Heat kernel asymptotics at the cut locus for Riemannian and sub-Riemannian manifolds

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Joint work with

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- Jacek Jendrej (CMLS, École Polytechnique)
- Robert W. Neel (Lehigh University)

→ References:

1. D.B., U.Boscain, R.Neel, *Small time asymptotics of the SR heat kernel at the cut locus*, Journal of Differential Geometry, 92 (2012), no.3, 373-416.
2. D.B., J.Jendrej, *Small time heat kernel asymptotics at the cut locus on surfaces of revolution*. Ann. Inst. Henri Poincaré-Anal. Non Linéaire 31 (2014), 281-295.
3. D.B., U.Boscain, G.Charlot, R.Neel, *On the heat diffusion for generic Riemannian and sub-Riemannian structures*, submitted.

Outline

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Introduction

(Hypo)-elliptic operators $\longleftrightarrow$ *(Sub)-Riemannian metrics*

Main motivation:

- understand the interplay between

$\rightarrow$ the analysis of the diffusion processes on the manifold (heat equation)

$\rightarrow$ the geometry of these spaces (distance, geodesics, curvature)

Problem: relating

- analytic properties of the heat kernel $p_t(x, y)$ (small time asymptotics)
- geometry underlying (properties of distance and geodesics joining x and y)

$\rightarrow$ In particular: what happens for $p_t(x, y)$ when $y \in Cut(x)$?

$\rightarrow$ What happens "generically"?

Heat equation on $\mathbb{R}^2$

- The classical heat equation on $\mathbb{R}^2$

$$\partial_t \psi(t,x) = (\partial_{x_1}^2 + \partial_{x_2}^2)\psi(t,x)$$

- The fundamental solution, or *heat kernel*, of this equation

$$p_t(x,y) = \frac{1}{4\pi t} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

→ Every solution such that $\psi(0,x) = \phi(x)$ is of the form

$$\psi(t,x) = \int_{\mathbb{R}^2} p_t(x,y)\phi(y)dy$$

→ $p_t(\cdot, y)$ corresponds to the solution with initial datum Dirac δ_y.

Heat equation on $\mathbb{S}^2$

- The heat equation on the sphere $\mathbb{S}^2$

$$\partial_t \psi(t,x) = \Delta \psi(t,x)$$

where Δ is the Laplace Beltrami operator $\rightarrow$ elliptic operator.

- It is natural to expect that

$$p_t(x,y) \sim \frac{1}{4\pi t} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

- This is true everywhere but at the antipodal point $\widehat{x}$, where

$$p_t(x,\widehat{x}) \sim \frac{1}{4\pi t^{3/2}} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

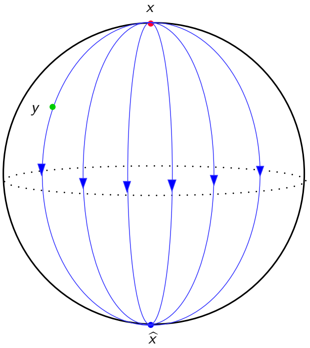
$\rightarrow$ Here and in what follows

$$f(t) \sim g(t) \qquad \Leftrightarrow \qquad f(t) = g(t)[C + o(1)], \quad C \neq 0$$

Heat vs Cut locus

Naive idea:
the heat diffuses along geodesics

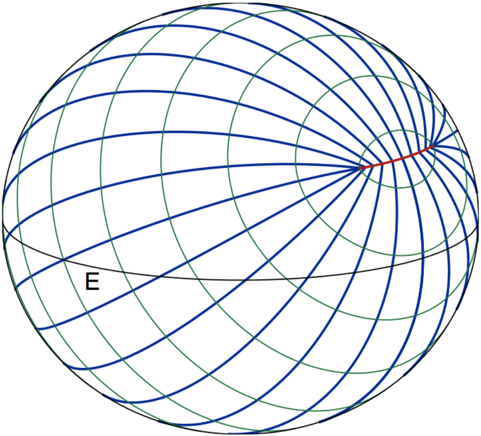
- only one optimal geodesic reaches y
- $\widehat{x}$ is the point where all geodesics meet
- $\widehat{x} = \mathrm{Cut}(x) = \mathrm{Conj}(x)$
- the function $x \mapsto d^2(x, \cdot)$ is not smooth at $\widehat{x}$



$\rightarrow$ even in this simple example it is easy to see how the structure of the geodesics is related with the heat kernel asymptotics.

