From Thermodynamics to Image Reconstruction: an invitation to Sub-Riemannian geometry

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Outline

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The origin of the name

Today is known under different names

- Sub-Riemannian geometry (Strichartz, 1986)
- Carnot-Carathéodory geometry (Gromov et al., 1981)

and not only these two

- Singular Riemanniann geometry (Brockett, 1981) → not really used anymore
- Nonholonomic geometry (Russian school) → from mechanics
- Sub-elliptic geometry → from PDEs

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? Why is it called Carnot-Carathéodory geometry?

Carnot meets Carathéodory

- 1824, Sadi Carnot → Carnot cycle: a theoretical thermodynamic cycle
  - 4 cycle engine (2 isothermal + 2 adiabatic curves)
  - isothermal = same temperature
  - adiabatic = no heat exchange

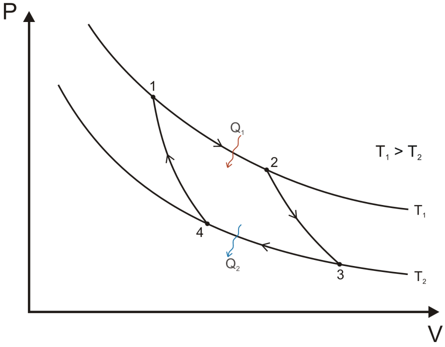

Image credits: https://en.wikipedia.org/wiki/Carnot_cycle

Carnot meets Carathéodory

Some years later

- 1909, Constantin Carathéodory → geometric foundations of themodynamics
  - theory of Pfaffian equations
  - reformulations of the second principle of thermodynamics
  - adiabatic process → horizontal curve for a SR structure

C. Carathéodory. Grundlagen der Thermodynamik. 355

Untersuchungen über die Grundlagen der Thermodynamik.

Von

C. Carathéodory in Hannover.

Carnot meets Carathéodory

"Untersuchungen über die Grundlagen der Thermodynamik," Math. Ann. **67** (1909), 355-386.

Examination of the foundations of thermodynamics

By

C. CARATHEODORY in Hannover

Translated by:

D. H. Delphenich

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The 2nd law of Theormodynamics

- Q = heat transfer, T = temperature
- heat transfer along a curve γ

$$Q = \int_\gamma \delta Q \qquad \rightarrow \text{ depends on } \gamma \text{ (or } \delta Q \text{ not exact)}$$

Theorem (Clausius, 1854)

For any reversible cycle R we have

$$\oint_R \frac{\delta Q}{T} = 0$$

- there exists a function S, called entropy such that

$$\delta Q = T dS$$

- Along a curve γ

$$\int_\gamma \frac{\delta Q}{T} = \int_\gamma dS = S_F - S_I$$

Adiabatic processes

Heat exanged along a curve γ

$$Q = \int_\gamma \omega, \qquad \omega = T dS$$

- adiabatic curve: no heat exchange $\rightarrow$

$$\omega(\dot\gamma) = 0$$

Axiom II, Carathéodory, 1909

In any arbitrary neighborhood of an arbitrarily give initial state of a system (of any number of thermodynamic coordinates), there is a state that is inaccessible by reversible adiabatic processes.

Axiom II: *In jeder beliebigen Umgebung eines willkürlich vorgeschriebenen Anfangszustandes gibt es Zustände, die durch adiabatische Zustandsänderungen nicht beliebig approximiert werden können.*

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Adiabatic processes and Pfaffian equations

Lemma from Pfaffian equations, Carathéodory, 1909

If any arbitrary neighborhood of an arbitrarily give initial state of a system , there is a state that is inaccessible by curves γ that satisfy

$$\omega(\dot{\gamma}) = 0$$

then necessarily ω is integrable, i.e., there exist functions T, S such that $\omega = TdS$.

4. Hilfssatz aus der Theorie der Pfaffschen Gleichungen.

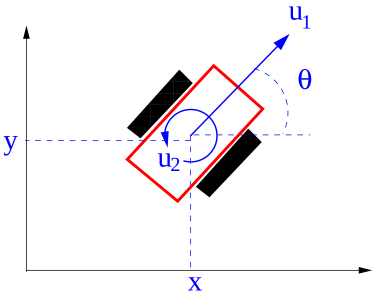
Ist eine Pfaffsche Gleichung

(16) $$dx_0 + X_1\,dx_1 + X_2\,dx_2 + \cdots + X_n\,dx_n = 0$$

gegeben, wobei die X_i endliche, stetige, differentiierbare Funktionen der x_i sind, und weiß man, daß es in jeder Umgebung eines beliebigen Punktes P des Raumes der x_i Punkte gibt, die man längs Kurven, welche dieser Gleichung genügen, nicht erreichen kann, so muß notwendig der Ausdruck (16) *einen Multiplikator besitzen, der ihn zum vollständigen Differentiale macht.*

Example: motion of a car robot

A car robot in the plane.



