On the Brunn-Minkovski inequality in sub-Riemannian geometry

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Joint work with

This is based on joint works with

- Luca Rizzi (Institut Fourier, Univ. Grenoble-Alpes)

→ Main references:

BR-17 DB, L. Rizzi,
Sub-Riemannian interpolation inequalities,
→ Preprint Arxiv, 2017

BR-18 DB, L. Rizzi,
Sub-Riemannian Bakry-Emery curvature: comparison and model spaces,
→ Soon on Arxiv!

Outline

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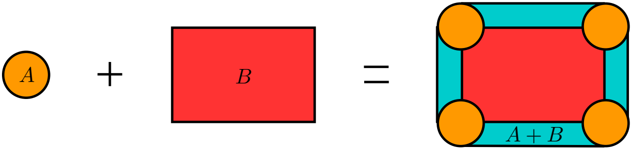
Euclidean Brunn-Minkowski

$A, B \subset \mathbb{R}^n$ non-empty measurable bounded sets

Minkowski sum: $A + B = \{z \mid z = a + b,\ a \in A,\ b \in B\}$



Brunn-Minkowski Inequality:

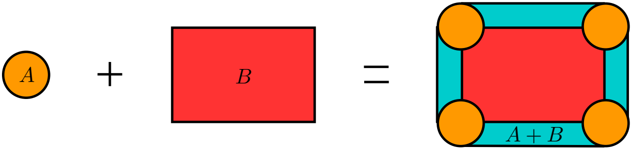
$$\mathrm{vol}(A+B)^{1/n} \geq \mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n}$$

Here vol is the Lebesgue measure in $\mathbb{R}^n$

Euclidean Brunn-Minkowski

$A, B \subset \mathbb{R}^n$ non-empty measurable bounded sets

Minkowski interpolation: $(1-t)A + tB = \{z \mid z = (1-t)a + tb,\ a \in A,\ b \in B\}$



Brunn-Minkowski Inequality:

$$\mathrm{vol}((1-t)A + tB)^{1/n} \geq (1-t)\mathrm{vol}(A)^{1/n} + t\mathrm{vol}(B)^{1/n} \qquad \forall t \in [0,1]$$

Here vol is the Lebesgue measure in $\mathbb{R}^n$

Functional inequalities

Geometric inequalities have often a functional counterpart

Theorem ($+\infty$-mean Borell-Brascamp-Lieb inequality)

Fix $t \in [0,1]$. Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be non-negative and integrable. Assume that for every $x, y \in \mathbb{R}^n$

$$h((1-t)x + ty) \geq \max\{f(x), g(y)\}. \tag{1}$$

Then,

$$\|h\|_{L^1}^{1/n} \geq (1-t)\|f\|_{L^1}^{1/n} + t\|g\|_{L^1}^{1/n}, \tag{2}$$

- one could restrict to $(x, y) \in A \times B$
- $A, B \subset \mathbb{R}^n$ Borel subsets such that $\int_A f\, dm = \|f\|_{L^1}$ and $\int_B g\, dm = \|g\|_{L^1}$.

→ generalized to other p-mean inequalities
(from Prékopa-Leindler to Borell-Brascamp-Lieb)

Generalization to Riemannian: a necessary condition

Denote $Z_t(A,B) := (1-t)A + tB$ the t-interpolating set

Brunn-Minkowski Inequality:

$$\mathrm{vol}(Z_t(A,B))^{1/n} \geq (1-t)\mathrm{vol}(A)^{1/n} + t\mathrm{vol}(B)^{1/n} \qquad \forall t \in [0,1]$$

- notice for $A = \{x\}$ and $B = \mathcal{B}_r(y)$ a ball.

$$\mathrm{vol}(Z_t(x,\mathcal{B}_r(y))) \geq t^n \mathrm{vol}(\mathcal{B}_r(y)) \qquad \forall t \in [0,1]$$

- in general this implies a control on the ratio

$$\frac{\mathrm{vol}(Z_t(x,\mathcal{B}_r(y)))}{\mathrm{vol}(\mathcal{B}_r(y))} \geq t^n$$

