14/04.

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LECTURE no 2 (A. Agrachev)

Recall that we showed that singular (abnormal) extremals are critical points of a map which is defined totally intrinsic through the boundary / evaluation / endpoint map.

Recall that given the curve $\gamma(t)$ we produce a lift $p(t)$ which is defined through the flow $\Phi_{\tau,t}$ which is related to the choice of the frame.

Let us show that indeed abnormal / singular depends only on the distribution.

We have that

$$\boxed{D_{\bar u} E^1_{q_0} = \left.\frac{\partial}{\partial u}\right|_{u=\bar u} Ev^1}$$

at q_0

(*) Recall that

$$E^1_{q_0} = Ev^1\Big|_{\Omega_{q_0}}$$

endpoint   evaluation map.

In "coordinates"

$$Ev^1 : (q_0, u(\cdot)) \longrightarrow \gamma_u(q_0, 1)$$

$$E^1_{q_0} = Ev^1(q_0, \cdot) : u(\cdot) \longmapsto \gamma_u(q_0, 1)$$

In particular $\dfrac{\partial}{\partial q_0} Ev^1 = (\Phi_{0,1})_* .$

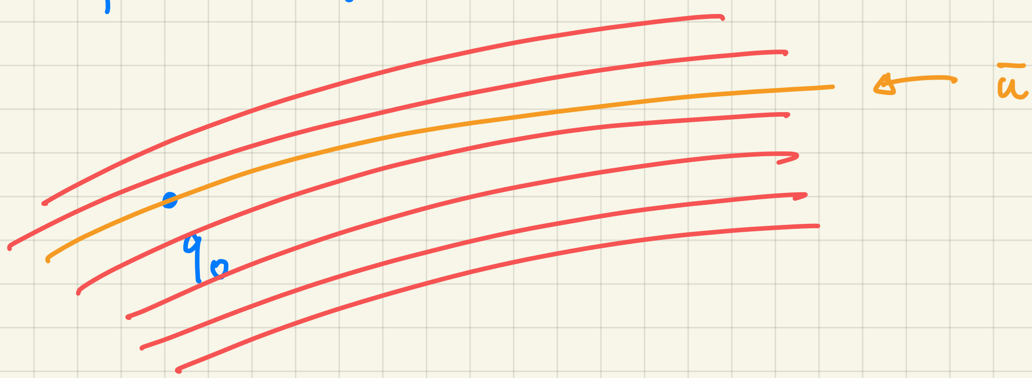
at $\bar u$

associated to the good $\bar u$.

(notice that $\frac{\partial}{\partial q_0} Ev^1 : T_{\bar{\gamma}(0)}M \to T_{\bar{\gamma}(1)}M$)

notice that $\frac{\partial}{\partial q_0} Ev^1 = id : T_{\underset{q_0}{\underset{\shortparallel}{\bar{\gamma}(0)}}}M \to T_{\underset{q_0}{\underset{\shortparallel}{\bar{\gamma}(0)}}}M$

($\Phi_{0,0} = id$ by construction.)



So now recall that $\bar{\gamma}$ singular trajectory if and only if $\exists\, \lambda_1$ such that

$$\lambda_1 \frac{\partial}{\partial u} Ev^1 = 0$$

the partial derivatives depends on coordinates. We want to write in terms of complete diff.

recall

$$\lambda_t = \Phi_{t,1}^{*} \lambda_1$$

$$= \lambda_1 \circ \Phi_{t,1\,*}$$

as composition of linear maps.

thanks to this identity.

$$(*) \quad \boxed{\lambda_1 \, D_{\bar{\gamma}} Ev^1 = \lambda_0 \, D_{\bar{\gamma}} Ev^0}$$

Now this equation (*) is coordinate independent!

Similarly one can prove the following claim

Exercise The curve λ_t, $0 \le t \le 1$, is a singular (abnormal) extremal corresponding to $\bar{\gamma}$ if and only if

$$(**) \quad \boxed{\lambda_t \, D_{\bar{\gamma}} Ev^t = \lambda_s \, D_{\bar{\gamma}} Ev^s} \quad \forall \; 0 \le t, s \le 1.$$

notice that (*) is (**) specified for $\begin{matrix} t=0 \\ s=1 \end{matrix}$.
(for $s=t$ is a tautology)

Now the identity (**) is coordinate independent

EXAMPLES

1 (Constant curves) for any metric these have length zero $\Rightarrow$ minimizers.

A constant curve $\gamma(t) = q_0$ $\forall t \in [0,1]$.
is singular if and only if $\Delta_{q_0} \neq T_{q_0}M$.

Indeed γ is associated with control $u=0$.
Hence we have (check as an exercise!)

