Volume geodesic distortion and Ricci curvature in sub-Riemannian geometry

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References

This is a joint work with

- Andrei Agrachev (SISSA, Trieste)
- Elisa Paoli (PhD at SISSA, 2015)

→ Main reference:

ABP-16 Andrei Agrachev, DB, Elisa Paoli
Volume geodesic distortion and Ricci curvature for Hamiltonian flows,
[Arxiv Preprint, 27 pp.]

→ Other references:

ABR-13 Andrei Agrachev, DB, Luca Rizzi
Curvature: a variational approach
[Memoirs of the AMS, in press.]

Outline

1. Motivation and introduction
2. Riemannian case
3. Main result and assumptions

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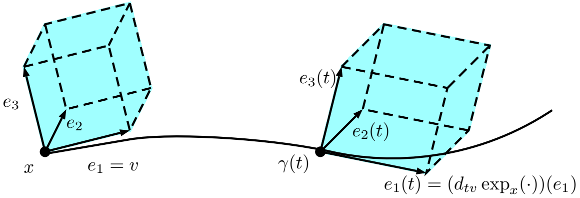
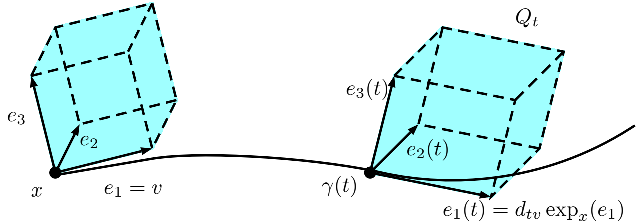
Introduction

One of the possible ways of introducing Ricci curvature in Riemannian geometry is by computing the variation of the Riemannian volume under the geodesic flow.

- fix x on a Riemannian manifold (M, g) and a tangent unit vector $v \in T_xM$
- $\gamma(t) = \exp_x(tv)$ geodesic starting at x with initial tangent vector v.
- an orthonormal basis $e_1, \ldots, e_n$ in T_xM
- $e_i(t) = (d_{tv} \exp_x)(e_i)$ ($\rightarrow$ Jacobi fields)



- Let Q_t be the parallelotope with edges $e_i(t)$
- vol_g is the canonical Riemannian volume



The volume of the time-dependent parallelotope Q_t has the following expansion for $t \to 0$,

$$\mathrm{vol}_g(Q_t) = 1 - \frac{1}{6}\mathrm{Ric}^g(v,v)t^2 + O(t^3), \tag{1}$$

Measure contraction along geodesics

- (M, g) is a complete, connected Riemannian manifold
- μ smooth volume form on M.

- Fix $\Omega \subset M$ be a bounded, measurable set, with $0 < \mu(\Omega) < +\infty$
- let $\Omega_{x,t}$ the set of t-intermediate points between Ω and x $\;(t \in [0, 1])$.

- Understand the behavior of $\mu(\Omega_{x_0,t})$
- Assume $\Omega = \exp_{x_0}(A)$, then $\Omega_{x_0,t} = \exp_{x_0,t}(A)$

$$\mu(\Omega_{x_0,t}) = \int_{\Omega_{x_0,t}} \mu = \int_A \exp^*_{x_0,t} \mu$$

$\rightarrow$ Idea: study for infinitesimal A, i.e. the asymptotic of $\exp^*_{x,t} \mu$

Measure contraction along geodesics

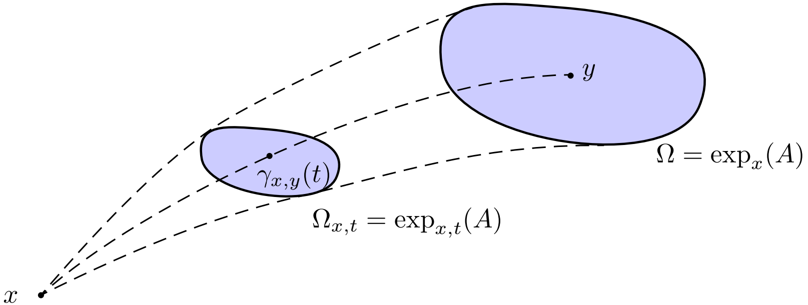


Figure: t-intermediate points between Ω and x.

