Unified synthetic curvature bounds

for Riemannian and sub-Riemannian geometry

Davide Barilari
Dip. Matematica "Tullio Levi-Civita", UNIPD
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Università degli Studi di Padova

Joint work with

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This is based on a joint work with

- Andrea Mondino (Oxford)
- Luca Rizzi (SISSA, Trieste)

Main reference:

BMR-24 DB, A.Mondino, L.Rizzi,
Unified synthetic Ricci curvature lower bounds for Riemannian and sub-Riemannian structures,
Memoirs of the AMS, to appear, 153 pp.

→ Other references:

BR-20 DB, L.Rizzi, *Bakry-Emery curvature in SR geometry*, Mathematische Annalen, 2020

BR-19 DB, L.Rizzi, *SR Interpolation inequalities*, Inventiones Mathematicae, 2019

A long story short: part 1

1960s Comparison theorems in Riemannian geometry: bounds on the (Ricci) curvature implies bounds on the geometry

1997 Cheeger-Colding theory: extension to Ricci limits (singular spaces)

2006 Lott-Sturm-Villani theory : synthetic notion of curvature (Ricci) bounds in metric spaces $CD(K, N)$

Nice properties

- consistent with classical Riemannian theory
- contains limits (compactness, stability)
- unified viewpoint (through optimal transport)

A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z}$$

- $(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

→ The Riemannian Hamiltonian is degenerate for $\varepsilon \to 0$:

$$H_\varepsilon(p,x) = \frac{1}{2}\sum_{i,j=1}^{3} g_\varepsilon^{ij}(x) p_i p_j$$

- $g^{ij}(x) = \lim_{\varepsilon\to 0} g_\varepsilon^{ij}(x)$ is ≥ 0 but not invertible at any x
- it is like if the "inverse" $g_{ij}(x)$ has one eigenvalue $= +\infty$.

Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z}$$

- crucial point $[X_1, X_2] = \partial_z$.

As metric spaces $(\mathbb{R}^3, d^\varepsilon) \to (\mathbb{R}^3, d_{SR})$ (in the Gromov-Hausdorff sense)

- $D^\varepsilon = \mathrm{span}\{X_1, X_2, X_3^\varepsilon\} \to D = \mathrm{span}\{X_1, X_2\}$
- $\mathrm{Ric}^\varepsilon(v) \to -\infty$ for all $v \in D$

The sequence of curvatures is unbounded from below

A long story short: part 2

Other observations

- sub-Riemannian manifolds are limit of Riemannian manifolds
- the curvature is unbounded at the limit
- sub-Riemannian are not CD (cf. talks G.Stefani, M.Magnabosco)

2009 Juillet : the SR Heisenberg group does not satisfy $CD(K, N)$

- Juillet : the SR Heisenberg group satisfies $MCP(0, 5)$

2016 Balogh, Kristaly, Sipos : SR Heisenberg has interpolation inequalities

2019 DB, Rizzi: SR manifolds admit interpolation inequalities

Question:

is it possible a theory containing both Riem and sub-Riem geometry?

Grande unification - C. Villani, 2017

Séminaire BOURBAKI Janvier 2017
69ème année, 2016-2017, n$^{\mathrm{o}}$ 1127

INÉGALITÉS ISOPÉRIMÉTRIQUES DANS LES ESPACES MÉTRIQUES MESURÉS [d'après F. Cavalletti & A. Mondino]

par **Cédric VILLANI**

— — —

• incorporer les géométries sous-riemanniennes, du style de l'espace de Heisenberg. Si Baudoin–Bonnefont-Garofalo [BBG] ont proposé de gérer ces espaces par des familles d'inégalités fonctionnelles à la Bakry–Émery, en revanche Balogh–Krystály–Sipos [BKS] ont montré tout récemment que l'on pouvait les traiter par transport, de façon similaire aux espaces CD et CD*, grâce à l'emploi de coefficients de distortion bien choisis, qui *ne se comparent pas* aux coefficients $\beta^{K,N}$ ou $\overset{*}{\beta}{}^{K,N}$, et permettent d'obtenir des inégalités isopérimétriques optimales.

Un travail important de clarification reste à mener, mais on peut espérer ainsi une « grande unification » synthétique des bornes de courbure-dimension dans trois larges classes de géométries : riemanniennes, finslériennes, sous-riemanniennes.

