Comparison theorems for conjugate points in sub-Riemannian geometry

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Joint work with

Joint work with Luca Rizzi (CMAP, École Polytechnique)

→ Main Reference:

1. *Comparison theorems for conjugate points in sub-Riemannian geometry* (with L. Rizzi). Submitted. Preprint ArXiv.

→ Other references:

2. *The curvature: a variational approach* (with A. Agrachev and L. Rizzi). Submitted. Preprint ArXiv.
3. *Curvature for contact sub-Riemannian manifold* (with A. Agrachev and L. Rizzi). In preparation.

Outline

Outline

What do we mean by comparison theorem?

Let M be a Riemannian manifold:

Comparison between a property on M w.r.t. some model space:

- local property = sectional curvature, Ricci curvature
- model spaces = space forms ($\mathbb{R}^n$, S^n, H^n)

Many examples of these results:

- Bonnet-Myers theorem → diameter
- Bishop-Gromov inequality → volumes
- Spectral Gap inequality → first eigenvalue of Laplacian
- and also many geometric inequalities (Poincaré, Li-Yau, Sobolev, etc.)

In this talk we will focus on comparison on conjugate points.

Examples of comparison theorems

- M a Riemannian manifold.
- $\mathrm{Sec}(v, w)$ = sectional curvature of the plane $v \wedge w = R(v, w, v, w)$.
- $\mathrm{Ric}(v) = \mathrm{trace}\ \mathrm{Sec}(v, \cdot)$.

Theorem (Riemannian comparison for conjugate points)

Let γ be a unit speed geodesic:

(L) *If for all t and unit $v \perp \dot{\gamma}(t)$*

$$\mathrm{Sec}(\dot{\gamma}(t), v) \geq \kappa > 0$$

then $\gamma(t)$ has a conjugate point at time $t_c(\gamma) \leq \pi/\sqrt{\kappa}$.

(U) *If for all t and unit $v \perp \dot{\gamma}(t)$*

$$\mathrm{Sec}(\dot{\gamma}(t), v) \leq 0$$

then $\gamma(t)$ has no conjugate points, i.e. $t_c(\gamma) = +\infty$.

Examples of comparison theorems

- M a Riemannian manifold.
- $\mathrm{Sec}(v, w) =$ sectional curvature of the plane $v \wedge w = R(v, w, v, w)$.
- $\mathrm{Ric}(v) = \mathrm{trace}\ \mathrm{Sec}(v, \cdot)$.

Theorem (Riemannian comparison for conjugate points)

Let γ be a unit speed geodesic:

(AL) *If for all t*

$$\mathrm{Ric}(\dot\gamma(t)) \geq \kappa > 0$$

then $\gamma(t)$ has a finite first conjugate time $t_c(\gamma) \leq \pi/\sqrt{\kappa}$.

(U) *If for all t and unit $v \perp \dot\gamma(t)$*

$$\mathrm{Sec}(\dot\gamma(t), v) \leq 0$$

then $\gamma(t)$ has no conjugate points, i.e. $t_c(\gamma) = +\infty$.

$\rightarrow$ Proof: uses theory of Jacobi fields.

Some ideas

The first conjugate time $t_c(\gamma)$ is the infimum of $T > 0$ such that there exists a Jacobi field

$$J(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \gamma_s(t)$$

such that $J(0) = J(T) = 0$.

- Jacobi equation for Jacobi fields

$$\ddot{J}_i(t) + R_{ik}(t) J_k(t) = 0$$

where $J_1(t), \ldots, J_n(t)$ are n independent Jacobi fields along the geodesics and

$$R_{ij}(t) = \mathrm{Riem}(\dot{\gamma}(t), f_i(t), \dot{\gamma}(t), f_j(t))$$

where $f_1(t), \ldots, f_n(t)$ is parallely transported frame along γ.

- When M has constant curvature $R(t) = \kappa \mathbb{I}$ and one gets the solutions $x(t)$ of the equation

$$\ddot{x} + \kappa x = 0$$

Some ideas

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- Jacobi equation for Jacobi fields

$$\ddot{J}(t) + R(t)J(t) = 0$$

where $J(t) = (J_1(t), \dots, J_n(t))$ are n independent Jacobi fields along the geodesics and

$$\boxed{R(t) = \mathrm{Riem}(\dot{\gamma}(t), \cdot, \dot{\gamma}(t), \cdot)}$$

is the directional curvature written in a parallely transported frame.

