On the regularity of the value function for a class of affine optimal control problems

Davide Barilari
IMJ-PRG, Université Paris Diderot

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This is a joint work with

- Francesco Boarotto (CMAP, École Polytechnique)

  - D.B., F. Boarotto, *On the set of points of smoothness for the value function of affine optimal control problems*, ArXiv preprint 2016.

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- Luca Rizzi (Institut Fourier, Univ. Grenoble Alpes)

  - D.B., L. Rizzi, *Sub-Riemannian interpolation inequalities : ideal case*, ArXiv preprint 2017.

Outline

1. Affine control systems with action-like cost
2. On the set of points of smoothness of the value function
3. Characterization of cut locus : sub-Riemannian case without abnormal
4. An open question

Outline

Affine optimal control problems

(Dynamic) Consider a smooth (C^∞) affine control system on a manifold M

$$\dot{x} = F(x,u) = X_0(x) + \sum_{i=1}^{k} u_i X_i(x), \qquad x \in M, u \in \mathbb{R}^k.$$

- fix a control $u \in L^2([0,T],\mathbb{R}^k)$ ($\rightarrow$ control set unbounded).
- denote by $x_u(\cdot)$ the solution of the Cauchy problem with $x(0) = x_0$
- $A_{x_0}^T$ is the attainable set from x_0 in time $T > 0$

(Cost) Given a smooth function $L : M \times \mathbb{R}^k \to \mathbb{R}$ we define the *cost at time T*

$$C_T(u) := \int_0^T L(x_u(t), u(t))dt,$$

Definition

For a fixed $x_0 \in M$ and $T > 0$, we define the *value function*

$$S_{x_0}^T(x) = \inf \left\{ C_T(u) \,|\, u \text{ admissible}, x_u(0) = x_0, x_u(T) = x \right\},$$

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- $A_{x_0}^T$ is the attainable set from x_0 in time $T > 0$

(Cost) In what follows we will mainly deal with action-like cost

$$C_T(u) := \frac{1}{2}\int_0^T \|u(t)\|^2 - Q(x_u(t))dt$$

Definition

For a fixed $x_0 \in M$ and $T > 0$, we define the *value function*

$$S_{x_0}^T(x) = \inf\left\{C_T(u) \,|\, u \text{ admissible}, x_u(0) = x_0, x_u(T) = x\right\},$$

Assumptions

$$\dot{x} = X_0(x) + \sum_{i=1}^{k} u_i X_i(x), \qquad x \in M, u \in \mathbb{R}^k. \tag{1}$$

$$C_T(u) := \int_0^T \|u(t)\|^2 - Q(x_u(t))dt \tag{2}$$

(A1) The *weak bracket generating* condition is satisfied, namely

$$\mathrm{Lie}_x \left\{ (\mathrm{ad}\, X_0)^i X_j, i \in \mathbb{N}, j = 1, \ldots, k \right\} = T_x M,$$

[the vector field X_0 is not included in the generators of the Lie algebra]

(A2) For every bounded family $\mathcal{U}$ of admissible controls, there exists a compact subset $K_T \subset M$ (depending on $\mathcal{U}$) such that $x_u(t) \in K_T$, for every $u \in \mathcal{U}, t \in [0, T]$.

(A3) The potential Q is a smooth function bounded from above.

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(A3) The potential Q is a smooth function bounded from above.

- (A1) guarantees that $\mathrm{int}\left(A^T_{x_0}\right) \neq \emptyset$. [Jurdjevic-Sussmann, '72]
- (A2) is a completeness/compactness assumption on the dynamical system
- (A2)+(A3), guarantees the existence of optimal controls.

→ (A2) and (A3) are automatically satisfied when M is compact.

→ For M not compact, (A2) can be replaced by growth condition on vector fields (sublinear / other).

Motivation : a framework for geometry (and analysis)

This setting includes many different geometric structures such as

- Riemannian structures (or mechanical systems on Riemannian manifolds)
- sub-Riemannian structures
- smooth Finsler structures (or even sub-Finsler)

The (sub-)Riemannian case corresponds to the case when

- the system is driftless with zero potential ($X_0 = 0, Q = 0$)
- $k < n$ ($k = n$ corresponds to Riemannian)
- the cost is quadratic $L(x, u) = \frac{1}{2}|u|^2$

$\rightarrow$ The cost is induced by a scalar product such that $X_1, \dots, X_k$ are orthonormal.