- $\Phi_{\tau,t} = id.$
- $D_0 E^1_{q_0}[w] = \int_0^1 \sum_{i=1}^{k} w_i(t) X_i(q_0)\, dt$

notice that $\operatorname{im} D_0 E^1_{q_0} = \Delta_{q_0}$.
so that λ exists $\Leftrightarrow$ $\Delta_{q_0} \neq T_{q_0}M$.

Notice that when $\Delta_{q_0} = T_{q_0}M$ (Riemann. geom) then <u>no</u> critical point.

Genuine SR structure $\boxed{k < n} \Rightarrow$ constant loop is singular

BEFORE GOING TO NEXT EXAMPLE NEED SOME NOTIONS : POISSON BRACKETS !

coordinates

Let $g, h : T^*M \to \mathbb{R}$ $T^*M = \{(p,q)\}$

$q \in M, \quad p \in T_q^*M$.

we introduce the

Poisson Bracket $\{h, g\} : T^*M \to \mathbb{R}$

$$\{h,g\}(p,q) = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}$$

The Poisson Bracket extends the Lie Bracket in the following sense:

$$\boxed{\begin{array}{l} \text{if} \quad h(p,q) = \langle p, X(q)\rangle \qquad g(p,q) = \langle p, Y(q)\rangle \\ \text{then} \qquad \{h,g\}(p,q) = \langle p, [X,Y](q)\rangle \end{array}}$$

Morally we can see $Vec(M) \hookrightarrow C^\infty(T^*M)$

include as linear functions of p.

"the Poisson bracket for such a f. is the Lie bracket".

Given an Hamiltonian $h : T^*M \to \mathbb{R}$ we define the Hamiltonian vector field

$$\boxed{\vec{h} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i}}$$

then we have $\boxed{\vec{h}\, g = \{h, g\}}$ ← exercise: check this!

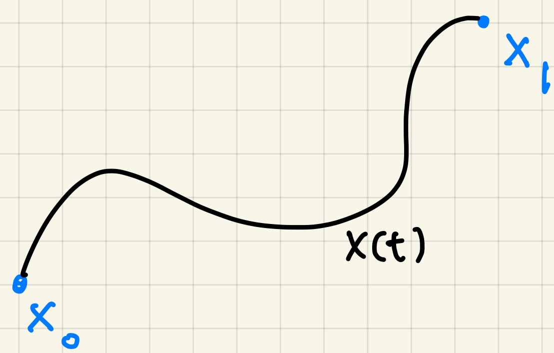
differentiate the function g along the vector field $\vec{h}$

Next we go to the next class of examples

2 Isoperimetric problems on $\mathbb{R}^2$

Let $\mathbb{R}^2 = \{(x_1, x_2)\}$ and we consider curves That solves the problem.

Given $a : \mathbb{R}^n \to \mathbb{R}^n$
and a constant $c \in \mathbb{R}$
we consider the problem



$$\inf \left\{ \ell(x(\cdot)) \;\middle|\; \int_0^1 \underbrace{< a(x(t)), \dot{x}(t) >}_{\sum_{i=1}^{2} a_i(x(t))\, \dot{x}_i(t)} dt = c \right\}$$

interpretation in a 3D space.

Let us consider the curve $(x(t), y(t)) := \gamma(t)$

where

$$y(t) = \int_0^t \langle a(x(s)), \dot{x}(s) \rangle \, ds$$

the constant we fixed.

then we have $y(0) = 0$ and $y(1) = c$

Let $M = \mathbb{R}^2 \times \mathbb{R}$ admissible curves $\gamma(t)$ are solutions to the diff. eq.

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{y} = u_1 a_1(x) + u_2 a_2(x) \end{cases} \iff \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ a_1(x) \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ a_2(x) \end{pmatrix}$$

so it is like $X_1 = \begin{pmatrix} 1 \\ 0 \\ a_1(x) \end{pmatrix}$ $\quad X_2 = \begin{pmatrix} 0 \\ 1 \\ a_2(x) \end{pmatrix}$

Write $q = (x, y)$ then admissible curves are solutions to $\boxed{\dot{q} = u_1 X_1(q) + u_2 X_2(q)}$

$$\boxed{\text{Horizontal curves that connect } (x_0, y_0) \text{ with } (x_1, y_1)}$$

$\updownarrow$ 1-1 correspondence

$$\boxed{\text{Plane curves } x(t) \text{ in } \mathbb{R}^2 \text{ such that } \int_0^1 \langle a(x(t)), \dot{x}(t) \rangle \, dt = y_1 - y_0}$$

RK: invariance with respect to third variable.

Exercise, Show that the Heisenberg group case corresponds to a sp. case of this. For which a?