Perturbation of the sphere: ellipsoid of revolution

- A complete proof on cut and conjugate locus has been proved only in 2004.
- (even if first works about geodesics on ellipsoids dates back to Jacobi)

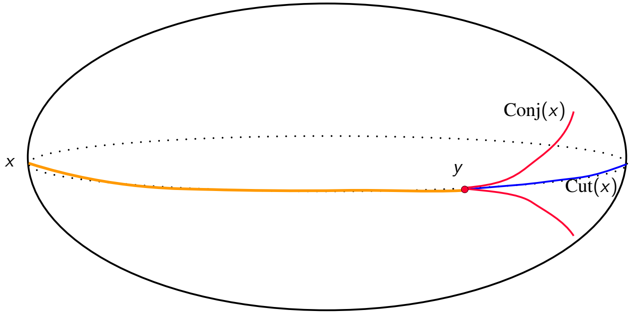


From Wikipedia:

Surfaces of revolution

For a metric on S^2 of the form $dr^2 + m^2(r)d\theta^2$ such that

- symmetric w.r.t. the equator
- non-singularity condition at the equator [i.e. $K'' \neq 0$]
- Typical example: ellipsoid of revolution



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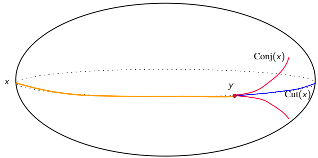
Theorem (D.B., J.Jendrej, '13)

Fix $x \in M$ along the equator and let y be a cut-conjugate point with respect to x. Then we have

$$p_t(x,y) \sim \frac{1}{t^{5/4}} e^{-d^2(x,y)/4t}, \qquad \text{for } t \to 0.$$



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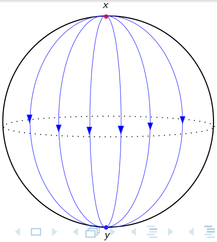
We have just said that on S^2

$$p_t(x,y) \sim \frac{1}{t^{3/2}} e^{-d^2(x,y)/4t}$$

$$x = \text{nord},\ y = \text{sud}$$



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Theorem (D.B., Jendrej)

Fix $x \in M$ along the equator and let y be a cut-conjugate point with respect to x. Then we have

$$p_t(x,y) \sim \frac{1}{t^{1+1/4}} e^{-d^2(x,y)/4t}, \qquad \text{for } t \to 0.$$

For the standard sphere S^2

$$p_t(x,y) \sim \frac{1}{t^{1+1/2}} e^{-d^2(x,y)/4t}$$

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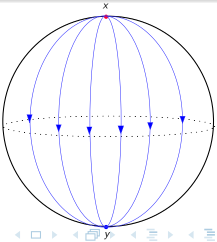


Outline

Sub-Riemannian geometry

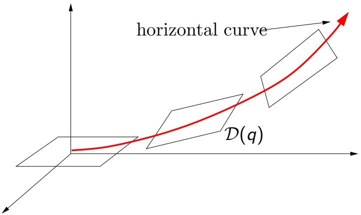
Definition

A *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$, where

- (i) M manifold, C^∞, dimension $n \geq 3$;
- (ii) $\mathcal{D}$ vector distribution of rank $k < n$, i.e. $\mathcal{D}_x \subset T_xM$ subspace k-dim. that is bracket generating: $\boxed{Lie_x\mathcal{D} = T_xM}$.
- (iii) $\langle\cdot,\cdot\rangle_x$ inner product on $\mathcal{D}_x$, smooth in x.

- A curve $\gamma : [0, T] \to M$ is horizontal if $\dot\gamma(t) \in \Delta_{\gamma(t)}$
- For a horizontal curve $\gamma : [0, T] \to M$ its length is

$$\ell(\gamma) = \int_0^T \sqrt{\langle\dot\gamma(t), \dot\gamma(t)\rangle}\, dt.$$



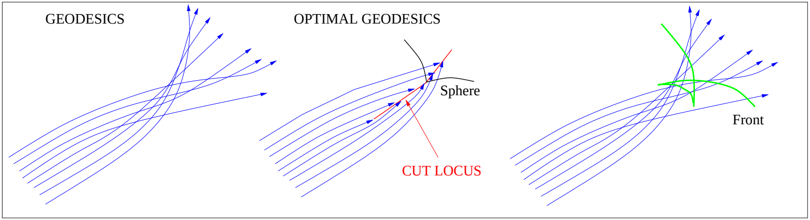
We can define the sub-Riemannian distance as

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y, \gamma \text{ horizontal}\}.$$

- The bracket generating condition implies

(i) $d(x, y) < +\infty$ for all $x, y \in M$.
(ii) topology (M, d) = manifold topology.