Position $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$ in a three dimensional space.
Three possible motions

- X - movement in the direction of the car
- Y - rotation
- Z - movement in the direction orthogonal to the car

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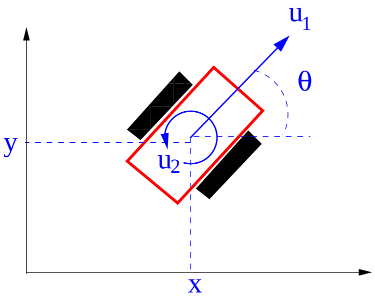

Position $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$ in a three dimensional space.
Three possible motions

- $X = \cos\theta\,\partial_x + \sin\theta\,\partial_y$
- $Y = \partial_\theta$
- $Z = -\sin\theta\,\partial_x + \cos\theta\,\partial_y$

Example: motion of a car robot

- $X = \cos\theta\,\partial_x + \sin\theta\,\partial_y$
- $Y = \partial_\theta$
- $Z = -\sin\theta\,\partial_x + \cos\theta\,\partial_y$

The condition $\omega(\dot\gamma) = 0$ means that we have only two admissible motions

- assume $\omega = d\theta$ then
  - ω is integrable
  - the two admissible movement X, Z commute
- assume $\omega = -\sin\theta\,dx + \cos\theta\,dy$ then
  - ω is not integrable
  - the two admissible movement X, Y do not commute

? When is it possible to reach every other configuration from a fixed one ?

Example: motion of a car robot

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Lie brackets

In the first case

→ the vector fields commute and we are constrained to an hypersurface of $\mathbb{R}^2 \times S^1$, where $\theta = const$

in the second case

→ The standard parking procedure suggest that we are able to go in the non admissible direction by a combination of admissible movements

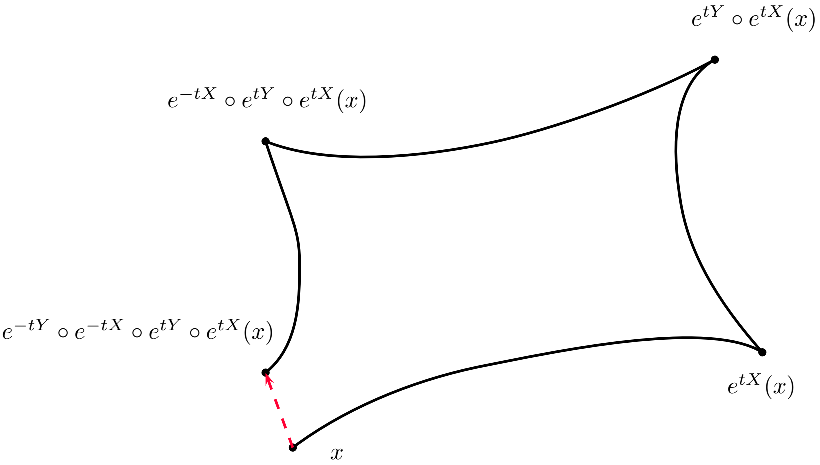
The Lie brackets of two vector fields measure whether the associated flows commutes or not

$$e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x) = x + t^2[X, Y](x) + o(t^2)$$

→ Indeed the commutator is the missing direction!

$$[X, Y] = [\cos\theta\,\partial_x + \sin\theta\,\partial_y, \partial_\theta] = -\sin\theta\,\partial_x + \cos\theta\,\partial_y = Z$$

Lie bracket



$$e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x) = x + t^2[X,Y](x) + o(t^2)$$

Vector distributions and Frobenius theorem

- the condition $\omega(\dot\gamma) = 0$ says that the curve is tangent to a family of hyperplanes

$$\dot\gamma(t) \in D_{\gamma(t)}, \qquad D_x = \ker \omega_x = \text{span}_x\{X, Y\}$$

- the distribution D does not change if we multiply ω by a function
- the Carathéodory condition says that $\frac{1}{T}\omega = dS$ is closed (indeed exact)

$\boxed{?}$ What for higher dimension and codimension?
Assume we are in dimension n with $k \leq n$ vector fields

$$D_x = \text{span}_x\{X_1, \ldots, X_k\}$$

where can we go with curves tangent to D?

Theorem (Frobenius, 1877)

If $[D, D] \subset D$ (D is Frobenius integrable) then we are constrained to a k dimensional submanifold. The ambient space is foliated by such submanifolds

Frobenius, G. "Über das Pfaffsche probleme", J. für Reine und Agnew. Math., 82 (1877) 230-315.

Ueber das *Pfaff*sche Problem.

(Von Herrn *Frobenius* in Zürich.)

Einleitung.