$(*)$ Given a smooth measure μ on M we can introduce a sub-Laplacian operator

$$\Delta_\mu = div_\mu \nabla = \sum_{i=1}^{k} X_i^2 + (div_\mu X_i) X_i$$

The (sub)-Riemannian case

$$\dot{x} = \sum_{i=1}^{k} u_i X_i(x) \qquad C_T(u) := \frac{1}{2}\int_0^T \|u(t)\|^2 dt \tag{3}$$

- (A1) is the classical bracket generating condition

$$\mathrm{Lie}_x \{X_j \mid j = 1, \dots, k\} = T_x M,$$

- we have $A_{x_0}^T = M$ and

$$S_{x_0}^T(x) = \frac{1}{2T} d_{SR}^2(x_0, x)$$

  where d_{SR} is the Carnot-Carathéodory distance
- (A2) is naturally replaced by (M, d_{SR}) complete metric space.
- (A3) is automatic since $Q = 0$

Rashevsky-Chow Theorem

The value function $S_{x_0}^T(x) = \frac{1}{2T} d_{SR}^2(x_0, x)$ is continuous on M

(Some) other results

$$\dot{x} = X_0(x) + \sum_{i=1}^{k} u_i X_i(x) \qquad C_T(u) := \frac{1}{2}\int_0^T \|u(t)\|^2 - Q(x_u(t))dt \qquad (4)$$

with some similar assumptions

- [Trélat '00] Continuity and subanaliticity with no abnormals ($Q = 0$, C^ω)
- [Cannarsa-Rifford '08] Semiconcavity with no abnormals (for Tonelli L)
- [Agrachev-Lee '10] Continuity (growth on L, horizontal step 3 condition)
- [Cannarsa-Frankowska et al.] Regularity along optimal trajectories
- many others

Regularity of $S^T_{x_0}$: heuristics

- $S^T_{x_0}$ is smooth at "good points" (reached by a unique "good" minimizer)
- can be not continous at "bad points" (for instance when have abnormals).

→ Can we have a qualitative understanding of the good set?

Regularity of $d^2_{SR}(x_0, \cdot)$ in SR geometry

Let $x_0 \in M$ and $S^T_{x_0}(x) = \frac{1}{2T} d^2_{SR}(x_0, x)$. Define the set

$$\Sigma(x_0) = \{x \in M \mid \exists! \text{ not abnormal non-conjugate minimizer from } x_0 \text{ to } x\}$$

Theorem (Agrachev '09, Trélat-Rifford '05)

$\Sigma(x_0)$ is open and dense in M. Moreover $d^2_{SR}(x_0, \cdot)$ is smooth on $\Sigma(x_0)$

- $d^2_{SR}(x_0, \cdot)$ never smooth on diagonal, i.e., $x_0 \notin \Sigma(x_0)$.

→ Big open question: is $\mathrm{measure}(M \setminus \Sigma(x_0)) = 0$?

- crucial that $d^2_{SR}(x_0, \cdot)$ is continuous, even with "bad" abnormals.

1. Can we extend to our setting?
2. Can we characterize $\mathrm{Cut}(x_0) = M \setminus \Sigma(x_0)$? (→ in SR case)

Outline

End-point map

Fix a point $x_0 \in M$ and $T > 0$.

- The *end-point map at time T* is the smooth map

$$E_{x_0}^T : \mathcal{U} \to M, \qquad u \mapsto x_u(T),$$

  where $\mathcal{U} \subset L^2([0,T],\mathbb{R}^k)$ is the open subset s.t. $x_u(t)$ is defined on $[0,T]$.
- The attainable set $A_{x_0}^T = E_{x_0}^T(\mathcal{U})$.

→ Value function rewritten via the end-point map

$$S_{x_0}^T(x) = \inf\{C_T(u) \mid E_{x_0}^T(u) = x\} = \inf_{(E_{x_0}^T)^{-1}(x)} C_T$$

In general $E_{x_0}^T$ is smooth but

- $d_u E_{x_0}^T$ is not surjective
- the set $(E_{x_0}^T)^{-1}(x)$ is not a smooth manifold.
- $S_{x_0}^T$ is lower semicontinous (→ recall: in general not continuous)

Lagrange multipliers rule

A necessary condition for a constrained critical point for $\inf_{(E_{x_0}^T)^{-1}(x)} C_T$.