Question: Regularity of d^2? Relation with minimizing admissible curves?



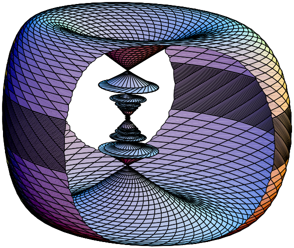
For a minimizing curve we can define

- Conjugate locus: where geodesics lose local optimality
- Cut locus: where geodesics lose global optimality (and d^2 is not smooth)

Regularity of d^2

Consider geodesics starting from $x \in M$

- geodesics lose optimality arbitrarily close to x
- $\mathfrak{f}(\cdot) = \frac{1}{2} d^2(x, \cdot)$ is not smooth at x



- $\mathfrak{f} : M \to \mathbb{R}$ is C^∞ on an open and dense set $\Sigma(x)$ [A.Agrachev, 2009]

$$\boxed{x \notin \Sigma(x)} \qquad \text{and} \qquad \boxed{\mathrm{Cut}(x) \subset M \setminus \Sigma(x)}$$

$\Sigma(x) = \{y \in M \mid \exists!$ non-abnormal and non-conjugate minimizer from x to $y\}$

$\to$ for simplicity: assume no minimizing abnormal extremals.

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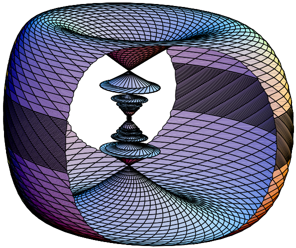

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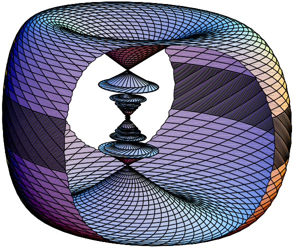

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$\rightarrow$ for simplicity: assume no minimizing abnormal extremals.

Conjugate points and Exponential map

- Normal minimizer are projection of the flow of $\vec{H}$.

Theorem (PMP)

Let M be a SR manifold and let $\gamma : [0, T] \to M$ be a minimizer. $\exists$ Lipschitz curve $\lambda : [0, T] \to T^*M$, with $\lambda(t) \in T^*_{\gamma(t)}M$, such that $\dot{\lambda}(t) = \overrightarrow{H}(\lambda(t))$.

- $\lambda(t) = e^{t\vec{H}}(\lambda_0) \to$ parametrized by initial covectors $\lambda_0 \in T^*_{x_0}M$
- $\gamma(t) = \pi(\lambda(t))$
- The *exponential map* starting from x_0 as

$$\mathrm{Exp}_{x_0} : T^*_{x_0}M \to M, \qquad \mathrm{Exp}_{x_0}(\lambda_0) = \pi(e^{\vec{H}}(\lambda_0)).$$

- $\mathrm{Exp}_{x_0}(t\lambda_0) = \gamma(t)$. $\quad (\to$ by homogeneity of $H)$

Fact:

- $\bar{t}$ *first conjugate time* along $\gamma \;\Rightarrow\; \bar{t}\lambda_0$ is a critical point of Exp_{x_0}.

SR Laplacian

We introduce the SR Laplacian operator Δ to define

$$\partial_t \psi(t,x) = \Delta \psi(t,x)$$

→ If $X_1, \dots, X_k$ is an orthonormal basis for $\mathcal{D}$ we set

$$\Delta \phi = div(\nabla \phi), \qquad \nabla \phi = \sum_i X_i(\phi) X_i$$

$$\Delta = \sum_{i=1}^{k} X_i^2 + (div\, X_i) X_i$$

→ sum of squares + 1st order term that depends on the volume

We need to fix a volume μ!

