of C^∞ vector fields on M.

$$\boxed{[X,Y] = \frac{d}{dt}\Big|_{t=0} e^{-tX}_* Y}$$

Lie bracket between X, Y.

if we treat vector fields as 1st order diff operators

$$[X,Y] = XY - YX$$

Recall that vector fields act on functions

$$X \in \text{Vec}(M) \ , \ a \in C^\infty(M) \qquad (Xa)(q) = \langle d_q a, X(q) \rangle .$$

$$= \frac{d}{dt}\Big|_{t=0} a(e^{tX}(q)) .$$

We define then

$$\mathrm{Lie}_q \Delta = \mathrm{span}\{ [X_{i_1}, \ldots, [X_{i_{j-1}}, X_{i_j}]\ldots] \mid j \in \mathbb{N} \}\big|_q$$

def We say that Δ is bracket-generating if

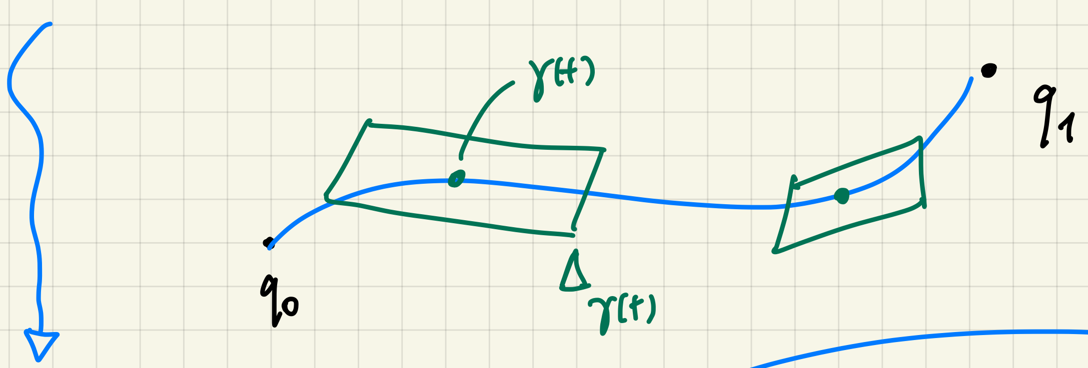
$$\mathrm{Lie}_q \Delta = T_q M \quad \forall q \in M$$

(ie. $\dim \mathrm{Lie}_q \Delta = n \quad \forall q \in M$)

We recall the Rashevski-Chow theorem

Theorem (Rashevski-Chow) If Δ is bracket-generating (and M is connected) then $\forall\, q_0, q_1 \in M$ there exists $\gamma : [0,T] \longrightarrow M$ horizontal such that $\gamma(0) = q_0$, $\gamma(T) = q_1$.

Remark here horizontal means Lipschitz as in 1st part but the class can be improved for instance to piecewise smooth.



We will use different class of curves later. [curves whose derivatives are dense in L^2]

GOAL : study abnormal extremals

↳ they do not depend on the metric.] only distrib.

then if we have a metric we can ask if they are length-minimizers.

Recall that horizontal curves

$$\boxed{\dot{\gamma}(t) = \sum_{i=1}^{k} u_i(t)\, X_i(\gamma(t))} \qquad u=(u_1, \dots, u_k)$$

(a) in the first part we choose $u \in L^\infty([0,T], \mathbb{R}^k)$

(b) here is good to take $u \in L^2([0,T], \mathbb{R}^k)$

Case (a) ⤳ the solution is lipschitz

(b) ⤳ the solution is H^1 (Sobolev space)

We introduce the set of curves

in the sense of control.

$$\Omega := \{ \gamma : [0,1] \to M \mid \dot{\gamma}(t) \in \Delta_{\gamma(t)},\ \dot{\gamma} \in L^2 \}$$

this is an Hilbert manifold.

(actually the important point here is the topology in the space Ω than the space itself)

This is a convenient setting to study our pb.