- assume y regular value of $\exp_x$
- Ω small such that $\Omega = \exp_x(A)$.
- $\Omega_{x,t} = \{\gamma_{x,y}(t) \mid y \in \Omega\}$

Measure contraction along geodesics

Two basic examples

(R) For a Riemannian structure (M^n, g), it is well known that

$$\mu(\Omega_{x,t}) \sim t^n, \qquad \text{for } t \to 0,$$

[here $f(t) \sim g(t)$ means $f(t) = g(t)(C + o(1))$ for $t \to 0$ and $C > 0$]

(SR) In the 3D Heisenberg group it follows from [Juillet '09] that

$$\mu(\Omega_{x,t}) \sim t^5, \qquad \text{for } t \to 0,$$

$$5 \qquad \neq \qquad \text{top. dim. } (= 3) \qquad \neq \qquad \text{metric dim. } (= 4).$$

→ different dimensional invariant

→ associated with behavior of geodesics based at x_0

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- Understand the behavior of $\mu(\Omega_{x,t})$
- Assume $\Omega = \exp_x(A)$, then $\Omega_{x,t} = \exp_{x,t}(A)$

$$\mu(\Omega_{x,t}) = \int_{\Omega_{x,t}} \mu = \int_A \exp_{x,t}^* \mu$$

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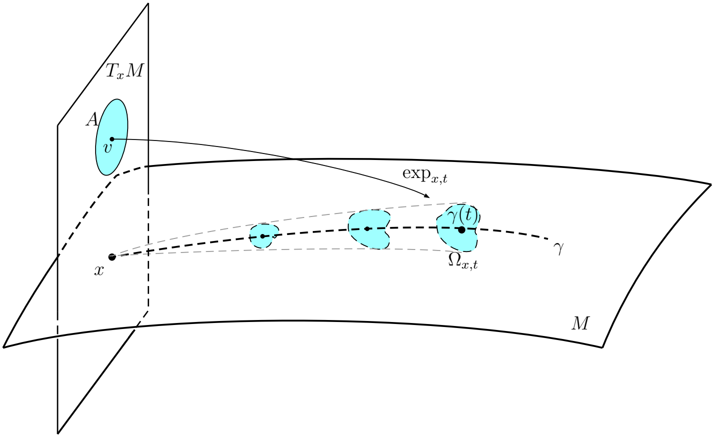

Given μ smooth volume on M

- $\exp_{x,t}^* \mu$ is a (t-dependent) measure that lives on T_xM
- we compare it with a fixed volume form there

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3. Main result and assumptions

Riemannian manifolds

Let (M, g) be a Riemannian manifold. Fix $x \in M$

- g_{ij} the coefficients of the metric
- $\exp_x : T_xM \to M$ be the exponential map
- vol$_g = \sqrt{\det g_{ij}}\, dx_1 \cdots dx_n$= Riemannian volume

→ From the classical formula in normal coordinates

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{ijkl} x^k x^l + o(|x|^2)$$

- one obtains the expansion

$$\sqrt{\det g_{ij}(\exp_x(tv))} = 1 - \frac{1}{6} \mathrm{Ric}^g(v, v) t^2 + o(t^2)$$

Using that

$$\phi^*(f\omega) = (f \circ \phi)\phi^*(\omega)$$

with $\phi = \exp_x : T_xM \to M$

Riemannian manifolds II

$$(\exp_x^* \mathrm{vol}_g)(tv) = \sqrt{\det g_{ij}(\exp_x(tv))}\, \underbrace{\exp_x^*(dx_1 \cdots dx_n)}_{\text{volume on } T_xM}$$