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$$\Delta_\mu \phi = div_\mu(\nabla \phi), \qquad \nabla \phi = \sum_i X_i(\phi) X_i$$

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Heat equation

The sub-Riemannian heat equation on a *complete* manifold M

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(t,x) = \Delta\psi(t,x), & \text{in } (0,\infty)\times M, \\ \psi(0,x) = \varphi(x), & x \in M, \quad \varphi \in C_0^\infty(M). \end{cases} \qquad (*)$$

Theorem (Hörmander)

If $\{X_1,\ldots,X_k\}$ are bracket generating, then Δ is hypoelliptic.

The problem $(*)$ has unique solution for $\varphi \in C_0^\infty(M)$

$$\psi(t,x) := e^{t\Delta}\varphi(x) = \int_M p_t(x,y)\varphi(y)d\mu(y), \qquad \varphi \in C_0^\infty(M),$$

where $p_t(x,y) \in C^\infty$ is the *heat kernel* associated with Δ.

Results on the asymptotic of $p_t(x, y)$

Fix $x, y \in M$, $\dim M = n$:

Theorem (Main term, Leandre, '87)

$$\lim_{t \to 0} 4t \log p_t(x, y) = -d^2(x, y) \quad (1)$$

Theorem (Smooth points, Ben Arous, '88)

Assume $y \in \Sigma(x)$, then

$$p_t(x, y) \sim \frac{1}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t}\right) \quad (1)$$

Facts

1. In Riemannian geometry $x \in \Sigma(x)$, in sub-Riemannian it is not true!
2. The on-the-diagonal expansion indeed is different.

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Theorem (On the diagonal, Ben Arous, '89)

We have the expansion

$$p_t(x, x) \sim \frac{1}{t^{Q/2}} \qquad (1)$$

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Theorem (Smooth points, Ben Arous, '88)

Assume $y \in \Sigma(x)$, *then*

$$p_t(x, y) \sim \frac{1}{t^{n/2}} \exp\left(-\frac{d^2(x, y)}{4t}\right) \qquad (1)$$

Questions

1. What happens in (1) if $y \in Cut(x)$?
2. Can we relate the expansion of $p_t(x, y)$ with the properties of the geodesics joining x to y?

Cut/Conjugacy *vs* Asymptotics

Theorem (D.B., Boscain, Neel,'12)

Let M be an n-dimensional complete SR manifold, μ smooth volume. Let $x \neq y$ and assume that every optimal geodesic joining x to y is strongly normal.

- *If x and y are not conjugate*

$$p_t(x,y) = \frac{C}{t^{n/2}} e^{-d^2(x,y)/4t}(1+O(t)),$$

- *If x and y are conjugate along at least one minimal geodesic*

$$\frac{C}{t^{(n/2)+(1/4)}} e^{-d^2(x,y)/4t} \leq p_t(x,y) \leq \frac{C'}{t^{n-(1/2)}} e^{-d^2(x,y)/4t},$$

→ we can detect only points that are cut and conjugate.

→ If we are cut but not conjugate the constant C changes.

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Case of a 2-dim surface

The theorem in the case of a 2-dim Riemannian surface says that

- If x and y are not conjugate

$$p_t(x,y) = \frac{C}{t} e^{-d^2(x,y)/4t}(1 + O(t)),$$

- If x and y are conjugate along at least one minimal geodesic

$$\frac{C}{t^{5/4}} e^{-d^2(x,y)/4t} \leq p_t(x,y) \leq \frac{C'}{t^{3/2}} e^{-d^2(x,y)/4t},$$

$\rightarrow$ all cases are between the ellipsoid and the sphere.

$\rightarrow$ they correspond to the "minimal" and "maximal" degeneration for a conjugate point on a surface.

Refinement

If $\gamma(t) = \mathrm{Exp}_x(t\lambda)$ joins x and y we say that

- γ is conjugate of order r if $rank(D_\lambda \mathrm{Exp}_x) = n - r$

Theorem (D.B., Boscain, Charlot, Neel,'13)

Let M be an n-dimensional complete SR manifold, μ smooth volume. Let $x \neq y$ and assume that the only optimal geodesic joining x to y is conjugate of order r.