Outline

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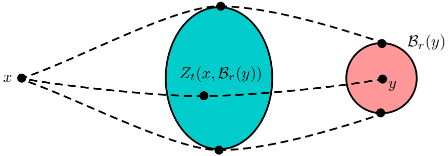
Distortion coefficient

(M, g) Riemannian manifold, vol Riemannian volume measure

Distortion coefficient

$$\beta_t(x,y) := \limsup_{r\to 0} \frac{\mathrm{vol}(Z_t(x,\mathcal{B}_r(y)))}{\mathrm{vol}(\mathcal{B}_r(y))}, \qquad \forall (x,y) \notin \mathrm{cut}(M),\ t \in [0,1]$$



- $\beta_t(x,y) = t^n$ in $\mathbb{R}^n$ by homothethy
- if v is the vector such that $\exp_x(v) = y$.

$$\mathrm{vol}(Z_t(x,\mathcal{B}_r(y))) = \mathrm{vol}(\mathcal{B}_r(y))t^n \left(1 - \frac{1}{6}\mathrm{Ric}(v)t^2 + o(t^2)\right)$$

Comparison for distortion

If the space is positively curved we have a lower bound on distortion

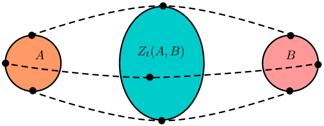
if $\mathrm{Ric} \geq 0$ then for all $t \in [0,1]$

$$\beta_t(x,y) \geq t^n$$

it also implies a Brunn-Minkowski inequality:

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq (1-t)\mathsf{m}(A)^{1/n} + t\mathsf{m}(B)^{1/n}$$

which is stronger : take $A = \{x\}$ and $B = B(x,r)$ with $r \to 0$



Comparison for distortion

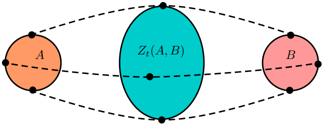
If the space is positively curved we have a lower bound on distortion

if $\mathrm{Ric} \geq 0$ then for all $t \in [0,1]$

$$\beta_t(x,y) \geq \overline{\beta}_t^{0,n} = t^n$$

under $\mathrm{Ric} \geq 0$ one has a Brunn-Minkowski inequality:

$$\mathsf{m}(Z_t(A,B))^{1/n} \geq (\overline{\beta}_{1-t}^{0,n})^{1/n}\mathsf{m}(A)^{1/n} + (\overline{\beta}_t^{0,n})^{1/n}\mathsf{m}(B)^{1/n}$$



Comparison: the Riemannian case

- Distortion coefficients are in general difficult to compute,
- a bound on the geometry gives a bound in terms of model spaces.

Theorem

Let (M, g) be a N-dimensional Riemannian, with $\mathrm{m} = \mathrm{vol}_g$ Riemannian volume. Assume that $\mathrm{Ric}_g \geq K$. Then for $(x, y) \notin \mathrm{Cut}(M)$ we have

$$\beta_t(x, y) \geq \overline{\beta}_t^{K,N}(d(x, y)), \qquad \forall\, t \in [0, 1]. \tag{1}$$

- $\overline{\beta}_t^{K,N}$ = distortion coefficient of constant curvature K and dim N.

$$\overline{\beta}_t^{K,N}(\theta) = t \left(\frac{\sin(t\theta\sqrt{K/N-1})}{\sin(\theta\sqrt{K/N-1})} \right)^{N-1}, \qquad \text{for } K > 0, \tag{2}$$

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$$\overline{\beta}_t^{K,N}(\theta) = \frac{t}{1}\left(\frac{\sin(t\theta\sqrt{K/N-1})}{\sin(1\theta\sqrt{K/N-1})}\right)^{N-1}, \qquad \text{for } K > 0, \tag{2}$$

$$\overline{\beta}_t^{K,N}(\theta) \sim t^N$$

$CD(K, N)$ condition

The following inequality depends only on geodesics and measure

Brunn Minkovski condition

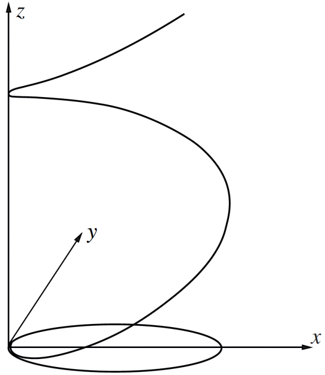
$$\mathsf{m}(Z_t(A, B))^{1/N} \geq (\overline{\beta}_{1-t}^{K,N})^{1/N}\mathsf{m}(A)^{1/N} + (\overline{\beta}_t^{K,N})^{1/N}\mathsf{m}(B)^{1/N}$$

Could be used as a "definition" of curvature bounds on m.m.s. (X, d, m).

- actually defined by optimal transport
- unifies Riemannian and Finsler
- stability and compactness (Ricci limits)
- it implies the BM above (hence comparison on dist coeff)
- note that $(\overline{\beta}_t^{K,N})^{1/N} \sim t$, the weights are "linear"

Outline

Heisenberg geodesics



$$\begin{cases} x(t,\theta,a) = \frac{1}{a}\left(\cos(at+\theta) - \cos(\theta)\right) \\ y(t,\theta,a) = \frac{1}{a}\left(\sin(at+\theta) - \sin(\theta)\right) \\ z(t,\theta,a) = \frac{1}{2a^2}\left(at - \sin(at)\right) \end{cases}$$

- [Juillet, '09] proved the sharp inequality

$$\beta_t(x,y) \geq t^5$$

$\rightarrow$ Here $5 =$ geodesic dimension of SR manifold [Agrachev-DB-Rizzi '13]