→ with respect to the exponential map at time t

$$\exp_{x,t}(v) := \exp_x(tv), \qquad \exp_{x,t}^* = t^n \exp_x^*$$

$$(\exp_{x,t}^* \mathrm{vol}_g)(v) = t^n \left(1 - \frac{1}{6}\mathrm{Ric}^g(v,v)t^2 + o(t^2)\right) \widehat{\mathrm{vol}}_x$$

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Weighted Riemannian manifolds

- (M, g, μ) with $\mu = e^{\psi}\,\mathrm{vol}_g$ and $\psi : M \to \mathbb{R}$ smooth.
- $\gamma(t) = \exp_{x,t}(v)$

$$(\exp_{x,t}^* \mu)(v) = t^n e^{\psi(\gamma(t))}\left(1 - \frac{1}{6}\mathrm{Ric}^g(v,v)t^2 + o(t^2)\right)\widehat{\mathrm{vol}}_x$$

Writing

$$\psi(\gamma(t)) = \psi(x) + \int_0^t \underbrace{\langle \nabla\psi(\gamma(s)), \dot\gamma(s)\rangle}_{\rho(\dot\gamma(t))}\, ds, \qquad \widehat{\mu}_x = e^{\psi(x)}\widehat{\mathrm{vol}}_x$$

$$(\exp_{x,t}^* \mu)(v) = t^n e^{\int_0^t \rho(\dot\gamma(s))ds}\left(1 - \frac{1}{6}\mathrm{Ric}^g(v,v)t^2 + o(t^2)\right)\widehat{\mu}_x$$

Goal

We want to extend this result to Hamiltonian quadratic on fibers

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k}(p \cdot X_i(x))^2 + p \cdot X_0(x) + \frac{1}{2}Q(x)$$

$\rightarrow$ are associated with the following optimal control problem:

$$\dot{x} = X_0(x) + \sum_{i=1}^{k} u_i X_i(x) \tag{2}$$

$$J_T(u) = \frac{1}{2}\int_0^T |u(s)|^2 - Q(x_u(s))ds \to \min \tag{3}$$

- $X_0, X_1, \ldots, X_k$ smooth vector fields
- Q is a smooth potential

Goal

We want to extend this result to Hamiltonian quadratic on fibers

$$H(p,x) = \frac{1}{2}\sum_{i=1}^{k}(p\cdot X_i(x))^2 + \cancel{p\cdot X_0(x) + \frac{1}{2}Q(x)}$$

→ are associated with the following optimal control problem:

$$\dot{x} = \cancel{X_0(x)} + \sum_{i=1}^{k} u_i X_i(x) \tag{4}$$

$$J_T(u) = \frac{1}{2}\int_0^T |u(s)|^2 - \cancel{Q(x_u(s))}\,ds \to \min \tag{5}$$

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$\rightarrow$ are associated with the following optimal control problem:

$$\dot{x} = \sum_{i=1}^{k} u_i X_i(x) \tag{6}$$

$$J_T(u) = \frac{1}{2}\int_0^T |u(s)|^2 ds \rightarrow \min \tag{7}$$

- $X_1, \ldots, X_k$ smooth vector fields

$\rightarrow$ $\dim \mathrm{Lie}\{X_1, \ldots, X_k\}(x) = n$, for all $x \in M$

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The Hamiltonian viewpoint

In this case we introduce the exponential map on the cotangent space

$$\exp_{x,t} : T_x^*M \to M, \qquad \exp_{x,t} = \pi \circ e^{t\vec{H}}\big|_{T_x^*M}$$

- $\pi : T^*M \to M$ canonical projection
- $\vec{H}$ the Hamiltonian vector field associated to H