- *Then there exist positive constants, such that for small t*

$$\frac{C}{t^{\frac{n}{2}+\frac{r}{4}}} e^{-d^2(x,y)/4t} \leq p_t(x,y) \leq \frac{C'}{t^{\frac{n}{2}+\frac{r}{2}}} e^{-d^2(x,y)/4t},$$

$\rightarrow$ This result can give estimates on the order of conjugacy of a point in the cut locus once you know the heat kernel (roughly, how much it is symmetric)

Example: Heisenberg

In the Heisenberg group the Heat kernel is explicit (here $q = (x, y, z)$)

$$p_t(0, q) = \frac{1}{(4\pi t)^2} \int_{-\infty}^{\infty} \frac{\tau}{\sinh \tau} \exp\left(-\frac{x^2 + y^2}{4t} \frac{\tau}{\tanh \tau}\right) \cos\left(\frac{z\tau}{t}\right) d\tau.$$

and gives the asymptotics for cut-conjugate points $\zeta = (0, 0, z)$

$$p_t(0, \zeta) \sim \frac{1}{t^2} \exp\left(-\frac{\pi z}{t}\right) = \frac{1}{t^2} \exp\left(-\frac{d^2(0, \zeta)}{4t}\right)$$

Remark: The fact that $\frac{4}{2} > \frac{3}{2}$ confirm the fact that the points $\zeta = (0, 0, z)$ are not smooth points. What is the meaning?

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Example: Heisenberg

In the Heisenberg group we had the asymptotics for cut-conjugate points $\zeta = (0, 0, z)$

$$p_t(0,\zeta) \sim \frac{1}{t^{\frac{4}{2}}} \exp\left(-\frac{\pi z}{t}\right) = \frac{1}{t^{\frac{4}{2}}} \exp\left(-\frac{d^2(0,\zeta)}{4t}\right)$$

Remark: This a consequence of the fact that there exists a one parametric family of optimal trajectories (varying the angle), hence the hinged energy function is actually a function of two variables, being constant on the midpoints.

Idea of the proof: What happens at non good point?

Let $x, y \in M$ with $y \in Cut(x)$ and write

$$p_t(x, y) = \int_M p_{t/2}(x, z) p_{t/2}(z, y) d\mu(z)$$

Idea: $z \in \Sigma(x) \cap \Sigma(y)$ and apply Ben-Arous expansion

$$p_{t/2}(x, z) p_{t/2}(z, y) \sim \frac{1}{t^n} \exp\left(-\frac{d^2(x, z) + d^2(z, y)}{4t}\right)$$

This led to the study of an integral of the kind

$$p_t(x, y) = \frac{1}{t^n} \int_M c_{x,y}(z) \exp\left(-\frac{h_{x,y}(z)}{2t}\right) d\mu(z)$$

where $h_{x,y}$ is the hinged energy function

$$h_{x,y}(z) = \frac{1}{2}\left(d^2(x, z) + d^2(z, y)\right).$$

$\rightarrow$ the asymptotic is given by the behavior of $h_{x,y}$ near its minimum. (Laplace method)

Properties of $h_{x,y}$ hinged energy function

Lemma

Let Γ be the set of midpoints of the minimal geodesics joining x to y. Then $\min h_{x,y} = h_{x,y}(\Gamma) = d^2(x,y)/4$.

- A minimizer is called strongly normal if any piece of it is not abnormal.

Theorem (D.B., Boscain, Neel,'12)

Let γ be a strongly normal minimizer joining x and y. Let z_0 be its midpoint. Then

(i) *y is conjugate to x along $\gamma \Leftrightarrow Hess_{z_0} h_{x,y}$ is degenerate.*

(ii) *The dimension of the space of perturbations for which γ is conjugate is equal to* $\dim(\ker Hess_{z_0} h_{x,y})$.

Remark: *Hess* $h_{x,y}$ is never degenerate along the direction of the geodesic!

Hinged vs Asymptotics

- To have the precise asymptotic one need that the expansion of $h_{x,y}$ is diagonal in some coordinates.