Das *Pfaff*sche Problem ist nach den Vorarbeiten *Jacobis* (dieses Journal Bd. 2, S. 347; Bd. 17, S. 128; Bd. 29, S. 236) hauptsächlich von Herrn *Natani* (dieses Journal Bd. 58, S. 301) und von *Clebsch* (dieses Journal Bd. 60, S. 193, Bd. 61, S. 146) zum Gegenstand eingehender Untersuchungen gemacht worden. In seiner ersten Arbeit führt *Clebsch* die Lösung der Aufgabe auf die Integration mehrerer Systeme homogener linearer partieller Differentialgleichungen zurück mittelst einer indirecten Methode, von der er später (Bd. 61, S. 146) selbst sagt, dass sie nicht vollständig geeignet sei, die Natur der betreffenden Gleichungen ins rechte Licht zu setzen. Desshalb hat er in der zweiten Arbeit die Aufgabe auf einem andern directen Wege angegriffen, aber nur solche Differentialgleichungen

$$X_1\,dx_1 + X_2\,dx_2 + \cdots + X_p\,dx_p = 0$$

behandelt, für welche die Determinante der Grössen

$$a_{\alpha\beta} = \frac{\partial X_\alpha}{\partial x_\beta} - \frac{\partial X_\beta}{\partial x_\alpha}$$

von Null verschieden ist.

Non-integrability or non-Frobenius

If we want to hope to join a full neighborhood of a point we need to have a "non-integrability" condition on D.

- if $D = \ker \omega$ then

$$D \text{ is (Frobenius) integrable} \iff \omega \text{ is (Carathéodory) integrable}$$

- ω and $f\omega$ defined the same D (we take $\ker$)
- Cartan formula

$$\mathrm{d}\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

- If $\mathrm{d}\omega = 0$ (ω closed) and $X, Y \in D$ then $[X, Y] \in D$

? What guarantees we can reach a full dimensional neighborhood in general ?

- It is reasonable to expect that if the Lie brackets span all missing directions we can!

$\rightarrow$ just iterate the above procedure

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Chow Theorem

Assume M is a n-dimensional manifold and

$$D = \mathrm{span}\{X_1, \dots, X_k\}$$

→ Hörmander condition:

$$\mathrm{Lie}_x D = \mathrm{span}\{[X_1, [\dots [X_{j-1}, X_j]]](x) \mid i = 1, \dots, k,\ j \in \mathbb{N}\} = T_x M, \quad \forall\, x \in M,$$

Theorem (Chow '39, Rashevsky '38)

If D satisfies the Hörmander condition then from every point we can reach a full neighborhood by curves that are tangent to D.

Chow, W.L. (1939), "Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung", Mathematische Annalen 117: 98-105.

Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.

Von

Wei-Liang Chow in Shanghai (China).

C. Carathéodory hat bei seiner Begründung des zweiten Hauptsatzes der Thermodynamik [1]) den folgenden Satz über eine Pfaffsche Gleichung bewiesen: *Wenn eine Pfaffsche Gleichung in jedem Punkte die Eigenschaft hat, daß es in jeder Umgebung von ihm Punkte gibt, die sich nicht durch eine Integralkurve der Gleichung mit ihm verbinden lassen, dann ist die Gleichung vollständig integrierbar.* Dabei ist unter einer Integralkurve einer Pfaffschen Gleichung $\sum_{j=1}^{n} \alpha_j (x_1, \ldots, x_n)\, d x_j = 0$ eine stückweise stetig differenzierbare Kurve [2]) $x_j(t)$ zu verstehen, deren jedes stetig differenzierbare Stück (auch in den Endpunkten) der Gleichung $\sum_{j=1}^{n} \alpha_j (x_1(t), \ldots, x_n(t)) \frac{d x_j}{d t} = 0$ genügt. Diesen Satz, den Carathéodory durch eine geometrische Konstruktion der Integralhyperflächen bewiesen hat, werden wir nun in einer ganz anderen Weise beweisen und gleichzeitig auf Systeme von Pfaffschen Gleichungen verallgemeinern. Unsere Methode besteht darin, daß wir zuerst in bekannter Weise das Pfaffsche System auf ein System von linearen partiellen Differentialgleichungen erster Ordnung zurückführen und dann die von einem Punkte

Outline

Sub-Riemannian geometry

Definition

A *sub-Riemannian manifold* is a triple $(M, D, \langle\cdot,\cdot\rangle)$, where

(i) M manifold, C^∞, dimension $n \geq 3$;

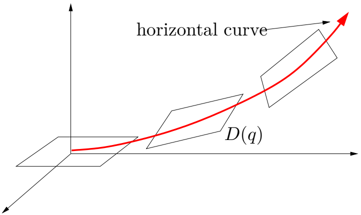
(ii) D vector distribution of rank $k < n$, i.e. $D_x \subset T_xM$ subspace k-dim. that satisfies Hörmander condition: $\boxed{\text{Lie}_x D = T_xM}$.

(iii) $\langle\cdot,\cdot\rangle_x$ inner product on D_x, smooth in x.

- A curve $\gamma : [0,T] \to M$ is horizontal if $\dot\gamma(t) \in \Delta_{\gamma(t)}$

- For a horizontal curve $\gamma : [0,T] \to M$ its length is

$$\ell(\gamma) = \int_0^T \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle}\, dt.$$



We can define the sub-Riemannian distance as

$$d(x,y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y, \gamma \text{ horizontal}\}.$$

Theorem (Rashevsky-Chow, 1938)

If $D = \text{span}\{X_1, \dots, X_k\}$ satisfies the Hörmander condition, then

- $d(x,y) < +\infty$ *for every* $x, y \in M$.
- d *induces the manifold topology.*

Question: Regularity of d? Relation with minimizing admissible curves?