No $CD(K,N)$ is satisfied in Heisenberg [Juillet, '09]

$$\mathsf{m}(Z_t(A,B))^{1/5} \not\geq (1-t)\mathsf{m}(A)^{1/5} + t\mathsf{m}(B)^{1/5}$$

- [Balogh et al. '16] the modified Brunn-Minkowski inequality:

$$\mathsf{m}(Z_t(A,B))^{1/3} \geq (1-t)^{5/3}\mathsf{m}(A)^{1/3} + t^{5/3}\mathsf{m}(B)^{1/3}$$

- No $CD(K,N)$ is satisfied by any SR manifold [Rizzi-Stefani '23, Magnabosco-Rossi '22, Ambrosio-Stefani '20, Juillet '18]

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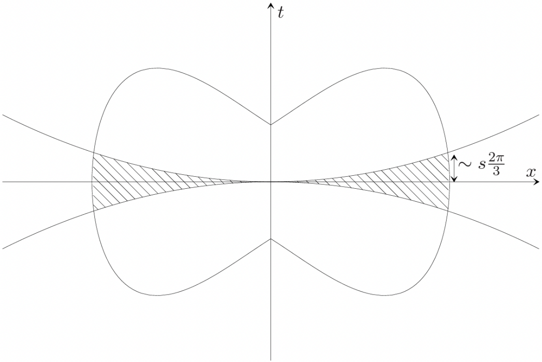
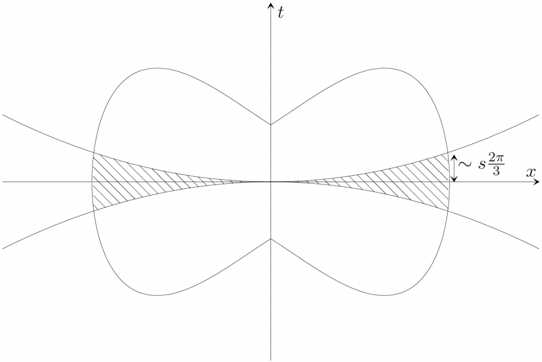
$$\mathsf{m}(Z_t(x,B)) \geq t^{5}\mathsf{m}(B)$$



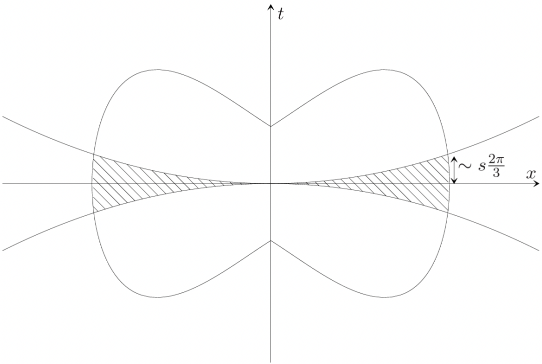
The set $\delta_{1/s}(Z_s(x,B))$ for s small [picture from Juillet '09]

$$t^{-4}\mathsf{m}(Z_t(x,B)) \geq t^{1}\mathsf{m}(B)$$



The set $\delta_{1/s}(Z_s(x,B))$ for s small [picture from Juillet '09]

$$\mathsf{m}(\delta_{1/t} Z_t(x,B)) \geq t^1 \mathsf{m}(B)$$



The set $\delta_{1/s}(Z_s(x,B))$ for s small [picture from Juillet '09]

Equivalence of inequalities: ideal case

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Theorem (DB, Rizzi, 2019)

Let (M, D, g) be a n-dim ideal sub-Riemannian manifold, m smooth measure. Let $N > 0$. The following are equivalent:

(i) bound on the distortion coefficient:

$$\beta_t(x, y) \geq t^N$$

(ii) the measure contraction property :

$$\mathsf{m}(Z_t(x, B)) \geq t^N \mathsf{m}(B)$$

(iii) the modified Brunn-Minkowski inequality:

$$\mathsf{m}(Z_t(A, B))^{1/n} \geq (1-t)^{N/n} \mathsf{m}(A)^{1/n} + t^{N/n} \mathsf{m}(B)^{1/n}$$

$\rightarrow$ with this [Julliet '09] implies [Balogh et al. '16]

interpolation inequalities for densities

- dimensional parameter $n \in [1, +\infty)$,

For any $\mu_0 \in \mathcal{P}_{bs}(X, \mathsf{d}, \mathfrak{m})$, $\mu_1 \in \mathcal{P}^*_{bs}(X, \mathsf{d}, \mathfrak{m})$ there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$, such that $\mu_t \ll \mathfrak{m}$ for all $t \in (0, 1]$, and letting $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}$:

$$\frac{1}{\rho_t(\gamma_t)^{1/n}} \geq \frac{\beta_{1-t}^{(X,\mathsf{d},\mathfrak{m})}(\gamma_1, \gamma_0)^{1/n}}{\rho_0(\gamma_0)^{1/n}} + \frac{\beta_t^{(X,\mathsf{d},\mathfrak{m})}(\gamma_0, \gamma_1)^{1/n}}{\rho_1(\gamma_1)^{1/n}}, \tag{3}$$

for all $t \in (0, 1)$ and ν-a.e. $\gamma \in \mathrm{Geo}(X)$

- induced by $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$
- the first term in the right hand side of (3) is omitted if $\mu_0 \notin \mathcal{P}_{ac}(X, \mathfrak{m})$