RK Once the control $u \in L^2([0,1], \mathbb{R}^k)$ then we can solve the ODE

$$\dot{q} = \sum_{i=1}^{k} u_i(t) X_i(q)$$

need completeness to guarantee the solution on $[0,1]$ cf discussion (**)

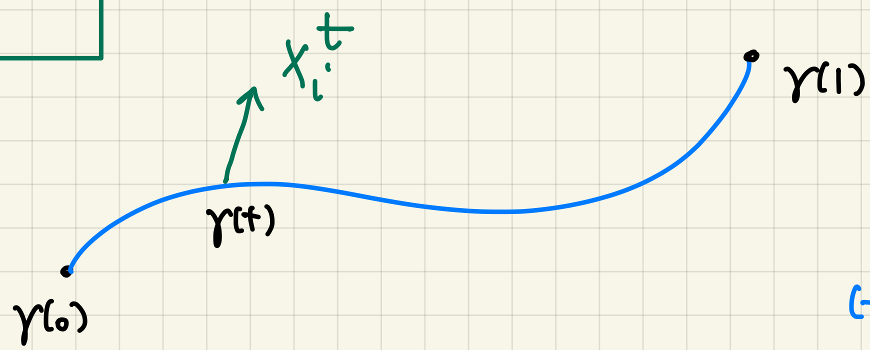
so that we can parametrize all horizontal curves by pairs

- $u \in L^2$ control.
- $q_0 \in M$ base point

If you take local coordinates then curves are parametrized as $\underset{\substack{\uparrow \\ \text{initial pt}}}{\mathbb{R}^n} \times \underset{\substack{\uparrow \\ \text{control.}}}{L^2([0,1], \mathbb{R}^k)}$

$\boxed{\text{structure of Hilbert manifold of } \Omega}$

Take $\bar{\gamma} \in \Omega$



(**).

we can define a moving basis (moving frame) $X_1^t, \dots, X_k^t$

$$\begin{cases} \dot{\bar{\gamma}} = \sum_{i=1}^{k} \bar{u}_i(t) X_i^t(\bar{\gamma}(t)) \\ \bar{\gamma}(0) = x_0 \end{cases} \quad (1)$$

$$\begin{cases} \dot{q} = \sum_{i=1}^{k} u_i(t) X_i^t(q) \\ q(0) = q_0 \end{cases} \quad (2)$$

you parametrize
$|q_0 - x_0|$ small
$\|u - \bar{u}\|_{L^2}$ small.

if the solution of (1) is def on $[0,1]$ then for

·) $|q_0 - x_0|$ small enough

·) $\|u - \bar{u}\|_{L^2}$ small enough

the solution of (2) is also def on $[0,1]$.

this is the "concrete" Hilbert manifold structure of the space Ω, locally near a curve $\bar{\gamma} \in \Omega$ it is parametrized by $\mathbb{R}^n \times L^2([0,T), \mathbb{R}^k)$

A neigh $\mathcal{O}_{\bar{\gamma}} \subset \Omega \Rightarrow \mathcal{O}_{\gamma} \simeq \mathcal{O}_{q_0} \times L^2([0,1], \mathbb{R}^k)$

$$\simeq \mathcal{O}_{\bar{\gamma}(0)} \times L^2([0,1], \mathbb{R}^k)$$

[we can center our coordinates at every point on the curve $\bar{\gamma}$]

$$\simeq \mathcal{O}'_{\bar{\gamma}(t)} \times L^2([0,1], \mathbb{R}^k)$$

you have to fix the control and $\bar{\gamma}$ at some point not necessary zero. ← kind of change of coordinates.

Now we can introduce the evaluation map (at time $t \in [0,1]$).

def $\quad Ev^t : \Omega \longrightarrow M$

$$\gamma \longmapsto \gamma(t)$$

this map is smooth (follows from ODE properties)

Moreover Ev^t is a submersion ($D_{\bar{\gamma}} Ev^t$ is su for every $\bar{\gamma}$)

↑ differential is surjective at every point.

Indeed if you write the map in coordinates (essentially because we can change initial conditions arbitrarily).

(idea: in coordinates $Ev^t : \mathbb{R}^n \times L^2 \longrightarrow \mathbb{R}^n$
and $Ev^t|_{\mathbb{R}^n \times \{0\}} = id$.)