We can reach an n-dim BUT we have only k-dim set of velocities!! $(k < n)$
*** We cannot parametrize final points (geodesics) by their initial velocity! ***

$\rightarrow$ NO second order equation for geodesics

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}_i\dot{x}_j = 0$$

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Orthonormal frames and geodesics

Locally, the pair (D, g) can be given by assigning a set of k smooth vector fields, called a *local orthonormal frame*, spanning D and that are orthonormal

$$D_x = \mathrm{span}\{X_1(x), \ldots, X_k(x)\}, \qquad \langle X_i(x), X_j(x)\rangle = \delta_{ij}.$$

If we use this frame as a coordinate set

$$\dot\gamma(t) = \sum_{i=1}^{k} u_i(t) X_i(\gamma(t))$$

the norm of the tangent vector is computed as follows

$$\langle \dot\gamma(t), \dot\gamma(t)\rangle = \sum_{i,j=1}^{k} u_i(t) u_j(t) \langle X_i(\gamma(t)), X_j(\gamma(t))\rangle = \sum_{i=1}^{k} u_i(t)^2$$

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The problem of finding geodesics, i.e. curves that minimize the length between two given points x_0, x_1, is rewritten as the optimal control problem

$$\begin{cases} \dot{\gamma} = \displaystyle\sum_{i=1}^{k} u_i \, X_i(\gamma) \\ \displaystyle\int_0^T \sqrt{\sum_{i=1}^{k} u_i^2} \to \min \\ \gamma(0) = x_0, \quad \gamma(T) = x_1 \end{cases}$$

Orthonormal frames and geodesics

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$$\begin{cases} \dot\gamma = \displaystyle\sum_{i=1}^{k} u_i \, X_i(\gamma) & \longleftarrow \text{linear in } u \\ \displaystyle\frac{1}{2}\int_0^T \sum_{i=1}^{k} u_i^2 \to \min & \longleftarrow \text{quadratic in } u \\ \gamma(0) = x_0, \quad \gamma(T) = x_1 \end{cases}$$

Geodesics - Dual metric

- Assume $D = \mathrm{span}\{X_1, \ldots, X_k\}$ and X_i are orthonormal
- One can build the sub-Riemannian Hamiltonian $H \in C^\infty(T^*M)$

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k}(p \cdot X_i(x))^2.$$

→ In the Riemannian case $H(p,x) = \frac{1}{2}g^{ij}(x)p_i p_j$

→ It is like the inverse metric g^{ij} is defined but not invertible (has some zero eigenvalues)

- The Hamiltonian vector field $\vec{H}$ defined by

$$\vec{H}(p,x) = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}$$

→ $\vec{H}$ defines the geodesic flow in T^*M

Theorem

Assume $(x(t), p(t)) \in T^*_{x(t)}M$ *is a solution of*

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,p), \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,p), \end{cases}$$

Then $x(t)$ *locally minimizes the sub-Riemannian distance.*

- The system corresponds to the Euler-Lagrange equation (1st order necessary condition for optimality)
- when H is Riemannian, i.e., $H(x,p) = \frac{1}{2} g^{ij}(x) p_i p_j$

$$\begin{cases} \dot{x}_j = g^{ij} p_i, \\ \dot{p}_k = -\dfrac{1}{2} \dfrac{\partial g^{ij}}{\partial x_k} p_i p_j, \end{cases} \quad \Leftrightarrow \quad \ddot{x}^k + \Gamma^k_{ij} \dot{x}_i \dot{x}_j = 0$$

- 2nd order on M $\Leftrightarrow$ 1rd order on TM. In general it is not possible!

Exponential map

The SR exponential map

$$\exp_{x_0} : T^*_{x_0}M \to M, \qquad \exp_{x_0}(p_0) = x(1)$$

where $(x(t), p(t))$ solves the system with $x(0) = x_0$ and $p(0) = p_0$.

- geodesics starting from a point x_0 can be parametrized by

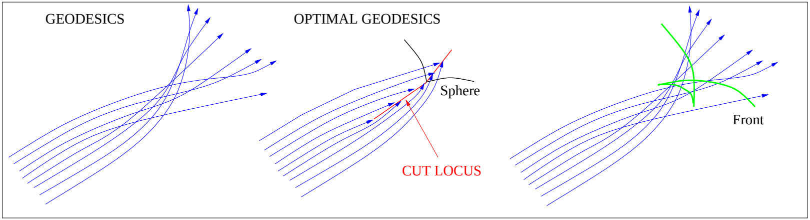
$$\Lambda_{x_0} = \left\{ p_0 \in T^*_{x_0}M : H(x_0, p_0) = \frac{1}{2} \right\} \longleftarrow \quad \text{“Unit speed”}$$

- in the Riemannian case $\Lambda_{x_0} \simeq S^{n-1}$ is compact
- the sub-Riemannian case $\Lambda_{x_0} \simeq S^{k-1} \times \mathbb{R}^{n-k}$ is a non compact set.