Theorem (D.B.,Boscain,Neel,'12)

Assume that, in a neighborhood of the midpoints of the strongly normal geodesic joining x to y there exists coordinates such that

$$h_{x,y}(z) = \frac{1}{4}d^2(x,y) + z_1^{2m_1} + \ldots + z_n^{2m_n} + o(|z_1|^{2m_1} + \ldots + |z_n|^{2m_n})$$

Then for some constant $C > 0$

$$p_t(x,y) = \frac{1}{t^{n-\sum_i \frac{1}{2m_i}}} \exp\left(-\frac{d^2(x,y)}{4t}\right)(C + o(1)).$$

Note: $h_{x,y}$ non degenerate ($m_i = 2$) $\rightarrow$ the exponent is $n/2$

Hinged vs Asymptotics

- To have the precise asymptotic one need that the expansion of $h_{x,y}$ is diagonal in some coordinates.

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Remarks

Nevertheless there are at least two cases that simplifies the analysis

- If we have symmetry $\rightarrow$ a one parametric family of optimal trajectories then $h_{x,y}$ is constant along the trajectory of midpoints.
- If there is only one degenerate direction then $h_{x,y}$ is always diagonalizable

Lemma (Splitting Lemma - Gromoll, Meyer, '69)

Let $h : \mathbb{R}^n \to \mathbb{R}$ smooth such that $h(0) = dh(0) = 0$ and that $\dim\ker d^2h(0) = 1$. Then there exists coordinates such that

$$h(z) = z_1^2 + \ldots + z_{n-1}^2 + \psi(z_n), \qquad \textit{where} \qquad \psi(z_n) = O(z_n^4).$$

Outline

Exponential map as a Lagrangian map

- A fibration $\pi : E \to N$ is Lagrangian if E is a symplectic manifold and each fiber is Lagrangian.
- A Lagrangian map is a smooth map $f : M \to N$ between manifolds of the same dimension obtained by composition of a Lagrangian immersion $i : M \to E$ and a projection

$$f : M \xrightarrow{i} E \xrightarrow{\pi} N.$$

The exponential map Exp_{x_0} is a Lagrangian map

$$\mathrm{Exp}_{x_0} : T^*_{x_0}M \to M, \qquad Exp_{x_0} = \pi \circ e^{\vec{H}}|_{T^*_{x_0}M}$$

It is the composition of

- Lagrangian immersion $e^{\vec{H}} : T^*_{x_0}M \to T^*M$
- a projection $\pi : T^*M \to M$

Normal form of generic singularities of Lagrangian maps

Theorem (Arnold's school)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a generic Lagrangian singularity at x_0. Then there exist changes of coordinates around x_0 and $f(x_0)$ such that in the new coordinates $x_0 = f(x_0) = 0$ and:

- *if $n = 1$, f is the map*
  $x \mapsto x^2$ $\qquad (A_2)$
- *if $n = 2$ then f is the map*
  $(x, y) \mapsto (x^3 + xy, y)$ $\qquad (A_3)$
  or a suspension of the previous one;
- *if $n = 3$ then f is the map*
  $(x, y, z) \mapsto (x^4 + xy^2 + xz, y, z)$ $\qquad (A_4)$
  $(x, y, z) \mapsto (x^2 + y^2 + xz, xy, z)$ $\qquad (D_4^+)$
  $(x, y, z) \mapsto (x^2 - y^2 + xz, xy, z)$ $\qquad (D_4^-)$
  or a suspension of the previous ones;

Normal form of generic singularities of Lagrangian maps

Theorem (Arnold's school)

- *if $n = 4$ then f is the map*
  $(x, y, z, t) \mapsto (x^5 + xy^3 + xz^2 + xt, y, z, t)$ (A_5)
  $(x, y, z, t) \mapsto (x^3 + y^2 + x^2z + xt, xy, z, t)$ (D_5^+)
  $(x, y, z, t) \mapsto (-x^3 + y^2 + x^2z + xt, xy, z, t)$ (D_5^-)
  or a suspension of the previous ones;
- *if $n = 5$ then f is the map*
  $(x, y, z, t, u) \mapsto (x^6 + xy^4 + xz^3 + xt^2 + xu, y, z, t, u)$ (A_6)
  $(x, y, z, t, u) \mapsto (x^4 + y^2 + x^3z + xt^2 + xu, xy, z, t, u)$ (D_6^+)
  $(x, y, z, t, u) \mapsto (-x^4 + y^2 + x^3z + xt^2 + xu, xy, z, t, u)$ (D_6^-)
  $(x, y, z, t, u) \mapsto (x^2 + xyz + ty + ux, y^3 + x^2z, z, t, u)$ (E_6^+)
  $(x, y, z, t, u) \mapsto (x^2 + xyz + ty + ux, -y^3 + x^2z, z, t, u)$ (E_6^-)
  or a suspension of the previous ones.