$\rightarrow$ no second order equation for determinant of the jacobian

Heisenberg distortion coefficient

??? What if one computes explicitly the Heisenberg distortion

$$\beta_t^{\mathbb{H}}(x,y) = \overline{\beta}_t(a(x,y))$$

One can see that

$$\overline{\beta}_t(\theta) = \frac{\overline{s}(t\theta)}{\overline{s}(\theta)}, \quad \overline{s}(\theta) = \theta \sin\left(\frac{\theta}{2}\right)\left(\sin\left(\frac{\theta}{2}\right) - \frac{\theta}{2}\cos\left(\frac{\theta}{2}\right)\right)$$

where

- $a(x,y)$ is the vertical part of the covector joining x and y.

- the β is not a Riemannian model function
- it does not depend on the distance!

Outline

A link with Control Theory

Variational problems in $\mathbb{R}^n$ with minimization of a quadratic cost

$$\begin{cases} \dot{x} = Ax + Bu \\ \frac{1}{2}\int_0^1 (u^* u - x^* Q x) dt \longrightarrow \text{min} \end{cases}$$

Kalman condition: $\exists\, m \geq 0$ such that $\operatorname{rank}(B, AB, \dots, A^m B) = n$

$\rightarrow$ Optimal trajectories solve a Hamiltonian system:

$$H(p, x) = \frac{1}{2}(p^* B B^* p + 2p^* A x + x^* Q x)$$

$$\beta_t^{A,B,Q}(x, y) := \limsup_{r \to 0} \frac{|Z_t(x, \mathcal{B}_r(y))|}{|\mathcal{B}_r(y)|}, \qquad x, y \in \mathbb{R}^n$$

The Riemannian case

Definition (LQ distortion coefficients)

$$\beta_t^{A,B,Q}(x,y) := \limsup_{r\to 0} \frac{|Z_t(x,\mathcal{B}_r(y))|}{|\mathcal{B}_r(y)|}, \qquad x,y\in\mathbb{R}^n$$

- We use the Lebesgue measure on $\mathbb{R}^n$ & Euclidian balls
- One can replace $\mathcal{B}_r$ with sets "nicely shrinking" to points

$\rightarrow$ It does not depend on x,y

$$\beta_t^{A,B,Q} = \frac{\det N(t)}{\det N(1)}, \qquad \begin{cases} \dot N = BB^*M + AN \\ \dot M = -A^*M - QN \end{cases}$$

A basic example: the harmonic oscillator in $\mathbb{R}^n$

$$\dot{x} = u, \qquad \int_0^T |u|^2 - K|x|^2 dt \to \min$$

- no drift ($A = 0$), no constraint on velocity ($B = \mathbb{1}$),
- isotropic potential ($Q = K\mathbb{1}$)

$\Rightarrow \beta_t^{A,B,Q}$ = Riemannian distortion coefficients $\beta_t^{(K,n)}$ of curvature κ!

$$\beta_t^{A,B,Q} = \frac{\det N(t)}{\det N(1)}, \qquad \begin{cases} \dot{N} = BB^*M + AN \\ \dot{M} = -A^*M - QN \end{cases}$$

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which is the Jacobi field equation $\ddot{N} + QN = 0$.

The Heisenberg case

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A different example: “1D control of acceleration”, with potential

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = u_2 \end{cases} \qquad \int_0^T (u_1^2 + u_2^2) - \lambda x_1^2 dt \to \min$$

- drift $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, two controls $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- diagonal potential $Q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda \geq 0$

$\beta_t^{A,B,Q}$ is the solution of a costant coefficient matrix Riccati equation

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- diagonal potential $Q = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\lambda \geq 0$ $\qquad \theta = \frac{\sqrt{\lambda}}{2}$

$\beta_t^{A,B,Q}$ is the solution of a costant coefficient matrix Riccati equation

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SR comparison

Let $(x, y) \notin \mathrm{Cut}(M)$ and assume a unique length-minimizer joining

- associated with matrices A, B given by Lie bracket structure

$\rightarrow$ depend on trajectory but in Heisenberg one have same A, B for all

- associate a curvature along the length minimizer $\mathfrak{R}_{\gamma(t)}$

Theorem (DB-Rizzi, Math.Ann. 2020)

If there exists Q such that $\mathfrak{R}_{\gamma(t)} \geq Q$ for every $t \in [0, T]$, then

$$\beta_t(x, y) \geq \beta_t^{A,B,Q} \tag{4}$$

$\rightarrow$ it is a matrix comparison

- a scalar comparison exists but more than one scalar

Outline

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A new approach

Remind problems we have encountered in the Heisenberg group

- the model β is not a Riemannian one
- it does not depend on distance but another function of the geometry!