$\rightarrow$ when $n = 3$ and $k = 2$ is a cylinder $S^1 \times \mathbb{R}$

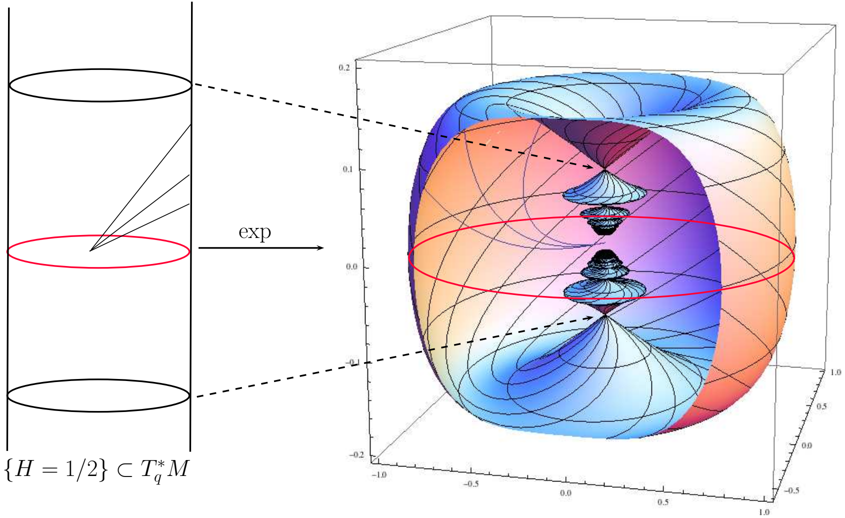
Features in SRG

One can introduce classical objects



- Sphere at time T: points reached optimally at time T (level sets of the distance)
- Front at time T: end points of geodesics at time T.
- Cut locus: set of points where geodesics lose global optimality (the distance is not smooth)

Case $n = 3$ and $k = 2$

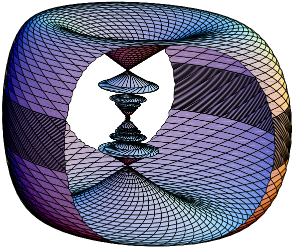


Basic features in SRG - 2

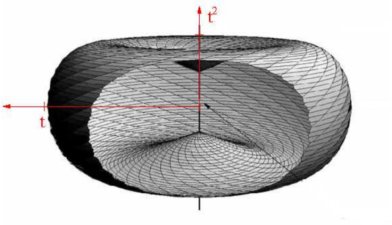
Consider geodesics from a fixed point $x_0 \in M$

- there are geodesics loosing optimality arbitrary close to x_0
- the function $x \mapsto d^2(x_0, x)$ is not smooth at $x = x_0$

$\downarrow$



Front at time T



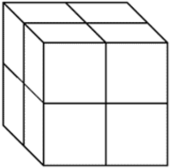
- Spheres are highly non isotropic.

The optimal synthesis is in general very difficult to obtain!

Hausdorff dimension and Hausdorff measure

For every subset of a metric space $\Omega \subset X$ there is a well defined notion of dimension which is defined only by the metric structure.
(sometimes it is called also fractal dimension)

Idea: the Hausdorff dimension of Ω is Q if the number $N(\varepsilon)$ of balls of radius at most ε required to cover Ω completely grows like $N(\varepsilon) \sim \varepsilon^{-Q}$ as $\varepsilon \to 0$.



$$N(\varepsilon) \sim \varepsilon^{-3}$$

$$dim_H X = 3$$

$$N(\varepsilon) \sim \varepsilon^{-\alpha}$$

$$dim_H X = \alpha = \log_3 2 \simeq 0.63$$

Hausdorff dimension: 3D case

A sub-Riemannian ball rescale of order two in the direction that is not in the distribution

- Ball-Box theorem (3D): for ε small enough

$$\mathsf{Box}(c\varepsilon) \subset B(x,\varepsilon) \subset \mathsf{Box}(C\varepsilon)$$

  where

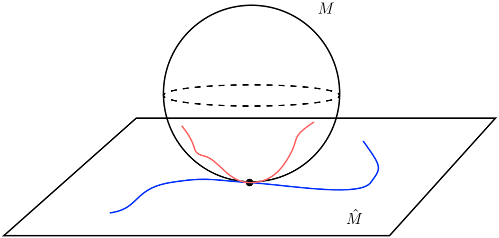
$$\mathsf{Box}(\varepsilon) = [-\varepsilon,\varepsilon] \times [-\varepsilon,\varepsilon] \times [-\varepsilon^2,\varepsilon^2]$$

- it is easy to see that the Hausdorff dimension is 4!
- there exists a formula for the general case in terms of the brackets (Mitchell)

Outline

Rolling manifolds

The rolling sphere consists of a two-sphere S^2, rolling without slip or twist over a plane $\mathbb{R}^2$.



The configuration space of this mechanical system has dimension 5.

- A point on the plane $\mathbb{R}^2$
- A point on the sphere
- An angle identifying the orientation between the two tangent spaces

$\rightarrow$ Only 2 movements are allowed!

Rolling manifolds

This naturally defines a SR structure on a 5-dimensional manifold, where the set of admissible velocities has dimension 2!