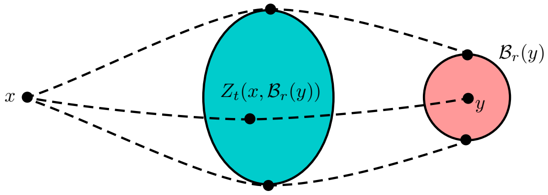
$\rightarrow$ measure contraction along geodesics, curvature

Distortion coefficient

(M, g) Riemannian manifold, vol Riemannian volume measure

Distortion coefficient

$$\beta_t(x,y) := \limsup_{r\to 0} \frac{\mathrm{vol}(Z_t(x,\mathcal{B}_r(y)))}{\mathrm{vol}(\mathcal{B}_r(y))}, \qquad \forall x,y \in M,\ t\in[0,1]$$

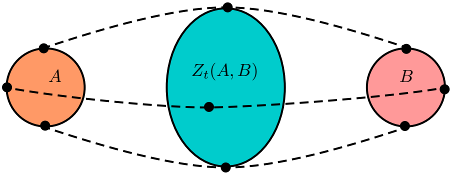


- $\beta_1(x,y) = 1$ and $\beta_0(x,y) = 0$. Important: $\beta_t(x,y) \sim t^n$ for $t \to 0$.
- $\beta_t(x,y)$ depends on the geodesics joining x with y
- Computable in terms of Jacobi fields.

Riemannian Brunn-Minkowski

(M, g) complete Riem. manifold, A, B non-empty Borel sets

$$Z_t(A, B) := \{\gamma(t) \mid \gamma : [0, 1] \to M \text{ geodesic s.t. } \gamma(0) \in A,\ \gamma(1) \in B\}$$



Theorem (Cordero-Erausquin, McCann, Schmuckenschläger - 2001)

Assume (M, g) *complete Riem. manifold with* $\mathrm{Ric} \geq 0$. *Then*

$$\mathrm{vol}(Z_t(A, B))^{1/n} \geq (1 - t)\mathrm{vol}(A)^{1/n} + t\mathrm{vol}(B)^{1/n}$$

- If $\mathrm{Ric} \geq K$ the inequality holds with modified coefficients
- It can be used to *define* Ricci bounds for m.m.s. (Sturm, Lott-Villani, . . .)

A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z}$$

- $(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure for $\varepsilon > 0$ with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

The sequence of curvatures is unbounded from below:

- $D^\varepsilon = \mathrm{span}\{X_1, X_2, X_3^\varepsilon\} \to D = \mathrm{span}\{X_1, X_2\}$
- $\mathrm{Sec}^\varepsilon(v_1, v_2) \to -\infty$ for all $v_1, v_2 \in D$
- $\mathrm{Ric}^\varepsilon(v) \to -\infty$ for all $v \in D$

As metric spaces $(\mathbb{R}^3, d^\varepsilon) \to (\mathbb{R}^3, d_{SR})$ (in the Gromov-Hausdorff sense)

▶ Cannot prove directly BM by taking limits of Ricci bounded structures

Outline

Sub-Riemannian geometry

Sub-Riemannian structure

- M smooth, connected manifold
- $D \subseteq TM$ distribution of constant* rank $k \leq n$
  - Hörmander condition: $\mathrm{Lie}(D)|_x = T_x M$ for all $x \in M$
- g smooth scalar product on D

Admissible curve: $\gamma : [0,1] \to M$ such that $\dot{\gamma}(t) \in D_{\gamma_t}$

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$$

Sub-Riemannian distance: (or Carnot-Carathédory)

$$d_{SR}(x,y) = \inf\{\ell(\gamma) \mid \gamma \text{ admissible joining } x \text{ with } y\}$$

Chow-Rashevskii: $d_{SR} < +\infty$ and (M, d_{SR}) has the same topology of M

Brunn-Minkowski on the Heisenberg group

The standard Brunn-Minkowski inequality $\mathrm{BM}(0, N)$:

$$\mathrm{vol}(Z_t(A, B))^{\frac{1}{N}} \geq (1-t)\mathrm{vol}(A)^{\frac{1}{N}} + t\mathrm{vol}(B)^{\frac{1}{N}},$$