Idea: generalize in two ways

- the β can be more general

$$\overline{\beta}_t(\theta) = \frac{\overline{s}(t\theta)}{\overline{s}(\theta)}, \qquad \overline{s}(\theta) = c\theta^D + o(t^D)$$

- the metric measure space (X, d, m) is endowed with a gauge G
- background idea

$$\text{“}CD(\beta, n) \Rightarrow \beta_t(x, y) \geq \overline{\beta}_t(G(x, y))\text{''}$$

Which natural choice of $\overline{s}$?

Choice of $\overline{\beta}$ and $\overline{s}$

- the β can be more general

$$\overline{\beta}_t(\theta) = \frac{\overline{s}(t\theta)}{\overline{s}(\theta)}, \qquad \overline{s}(\theta) = c\theta^D + o(t^D)$$

$\rightarrow$ one can pick $\overline{s}$ as a solution of a matrix Riccati equation

Riccati distortion

$$\overline{\beta}_t^{A,B,Q} = \frac{\det N(t)}{\det N(1)}, \qquad \begin{cases} \dot{N} = BB^*M + AN \\ \dot{M} = -A^*M - QN \end{cases}$$

where A, B are in some normal form.

$\rightarrow$ D is computed explicitly

Choice of G

To develop the theory G can be any measurable function

$$G : X \times X \to [0, +\infty]$$

→ The theory is much more simple if G is continuous

- The natural G is Heisenberg is not continuous at the origin

Compatibility metric/gauge - *meek condition*

A natural compatibility for G is the following:

- along any geodesics γ_t joining $x = \gamma_0$ and $y = \gamma_1$

$$G(\gamma_0, \gamma_t) = tG(\gamma_0, \gamma_1)$$

One might think to the distance $d(\gamma_0, \gamma_t) = td(\gamma_0, \gamma_1)$

Natural gauge functions

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Let (M, D, g) be a sub-Riemannian manifold:

- fix a Riemannian extension g_R of g
- define $D(x, y) = \|\nabla_x^R d(\cdot, y)\|_R$,
- a natural gauge function is a function $\mathsf{G} : X \times X \to [0, +\infty]$, such that
  - $\mathsf{G}(x, y) = f(\mathsf{d}(x, y), \mathsf{D}(x, y))$ for all $x, y \in X$, where
  - f is continuous
  - f is 1-homogeneous : $f(\lambda a, \lambda b) = \lambda f(a, b)$ for all $\lambda > 0$

$\rightarrow$ In the Heisenberg group the gauge is $G = \sqrt{\mathsf{D}^2 - \mathsf{d}^2}$

$$
\begin{tikzcd}
\boxed{\text{step} \leq 2 \text{ (Def. 7.23)}} \arrow[r, Rightarrow, "\text{Theorem 7.24}"] \arrow[d, Rightarrow] & \boxed{\mathsf{G} \text{ is locally bounded}} \arrow[d, Rightarrow] \\
\boxed{\begin{matrix}\text{minimizing Sard} \\ \text{(Def. 7.13)}\end{matrix}} \arrow[r, Rightarrow, "\text{Theorem 7.24}"] \arrow[dr, Rightarrow, "\text{Theorem 7.18}"] \arrow[dd, Rightarrow, shift left=2, "\text{real-analytic}"] & \boxed{\begin{matrix}\mathsf{G} \text{ is locally bounded} \\ \text{almost everywhere}\end{matrix}} \\
 & \boxed{\begin{matrix}\mathsf{G} \text{ satisfies the} \\ \text{regularity for stability} \\ \text{of } \mathsf{MCP} \text{ (Def. 5.3)}\end{matrix}} \\
\boxed{\begin{matrix}*\text{-minimizing Sard} \\ \text{(Def. 7.13)}\end{matrix}} \arrow[uu, Rightarrow, shift left=2] \arrow[r, Rightarrow, "\text{Theorem 7.16}"] & \boxed{\mathsf{G} \text{ is meek (Def. 4.23)}} \\
\boxed{\text{ideal (Def. 7.19)}} \arrow[u, Rightarrow] \arrow[r, Rightarrow, "\text{Theorem 7.20}"] \arrow[d, Rightarrow] & \boxed{\begin{matrix}\mathsf{G} \text{ satisfies the} \\ \text{regularity for stability} \\ \text{of } \mathsf{CD} \text{ (Def. 5.12)}\end{matrix}} \\
\boxed{\begin{matrix}\text{no non-trivial Goh} \\ \text{geodesics}\end{matrix}} \arrow[r, Rightarrow, "\text{Remark 7.26}"] & \boxed{\begin{matrix}\mathsf{G} \text{ is locally bounded} \\ \text{away from diagonal}\end{matrix}}
\end{tikzcd}
$$

Some formal definitions

- For $\mu \in \mathcal{P}_2(X)$ Boltzmann-Shannon entropy

$$\mathrm{Ent}(\mu|\mathfrak{m}) := \int_X \rho \log \rho \, \mathfrak{m}, \qquad \text{if } \mu = \rho \, \mathfrak{m} \in \mathcal{P}_2(X) \cap \mathcal{P}_{ac}(X, \mathfrak{m}),$$

if $\rho \log \rho \in L^1(X, \mathfrak{m})$, otherwise $\mathrm{Ent}(\mu|\mathfrak{m}) := +\infty$.