Let us check the bracket-generating condition.

$$[X_1, X_2] = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \end{pmatrix} =: \begin{pmatrix} 0 \\ 0 \\ b(x) \end{pmatrix}$$

function of x (not of y) which we call b.

All the problem is characterized by the function b. We can write

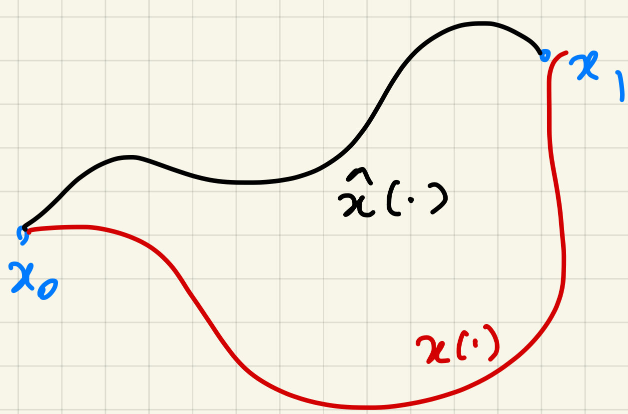
$$\int_0^1 \langle a(x(t)), \dot{x}(t) \rangle \, dt = \int_{x(\cdot)} A$$

integral of a 1-form over a curve.

where $A = a_1(x)\, dx_1 + a_2(x)\, dx_2$

RK! If we have two curves $x(\cdot)$, $\hat{x}(\cdot)$ connects two fixed points x_0, x_1 and such that

$$\int_{x(\cdot)} A = \int_{\hat{x}(\cdot)} A$$



Then adding an exact part to A the result does not change.

$$\int_{x(\cdot)} A + d\varphi = \int_{\hat{x}(\cdot)} A + d\varphi$$

the value of the integral change but we have the same minimizers.

Recall that in $\mathbb{R}^2$ exact forms $\Leftrightarrow$ closed forms.

Now if we consider the isoperimetric structures associated to the form A^1 and A^2 are equiv. if $A^1 - A^2$ is an exact form

$\Leftrightarrow$ $A^1 - A^2$ is a closed form

$\Leftrightarrow$ $dA^1 = dA^2$ (here d = exterior diff).

So the problem (up to translation in vert var) depends only on dA.

$$dA = d\left(a_1(x)\, dx_1 + a_2(x)\, dx_2 \right)$$

$$= -\frac{\partial a_1}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial a_2}{\partial x_1} dx_1 \wedge dx_2$$

so $\quad dA = b(x)\, dx_1 \wedge dx_2$

Now we come back to our questions.

<u>Exercise</u> Prove that $\Delta = \operatorname{span}\{X_1, X_2\}$ is bracket-generating distribution if and only if at least one partial derivative of b is not zero everywhere.

$$\frac{\partial^{i_1 \dots i_k}}{\partial x_{i_1} \dots \partial x_{i_k}} b(x) \neq 0 \quad \forall x \in \mathbb{R}^2 \quad i_j \in \{1,2\}.$$

(!) In the Heisenberg group $b(x) = \text{const} \neq 0$. so already 1 bracket is sufficient to span $\mathbb{R}^3$.

Now we compute abnormal curves, we use a very general method.

Recall that abnormals are characterized through the equations for $\lambda(t)$, $0 \leq t \leq 1$.

$$\lambda = (\underbrace{p_{x_1}, p_{x_2}, p_y}_{p}, \underbrace{x_1, x_2, y}_{q}) = (p, q)$$

We have
$$\boxed{\begin{aligned} \dot{\lambda}(t) &= u_1 \vec{h}_1(\lambda(t)) + u_2 \vec{h}_2(\lambda(t)) \\ h_1(\lambda(t)) &= h_2(\lambda(t)) = 0 \end{aligned}}$$
abnormal equations

the second line can be differentiated and we get that $0 = \frac{d}{dt} h_1(\lambda(t)) = \frac{d}{dt} h_2(\lambda(t))$

$$0 = \frac{d}{dt} h_1(\lambda(t)) = \left(u_1 \vec{h}_1 + u_2 \vec{h}_2\right)(h_1)$$

omit the evaluation at $\lambda(t)$

$$= \{u_1 h_1 + u_2 h_2, h_1\}$$

use Poisson brackets. and linearity

$$= u_1 \underbrace{\cancel{\{h_1, h_1\}}}_{=0} + u_2 \{h_2, h_1\}$$

skew-symmetry

linear in p associated to $[X_2, X_1] = b(x)\frac{\partial}{\partial y}$.

$$= -u_2(t) b(x)\, p_y$$

Similarly we have

$$0 = \frac{d}{dt} h_2(\lambda(t)) = \dots = + u_1(t)\, b(x)\, p_y$$

We can conclude

$$\boxed{\begin{cases} u_2(t)\, b(x(t))\, p_y(t) = 0 \\ u_1(t)\, b(x(t))\, p_y(t) = 0 \end{cases}}$$

along the trajectory $\begin{pmatrix} x(t), y(t) \\ p_x(t), p_y(t) \end{pmatrix}$ assoc. to controls $u_1(t)$ $u_2(t)$.