- When M has constant curvature $R(t) = \kappa \mathbb{I}$ and one gets the solutions $x(t)$ of the equation

$$\ddot{x} + \kappa x = 0$$

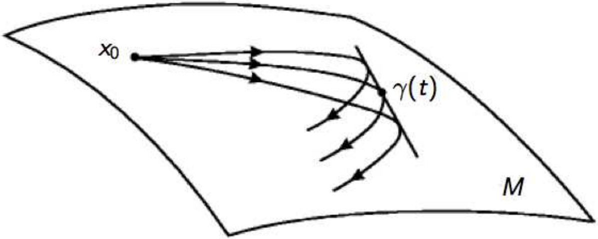


Figure: Conjugate points: where we lose local optimality

Motivation

We want to expand these ideas to sub-Riemannian geometry.

→ Difficulties

- No canonical connection and/or parallel transport
- Definition of sub-Riemannian curvature (sectional, Ricci)
- What are model spaces?

→ Main ideas:

- Sub-Riemannian problem is an affine optimal control problem
- Models: Linear-Quadratic problem with potential

→ Potential plays the role of the curvature

- Write the analogue of Jacobi equation
- Try to simplify them as much as possible → curvature

Why LQ optimal control problems?

Optimal control problem in $M = \mathbb{R}^n$ with k controls:

$$\dot{x} = Ax + Bu, \qquad \leftarrow \text{Kalman condition}$$

$$J_T(x_u(\cdot)) = \frac{1}{2}\int_0^T \left(|u|^2 - x^* Q x\right) dt \to min$$

The Hamiltonian function $H : T^*\mathbb{R}^n \to \mathbb{R}$ is

$$H(p, x) = \frac{1}{2} p^* B B^* p + p^* A x + \frac{1}{2} x^* Q x$$

Hamilton equations

$$(*) \begin{cases} \dot{p} = -A^* p - Qx \\ \dot{x} = BB^* p + Ax \end{cases}$$

The conjugate time t_c is the smallest $T > 0$ such that $\exists$ solution of $(*)$ such that $x(0) = x(T) = 0$

Why LQ optimal control problems?

Facts

- t_c depends only on A, B, Q.
- for $t < t_c$ there exists a unique optimal solution joining x_0 and x_1 in time t.
- for $t > t_c$ there are no optimal solution joining x_0 and x_1 in time t.

Example. Consider the case of a free particle in $\mathbb{R}^n$ with potential

$$\dot{x} = u, \qquad J_T(x_u(\cdot)) = \frac{1}{2}\int_0^T |u|^2 - x^* Q x \, dt.$$

In this case the Hamilton equations are equivalent to ($A = 0$ and $B = \mathbb{I}$)

$$\begin{cases} \dot{p} = -Qx \\ \dot{x} = p \end{cases} \qquad \Leftrightarrow \qquad \ddot{x} + Qx = 0$$

- These are precisely the equation of a Riemannian Jacobi field
- If $Q = \kappa \mathbb{I}$ we get the conjugate time $t_c = \pi/\sqrt{\kappa}$.

$\rightarrow$ The potential Q represents the directional curvature.

What to do: main ideas

Consider a SR geodesic $\gamma(t)$ (+ some assumptions on the geodesic)

We associate with it

- A "directional curvature" $\mathfrak{R}_{\gamma(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}$

$\rightarrow$ suitable adaptation of the Jacobi fields/equations

- a LQ control problem with $k = \dim \mathcal{D}$ control.

$\rightarrow$ Related to the linearization of the control system along the geodesic

$+$ A quadratic cost with potential Q that represents the bound for $\mathfrak{R}_{\gamma(t)}$.

Such that in the Riemannian case:

- $\mathfrak{R}_{\gamma(t)}(v) = \mathrm{Sec}(v, \dot{\gamma}(t))$
- $\dot{x} = u$ and $J_T = \frac{1}{2} \int u^2 - x^* Q x \, dt$

Outline

Affine optimal control problems

(Dynamic) Let us consider a smooth affine control system on a manifold M

$$\dot{x} = f(x,u) = X_0(x) + \sum_{i=1}^{k} u_i X_i(x), \qquad x \in M, u \in \mathbb{R}^k.$$

- we call $\mathcal{D}_x = \mathrm{span}_x\{X_1, \dots, X_k\}$ the *distribution*.
- we assume $Lie_x\{(ad^j X_0)X_i, i = 1, \dots, k, j \in \mathbb{N}\} = T_x M$ for all $x \in M$.