Theorem (Juillet - 2009)

The Heisenberg group $\mathbb{H}_3$, equipped with Lebesgue measure:

- *satisfy the* $\mathrm{MCP}(0, N)$ *for* $N \geq 5$
- *does not satisfy any* $\mathrm{BM}(0, N)$

$\Rightarrow$ **Geodesic dimension** (Agrachev, DB, Rizzi - 2013)

Theorem (Balogh, Kristály, Sipos - 2016)

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Towards SR interpolation inequalities

For the Heisenberg group ($\rightarrow$ and higher dimensional versions):

- Juillet $\Rightarrow$ standard BM is not the right one
- Balogh-Kristály-Sipos $\Rightarrow$ some modified BM holds

Do *general* sub-Riemannian structures support interpolation inequalities?
(with weights that may depend on geometry)

Main results

- interpolation inequalities for ideal sub-Riemannian structures
- new examples of sharp BM (Grushin plane, some Carnot groups)
- regularity results for the sub-Riemannian distance

Main assumption: ideal structures

Definition (Ideal structure)

A sub-Riemannian structure is ideal if (M, d_{SR}) is complete and it admits **no singular minimizing geodesics**

- singular minimizer: cf talk by Ludovic Rifford
- True for the generic sub-Riemannian structure with $\mathrm{rank}\, D \geq 3$
  $\rightarrow$ [Chitour, Jean, Trélat - 2006]
- True for all contact structures

In this case, geodesics are described by a Hamiltonian flow on T^*M

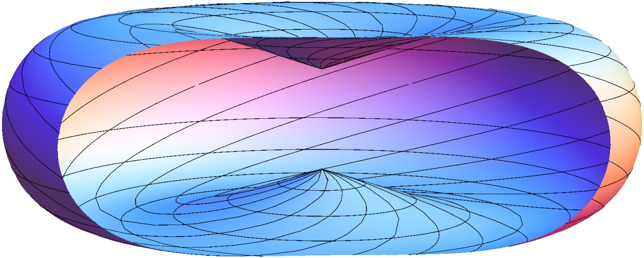
- H is quadratic on fibers but degenerate

$$H(p, x) = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij}(x) p_i p_j, \qquad g^{ij}(x) \quad \text{is degenerate}$$

- not immediate replace Levi-Civita connection / tensor curvature (in general)

The Heisenberg sphere

Even without singular minimizers things are not trivial



Sub-Riemannian spheres are not smooth, even for small radii

Asymptotics of distortion coefficients

Sub-Riemannian distortion coefficient

$$\beta_t(x,y) := \limsup_{r\to 0} \frac{\mathsf{m}(Z_t(x,\mathcal{B}_r(y)))}{\mathsf{m}(\mathcal{B}_r(y))}, \qquad \forall x,y \in M,\ t\in[0,1]$$

- **Riemannian case**: $\beta_t(x,y) \sim t^n$

Theorem (Agrachev, DB, Rizzi - 2013)

Fix $x \in M$. Then for a.e. $y \in M$ one has

$$\beta_t(x,y) \sim t^{\mathcal{N}(x)},$$

for some $\mathcal{N}(x) \in \mathbb{N}$ such that $\mathcal{N}(x) > n$.

- $\mathcal{N}(x)$ is the **geodesic dimension at** x
- definable in terms of directional Lie brackets
- it is bigger also than the Hausorff dimension

Sub-Riemannian BM with weights

Given $A, B \subset M$ Borel and $t \in [0,1]$

$$\beta_t(A,B) := \inf\{\beta_t(x,y) \mid (x,y) \in A \times B\}$$

Theorem (Barilari, R. - 2017)

Let (M, D, g) be an ideal n-dim sub-Riemannian manifold, m *smooth measure. For all $A, B \subset M$ Borel and $t \in [0,1]$*

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq \beta_{1-t}(B,A)^{1/n}\mathsf{m}(A)^{1/n} + \beta_t(A,B)^{1/n}\mathsf{m}(B)^{1/n}$$

- Particular case of more general sub-Riemannian interpolation inequalities
- functional inequalities à la Borell-Brascamp-Liebb
- $\beta_t(x,y)$ **explicitly computable** in terms of Hamiltonian flow

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- Particular case of more general sub-Riemannian interpolation inequalities
- difficulties: absence of standard Jacobi fields, degenerate Hamiltonian
- $\beta_t(x,y)$ **explicitly computable** in terms of Hamiltonian flow

$\rightarrow$ notice that IF $\beta_t(x,y) \geq t^n$ then linear weights in t, but ...