<u>Claim</u> : if $b(x) \neq 0$ everywhere then $u_1(t) = u_2(t) = 0$.
(for all t)

indeed if $p_y = 0$ we immediately from

$$0 = h_1 = p_{x_1} - a\, p_y$$
$$0 = h_2 = p_{x_2} + a\, p_y$$

$\Rightarrow p_y = 0 = p_{x_1} = p_{x_2}$ not possible that $\underline{\underline{p = 0}}$

So we have that abnormal curves necessarily lives in the set $\{b=0\}$

Proposition A curve $(x(t), y(t))$ is abnormal if and only if $b(x(t)) \equiv 0$

pf $\Rightarrow)$ previous discussion

$\Leftarrow)$ if $b(x(t)) = 0$ then take
$p(t) = (a_1(x(t)), a_2(x(t)), -1)$. check!

More comments : interpretation as a charged particle in a magnetic field

Indeed A = magnetic potential
dA = magnetic field. $= b\, dx_1 \wedge dx_2$

Abnormal appears where magnetic field change sign.

Abnormal curve $x(t) \in b^{-1}(0)$.

If $b^{-1}(0)$ is a regular level set then this set is a 1d curve

generic by Sard lemma condit.

But in general it might be difficult set.

MORE SYMPLECTIC GEOMETRY

(and another charact of the abnormal equation)

Let M be smooth manifold.

$\Delta \subset TM$ distribution

$\Delta^\perp \subset T^*M$ subbundle of T^*M

$$\Delta_q^\perp = \{ \lambda \in T_q^*M \mid \langle \lambda, \Delta_q \rangle = 0 \} = \Delta^\perp \cap T_q^*M .$$

$$\Delta^\perp = \bigcup_{q \in M} \Delta_q^\perp \quad \text{dual object to } \Delta$$

Notice that $h_1(\lambda(t)) = \dots = h_k(\lambda(t)) = 0$ for the vector fields $X_1, \dots, X_k$ means exactly that $\lambda(t) \in \Delta^\perp_{\gamma(t)}$ $\quad \gamma(t) = \pi(\lambda(t))$

Let us introduce the tautological form in T^*M

s 1-form in T^*M

For $\lambda \in T^*M$ $\quad s_\lambda : T_\lambda(T^*M) \longrightarrow \mathbb{R}$ $\quad$ linear form on T^*M at λ

$$\left(s_\lambda \in T^*_\lambda(T^*M) \right)$$

defined as $\boxed{s_\lambda(w) = \langle \lambda, \pi_* w \rangle}$ $\quad (*)$

where $\pi : T^*M \to M$ is the projection

$\pi_* : T_\lambda(T^*M) \to T_q M$ where $q = \pi(\lambda)$.

Formula (*) is well defined

$$
\begin{tikzcd}
T_\lambda(T^*M) \arrow[r, "\pi_*"] \arrow[dr, "s_\lambda"'] & T_q M \arrow[d, "\lambda"] \\
& \mathbb{R}
\end{tikzcd}
$$

$w \in T_\lambda(T^*M)$, $\quad T_qM \ni \pi_* w$, $\quad \mathbb{R} \ni \langle \lambda, \pi_* w \rangle$

Shortly

$$\boxed{s_\lambda = \lambda \circ \pi_*}$$

Exercise In coordinates ($M = \mathbb{R}^n$) we have that if $\lambda = \sum_{i=1}^{n} p_i \, dq_i$

then $s_\lambda = \sum_{i=1}^{n} p_i \, dq_i$

same expression
meaning of "tautological"

Next we introduce the symplectic structure on T^*M is the 2-form $\sigma = ds$

in coordinates $\sigma = \sum_{i=1}^{n} dp_i \wedge dq_i$

closed 2-form
non degenerate

We formulate and we continue later.

Theorem Consider $\sigma|_{\Delta^\perp}$ restriction of σ to $\Delta^\perp \subset T^*M$

$$\lambda(t) \text{ is abnormal} \iff \begin{cases} \lambda(t) \in \Delta^\perp \\ \dot\lambda(t) \in \operatorname{Ker} \sigma|_{\Delta^\perp} . \end{cases}$$