Natural questions:

1. Is it possible to reach every other configuration from a fixed one ?
2. What are "geodesics" for this problem?

Answers

1. Yes, it is possible (easy with combination of admissible motion)
2. Projections on $\mathbb{R}^2$ are Euler's elasticae.

? What happens if we do the same with two convex surfaces M and M'? ?

Hint: try with two spheres of radius R and R' $\rightarrow$ It is possible if and only if $R \neq R'$!

Rolling manifolds

This naturally defines a SR structure on a 5-dimensional manifold, where the set of admissible velocities has dimension 2!

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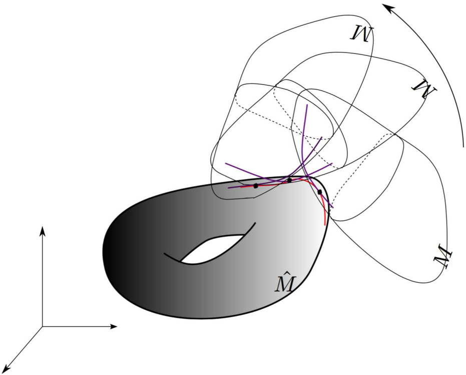


Image credits: Yacine Chitour

Controllability: 2D-case ($n = 2$)

Theorem (Agrachev-Sachkov, 1999)

Let $K, \hat{K}$ be Gaussian curvatures of $M, \hat{M}$, resp. Given $q_0 = (x, \hat{x}; \theta) \in Q$

- *if $K(x) - \hat{K}(\hat{x}) \neq 0$ for some $(x, \hat{x}; \theta)$, then one can reach locally a 5-dimensional set of configurations.*

The case of two arbitrary n-dimensional manifold: a full new theory developed

- Y.Chitour, P. Kokkonen
- F.S. Leite, I. Markina, M. Godoy, E. Grong

$\rightarrow$ Rolling and Cartan connections

$\rightarrow$ Sub-Riemannian Holonomy groups

Cartan at ICM

- 1928, Élie Cartan → ICM in Bologna, Italy
  - theory of (several) Pfaffian equations
  - description as a non holonomic mechanical system
  - firs link with idea of "rolling"

E. Cartan (Paris - Francia)

SUR LA REPRÉSENTATION GÉOMÉTRIQUE
DES SYSTÈMES MATÉRIELS NON HOLONOMES

1. - Considérons un système matériel dont la position dépend de n paramètres $q_1, q_2, ..., q_n$; nous supposerons qu'il est assujetti à des liaisons non holonomes indépendantes du temps, définies par $n-m$ équations aux différentielles totales indépendantes

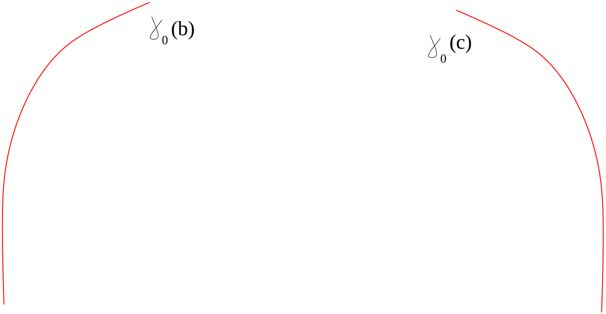
$$(1) \qquad \begin{cases} a_{11} dq_1 + ... + a_{1n} dq_n = 0, \\ \dots\dots\dots\dots\dots\dots \\ a_{n-m,1} dq_1 + ... + a_{n-m,n} dq_n = 0, \end{cases}$$

n'admettant aucune combinaison intégrable.

Outline

Application: Reconstruction of a curve - 1

Consider a smooth curve $\gamma_0 : [a,b] \cup [c,d] \to \mathbb{R}^2$, interrupted in $[b,c]$.

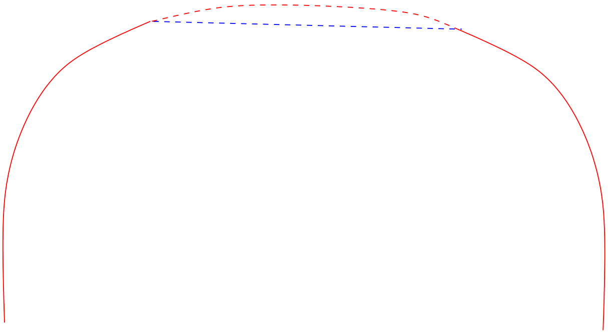


How does the brain reconstruct the image?

Application: Image reconstruction - 2

To reconstruct the curve the brain takes the decision that minimize some cost.

- it is not the Euclidean lenght of the curve



- but rather some costs that takes into account of the directions and curvature.