Equivalence of inequalities

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq \beta_{1-t}(B,A)^{1/n}\mathsf{m}(A)^{1/n} + \beta_t(A,B)^{1/n}\mathsf{m}(B)^{1/n}$$

- Interesting case: $\beta_t(x,y) \geq t^N$ for some N ($\rightarrow$ hence $N \geq \mathcal{N}(x)$)

Corollary

Let (M,D,g) be an ideal n-dim sub-Riemannian manifold, m smooth measure. Let $N > 0$. The following are equivalent:

(i) bound on the distortion coefficient:

$$\beta_t(x,y) \geq t^N$$

(ii) the modified Brunn-Minkowski inequality:

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq (1-t)^{N/n}\mathsf{m}(A)^{1/n} + t^{N/n}\mathsf{m}(B)^{1/n}$$

(iii) the measure contraction property $\mathrm{MCP}(0,N)$:

$$\mathsf{m}(Z_t(x,B)) \geq t^N\mathsf{m}(B)$$

Application to some Carnot groups

Theorem (Rifford - 2014, Rifford, Badreddine - 2018)

There exists $N > 0$ such that

$$\beta_t(x, y) \geq t^N \qquad \forall t \in [0, 1]$$

a) for every compact 2-step sub-Riemannian manifold $(\to D + [D, D] = TM)$
b) a class of 3-step Carnot group $(\to D + [D, D] + [X, [D, D]] = TM)$

Conjecture: for Carnot groups **best exponent** = geodesic dimension?

Theorem (Barilari, R. - 2017)

For any generalized H-type Carnot group of dimension n and rank k, equipped with the Lebesgue measure, for all Borel subsets A, B we have the ***sharp*** *inequality*

$$\mathrm{vol}(Z_t(A, B))^{\frac{1}{n}} \geq (1-t)^{\frac{k+3(n-k)}{n}} \mathrm{vol}(A)^{\frac{1}{n}} + t^{\frac{k+3(n-k)}{n}} \mathrm{vol}(B)^{\frac{1}{n}},$$

$\to$ not necessarily ideal (tensorization: Ritoré-Nicolàs - 2017)

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Application to the Grushin plane

Rank-varying structure on $M = \mathbb{R}^2$, equipped with Lebesgue measure

$$X_1 = \partial_x, \qquad X_2 = x\partial_y$$

Well defined geodesic m.m.s. (almost Riemannian, with $\mathrm{Curv} \to -\infty$).

Theorem (Barilari, R. - 2017)

The distortion coefficient of Grushin satisfies the following sharp inequality

$$\beta_t(x,y) \geq t^{5}, \qquad \forall t \in [0,1]$$

which is equivalent to the Brunn-Minkowski inequality:

$$\mathrm{vol}(Z_t(A,B))^{1/2} \geq (1-t)^{5/2}\mathrm{vol}(A)^{1/2} + t^{5/2}\mathrm{vol}(B)^{1/2}$$

- **Gap** between the geodesic dimension and the best N

$$\mathcal{N}(x) = \begin{cases} 2 & \text{in the Riemannian region} \\ 4 & \text{otherwise} \end{cases}$$

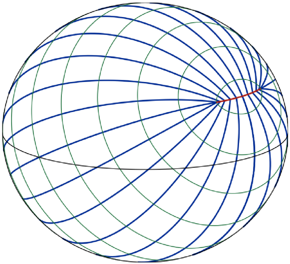
Regularity of distance

(M, D, g) complete (sub-)Riemannian structure. Fix $x \in M$.

Theorem (Agrachev - 2009, Rifford-Trélat - 2006)

The set of points where $d_{SR}^2(x, \cdot)$ is smooth is open and dense in M.