$$\vec{H} = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}$$

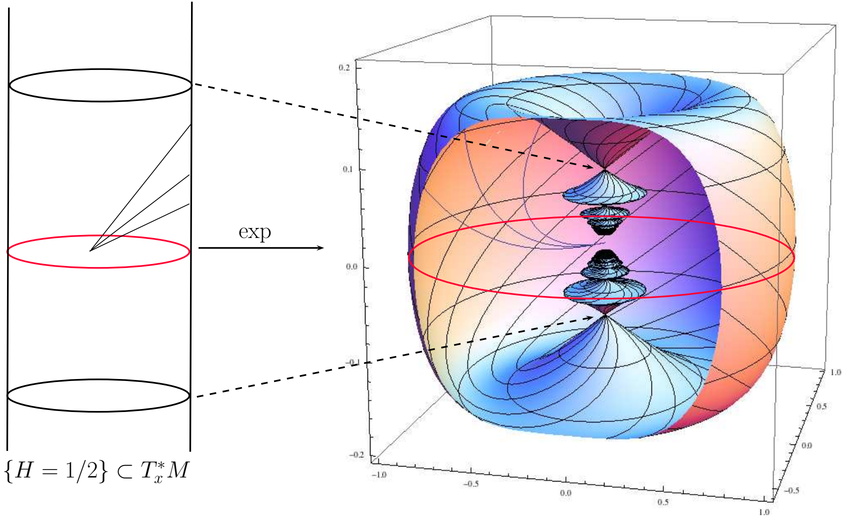
Comments:

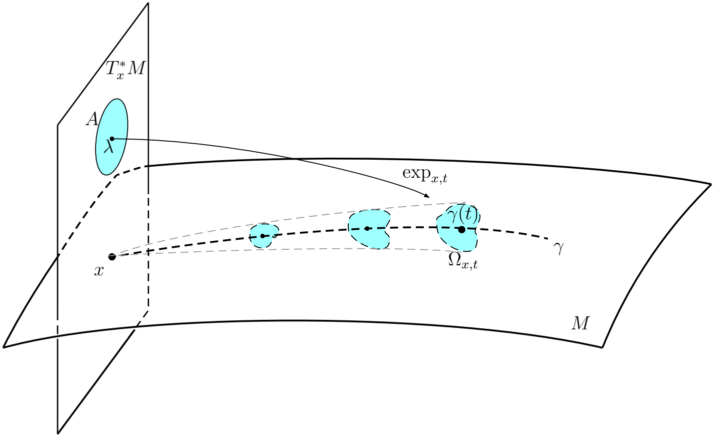
- in the Riemannian case the two approach are equivalent

→ (Using the canonical isomorphism $\mathfrak{i} : TM \to T^*M$ given by the metric g)

- in the sub-Riemannian one only the cotangent viewpoint survives!

Case $n = 3$ and $k = 2$





Given μ smooth volume on M

- $\widehat{\mu}_x$ = induced volume form on T_xM
- $\widehat{\mu}_x^*$ = induced volume form on T_x^*M dual to $\widehat{\mu}_x$, i.e. $\langle \widehat{\mu}_x^*, \widehat{\mu}_x \rangle = 1$

Main result

In the sub-Riemannian case we have the following theorem

Theorem (Agrachev, DB, Paoli, '16)

For any geodesic $\gamma(t) = \exp_{x,t}(\lambda)$ *ample and equiregular we have for* $t \to 0$

$$(\exp_{x,t}^* \mu)(\lambda) = C_\lambda t^{\mathcal{N}(\lambda)} e^{\int_0^t \rho(\dot\gamma(s))ds} \left(1 - \frac{1}{6}\mathrm{tr}(\mathcal{R}_\lambda) t^2 + o(t^2)\right) \widehat{\mu}_x^*$$

where

- C_λ *is a positive constant*
- $\mathcal{N}(\lambda)$ *is an integer*
- $\mathcal{R}_\lambda : D_{x_0} \to D_{x_0}$ *is a symmetric operator.*
- *Interaction volume-geodesic flow*

$$\rho(\lambda)\widehat{\mu}_x^* = \frac{d}{dt}\bigg|_{t=0} \log(t^{-\mathcal{N}(\lambda)} (\exp_{x,t}^* \mu)(\lambda))$$