Reconstructing a curve

We can use the model of the car robot

- $X = \cos\theta\,\partial_x + \sin\theta\,\partial_y$
- $Y = \partial_\theta$
- $Z = -\sin\theta\,\partial_x + \cos\theta\,\partial_y$

$$\dot\gamma(t) = u(t)X(\gamma(t)) + v(t)Y(\gamma(t))$$

$$\int_0^T u(t)^2 + v(t)^2 dt \to \min$$

$\rightarrow$ to be precise it is not on $\mathbb{R}^2 \times S^1$ but rather on $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$

- Petitot ('99), Citti-Sarti ('03): model of the V1 cortex (after Nobel prizes Hubel and Wiesel '59)
- Agrachev-Boscain-Gauthier-Prandi et al., from 2012
- Sachkov et al, Duits et al.

Reconstruction of level sets, Sachkov et al.

Original

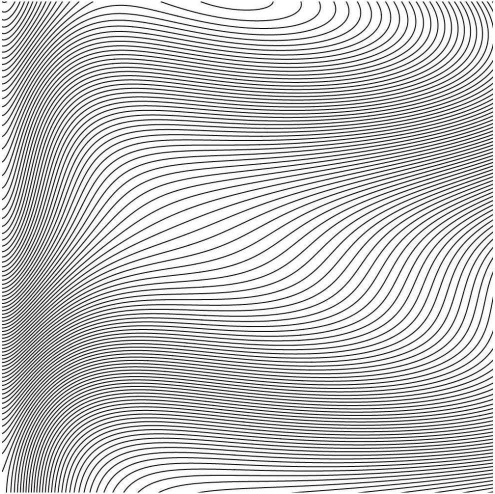


Image credits: Yuri Sachkov

Reconstruction of level sets, Sachkov et al.

corrupted

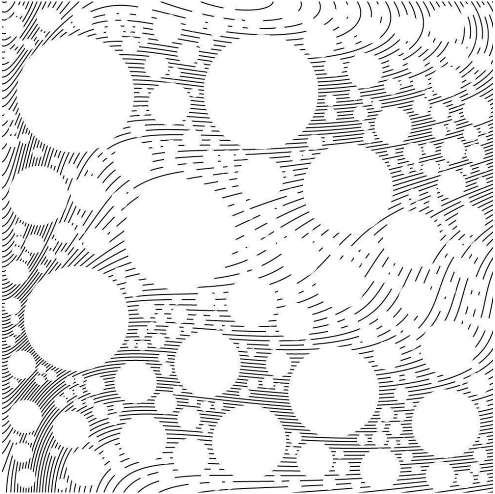


Image credits: Yuri Sachkov

Reconstruction of level sets, Sachkov et al.

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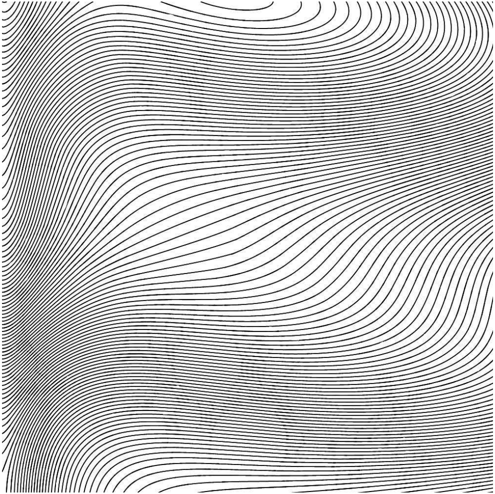


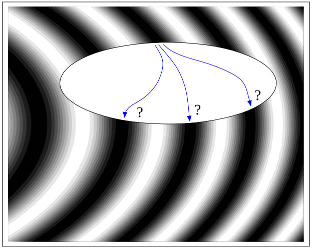
Image credits: Yuri Sachkov

Application: Image reconstruction - 3

Complex image (not a simple contour):

$\Downarrow$

Then all possible paths are activated as a Brownian motion



$$dx = X_1 dW_1 + X_2 dW_2 \qquad \longleftrightarrow \qquad \partial_t \psi(t,x) = (X_1^2 + X_2^2)\psi(t,x)$$

$$dP(x,t) = \psi(t,x)dx$$

The operator

$$\mathcal{L} = X_1^2 + X_2^2$$

is the sub-Laplacian of the sub-Riemannian structure.
The image is reconstructed by means of the sub-Riemannian Heat Equation

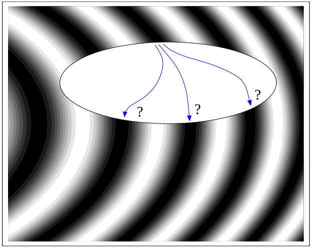
highly non-isotropic diffusion

Application: Image reconstruction - 3

Complex image (not a simple contour):

⇓

Then all possible paths are activated as a Brownian motion



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The operator

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(hypo-elliptic operators, under Lie bracket cond. $\Longrightarrow$ solutions are smooth)

→ This algorithm does not use the knowledge of where the image is corrupted

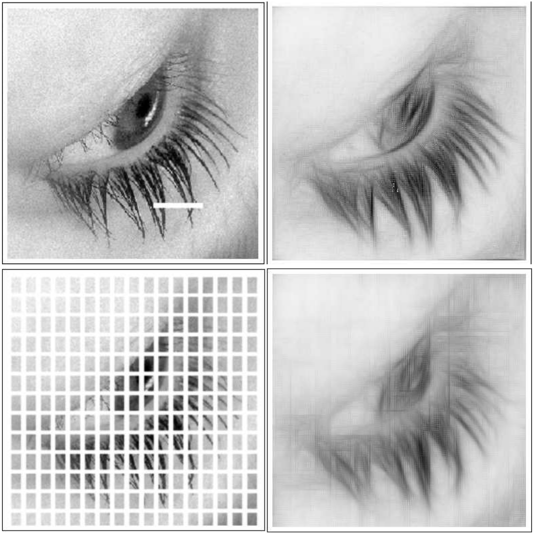


Image credits: Ugo Boscain

Dynamic restoration - Boscain, Gauthier, Prandi et al.