The **cut locus** $\mathrm{cut}(x)$ is the complement of the set of smooth points



Regularity of distance

The proof of the main inequality implies the following characterization

Theorem (Barilari, R. - 2017)

Let (M, D, g) be an ideal sub-Riemannian manifold. Let $y \neq x$. Then $y \in \mathrm{cut}(x)$ if and only if $f = d^2_{SR}(x, \cdot)$ fails to be semiconvex at y:

$$\inf_{0<|v|<1} \frac{f(y+v) + f(y-v) - 2f(y)}{|v|^2} = -\infty$$

→ "one cannot put a parabola below the graph of the distance"

- Extends an analogue result in the Riemannian case [CEMS,2001]
- Differentiability of transport map [FR,2008]
- Sharp → there are non-ideal structures where $d^2_{SR}(x, \cdot)$ is locally semiconvex at the cut locus (it fails to be semiconcave)

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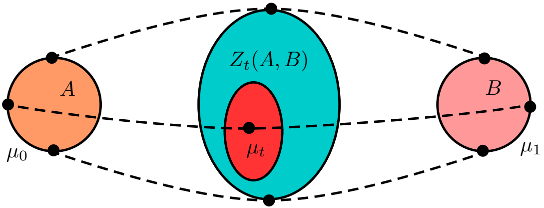
- Sharp → there are non-ideal structures where $d^2_{SR}(x, \cdot)$ is locally semiconvex at the cut locus (it fails to be semiconcave) → role of abnormal minimizers

Outline

Idea of the proof

Step 0. Optimal transport problem: $\mu_0, \mu_1 \in \mathcal{P}_c(M)$

$$\inf_{T_\sharp \mu_0 = \mu_1} \frac{1}{2} \int_M d^2_{SR}(x, T(x)) d\mathsf{m}(x)$$



- Maps $T : M \to M$ that realizes the inf are **optimal transport maps**
- Points are transported along geodesics $x \mapsto T(x)$
- To prove BM $\Rightarrow$ choose $(\mu_0, \mu_1) = (\chi_A, \chi_B)$. The interpolating measure measure μ_t gives a lower bound for the measure of $Z_t(A, B)$

Idea of the proof

Step 1. The optimal transport problem is well defined on ideal structures (Ambrosio-Rigot 2004, Agrachev-Lee 2008, Figalli-Juillet 2008, Figalli-Rifford 2010)

Theorem (Figalli, Rifford - 2010)

Let $\mu_0 \in \mathcal{P}_c^{ac}(M)$, $\mu_1 \in \mathcal{P}_c(M)$. *Assume* $\mathrm{supp}(\mu_0) \cap \mathrm{supp}(\mu_1) = \emptyset$.

- *There exists a unique optimal transport map* $T : M \to M$ *such that* $T_\sharp \mu_0 = \mu_1$, *given by*

$$T(x) = \exp_x(d_x\psi),$$

  where $\psi : M \to \mathbb{R}$ *is locally semiconvex.*

- *For* μ_0*-a.e.* $x \in M$ *there exists a unique geodesic joining* x *with* $T(x)$*:*

$$T_t(x) = \exp_x(t d_x\psi), \qquad \forall t \in [0, 1].$$

- **ideal** $\Rightarrow$ semiconvexity of ψ

Idea of the proof

Step 2. Geodesics interpolation between μ_0 and μ_1 at time $t \in [0,1]$:

$$\mu_t := (T_t)_\sharp \mu_0, \qquad \text{with} \qquad T_t(x) = \exp_x(t d_x \psi)$$

- ψ Alexandrov second differentiability theorem $\Rightarrow T_t(x)$ is m-a.e. differentiable

$$\text{if } |\det(d_x T_t)| > 0 \text{ m-a.e.} \qquad \mu_t = \rho_t \mathsf{m}, \qquad \rho_t(T_t(x)) = \frac{\rho_0(x)}{|\det(d_x T_t)|}$$

Step 3. The differential $d_x T_t : T_x M \to T_{T_t(x)} M$

$$d_x T_t = \pi_* \circ e_*^{t\vec{H}} \circ d_x^2 \psi$$

- use the natural symplectic structure on T^*M and Darboux moving frames
- avoid the classical machinery of connection and parallel transport