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where

- C_λ *is a positive constant (*$\to C_\lambda = 1$ *in Riemannian case)*
- $\mathcal{N}(\lambda)$ *is an integer (*$\to \mathcal{N}(\lambda) = n$ *in Riemannian case)*
- $\mathcal{R}_\lambda : D_{x_0} \to D_{x_0}$ *is a symmetric operator. (*$\to \mathcal{R}_\lambda(v) = R^g(v, \dot\gamma)\dot\gamma$*)*
- *Interaction volume-geodesic flow*

$$\rho(\lambda) = \langle \nabla\psi(x), v\rangle\,, \qquad \lambda = \mathrm{i}(v)$$

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$$(\exp^*_{x,t}\mu)(\lambda) = C_\lambda t^{\mathcal{N}(\lambda)} \underbrace{e^{\int_0^t \rho(\dot\gamma(s))ds}}_{\text{volume-dynamics}} \underbrace{\left(1 - \frac{1}{6}\mathrm{tr}(\mathcal{R}_\lambda)t^2 + o(t^2)\right)}_{\text{dynamics}} \widehat{\mu}^*_x$$

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Geodesic growth vector

Let $\gamma(t) = \exp_{x,t}(\lambda)$. Let $\mathsf{T} \in \Gamma(D)$ an admissible extension of $\dot\gamma$

Geodesic flag

$$\mathcal{F}^1 = D, \qquad \mathcal{F}^{i+1} = \mathcal{F}^i + [\mathsf{T}, \mathcal{F}^i]$$

This defines a flag → well-defined along $\gamma(t)$

$$\mathcal{F}^1_{\gamma(t)} \subset \mathcal{F}^2_{\gamma(t)} \subset \ldots \subset T_{\gamma(t)}M \qquad (*)$$

→ does not depend on the choice of T

Geodesic growth vector

$$\mathcal{G}_{\gamma(t)} = \{k_1(t), k_2(t), \ldots\}, \qquad k_i(t) = \dim \mathcal{F}^i_{\gamma(t)}$$

→ in general different from the growth vector of D.

Ample and equiregular geodesics

A normal geodesic is

- ample at t if $\exists\, m = m(t) > 0$ s.t. $\mathcal{F}^m_{\gamma(t)} = T_{\gamma(t)}M$
- equiregular if $k_i(t) = \dim \mathcal{F}^i_{\gamma(t)}$ does not depend on t

- "Microlocal bracket generating condition". $\mathcal{G}_\gamma = \{k_1, \ldots, k_m\}$
- γ is not abnormal and no conjugate points for t small enough

$$\underbrace{\mathcal{A}^e}_{\text{ample+equireg}} \subset \underbrace{\mathcal{A}}_{\text{ample}} \subset T^*M$$

Theorem

- *The set $\mathcal{A}^e$ is nonempty, open and dense in T^*M.*
- *The set $\mathcal{A} \cap T^*_xM$ is nonempty, open and dense in T^*_xM, for all x.*

The main terms

Assume $\mathcal{G}_\gamma = \{k_1, \dots, k_m\}$

- The constant C_λ is explicit
  - $0 < C_\lambda \leq 1$
  - depends only on $\{k_1, \dots, k_m\}$
- The integer $\mathcal{N}(\lambda)$
  - satisfies

$$\mathcal{N}(\lambda) = \sum_{i=1}^{m} (2i-1)(k_i - k_{i-1})$$

  - is a sort of geodesic dimension
- in the contact case every non const. geodesic is ample equiregular

$$\mathcal{G}_\gamma = \{2n, 2n+1\}$$

$$C_\lambda = \frac{1}{12}, \qquad \mathcal{N}(\lambda) = 2n + 3 \cdot 1 = 2n + 3$$

The invariant ρ

It is defined by the identity

$$\rho(\lambda)\widehat{\mu}_x^* = \frac{d}{dt}\Big|_{t=0} \log(t^{-\mathcal{N}(\lambda)}(\exp_{x,t}^* \mu)(\lambda))$$

- We can define a canonical n-form ω defined only along $\gamma(t)$

Theorem

Let T *be any admissible extension of* $\dot{\gamma}$*. Then for every* $\lambda \in T_x^*M$

$$\rho(\lambda) = (\mathrm{div}_\mu \mathsf{T} - \mathrm{div}_\omega \mathsf{T})|_x. \quad (8)$$