Can we do more by using the information of where the image is corrupted to work on images as ? (85% of corrupted pixels)

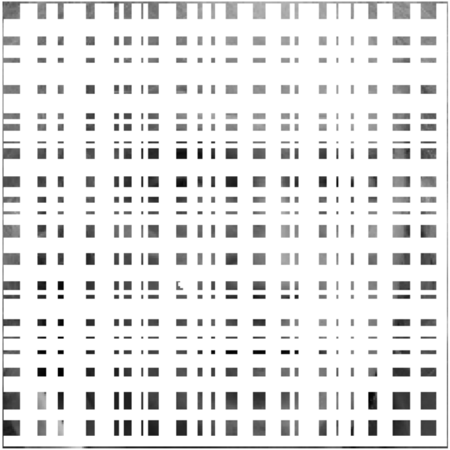


Image credits: Ugo Boscain

Using the information: Several steps

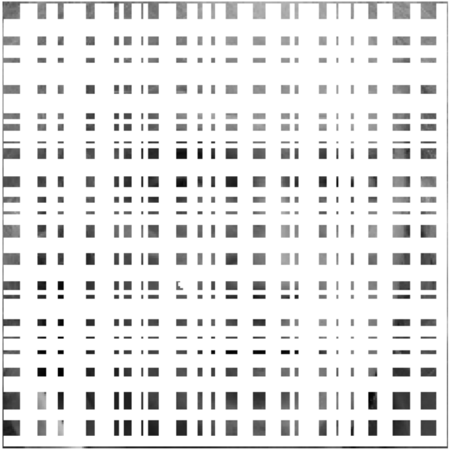
S0 one divides the pixels in “bad (corrupted)” and “good” (non corrupted)

S1 one makes a diffusion for δt using the previous method

S2 each good point is “averaged” with the original point

S3 bad points close to good points that got a sufficient mass are “upgraded” to good points.

S4 repeat from S1



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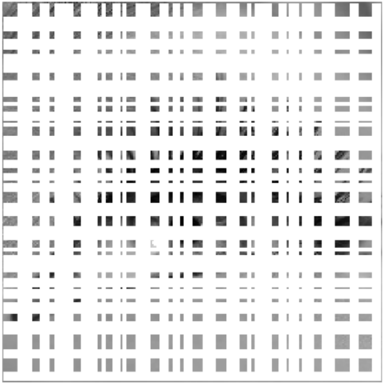


Image credits: Ugo Boscain

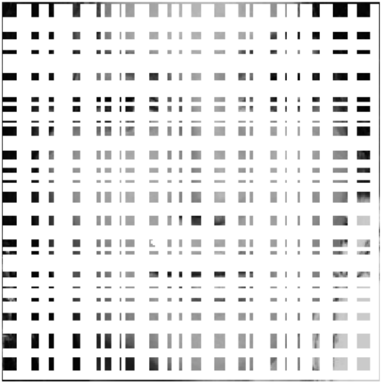
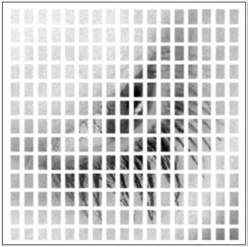
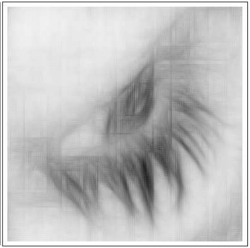


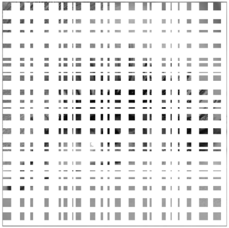
Image credits: Ugo Boscain

The visual cortex is doing “pure” hypoelliptic diffusion?

V1 is able to reconstruct an image like:



(pure hypoell. diffusion)

but not an image like



(hyp. diff. + dyn. restor.)

Image credits: Ugo Boscain

Sub-Riemannian Trimester - Paris 2014 - Proceedings

EMS Lecture Notes in Mathematics

- Volume 1
  - F. Serra Cassano
  - F. Baudoin
  - N. Garofalo
- Volume 2
  - A.Agrachev, D.Barilari, U.Boscain
  - L. Ambrosio, R. Ghezzi
  - P. Fritz, P. Gassiat
  - A. Thalmaier
  - M. Zhitomirskii

Geometric measure Theory, PDEs, Differential Geometry, Rough Paths, Diffusion and Stochastic processes, etc.