Step 4. Jacobian inequality: i.e., interpolation inequality for $\det d_x T_t$

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Concentration inequality

The proof implies:

- $T(x) \notin \mathrm{cut}(x)$ for μ_0-a.e. $x \in M$
- $\det(d_x T_t) > 0$ for all $t \in [0,1)$ and $\mu_t = \rho_t \mathsf{m}$
- The Jacobian inequality holds on the whole $[0,1]$

Theorem (Barilari, R. - 2017)

Let (D, g) be an ideal sub-Riemannian structure on M, and $\mu_0, \mu_1 \in \mathcal{P}_c^{ac}(M)$. Let $\rho_t = d\mu_t/d\mathsf{m}$. For all $t \in [0,1]$, it holds

$$\frac{1}{\rho_t(T_t(x))^{1/n}} \geq \frac{\beta_{1-t}(T(x),x)^{1/n}}{\rho_0(x)^{1/n}} + \frac{\beta_t(x,T(x))^{1/n}}{\rho_1(T(x))^{1/n}}, \qquad \mu_0 - \text{a.e. } x \in M.$$

If μ_1 is not absolutely continuous, an analogous result holds, provided that $t \in [0,1)$, and that the second term on the right hand side is omitted.

▶ Borell-Brascamp-Lieb, p-mean inequality, Brunn-Minkowski follow

Outline

Comparison: the Riemannian case

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq \beta_{1-t}(B,A)^{1/n}\mathsf{m}(A)^{1/n} + \beta_t(A,B)^{1/n}\mathsf{m}(B)^{1/n}$$

- Distortion coefficients are in general difficult to compute,
- a bound on the geometry gives a bound in terms of model spaces.

Theorem

Let (M,g) be a n-dimensional Riemannian, with $\mathsf{m} = \mathrm{vol}_g$ Riemannian volume. Assume that $\mathrm{Ric}_g \geq K$. Then for $(x,y) \notin \mathrm{cut}(M)$ we have

$$\beta_t(x,y) \geq \beta_t^{(K,n)}(x,y), \tag{3}$$

- $\beta_t^{(K,n)}(x,y)$ distortion coefficient of model: constant sectional curvature K and dimension n.

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- Assume the Riemannian manifold (M, g) endowed with arbitrary smooth measure $\mathsf{m} = e^{-V}\mathrm{vol}_g$
- Bakry-Emery Ricci tensor

$$\mathrm{Ric}_g^{\mathsf{m},N} := \mathrm{Ric}_g + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N-n}, \tag{4}$$

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$$\beta_t(x,y)^{\frac{1}{N}} \geq \beta_t^{(K,n)}(x,y)^{\frac{1}{n}}. \tag{5}$$

- This inequality is weaker than the one is possible to obtain ($\rightarrow$ the one defining $\mathrm{CD}(K, N)$ spaces)
- i.e., the latter *cannot* be obtained plugging this inequality into the main one.
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Explicit formula for the coefficient appearing in the right-hand side of (3)

$$\beta_t^{(K,n)}(x,y) = \begin{cases} t\left(\frac{\sin(t\alpha)}{\sin(\alpha)}\right)^{n-1} & \text{if } K>0,\\ t^n & \text{if } K=0,\\ t\left(\frac{\sinh(t\alpha)}{\sinh(\alpha)}\right)^{n-1} & \text{if } K<0, \end{cases} \qquad \alpha = \sqrt{\frac{|K|}{n-1}}d(x,y). \tag{6}$$

- only depends on $d(x,y)$
- the $(n-1)$ $\rightarrow$ no curvature in direction of the geodesic
- Jacobi equation in parallel transported frame

$$\ddot{J}_i + R_{ij}(t)J_j = 0$$

  where $R_{ij}(t) = R(X_i, \dot\gamma, \dot\gamma, X_j)$.
- constant curvature $R = \mathrm{diag}(K, K, \dots, K, 0)$

The problem of models

When do we have $\beta_t(x, y) \geq \beta_t^{\mathrm{model}}$?