- Let $\mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$ be the finiteness domain of the entropy and

$$\mathcal{P}_{bs}(X, \mathsf{d}, \mathfrak{m}) := \{\mu \in \mathcal{P}(X, \mathsf{d}) \mid \ \mathrm{supp}\ \mu \text{ bounded}, \mathrm{supp}\ \mu \subseteq \mathrm{supp}\ \mathfrak{m}\},$$
$$\mathcal{P}^*_{bs}(X, \mathsf{d}, \mathfrak{m}) := \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m})) \cap \mathcal{P}_{bs}(X, \mathsf{d}, \mathfrak{m}).$$

- a “dimensional” entropy [Erbar-Kuwada-Sturm],

$$\mathrm{U}_n(\mu|\mathfrak{m}) := \exp\left(-\frac{\mathrm{Ent}(\mu|\mathfrak{m})}{n}\right), \qquad n \in [1, +\infty), \tag{5}$$

with $\mathrm{U}_n(\mu|\mathfrak{m}) := 0$ if $\mu \notin \mathrm{Dom}(\mathrm{Ent}(\cdot|\mathfrak{m}))$.

Let $\mathsf{s} : [0, +\infty) \to \mathbb{R}$ be a continuous function and $N \in [1, +\infty)$ such that for some $c > 0$

$$\mathsf{s}(\theta) = c\,\theta^D + o(\theta^D) \qquad \text{as } \theta \to 0, \tag{6}$$

→ The parameter D will be a sharp dimensional upper bound.

$$\Theta := \inf\{\theta > 0 \mid \mathsf{s}(\theta) = 0\}. \tag{7}$$

$\Theta > 0$ will be a sharp upper bound on the gauge function

Define the *distortion coefficient* $\beta_{(\cdot)}(\cdot) : [0, 1] \times [0, +\infty] \to [0, +\infty]$ as

$$(t, \theta) \in [0, 1] \times [0, +\infty] \mapsto \beta_t(\theta) := \begin{cases} t^D & \theta = 0, \\ \dfrac{\mathsf{s}(t\theta)}{\mathsf{s}(\theta)} & 0 < \theta < \Theta, \\ \liminf\limits_{\phi \to \mathcal{D}^-} \dfrac{\mathsf{s}(t\phi)}{\mathsf{s}(\phi)} & \theta \geq \Theta. \end{cases} \tag{8}$$

CD(β, n) condition

Let $n \in [1, +\infty)$, and β as previously. A gauge metric measure space $(X, \mathsf{d}, \mathfrak{m}, \mathsf{G})$ satisfies:

- for all $\mu_0 \in \mathcal{P}_{bs}(X, \mathsf{d}, \mathfrak{m})$, $\mu_1 \in \mathcal{P}^*_{bs}(X, \mathsf{d}, \mathfrak{m})$ $\operatorname{supp}\mu_0 \cap \operatorname{supp}\mu_1 = \emptyset$, there exists a W_2-geodesic $(\mu_t)_{t\in[0,1]}$ connecting them, $\forall\, t \in (0,1)$,

$$\mathrm{U}_n(\mu_t|\mathfrak{m}) \geq \exp\left(\frac{1}{n}\int_{\mathrm{Geo}(X)} \log \beta_{1-t}\big(\mathsf{G}(\gamma_1, \gamma_0)\big)\, \nu(\mathrm{d}\gamma)\right) \mathrm{U}_n(\mu_0|\mathfrak{m}) + \exp\left(\frac{1}{n}\int_{\mathrm{Geo}(X)} \log \beta_t\big(\mathsf{G}(\gamma_0, \gamma_1)\big)\, \nu(\mathrm{d}\gamma)\right) \mathrm{U}_n(\mu_1|\mathfrak{m}), \quad (9)$$

with the convention that $\infty \cdot 0 = 0$.

MCP(β) condition

- MCP(β) if for any $\bar{x} \in \operatorname{supp} \mathfrak{m}$ and $\mu_1 \in \mathcal{P}^*_{bs}(X, \mathsf{d}, \mathfrak{m})$ with $\bar{x} \notin \operatorname{supp} \mu_1$ there exists a W_2-geodesic $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}_2(X, \mathsf{d})$ from $\mu_0 = \delta_{\bar{x}}$ to μ_1 such that, $\forall\, t \in (0,1)$,

$$\mathrm{U}_n(\mu_t|\mathfrak{m}) \geq \exp\left(\frac{1}{n}\int_X \log \beta_t\big(\mathsf{G}(\bar{x}, x)\big)\, \mu_1(\mathrm{d}x)\right) \mathrm{U}_n(\mu_1|\mathfrak{m}), \quad (10)$$

  for some (and then every) $n \geq 1$.