Question: which ones can appear as optimal singualities?
(i.e. as normal forms of Riemannian exponential maps at a cut-conjugate point?)

A3 singularity vs Exponential map

Let us consider the A3 singularity

$$\Phi : (x, y) \mapsto (x^3 + xy, y)$$

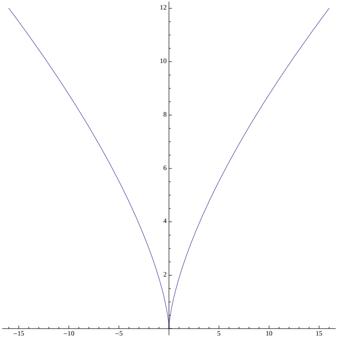
The set of critical points is

$$C = \{\det D\Phi = 0\} \Leftrightarrow \{3x - y^2 = 0\} \Leftrightarrow \{(t, 3t^2), t \in \mathbb{R}\}$$

The image of this set

$$\Phi(C) = \{(-2t^3, 3t^2)\} = \{y^3 = (27/4)x^2\}$$

It corresponds to the cut-conjugate point on the ellipsoid!



Lagrangian generic vs Riemannian generic

Let M be a smooth manifold and $\mathcal{G}$ be the set of all complete Riemannian metrics endowed with the C^∞ Whitney topology.

- We say that *for a generic Riemannian metric on M the property (P) holds* if the property (P) is satisfied on an open and dense subset of the set $\mathcal{G}$.

$\rightarrow$ Singularities of generic Riemannian exponential maps are generic Lagrangian singularities.

- Weinstein ('68), Wall ('76) and Janesko-Mostowski ('95).

Theorem

Let M be a smooth manifold with $\dim M \leq 5$, and fix $x \in M$. For a generic Riemannian metric on M, the singularities of the exponential map Exp_x are those listed in the previous Theorem.

Elimination of singularities

→ One can eliminate all the singularities but three of them if one restricts to optimal ones (i.e. along minimizing geodesics)

Theorem (DB, U.Boscain, G.Charlot, R.Neel)

Let M be a smooth manifold, $\dim M \leq 5$, *and $x \in M$. For a generic Riemannian metric on M and any minimizing geodesic γ from x to y we have that γ is*

- *either non-conjugate,*
- *A_3-conjugate,*
- *or A_5-conjugate.*

Notice that

- A_3 appears only for $\dim M \geq 2$
- A_5 can only appear for $\dim M \geq 4$.

→ in dimension 2 and 3 there is only "one kind" of generic cut-conjugate point.

Consequences

Corollary

Let M be a smooth manifold, $\dim M = n \leq 5$, *and $x \in M$. For a generic Riemannian metric on M the only possible heat kernel asymptotics are:*

(i) *No minimal geodesic from x to y is conjugate*

$$p_t(x,y) = \frac{C + O(t)}{t^{\frac{n}{2}}} \exp\left(-\frac{d^2(x,y)}{4t}\right),$$

(ii) *At least one min. geod. is A_3-conjugate but none is A_5-conjugate*

$$p_t(x,y) = \frac{C + O(t^{1/2})}{t^{\frac{n}{2}+\frac{1}{4}}} \exp\left(-\frac{d^2(x,y)}{4t}\right),$$

(iii) *At least one min. geod. is A_5-conjugate*

$$p_t(x,y) = \frac{C + O(t^{1/3})}{t^{\frac{n}{2}+\frac{1}{6}}} \exp\left(-\frac{d^2(x,y)}{4t}\right).$$

→ consistent with the results obtained on surfaces of revolution.

What is possible for non generic surfaces?