(Cost) Given a Tonelli Lagrangian $L : M \times \mathbb{R}^k \to \mathbb{R}$ we define the *cost at time T* as the functional

$$J_T(u) := \int_0^T L(\gamma_u(t), u(t))dt,$$

For two given points $x_0, x_1 \in M$ and $T > 0$, we define the value function

$$S_T(x_0, x_1) = \inf\{J_T(u) \,|\, u \text{ admissible}, \gamma_u(0) = x_0, \gamma_u(T) = x_1\},$$

Sub-Riemannian geometry

The (sub-)Riemannian case corresponds to the case when

- the system is driftless ($X_0 = 0$)
- $k < n$ ($k = n$ corresponds to Riemannian)
- the cost is quadratic
- Hörmander condition: $\text{Lie}_x\{X_1, \dots, X_k\} = T_x M$ for all $x \in M$

$$\dot{x} = \sum_{i=1}^{k} u_i X_i(x), \qquad x \in M, u \in \mathbb{R}^k.$$

$$J_T(u) := \frac{1}{2}\int_0^T \|\dot{\gamma}(t)\|^2 dt, \qquad S_T(x_0, x_1) = \frac{1}{2T} d^2(x_0, x_1)$$

→ The cost is induced by a scalar product such that $X_1, \dots, X_k$ are orthonormal.
→ $d(\cdot, \cdot)$ Carnot-Caratheodory distance, d is finite and continuous.
→ maximized Hamiltonian

$$H(p, x) = \frac{1}{2}\sum_{i=1}^{k} \langle p, X_i(x) \rangle^2$$

Exponential map

Two kind of extremals

- Abnormals: critical point of the end point map.
- Normals: projection of the flow of $\vec{H}$.

Theorem (PMP)

Let M be a SR manifold and let $\gamma : [0, T] \to M$ be a normal minimizer. $\exists$ Lipschitz curve $\lambda : [0, T] \to T^*M$, with $\lambda(t) \in T^*_{\gamma(t)}M$, such that $\dot{\lambda}(t) = \overrightarrow{H}(\lambda(t))$.

- $\lambda(t) = e^{t\vec{H}}(\lambda_0)$ $\to$ parametrized by initial covectors $\lambda_0 \in T^*_{x_0}M$
- $\gamma(t) = \pi(\lambda(t))$

The *exponential map* starting from x_0 as

$$\mathrm{Exp}_{x_0} : \mathbb{R}^+ \times T^*_{x_0}M \to M, \qquad \mathrm{Exp}_{x_0}(t, \lambda_0) = \pi(e^{t\vec{H}}(\lambda_0)) = \gamma(t).$$

Geodesic growth vector

Let γ be a normal geodesic. Let $T \in X_0 + \mathcal{D}$ an admissible extension of $\dot\gamma$

Geodesic flag

$$\mathcal{F}^i_\gamma(t) = \mathrm{span}\{\underbrace{[T,\dots,[T}_{j\leq i-1 \text{ times}},X]]|_{\gamma(t)} \mid \forall X \in \Gamma(\mathcal{D}), \quad j=0,\dots,i-1\}$$

For all t this defines a flag

$$\mathcal{F}^1_\gamma(t) \subset \mathcal{F}^2_\gamma(t) \subset \dots \subset T_{x_0}M$$

- Does not depend on the choice of T
- $\mathcal{F}^1_\gamma(t) = \mathcal{D}_{\gamma(t)}$.

Geodesic growth vector

$$\mathcal{G}_\gamma(t) = \{k_1(t), k_2(t), \dots\}, \qquad k_i(t) = \dim \mathcal{F}^i_\gamma(t)$$

$\rightarrow$ For an LQ problem $k_i = \mathrm{rank}\{B, AB, \dots, A^{i-1}B\}$.

Ample and equiregular geodesics

A normal geodesic is

- equiregular if $\dim \mathcal{F}^i_\gamma(t)$ does not depend on t
- ample if $\exists m > 0$ s.t. $\mathcal{F}^m_\gamma(t) = T_{x_0}M$

- "Microlocal Hörmander condition". $\mathcal{G}_\gamma = \{k_1, \dots, k_m\}$

$\rightarrow$ Related with controllability of the linearised system around γ

- Ample $\Rightarrow \gamma$ is not abnormal (even $\gamma|_{[0,t]}$ for all t).
- the linearized system along γ is controllable for all $T > 0$.

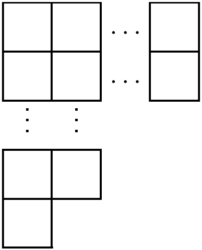
Let $\mathcal{G}_\gamma = \{k_1, k_2, \dots, k_m\}$

Lemma

For an equiregular ample geodesic the sequence $\{k_i - k_{i-1}\}_i$ is decreasing .