Some remarks

1. The MCP(β) condition does not depend on the value of n.
2. Non-absolutely continuous μ_0 are allowed, which by construction gives the implication CD(β, n) $\Rightarrow$ MCP(β).
3. Assuming that $(X, \mathsf{d}, \mathfrak{m})$ supports interpolation inequalities for densities with dimensional parameter n MCP(β) $\Rightarrow$ CD(β, n).

Compatibility with classical synthetic theories

The $\mathsf{CD}(\beta, n)$ and $\mathsf{MCP}(\beta)$ conditions satisfy the following compatibility properties:

- *Compatibility with Lott-Sturm-Villani's* CD:
  - choose $\beta = \beta_{K,N}$, and as gauge function the distance $\mathsf{G} = \mathsf{d}$,
  - for essentially non-branching m.m.s., $\mathsf{CD}(K, N)$ equal $\mathsf{CD}(\beta_{K,N}, N)$.
- *Compatibility with Ohta-Sturm's* MCP:
  - for essentially non-branching m.m.s. $\mathsf{MCP}(K, N)$ equal $\mathsf{MCP}(\beta_{K,N})$
- *Compatibility with Balogh-Kristály-Sipos*:
  - choose $\beta = \beta^{\mathbb{H}^d}$ and $\mathsf{G}(x, y) = a^{x,y}$
  - the Heisenberg group $\mathbb{H}^d$ satisfies the $\mathsf{CD}(\beta^{\mathbb{H}^d}, 2d + 1)$ condition
- *Compatibility with E. Milman's conditions*:
  - for gauge function $\mathsf{G} = \mathsf{d}$
  - Milman $\mathsf{CGTD}(K, N, n)$ for $K \in \mathbb{R}$, $n \geq 1$ equals $\mathsf{CD}(\beta_{K,N}, n)$
  - Milman $\mathsf{QCD}(Q, K, N)$ choose $\beta = \frac{1}{Q}\beta^{\tau}_{K,N}$

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Compatibility with classical synthetic theories

The CD(β, n) and MCP(β) conditions satisfy the following compatibility properties:

- *Compatibility with Lott-Sturm-Villani's* CD:
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  - for essentially non-branching m.m.s. MCP(K, N) equal MCP$(\beta_{K,N})$
- *Compatibility with Balogh-Kristály-Sipos*:
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  - the Heisenberg group $\mathbb{H}^d$ satisfies the CD$(\beta^{\mathbb{H}^d}, 2d + 1)$ condition
- *Compatibility with E. Milman's conditions*:
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  - Milman QCD(Q, K, N) choose $\beta = \frac{1}{Q}\beta^{\tau}_{K,N}$

Compatibility with classical synthetic theories

The CD(β, n) and MCP(β) conditions satisfy the following compatibility properties:

- *Compatibility with Lott-Sturm-Villani's* CD:
  - choose $\beta = \beta_{K,N}$, and as gauge function the distance G = d,
  - for essentially non-branching m.m.s., CD(K, N) equal CD$(\beta_{K,N}, N)$.
- *Compatibility with Ohta-Sturm's* MCP:
  - for essentially non-branching m.m.s. MCP(K, N) equal MCP$(\beta_{K,N})$
- *Compatibility with Balogh-Kristály-Sipos*:
  - choose $\beta = \beta^{\mathbb{H}^d}$ and G$(x, y) = a^{x,y}$
  - the Heisenberg group $\mathbb{H}^d$ satisfies the CD$(\beta^{\mathbb{H}^d}, 2d + 1)$ condition
- *Compatibility with E. Milman's conditions*:
  - for gauge function G = d
  - Milman CGTD(K, N, n) for $K \in \mathbb{R}$, $n \geq 1$ equals CD$(\beta_{K,N}, n)$
  - Milman QCD(Q, K, N) choose $\beta = \frac{1}{Q}\beta^{\tau}_{K,N}$

Some results - MCP is sufficient

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For some interesting cases (e.g. if $\mathsf{G} = \mathsf{D}$) it holds $\mathsf{G} \geq \mathsf{d}$.

Theorem (Bonnet-Myers)

Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with gauge function G*, satisfying the* $\mathsf{CD}(\beta, n)$*. Assume that* $\mathsf{G} \geq \mathsf{d}$*. Then* $\operatorname{diam}(\operatorname{supp}\mathfrak{m}) \leq \Theta$*, and if* $\Theta < +\infty$ *then* $\operatorname{supp}\mathfrak{m}$ *is compact.*

The Riemannian case

$$\overline{\beta}_t^{K,N}(\theta) = t\left(\frac{\sin(t\theta\sqrt{K/N-1})}{\sin(\theta\sqrt{K/N-1})}\right)^{N-1}, \qquad \text{for } K > 0, \tag{11}$$