We define, given $q_0 \in M$

$$\Omega_{q_0} = (Ev^0)^{-1}(q_0)$$ ← curves with starting pt q_0

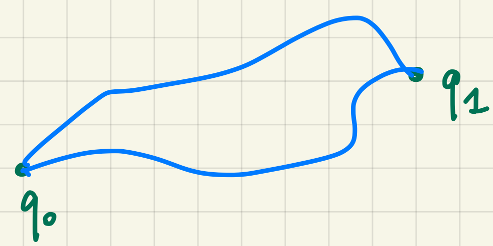
$$\Omega^{q_1} = (Ev^1)^{-1}(q_1)$$ ← curves with end point q_1

these are Hilbert submanifold of codimension n parametrized by controls

$\Omega_{q_0} \cap \Omega^{q_1}$ = generalized loop space
horizontal curves between q_0 & q_1

non empty by Rashevski - Chow.

Is it a manifold ??



To understand this space we have to study the following boundary map.

$$\boxed{\begin{aligned} \partial : \Omega &\longrightarrow M \times M \\ \gamma &\longmapsto (\gamma(0), \gamma(1)) \end{aligned}}$$

smooth map.

$$\Omega_{q_0} \cap \Omega^{q_1} = \partial^{-1}(q_0, q_1)$$

rk: $\partial = (Ev^0, Ev^1)$.

def A curve $\gamma \in \Omega$ is singular (or abnormal) if it is a critical point of the boundary map $\partial = (Ev^0, Ev^1)$.

this is an equivalent definition, it stresses that these curves do not depend on the metric.

It is convenient to replace these maps by what is called end-point maps.

def the end-point map is

$$E^t_{q_0} : \Omega_{q_0} \longrightarrow M$$

$$E^t_{q_0} := Ev^t \big|_{\Omega_{q_0}}$$

the restriction of the evaluation at time t for curves starting at q_0.

Rk: it is important here that Ω_{q_0} is a smooth manifold, which follows from Ev being a submersion.

notice

$$\boxed{E^t_{q_0} : u(\cdot) \longmapsto q(t) \quad \text{where} \quad \begin{cases} \dot q(t) = \sum_{i=1}^{k} u_i(t) X_i(q) \\ q(0) = q_0 \end{cases}}$$

end point map in coordinates

Now we consider the differential of the end-point map.

We have

$$\boxed{D_{\bar{u}} E^{t}_{q_0}[w] = \int_0^t \sum_{i=1}^k w_i(\tau)\left[(\Phi_{\tau,t})_* X_i\right](\bar{\gamma}(\tau))\,d\tau}$$

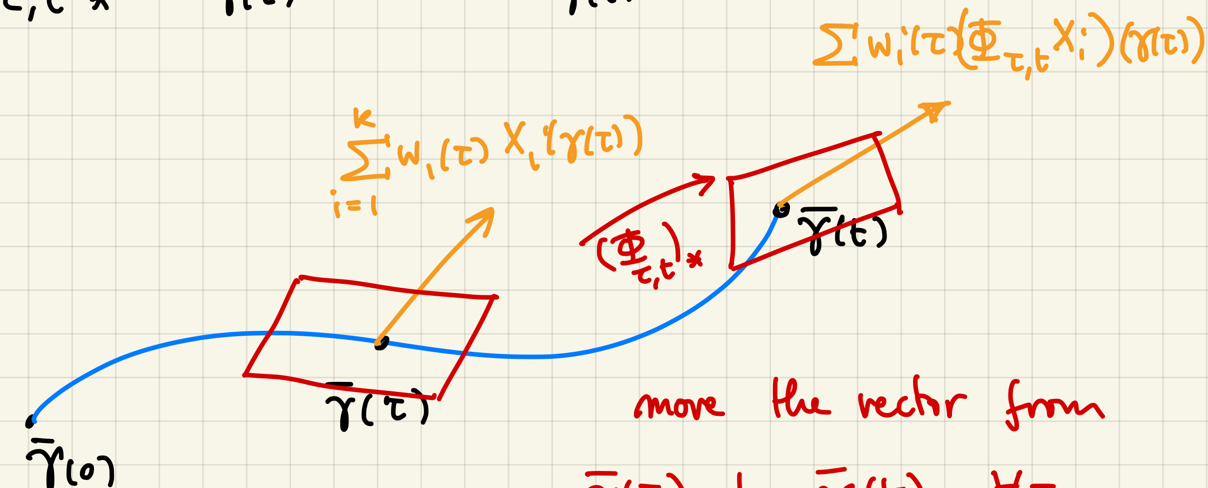
$w \in L^2([0,1],\mathbb{R}^k)$ ← tangent space

here $\Phi_{\tau,t} : q(\tau) \longrightarrow q(t)$ flow of $\bar{u}$

(and recall that $\bar{\gamma}(t) = \Phi_{0,t}(q_0)$.)