Theorem

Assume $u \in \mathcal{U}$ is a constr. crit. point, with $x = E_{x_0}^T(u)$. Then (at least) one of the two following statements hold true

(i) $\exists\, \lambda_T \in T_x^*M$ s.t. $\lambda_T \cdot d_u E_{x_0}^T = d_u C_T$,

(ii) $\exists\, \lambda_T \in T_x^*M \setminus \{0\}$ s.t. $\lambda_T \cdot d_u E_{x_0}^T = 0$.

$\rightarrow$ A control u (reps. the associated trajectory γ_u) is called

- *normal* in case (i),
- *abnormal* in case (ii).

A priori an optimal control u can be associated with two different covectors such that both (i) and (ii) are satisfied.

Hamiltonian and PMP

$$H(\lambda) = \max_{u \in \mathbb{R}^k} \left(\langle \lambda, F(x,u) \rangle - L(x,u) \right), \qquad \lambda \in T^*M,\ x = \pi(\lambda).$$

The maximum $\bar{u} = \bar{u}(\lambda)$ is characterized as the solution to the system

$$\langle \lambda, f_i(x) \rangle - \frac{\partial L}{\partial u_i}(x,u) = 0, \qquad i = 1, \dots, k.$$

Theorem (PMP, normal case)

Let $(u(t), \gamma_u(t))$ be a normal geodesic. Then there exists a Lipschitz curve $\lambda(t) \in T^*_{\gamma_u(t)}M$ such that $\dot{\lambda}(t) = \vec{H}(\lambda(t))$.

Fix $x_0 \in M$. The *exponential map* at time $T > 0$ and base point x_0 is the map $\exp^T_{x_0} : T^*_{x_0}M \to M$ defined by $\exp^T_{x_0}(\lambda_0) = \pi \circ e^{t\vec{H}}(\lambda_0)$.

→ a conjugate point is (the image of) a critical point of $\exp^T_{x_0}$.

If the value function is smooth

Lemma

Assume $S^T_{x_0}$ smooth at x. Then

(i) *there exists a unique minimizer $\gamma : [0, T] \to M$ joining x_0 to x in time T*

(ii) *$d_x S^T_{x_0} = \lambda_T$, the Lagrange multiplier associated with γ*

(iii) *γ is not abnormal and not conjugate*

- The functional $\Phi(v) = C_T(v) - S^T_{x_0}(E^T_{x_0}(v))$ is smooth and non negative.
- For every optimal u

$$0 = d_u\Phi = d_u C_T - d_x S^T_{x_0} \cdot d_u E^T_{x_0}.$$

- $\lambda_T = d_x S^T_{x_0}$ is the Lagrange multiplier of the normal trajectory.

→ For (i) and (ii) it is enough that $\partial_P S^T_{x_0}(x) \neq \emptyset$

$$\partial_P F(x) = \{\lambda = d_x\psi \in T^*_x M \mid \psi \in C^\infty \text{ and } F - \psi \text{ has local minimum at } x\}.$$

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Using only sub-differential

- Fix ψ a smooth function such that $\lambda = d_x\psi \in \partial_P S^T_{x_0}(x)$
- by definition the map $S^T_{x_0}(\cdot) - \psi(\cdot)$ has a local minimum at x,
- Then, set the smooth function

$$\Phi(v) = C_T(v) - \psi(E^T_{x_0}(v)).$$

- Let u be a minimal control such that $E^T_{x_0}(u) = x$. Then for v close to u

$$\begin{aligned} \Phi(v) = C_T(v) - \psi(E^T_{x_0}(v)) &\geq S^T_{x_0}(E^T_{x_0}(v)) - \psi(E^T_{x_0}(v)) \\ &\geq S^T_{x_0}(E^T_{x_0}(u)) - \psi(E^T_{x_0}(u)) \\ &= C_T(u) - \psi(E^T_{x_0}(u)) \\ &= \Phi(u) \end{aligned} \tag{5}$$

- one gets

$$0 = d_u\Phi = d_uC_T - d_x\psi \cdot d_uE^T_{x_0}.$$

Fair points

- $S_{x_0}^T$ is lower semicontinuous by a very general argument
- $\partial_P S_{x_0}^T(x) \neq \emptyset$ for a dense set of points $x \in \mathrm{int}\left(A_{x_0}^T\right)$.