Young diagram of the geodesic

Let γ be an ample, equiregular geodesic, with $\mathcal{G}_\gamma = \{k_1, k_2, \ldots, k_m\}$



- $k_1 = \dim \mathcal{D}_{\gamma(t)}$
- $k_i - k_{i-1}$: new "directions" obtained with Lie derivative in direction of $\dot\gamma$
- ample geodesics: $\#$ boxes $= \dim M$ ($\rightarrow$ generic condition)
- Length of the rows $\{n_1, \ldots, n_k\}$

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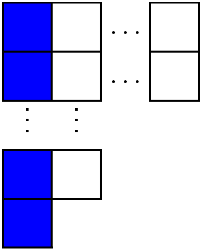

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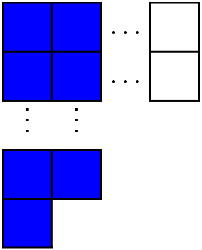

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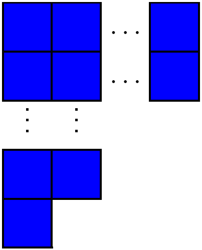

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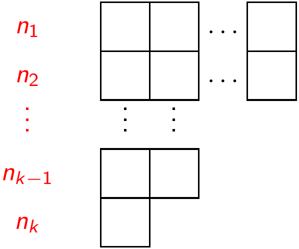

- $k_1 = \dim \mathcal{D}_{\gamma(t)}$
- $k_i - k_{i-1}$: new "directions" obtained with Lie derivative in direction of $\dot\gamma$
- ample geodesics: $\#$ boxes $= \dim M$ ($\rightarrow$ generic condition)
- Length of the rows $\{n_1, \dots, n_k\}$

For LQ problems: $\{n_1, \dots, n_k\} =$ Kronecker/controllability indices.

LQ models

Given an ample and equiregular geodesic with indices $n_1, \dots, n_k$

$\mathrm{LQ}(n_1, \dots, n_k; Q)$ is an LQ optimal control problem in $\mathbb{R}^n$ with

- k controls
- A, B corresponds to the Brunovsky normal form having indices $n_1, \dots, n_k$
  $\rightarrow$ coupling of k scalar equations $y^{(n_i)} = u_i$ for $i = i, \dots, k$.
- constant potential Q

We denote by $t_c(n_1, \dots, n_k; Q)$ its conjugate time

- a priori $t_c(n_1, \dots, n_k; Q)$ may be $+\infty$

$\rightarrow$ this always happens, for instance, when $Q = 0$.

Outline

Jacobi fields revisited

- $\gamma(t) = \pi(\lambda(t)) = \pi \circ e^{t\vec{H}}(\lambda_0)$, where $\lambda_0 \in T^*M$ initial covector of γ
- $\vec{H} \in \mathrm{Vec}(T^*M)$ Hamiltonian vector field

For any variation $\lambda_s \in T^*_{x_0}M$ of λ_0 we define the vector field along $\lambda(t)$:

$$X(t) := \frac{d}{ds}\bigg|_{s=0} e^{t\vec{H}}(\lambda_s) \in T_{\lambda(t)}(T^*M)$$

- $J(t) = \pi_* X(t)$ is a Jacobi field along the geodesic $\gamma(t) = \pi \circ \lambda(t)$

$$J(t) := \frac{d}{ds}\bigg|_{s=0} \gamma_s(t) = \frac{d}{ds}\bigg|_{s=0} \pi(e^{t\vec{H}}(\lambda_s)) \in T_{\gamma(t)}(M)$$

The first conjugate time $t_c(\gamma)$ is the smallest $T > 0$ such that there exists a Jacobi field along γ such that $J(0) = J(T) = 0$.

→ If γ not abnormal, then γ loses local optimality at time $t_c(\gamma)$

→ No connection needed.