- In the above example models are given by Riemannian space forms
- No reason to be good models also for the sub-Riemannian case
- do not depend only on $d(x, y)$ but on the whole trajectory

Problems in the SR case: What are models? What is curvature?

We propose an approach from the viewpoint of control theory:

- Curvature: invariant extracted from derivatives of the sub-Riemannian distance
- Models: simple optimal control problems

Linear Quadratic problems as models

Variational problems in $\mathbb{R}^n$

$$\dot{x} = Ax + Bu$$

with minimization of a quadratic cost

$$\frac{1}{2}\int_0^1 (u^*u - x^*Qx)dt \longrightarrow \text{min}$$

Bracket generating: $\exists m \geq 0$ such that $\operatorname{rank}(B, AB, \dots, A^m B) = n$

Optimal trajectories solve a Hamiltonian system:

$$H(p, x) = \frac{1}{2}(p^*BB^*p + 2p^*Ax + x^*Qx)$$

For all LQ problems that we use, minimizers exist and are unique.

LQ distortion

Definition (LQ distortion coefficients)

$$\beta_t^{A,B,Q}(x,y) := \limsup_{r\to 0} \frac{|Z_t(x,\mathcal{B}_r(y))|}{|\mathcal{B}_r(y)|}, \qquad x,y \in \mathbb{R}^n$$

- It does not depend on x, y (the Hamiltonian flow is linear)
- Very simple to compute

Example: Harmonic oscillator

No drift ($A = 0$), no constraint on velocity ($B = \mathbb{1}$), isotropic potential ($Q = \kappa\mathbb{1}$):

$$H(p,x) = \frac{1}{2}(|p|^2 + \kappa|x|^2)$$

$\Rightarrow \beta_t^{A,B,Q} =$ Riemannian distortion coefficients!

What is curvature? (sketchy)

Fact/definition:

To any SR geodesic γ (+ technical assumptions) we associate

- two constant matrices A and B $\rightarrow$ structure of Lie derivatives along geodesics
- a curvature operator, quadratic form $\mathfrak{R}_{\gamma(t)} : T_{\gamma(t)}M \to \mathbb{R}$ for $t \in [0, T]$.

- In the Riemannian case: $A = 0$, $B = I$, $\mathfrak{R}_{\gamma(t)}(X) = R^{\nabla}(\dot\gamma_t, X, X, \dot\gamma_t)$

Given the operator $\mathfrak{R}_{\gamma(t)}$ and a smooth measure m one can define a Bakry-Emery sub-Riemannian tensor

$$\mathfrak{R}^{\mathrm{m},N}_{\gamma(t)} = \mathfrak{R}_{\gamma(t)} - \frac{\dot\rho(t)}{k}\Pi_{\gamma(t)} - \frac{n}{N-n}\frac{\rho^2(t)}{k^2}\Pi_{\gamma(t)}. \tag{7}$$

Recall that $n = \dim M$ and $k = \dim D$.

- in Riemannian $\rho(t) = -g(\nabla V, \dot\gamma(t))$ for $\mathrm{m} = e^{-V}\mathrm{vol}_g$.

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Final comparison

In terms of the Bakry-Emery SR curvature $\mathfrak{R}^{\mathsf{m},N}_{\gamma(t)}$ we have the following comparison

Theorem

Let $(x,y) \notin \mathrm{cut}(M)$ and assume that the unique length-minimizer joining x and y is associated with matrices A, B.

(a) *If there exists $N > n$ and Q such that $\frac{1}{N}\mathfrak{R}^{\mathsf{m},N}_{\gamma(t)} \geq \frac{1}{n}Q$ for every $t \in [0,T]$, then*

$$\beta_t(x,y)^{\frac{1}{N}} \geq (\beta_t^{A,B,Q})^{\frac{1}{n}} \tag{8}$$

Assume now that $\rho = 0$.

(b) *If there exists Q such that $\mathfrak{R}_{\gamma(t)} \geq Q$ for every $t \in [0,T]$, then*

$$\beta_t(x,y) \geq \beta_t^{A,B,Q} \tag{9}$$

THANKS FOR YOUR ATTENTION !