→ A point $x \in \mathrm{int}\left(A_{x_0}^T\right)$ is said to be a *fair point* if there exists a unique optimal trajectory steering x_0 to x, which admits a normal lift.

- We call Σ_f the set of all fair points contained in the attainable set.

The set Σ_f of fair points is *dense* in $\mathrm{int}\left(A_{x_0}^T\right)$.

- the trajectory is normal but may be also abnormal
- when $\partial_P S_{x_0}^T(x) \neq \emptyset$, then the unique normal trajectory steering x_0 to x is not abnormal if and only if $\partial_P S_{x_0}^T(x)$ is a singleton.

Fair vs continuity

A general result guarantees that a lower semicontinuity functions has plenty of continuity points.

Lemma

The set Σ_c of continuity points of $S^T_{x_0}$ is a residual subset of $\operatorname{int}\left(A^T_{x_0}\right)$.

! a residual subset $=$ intersection of countably many sets with dense interiors.

$\rightarrow$ it could be $\Sigma_c \cap \Sigma_f = \emptyset$ $\leftarrow$

- in the sub-Riemannian case $\Sigma_c = M$ so $\Sigma_c \cap \Sigma_f = \Sigma_f$
- we need openness of the end-point map
- in SR the end-point is always open (Rashevsky-Chow) even with abnormals

$\rightarrow$ we need to locate good points of openness of end-point.

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$\rightarrow$ we need to locate good points of openness of end-point.

Tame points: openness

Let $x \in \mathrm{int}\left(A^T_{x_0}\right)$. We say that x is a *tame point* if for every optimal control u steering x_0 to x there holds

$$\mathrm{rank}\, d_u E^T_{x_0} = \dim M$$

We call Σ_t the set of tame points.

At tame points

- the end-point is open (at first order)
- one obtains continuity

Arguments of [Trélat, '00]

The set Σ_t of tame points is open. The value function $S^T_{x_0}$ is continous on Σ_t.

Tame points: density

Theorem (DB-Boarotto, '16)

The set Σ_t of tame points is open and dense in $\mathrm{int}\left(A_{x_0}^T\right)$.

- let $\mathcal{U}_x$ set of u minimizers control reaching x
- if $\min_{u\in\mathcal{U}_x} \mathrm{rank}\, d_u E_{x_0}^T = n$ then x is tame point

→ we have to control $\min_{u\in\mathcal{U}_x} \mathrm{rank}\, d_u E_{x_0}^T < n$

→ assume not dense : you can prove there exists O neighborhood

$$\Sigma_f \cap O \subset \exp_{x_0}^T(A), \qquad A = \text{ union of compact of positive codim.}$$

At tame points one has indeed better regularity that continuity

Proposition

Let $K \subset \Sigma_t$ compact subset of tame points. Then $S_{x_0}^T$ is Lipschitz on K.

Consequences

Introduce the subset of $A^T_{x_0}$

$$\Sigma(x_0) = \{x \in A^T_{x_0} \mid \exists! \text{ not abnormal non-conjugate minimizer from } x_0 \text{ to } x\}$$

Adapting arguments from the SR case to our setting one improves to

Theorem (DB-Boarotto, '16)

$\Sigma(x_0)$ is open and dense in $\mathrm{int}\left(A^T_{x_0}\right)$. Moreover $S^T_{x_0}$ is smooth on $\Sigma(x_0)$.

The previous analysis recovers also the following corollary:

Corollary

If there are no Goh abnormals then $S^T_{x_0}$ is continuous on $\mathrm{int}\left(A^T_{x_0}\right)$.