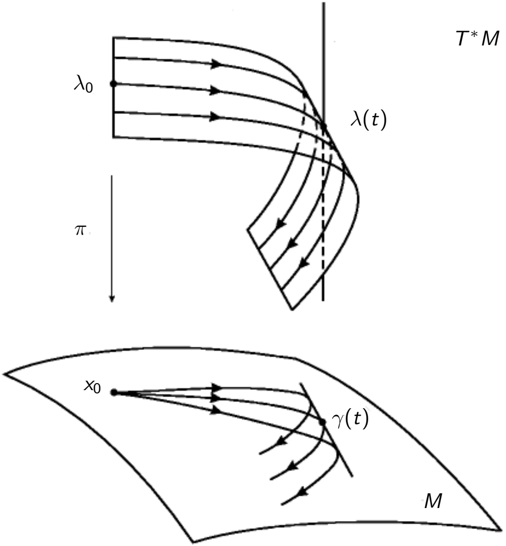

Figure: from "A.Agrachev, Y.Sachkov, *Control Theory from the geometric viewpoint.*"

Moving frame along the extremal

Aim: recover Jacobi equation, and generalize it to the sub-Riemannian setting

- σ is the symplectic form on T^*M

A frame along the extremal $\lambda(t)$:

$$E^i_{\lambda(t)}, F^j_{\lambda(t)} \in T_{\lambda(t)}(T^*M), \qquad i,j = 1, \dots, n$$

With the following properties:

- $\mathrm{ver}_{\lambda(t)} = \ker \pi_*|_{\lambda(t)} = \mathrm{span}\{E^i_{\lambda(t)},\ i = 1, \dots, n\}$
- It is a Darboux frame:

$$\sigma(E^i, E^j) = 0, \qquad \sigma(F^i, F^j) = 0, \qquad \sigma(E^i, F^j) = \delta_{ij}$$

$\rightarrow$ The projections $\pi_* F^i_{\lambda(t)}$ define a set of n vector fields along $\gamma(t) = \pi(\lambda(t))$.

Hamilton equations for the Jacobi fields

Jacobi field written in the moving frame along the extremal

$$X(t) = \sum_{i=1}^{n} p_i(t) E^i_{\lambda(t)} + x_i(t) F^i_{\lambda(t)}$$

The field $X(t)$ is associated with a curve $t \mapsto (p(t), x(t)) \in \mathbb{R}^{2n}$ such that

$$\dot{p} = -A_t^* p - Q_t x$$
$$\dot{x} = B_t B_t^* p + A_t x$$

for some matrices A_t, B_t, Q_t such that $\operatorname{rank} B_t = k$ and $Q_t = Q_t^*$

These are Hamilton equations in $\mathbb{R}^{2n}$ for the time-dependent Hamiltonian

$$H(p,x) = \frac{1}{2} p^* B_t B_t^* p + p^* A_t x + \frac{1}{2} x^* Q_t x$$

$\rightarrow$ The correspondence depends on the choice of the Darboux moving frame

Canonical frame

In the sub-Riemannian case, there exists a preferred choice:

- "Jacobi equation" = Hamilton equation for a LQ problem

Theorem (Agrachev-Zelenko 2002, Zelenko-Li 2009)

For any ample, equiregular geodesic $\gamma(t)$ with indices $n_1, \dots, n_k$ there exists a canonical moving frame along $\lambda(t)$ such that

- *A_t, B_t are constant, with A, B in Brunovski normal form*
- *Q_t has particular algebraic symmetries (equations as simple as possible)*

- This "replaces" the parallel transport along γ
- In the Riemannian case this procedure gives the equations

$$\begin{cases} \dot{p} = -Q_t x \\ \dot{x} = p \end{cases} \quad \Leftrightarrow \quad \ddot{x} + Q_t x = 0$$

Directional curvature

Denote $f_i(t) := \pi_* F^i_{\lambda(t)} \in T_{\gamma(t)}M$ the vector fields on γ.

$$T_{\gamma(t)}M = \mathrm{span}\{f_1(t), \dots, f_n(t)\}.$$

Sub-Riemannian directional curvature

The formula

$$\mathfrak{R}_{\gamma(t)}(f_i, f_j) := [Q_t]_{ij}$$

defines a well posed quadratic form

$$\mathfrak{R}_{\gamma(t)} : T_{\gamma(t)}M \times T_{\gamma(t)}M \to \mathbb{R}.$$

- In the Riemannian case

$$\mathfrak{R}_{\gamma(t)}(v) = \mathrm{Sec}(v, \dot\gamma(t))$$

- $\mathfrak{R}_{\gamma(t)}$ can be nicely expressed for contact manifold.