- depends only on the symbol of the structure along the geodesic

$\rightarrow$ a sort of microlocal nilpotent approximation

- $\rho : \mathcal{A}^e \to \mathbb{R}$ is a rational function
- $\rho(c\lambda) = c\rho(\lambda)$ for all $c > 0$

The invariant ρ in contact manifolds

- Let (M^{2d+1}, ω) be a contact manifold:
- Fix a metric g on $D = \ker \omega$. Then (M, D, g) is a *sub-Riemannian manifold*.
  - ◇ g be extended to TM by requiring that the *Reeb vector field* X_0 is orthogonal to D and of norm one.
- The *contact endomorphism* $J : TM \to TM$ is defined by:

$$g(X, JY) = d\omega(X, Y), \qquad \forall\, X, Y \in \Gamma(TM).$$

Theorem

Let $\gamma(t) = \exp_{x,t}(\lambda)$ be any non constant geodesic on a contact manifold. Then

$$\rho(\lambda) = \frac{d}{dt}\bigg|_{t=0} \log \|J\dot{\gamma}(t)\|.$$

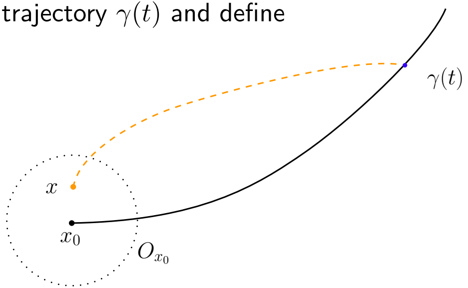
In particular, if $J^2 = -1$, then $\rho \equiv 0$.

Distance from a minimizer

Fix $x_0 \in M$ and a length-minimizer trajectory $\gamma(t)$ and define the geodesic cost

$$c_t : M \to \mathbb{R}, \quad t > 0$$

$$c_t(x) = -\frac{1}{2t}\mathsf{d}^2(x, \gamma(t))$$



Theorem (Agrachev, DB, Rizzi, '13)

Assume γ to be ample. Then

- *the function $(t, x) \mapsto c_t(x)$ is smooth on an open set $U \subset (0, \varepsilon) \times O_{x_0}$.*
- *$d_{x_0}\dot{c}_t = \lambda_0$ for any $t \in (0, \varepsilon)$.*

$\to$ $d^2_{x_0}\dot{c}_t : T_{x_0}M \to \mathbb{R}$ is a well defined family of quadratic forms.

Main result

Consider the restriction $d^2_{x_0}\dot{c}_t\big|_{D_{x_0}} : D_{x_0} \to \mathbb{R}$ to the distribution.

→ the scalar product $\langle\cdot,\cdot\rangle$ on D_{x_0} let us to define a family of symmetric operators

$$\mathcal{Q}_\lambda(t) : D_{x_0} \to D_{x_0}, \qquad d^2_{x_0}\dot{c}_t(v) = \langle \mathcal{Q}_\lambda(t)v, v\rangle\,, \quad v \in D_{x_0}$$

Theorem (Agrachev, DB, Rizzi, '13)

Assume the geodesic is ample. Then $\mathcal{Q}_\lambda(t)$ has a second order pole at $t=0$ and

$$\mathcal{Q}_\lambda(t) \simeq \frac{1}{t^2}\mathcal{I}_\lambda + \frac{1}{3}\mathcal{R}_\lambda + O(t), \qquad \text{for } t \to 0$$

where

- $\mathcal{I}_\lambda \geq \mathbb{I} > 0$
- *$\mathcal{I}_\lambda$ and $\mathcal{R}_\lambda$ are symmetric operators defined on D_{x_0}*

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where

- $\mathcal{I}_\lambda \geq \mathbb{I} > 0$ *[*$\to \operatorname{tr}(\mathcal{I}_\lambda) = \mathcal{N}(\lambda)$ *if equiregular]*
- $\mathcal{I}_\lambda$ *and* $\mathcal{R}_\lambda$ *are symmetric operators defined on* D_{x_0}

The Heisenberg case

The Heisenberg group $\mathbb{R}^3 = \{(x,y,z)\}$ with standard left-invariant structure

$$X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z$$

Every (non trivial) geodesic is

- ample and equiregular
- with geodesic growth vector $\mathcal{G} = (2,3)$.