Theorem (D.B.,Boscain,Charlot,Neel,'13)

For any integer $r \geq 3$, any positive real α, and any real β, there exists a smooth metric on S^2 and $x \neq y$ such that

$$p_t(x,y) = \frac{1}{t^{\frac{3}{2}-\frac{1}{2r}}} e^{-d^2(x,y)/4t}(\alpha + t^{1/r}\beta + o(t^{1/r})).$$

- the existence of such expansions is not so surprising.
- the "big-O" term is computed and cannot in general be improved.
- we do see expansions in fractional powers of t (and not integer)

Idea of the proof

Let $\gamma(t) = \text{Exp}_x(t\lambda_0)$ join x and y and conjugate

$$\text{Singularity of } \text{Exp}_x \text{ at } \lambda_0 \Leftrightarrow \text{Singularity of } h_{x,y} \text{ at midpoint } z_0$$

Use two crucial facts:

- If γ is minimizing there exists a variation $\lambda(s)$ such that $y(s) = \text{Exp}_x(\lambda(s))$ satisfies $y(s) - y = O(s^3)$ in a coordinate system.
- Assume $rank(D_\lambda \text{Exp}_x) = n - 1$. Then

$$h_{x,y}(z) = \frac{d^2(x,y)}{4} + z_1^2 + \ldots + z_{n-1}^2 + z_n^m$$

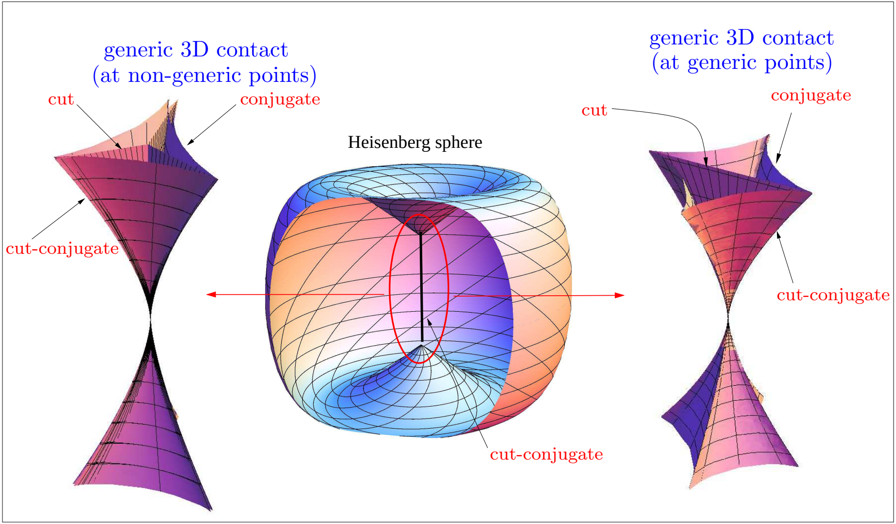
where $m = \max\{k \in \mathbb{N} \,|\, y(s) - y = s^k v + o(t^k), v \neq 0\}$ for all variations $y(s) = \text{Exp}_x(\lambda(s))$.

3D contact case

For the generic 3D contact case [Agrachev, Gauthier et al.,'96]

- close to the diagonal only singularities of type A_3 appear, accumulating to the initial point.
- The local structure of the conjugate locus is
  - either a suspension of a four-cusp astroid (at generic points)
  - or a suspension of a "six-cusp astroid" (along some special curves).
- for the four-cusp case, two of the cusps are reached by cut-conjugate geodesics,
- in the six-cusp case this happens for three of them.

$\rightarrow$ Notice that the conjugate locus at a generic point looks like a suspension of the first conjugate locus that one gets on a Riemannian ellipsoid


generic 3D contact
(at non-generic points)
cut
conjugate
cut-conjugate
Heisenberg sphere
cut-conjugate
generic 3D contact
(at generic points)
cut
conjugate
cut-conjugate

Theorem

Let M be a smooth manifold of dimension 3. Then for a generic 3D contact sub-Riemannian metric on M, every x, and every y (close enough to x) we have

(i) *If no minimal geodesic from x to y is conjugate then*

$$p_t(x,y) = \frac{C + O(t)}{t^{3/2}} \exp\left(-\frac{d^2(x,y)}{4t}\right),$$

(ii) *If at least one minimal geodesic from x to y is conjugate then*

$$p_t(x,y) = \frac{C + O(t^{1/2})}{t^{7/4}} \exp\left(-\frac{d^2(x,y)}{4t}\right),$$

Moreover, there are points y arbitrarily close to x such that case (ii) occurs.

- exponents of the form $N/4$, for integer N, were unexpected in the 90s literature for points close enough

Paris, 2014 - *www.cmap.polytechnique.fr/subriemannian*