Outline

Riemannian cut locus and semiconvexity

- (M, g) be a Riemannian manifold
- $\mathrm{Cut}(x_0)$ the cut locus from a point $x_0 \in M \rightarrow \mathrm{Cut}(x_0) = M \setminus \Sigma(x_0)$

Theorem (Cordero-McCann-Schmuckenschläger, '01)

Let (M, g) be a Riemannian manifold. Let $x \neq x_0$. Then $x \in \mathrm{Cut}(x_0)$ if and only if $d^2(x_0, \cdot)$ fails to be semiconvex at x, that is

$$\inf_{0<|v|<1} \frac{d^2(x_0, x+v) + d^2(x_0, x-v) - 2d^2(x_0, x)}{|v|^2} = -\infty. \tag{6}$$

the last equality understood in local coordinates

- it is proved in relation to the optimal transport problem
- it is implied by some jacobian inequality for the exp map
- $x \mapsto d^2(x_0, x)$ is everywhere semiconcave

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Abnormals and semiconcavity

Definition

A sub-Riemannian manifold M is *ideal* if the metric space (M, d_{SR}) is complete and there exists no non-trivial abnormal minimizers.

Theorem (Cannarsa-Rifford, '08)

Let M be an ideal sub-Riemannian manifold. Then $x \mapsto d^2_{SR}(x_0, x)$ is semiconcave out of the diagonal.

- the diagonal (constant curve) is always abnormal
- this results holds also in the general case for affine systems with our Lagrangian
- (hence our $S^T_{x_0}$ is semiconcave if there are no abnormals)

→ do analogue result of [C-McC-S, '01] holds in SR geometry?

Cut locus and semiconvexity

Define $\mathrm{Cut}(x_0) := M \setminus \Sigma(x_0)$.

Theorem (DB, Rizzi, '17)

Let M be an ideal sub-Riemannian structure. Let $x \neq x_0$. Then $x \in \mathrm{Cut}(x_0)$ if and only if $d^2(x_0, \cdot)$ fails to be semiconvex at x, that is

$$\inf_{0<|v|<1} \frac{d_{SR}^2(x_0, x+v) + d_{SR}^2(x_0, x-v) - 2d_{SR}^2(x_0, x)}{|v|^2} = -\infty. \qquad (7)$$

the last equality understood in local coordinates

- one implication is trivial
- the converse uses that d_{SR}^2 is semi-concave [Cannarsa-Rifford, '08]
- it is sharp in this form → not true for non-ideal structures

→ cf. open problem in last slide!

Cut locus and semiconvexity

Define $\mathrm{Cut}(x_0) := M \setminus \Sigma(x_0)$.

Theorem (DB, Rizzi, '17)

Let M be an ideal sub-Riemannian structure. Let $x \neq x_0$. Then $x \in \mathrm{Cut}(x_0)$ if and only if $d^2(x_0, \cdot)$ fails to be semiconvex at x, that is

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- it is sharp in this form $\to$ not true for non-ideal structures

$\to$ cf. open problem in last slide!

Idea of the proof

→ In this slide $f(x) := d^2_{SR}(x_0, x)$.

- By standard properties of semiconcave functions there exists $p \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that

$$f(x+v) - f(x) \leq p \cdot v + C|v|^2, \qquad \forall\, |v| < 1. \tag{8}$$

- If the infimum above is finite, that is there exists $K \in \mathbb{R}$ such that

$$f(x+v) + f(x-v) - 2f(x) \geq K|v|^2, \qquad \forall\, |v| < 1. \tag{9}$$

- Manipulating a little bit

$$f(x+v) \geq f(x) - (f(x-v) - f(x)) + K|v|^2,$$

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$$-\left(f(x-v) - f(x)\right) \geq p \cdot v - C|v|^2, \qquad \forall\, |v| < 1. \tag{8}$$

- If the infimum above is finite, that is there exists $K \in \mathbb{R}$ such that

$$f(x+v) + f(x-v) - 2f(x) \geq K|v|^2, \qquad \forall\, |v| < 1. \tag{9}$$

- Manipulating a little bit

$$f(x+v) \geq f(x) + p \cdot v + (K - C)|v|^2,$$

Idea of the proof

→ In this slide $f(x) := d^2_{SR}(x_0, x)$.