Outline

Microlocal comparison theorem

Theorem (DB, L.Rizzi, '14)

Let γ be an ample, equiregular geodesic, with indices $n_1, \ldots, n_k$. Then

(L) *if $\mathfrak{R}_{\gamma(t)} \geq Q_+$ for all t, then $t_c(\gamma) \leq t_c(n_1, \ldots, n_k; Q_+)$,*

(U) *if $\mathfrak{R}_{\gamma(t)} \leq Q_-$ for all t, then $t_c(n_1, \ldots, n_k; Q_-) \leq t_c(\gamma)$.*

- The first conjugate time of a LQ problem gives an estimate for the first conjugate time along the geodesic
- The LQ problem with Brunovsky normal form and constant potential is a model (i.e. we have equality)
- In SR case there are no example where the curvature $\mathfrak{R}_{\gamma(t)}$ is equal for all geodesics ($\rightarrow$ model spaces out of SR)
- We can "take out the direction of motion" (dimensional reduction)

Microlocal comparison theorem

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(U) *if $\mathfrak{R}_{\gamma(t)} \leq Q_-$ for all t, then $t_c(n_1, \dots, n_k; Q_-) \leq t_c(\gamma)$.*

Corollary (Constant curvature along γ)

Assume that $\mathfrak{R}_{\gamma(t)} = Q$ for all t, then $t_c(\gamma) = t_c(n_1, \dots, n_k; Q)$.

Corollary (Negative curvature)

Assume that $\mathfrak{R}_{\gamma(t)} \leq 0$ for all t, then $t_c(\gamma) = +\infty$

- ◇ These are matrix inequalities.
- ◇ Can be reduced to scalar with the "averaging" procedure. ($\rightarrow$ if I have time)

Conjugate points of LQ systems

Question: when does $t(n_1, \ldots, n_k; Q) < +\infty$?

Hamiltonian vector field of the LQ problem: $\vec{H}(p,x) = \begin{pmatrix} -A^* & -Q \\ BB^* & A \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}$

Theorem (Agrachev - Rizzi - Silveira, 2014)

The following are equivalent

- *LQ optimal control problem has finite conjugate time*
- *$\vec{H}$ has at least one Jordan block of odd size with purely imaginary eigenvalue.*

◇ computation of $t_c(n_1, \ldots, n_k, Q)$ reduces to an algebraic question

◇ there is no (evident) explicit formula for arbitrary Q and $n >> 1$.

◇ could be simplified with the "averaging" procedure. ($\rightarrow$ if I have time)

Example: Riemannian case

- For all γ we have $\mathcal{G}_\gamma = \{\dim M\} \Longrightarrow$ Indices: $\{1, 1, \ldots, 1\}$
- Moreover $\mathfrak{R}_{\gamma(t)}(v) = \mathrm{Sec}(\dot\gamma(t), v)$

Assume that $\mathfrak{R}_{\gamma(t)} = \mathrm{Sec}(\dot\gamma(t), v) \geq \kappa > 0$ for all unit $v \in T_{\gamma(t)}M$. Then

$$t_c(\gamma) \leq t_c(1, \ldots, 1; \kappa\mathbb{I}) = \pi/\sqrt{\kappa}$$

Indeed $\mathrm{LQ}(1, \ldots, 1; \kappa\mathbb{1})$ is the n-dimensional harmonic oscillator

$$H(p, x) = \frac{1}{2}(|p|^2 + \kappa|x|^2), \qquad t_c(1, \ldots, 1; \kappa) = \begin{cases} +\infty & \kappa \leq 0 \\ \frac{\pi}{\sqrt{\kappa}} & \kappa > 0 \end{cases}$$

Assume that $\mathfrak{R}_{\gamma(t)} = \mathrm{Sec}(\dot\gamma(t), v) \leq 0$ for all unit $v \in T_{\gamma(t)}M$. Then

$$t_c(\gamma) \geq t_c(1, \ldots, 1; 0) = +\infty.$$

Model example: Heisenberg group

- For all γ we have $\mathcal{G}_\gamma = \{2,3\} \Longrightarrow$ Kronecker indices: $\{2,1\}$
- Geodesic γ with initial covector $\lambda = (h_0, h_1, h_2)$.

→ Recall that $h_0 := \langle \lambda, Z \rangle$ is constant.

$$\mathfrak{R}_{\gamma(t)} = \begin{pmatrix} h_0^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: Q \qquad \text{constant along the extremal!}$$

$\mathrm{LQ}(2,1;Q)$ is a LQ problem in $\mathbb{R}^3$, with Hamiltonian

$$H(p,x) = \frac{1}{2}p_1^2 + p_2x_1 + \frac{1}{2}h_0^2x_1^2 \qquad t_c(2,1;Q) = \begin{cases} +\infty & h_0 = 0 \\ \frac{2\pi}{|h_0|} & h_0 \neq 0 \end{cases}$$

Let γ be a geodesic with initial covector λ, then $t_c(\gamma) = \begin{cases} +\infty & h_0 = 0 \\ \frac{2\pi}{|h_0|} & h_0 \neq 0 \end{cases}$

Model Example: SU(2) and SL(2)

- For all γ we have $\mathcal{G}_\gamma = \{2,3\} \Longrightarrow$ Kronecker indices: $\{2,1\}$
- Geodesic γ with initial covector $\lambda = (h_0, h_1, h_2)$.