and also $\Phi_{\tau,t}(\bar{\gamma}(\tau)) = \bar{\gamma}(t)$

$$(\Phi_{\tau,t})_* : T_{\bar{\gamma}(\tau)}M \longrightarrow T_{\bar{\gamma}(t)}M .$$



move the vector from $\bar{\gamma}(\tau)$ to $\bar{\gamma}(t)$ $\forall \tau$.
and then integrate.

One might convince of the formula by thinking to

$$E^{t}_{q_0}(\bar{u} + \varepsilon w_i) \quad \text{where} \quad w_i = \begin{cases} 0 & s \notin [\tau, \tau+\nu] \\ e_i & s \in [\tau, \tau+\nu] \end{cases}$$

$$= \bar{\gamma}(t) + \nu\varepsilon X_i(\bar{\gamma}(t)) + o(\varepsilon^2)$$

double check here!

const. vector.

We can charachterize singular trajectories

$\bar{u}$ is critical point for $E^1_{q_0}$
($\bar{\gamma}$ is a singular curve)

iff $D_{\bar{u}} E^1_{q_0}$ is not surjective

iff $\exists\, \lambda_1 \in T^*_{q_0} M \setminus \{0\}$ such that

$$\lambda_1 \circ D_{\bar{u}} E^1_{q_0} = 0 \qquad \left(\text{ie} \quad \lambda_1 \cdot D_{\bar{u}} E^1_{q_0} [w] = 0 \quad \forall w \right)$$

this means

recall $(\Phi_* X)(\Phi(q)) = \Phi_* (X(q))$.

$$\int_0^1 \sum_{i=1}^k w_i(t) < \lambda_1, \Phi_{1,t\,*} [X_i(\gamma(t))] > dt = 0 \qquad \forall w.$$

$$\underset{\text{all } w}{\Longleftrightarrow} \qquad < \lambda_1, (\Phi_{1,t})_* [X_i(\gamma(t))] > dt = 0 \qquad 0 \leq t \leq 1$$

$$\Longleftrightarrow \qquad < \underbrace{(\Phi_{1,t})^* \lambda_1}_{!! \; \lambda_t}, \; X_i(\bar{\gamma}(t)) > = 0 \qquad 0 \leq t \leq 1.$$

this is the abnormal condition of Part 1 of the course.

Rk $\lambda_t \in T^*_{\bar{\gamma}(t)} M$ is a curve of covectors!

One can write formulas directly in T^*M.

$$T^*M = \{(p,q)\} \quad \text{if} \quad M = \{(q)\}$$

$$\lambda \in T^*M \qquad \lambda = \sum_{i=1}^{m} p_i \, dq_i$$

$$h_i(\lambda) = \langle \lambda, X_i \rangle$$

linear hamiltonian associated with a vector field.

(ie. $h_i(p,q) = \langle p, X_i(q) \rangle$)

If we write our $\lambda_t = (p(t), q(t)) \in T^*M$ and we can say that λ_t satisfy the Hamiltonian system

$$(*) \quad \begin{cases} \dot{p}(t) = -\sum_{i=1}^{k} u_i \dfrac{\partial h_i}{\partial q} \\ \dot{q}(t) = \sum_{i=1}^{k} u_i \dfrac{\partial h_i}{\partial p} \end{cases}$$

← cf 1st part of the course.

Hence $\bar{u}$ is a singular control

($\bar{\gamma}$ is a singular curve)

iff $\exists\, p(t) \neq 0$ such that $(*)$ is satisfied.

and $h_i(p(t), q(t)) = 0 \quad \forall i = 1 \dots k \quad \forall t \in [0,1]$.