- By standard properties of semiconcave functions there exists $p \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that

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$$f(x+v) \geq \underbrace{f(x) + p \cdot v + (K-C)|v|^2}_{\phi},$$

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- One get that

There exists a function $\phi : M \to \mathbb{R}$, twice differentiable at x, such that

$$f(x) = \phi(x), \qquad \text{and} \qquad f(y) \geq \phi(y), \quad \forall y \in M.$$

Eliminating conjugate points

From optimal transport theory one is led to the following question:

Question from optimal transport [Figalli-Rifford, '10]

Assume that for $x \neq x_0 \in M$ there exists a function $\phi : M \to \mathbb{R}$, twice differentiable at x, such that

$$d^2_{SR}(x_0, x) = \phi(x), \qquad \text{and} \qquad d^2_{SR}(x_0, y) \geq \phi(y), \quad \forall y \in M.$$

Is it true that $x \notin \mathrm{Cut}(x_0)$?

$\rightarrow$ you are essentially asking if you can guarantee x is not conjugate.

Theorem (DB, Rizzi, '17)

It is true if M is an ideal sub-Riemannian manifold.

In transport one usually applies to situations in which ϕ is actually twice differentiable almost everywhere, such that the following map is well defined for a.e. $y \in M$.

$$T_t(y) = \exp_y(-t d_y \phi)$$

Theorem (DB, Rizzi, '17)

Under the same assumptions the linear maps $d_x T_t : T_x M \to T_{\gamma(t)} M$ satisfy for all fixed $s \in (0,1]$:

$$\det(d_x T_t)^{1/n} \geq \left(\frac{\det N_s(t)}{\det N_s(0)}\right)^{1/n} + \left(\frac{\det N_0(t)}{\det N_0(s)}\right)^{1/n} \det(d_x T_s)^{1/n}, \quad \forall\, t \in [0,s],$$

where $N_s(t)$ are Jacobi fields matrices.

- Both terms in the right hand side are non-negative for $t \in [0,s]$
- for $t \in [0,s)$, the first one is positive.
- In particular $\det(d_x T_t) > 0$ for all $t \in [0,1)$. But $\det(d_x T_1)$ might be zero.
- The final point is not conjugate $\to$ regularity of transport map.

Consequences

This has applications in:

- Borrell-Brascamp-Lieb and interpolation inequalities for optimal transport
- Brunn-Minkovski inequality
- recovers known results in Heisenberg group [Balogh et al., '16]
- do not require regular distributions

Theorem (DB-Rizzi '17, Grushin geodesic Brunn-Minkowski)

For all non-empty Borel sets $A, B \subset \mathbb{G}_2$, we have

$$\mathcal{L}^2(Z_t(A,B))^{1/2} \geq (1-t)^{5/2}\mathcal{L}^2(A)^{1/2} + t^{5/2}\mathcal{L}^2(B)^{1/2}, \qquad \forall\, t \in [0,1].$$

→ $Z_t(A,B)) = t$-intermediate points between A and B

- If one replaces the exponent 5 with a smaller one, the inequality fails for some choice of A, B.

Outline

1. Affine control systems with action-like cost
2. On the set of points of smoothness of the value function
3. Characterization of cut locus : sub-Riemannian case without abnormal
4. An open question

For a general, complete sub-Riemannian manifold M, let $x \in M$ and define :

$$\mathrm{SC}^-(x) := \{y \in M \mid \mathrm{d}_x^2 \text{ fails to be semiconcave at } y\},$$
$$\mathrm{SC}^+(x) := \{y \in M \mid \mathrm{d}_x^2 \text{ fails to be semiconvex at } y\},$$
$$\mathrm{Abn}(x) := \{y \in M \mid \exists \text{ abnormal minimizing geodesic joining } x \text{ to } y\}.$$
$$\mathrm{CutOpt}(x) := \{y \in M \mid \exists \text{ a geodesic joining } x \text{ to } y \text{ lose minimality}\}.$$

In the ideal case

- $\mathrm{Abn}(x) = \{x\} = \mathrm{SC}^-(x)$.
- $\mathrm{CutOpt}(x) = \mathrm{Cut}(x) \setminus \{x\} = \mathrm{SC}^+(x)$
- $\mathrm{Cut}(x) = \mathrm{CutOpt}(x) \cup \mathrm{Abn}(x) = \mathrm{SC}^+(x) \cup \mathrm{SC}^-(x)$.

Open questions

Are the following equalities true in general?

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THANKS FOR YOUR ATTENTION !