→ Recall that $h_0 := \langle \lambda, Z \rangle$ is constant.

$$\mathfrak{R}^{SU(2)}_{\gamma(t)} = \begin{pmatrix} h_0^2+1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{R}^{SL(2)}_{\gamma(t)} = \begin{pmatrix} h_0^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

→ We recover [Boscain, Rossi - 2008]:

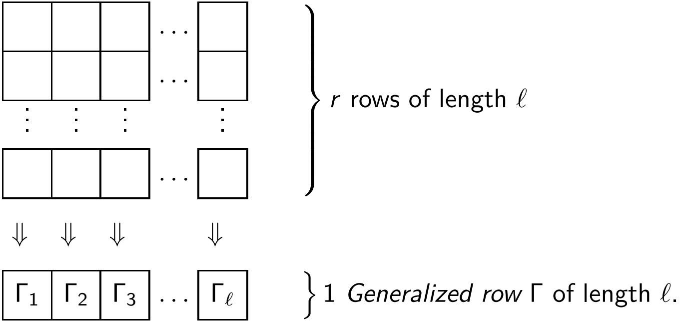
SU(2) Every geodesic has conjugate time $t_c(\gamma) = \dfrac{2\pi}{\sqrt{h_0^2+1}}$.

SL(2) Let γ be a geodesic with initial covector λ, then

$$t_c(\gamma) = \begin{cases} +\infty & |h_0| \leq 1 \\ \dfrac{2\pi}{\sqrt{h_0^2-1}} & |h_0| > 1 \end{cases}$$

Averaging - sub-Riemannian setting

- Collect all directions with the same controllability indices.



- Boxes, rows $\Longrightarrow$ *generalized* boxes, rows
- Average of $\mathfrak{R}_{\lambda(t)}$ w.r.t. directions in a gen. box $\Longrightarrow$ Ricci of the gen. box
- Riemannian case: 1 gen. box $\Longrightarrow$ 1 Ricci

Averaging - sub-Riemannian setting (2)

For a gen. row $\Gamma = \{\Gamma_1, \dots, \Gamma_\ell\}$, define the Ricci curvatures

$$\mathrm{Ric}_{\gamma(t)}(\Gamma_j) := \sum_{i \in \Gamma_j} \mathfrak{R}_{\gamma(t)}(f_i, f_i), \qquad j = 1, \dots, \ell$$

We have 1 comparison theorem for each gen. row

Theorem (DB, L.Rizzi, '14)

Let $\gamma(t)$ be an ample, equiregular geodesic. Assume that, for $\Gamma = \{\Gamma_1, \dots, \Gamma_\ell\}$

$$\frac{1}{r} \mathrm{Ric}_{\gamma(t)}(\Gamma_j) \geq \kappa_j, \qquad \forall j = 1, \dots, \ell$$

Then $t_c(\gamma) \leq t_c(\ell; Q)$, where $Q = diag\{\kappa_1, \dots, \kappa_\ell\}$

Sub-Riemannian Bonnet-Myers Theorem

- M complete, connected sub-Riemannian manifold
- All the minimizing geodesics have the same growth vector

Theorem (Sub-Riemannian Bonnet-Myers)

Assume that there exists a gen. row $\Gamma = \{\Gamma_1, \ldots, \Gamma_\ell\}$ *and constants* $\kappa_1, \ldots, \kappa_\ell$ *such that, for every geodesic,*

$$\frac{1}{r}\,\mathrm{Ric}_{\gamma(t)}(\Gamma_j) \geq \kappa_j, \qquad j = 1, \ldots, \ell$$

Then, if the polynomial

$$P_{\kappa_1,\ldots,\kappa_\ell}(x) = x^{2\ell} + \sum_{j=0}^{\ell-1} \kappa_{\ell-j} x^{2j} (-1)^{\ell-j-1}$$

has at least one simple imaginary root, the manifold is compact, has finite diameter $\leq t(\ell; \kappa_1, \ldots, \kappa_\ell)$. *Moreover its fundamental group is finite.*

Contact structures on 3D unimodular Lie Groups

- M is a unimodular, simply connected Lie group, $\dim M = 3$
- 1-form ω is the *contact form*. Distribution: $\Delta = \ker \omega$
- left-invariant sub-Riemannian structure $(\Delta, \langle \cdot | \cdot \rangle)$
- X_1, X_2 left-invariant orthonormal frame for $(\Delta, \langle \cdot | \cdot \rangle)$
- X_0 Reeb vector field: $X_0 \in \ker d\omega$, $\omega(X_0) = 1$
- Normalization $d\omega|_\Delta$ is the area element
- Structural constants: $[X_i, X_j] = \sum_{\ell=0}^{2} c_{ij}^{\ell} X_\ell$

Theorem (Agrachev, Barilari - 2012)

The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants: $\chi \geq 0$, $\kappa \in \mathbb{R}$.