If one fix two geodesics $\gamma_\lambda(t), \gamma_\eta(s)$ corresponding to two covectors λ, η

- $C(t,s) := \frac{1}{2} d^2(\gamma_\lambda(t), \gamma_\eta(s))$ is not C^2 at zero!

Still we can determine the main expansion

$$\mathcal{Q}_\lambda(t) \simeq \frac{1}{t^2}\mathcal{I}_\lambda + \mathcal{Q}_\lambda^{(0)} + O(t), \qquad \text{for } t \to 0$$

The Heisenberg case

We compute it on the orthonormal basis $v := \dot{\gamma}(0)$ and $v^\perp := \dot{\gamma}(0)^\perp$ for D_{x_0}.

The matrices representing $\mathcal{I}_\lambda$ and $\mathcal{R}_\lambda$ in the basis $\{v, v^\perp\}$ of D_{x_0} are

$$\mathcal{I}_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \qquad \mathcal{R}_\lambda = \frac{2}{5}\begin{pmatrix} 0 & 0 \\ 0 & h_z^2 \end{pmatrix}, \tag{9}$$

where λ has coordinates (h_x, h_y, h_z) dual to o.n. basis X, Y + Reeb Z

- anisotropy of the different directions on D_{x_0}
- curvature is always zero in the direction of $\dot{\gamma}$
- curvature of lines $\{h_z = 0\}$ is zero.
- curvature of lift of circles $\{h_z \neq 0\}$ is not bounded (nor constant)

$\rightarrow$ trace$(\mathcal{I}_\lambda) = 5 \leftrightarrow$ is related to MCP(0,5).

The invariant $\mathcal{R}_\lambda$ for contact manifolds

- there is a canonical linear connection, ∇ the *Tanno connection*
  - it is metric but not torsion free, extends Tanaka-Webster in CR geometry.
- The *Tanno tensors*

$$Q(X,Y) := (\nabla_Y J)X, \qquad \tau(X) = T^{\nabla}(X_0, X),$$

- $\tau = 0$ iff the Reeb is a Killing field
- $Q = 0$ iff the structure is CR

[Agrachev, DB, Rizzi, '15]

$$\operatorname{tr}(\mathcal{R}_\lambda) = \operatorname{Ric}^{\nabla}(\mathsf{T}) - \frac{3}{5} R^{\nabla}(\mathsf{T}, J\mathsf{T}, J\mathsf{T}, \mathsf{T}) + \frac{1}{5}\|Q(\mathsf{T},\mathsf{T})\|^2 - \frac{6}{5} g(\tau(\mathsf{T}), J\mathsf{T}) + \frac{8d-2}{20} h_0^2$$

SR Bonnet Myers for contact manifolds

Theorem (Agrachev, DB, Rizzi, '15)

Consider M a complete, SR contact structure of dimension $2d+1$, with $d>1$. Assume that for every horizontal unit vector X

$$\mathrm{Ric}^{\nabla}(X) - R^{\nabla}(X, JX, JX, X) \geq (2d-2)\kappa_1$$
$$\|Q(X,X)\|^2 \leq (2d-2)\kappa_2.$$

with $\kappa_1 > \kappa_2 \geq 0$. Then M is compact and

$$\mathrm{diam}_{SR}(M) \leq \frac{\pi}{\sqrt{\kappa_1 - \kappa_2}}$$

- first condition is the "partial trace" of the curvature on $\{X, JX\}^{\perp} \cap D$.
- for $\dim > 3$, the result was known only for $\tau = 0$ and $Q = 0$

THANKS FOR YOUR ATTENTION!