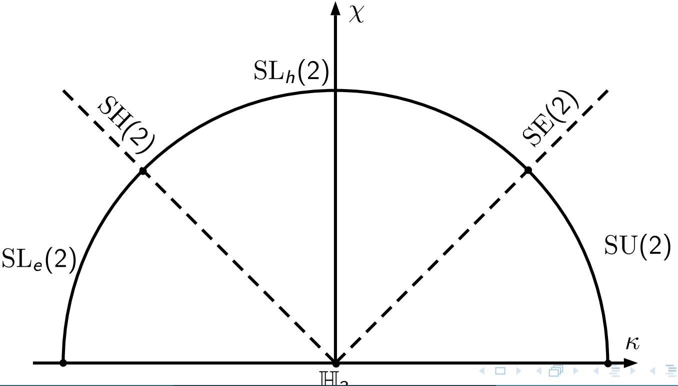
Up to rescaling $\chi^2 + \kappa^2 = 1$.

Contact structures on 3D unimodular Lie Groups

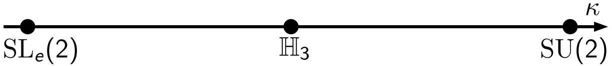
Theorem (Agrachev, Barilari - 2012)

The equivalence classes of isometric contact structures on 3D unimodular Lie groups are classified by two invariants $\chi, \kappa \in \mathbb{R}$.

Up to rescaling and reflections $\chi^2 + \kappa^2 = 1$ and $\chi \geq 0$.



Some known results (case $\chi = 0$)



$h_0 := \langle \lambda, X_0 \rangle$ is always a constant along the extremal

Theorem (Boscain, Rossi - 2008)

Let γ be a geodesic on $\mathrm{SL}(2)$, $\mathrm{SU}(2)$:

- $\mathrm{SL}(2)$ *($\kappa = -1$):* $t_c(\gamma) = \begin{cases} +\infty & h_0^2 \leq 1 \\ \frac{2\pi}{\sqrt{h_0^2 - 1}} & h_0^2 > 1 \end{cases}$
- $\mathrm{SU}(2)$ *($\kappa = 1$):* $t_c(\gamma) = \frac{2\pi}{\sqrt{h_0^2 + 1}}$

Some new results ($\chi > 0$)

Let $\chi > 0$. There exists a left-invariant orthonormal frame X_1, X_2 such that

$$[X_1, X_0] = (\chi + \kappa)X_2,$$
$$[X_2, X_0] = (\chi - \kappa)X_1,$$
$$[X_2, X_1] = X_0$$

Moreover the function $E : T^*M \to \mathbb{R}$ is a constant of the motion

$$E = \frac{h_0^2}{2\chi} + h_2^2, \qquad h_i(\lambda) := \langle \lambda, X_i \rangle$$

Theorem (Barilari, Rizzi - 2014)

Let M be a 3D unimodular Lie group with a left-invariant sub-Riemannian structure, with $\chi > 0$ and $\kappa \in \mathbb{R}$. Then there exists $\overline{E} = \overline{E}(\chi, \kappa)$ such that every length parametrised geodesic γ with $E(\gamma) \geq \overline{E}$ has a finite conjugate time.

Final Comments

Other results obtained:

- Proof of a Ricci-type "average" comparison result

→ Reduction to (more than one) scalar inequalities.

- Bonnet-Myers result (diameter estimate with t_c of LQ models).
- New results about conjugate points for unimodular 3D Lie groups

Technical points in the proofs

- Conjugate points = blow up time of a Riccati equation
- Comparison of solution for Matrix Riccati equations

→ This is highly extendable to other comparison results

- Difficult technical point: how to "average" ?

→ Collect all directions with the same controllability indices.

Good and bad points

◇ The method is quite general (no restriction on the sub-Riemannian structure)

◇ It could be very complicated to compute (and bound) $\mathfrak{R}_{\gamma(t)}$

THANKS FOR YOUR ATTENTION!