The first zero of $\mathsf{s}(\theta)$ is at

$$\Theta = \frac{\pi}{\sqrt{K/N-1}}$$

Some results - MCP is sufficient

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Theorem (Doubling)

Let (X, d, m, G) *satisfy* $CD(\beta, n)$. *Assume that*

- $\theta \mapsto \overline{\beta}_t(\theta)$ *is non decreasing*

then $\exists C > 0$ *such that for every* $x_0 \in \operatorname{supp}(m)$ *and* $r > 0$

$$m(B(x_0, 2r)) \leq C\, m(B(x_0, r))$$

- in Riemannian geometry $\theta \mapsto \overline{\beta}_t(\theta)$ non decreasing means $K > 0$

Theorem (Dimension estimate)

Let (X, d, m, G) *satisfy* $CD(\beta, n)$. *Then*

$$\dim_{Haus}(\operatorname{supp}(m)) \leq D$$

where $s(\theta) = c\theta^D + o(\theta^D)$.

Generalized Bishop-Gromov

Consider the following measure of "gauge balls" and "gauge spheres":

$$\mathrm{v}_{\mathsf{G}}(x_0, r) := \mathfrak{m}\Big(\{\mathsf{G}(x_0, x) \le r\} \cap B(x_0, \rho)\Big),$$
$$\mathrm{s}_{\mathsf{G}}(x_0, r) := \limsup_{\delta \downarrow 0} \frac{1}{\delta}\mathfrak{m}\Big(\{\mathsf{G}(x_0, x) \in (r - \delta, r]\} \cap B(x_0, \rho)\Big),$$

Theorem (DB-Mondino-Rizzi, '24)

Let $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s. endowed with*

- *a meek gauge function* G *satisfying the* $CD(\beta, n)$

Then the functions

$$r \mapsto \frac{\mathrm{s}_{\mathsf{G}}(x_0, r)}{\mathsf{s}(r)/r} \quad \text{and} \quad r \mapsto \frac{\mathrm{v}_{\mathsf{G}}(x_0, r) - \mathrm{v}_{\mathsf{G}}(x_0, 0)}{\int_0^r (\mathsf{s}(t)/t)\,\mathrm{d}t}$$

are monotone non-increasing for $r > 0$.

Gauge balls

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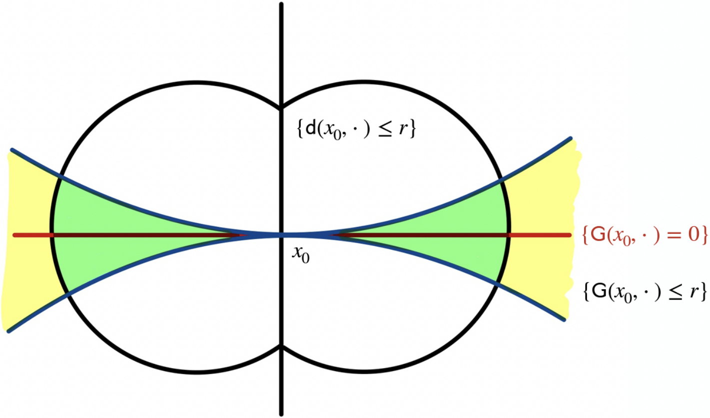
$$\mathrm{v}_{\mathsf{G}}(x_0, r) := \mathfrak{m}\left(\{x \in X \mid \mathsf{G}(x_0, x) \leq r,\ \mathsf{d}(x_0, x) \leq r\}\right),$$



Compatibility with SR theory

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Let (M, D, g) be a fat sub-Riemannian manifold: for all $X \in D$

$$D + [X, D] = TM$$

- fix a smooth measure m
- fix a Riemannian extension g_R of g
- as scalar gauge $G(x, y) = \|\nabla_x^R d(\cdot, y)\|_R$,

Theorem (Consistency)

Let (M, D, g) be a compact fat sub-Riemannian manifold of dim n. Then there exists $\overline{\beta}$ build from the class of Riccati distortion such that (M, D, m, G) satisfy $CD(\overline{\beta}, n)$.

$\rightarrow$ need *fat* to use the differential SR theory of curvature

Conclusions

One can define a new convergence for a sequence $(X_k, d_k, m_k, G_k)_{k\in\mathbb{N}}$

- we ask for a sort of L^1_{loc} convergence of gauge functions
- need regularity of limit G
- the low regularity of the gauge functions and their weaker convergence introduce new challenges in the proof.

$\rightarrow$ get stability and compactness results

Also:

- the Grushin plane satisfy CD(β, n) class of the Heisenberg group.
- canonical variations $(\mathbb{H}^1, \mathsf{d}_\varepsilon)$ within a single CD(β, n) class.
- convergence to the tangent cone of sub-Riemannian structures
- vector-valued gauges ($\rightarrow$ SR 3D Lie groups)

THANKS FOR YOUR ATTENTION

The Grushin plane

The Grushin plane $\mathbb{G}_2$ is the sub-Riemannian structure on $\mathbb{R}^2$ defined by the global generating frame

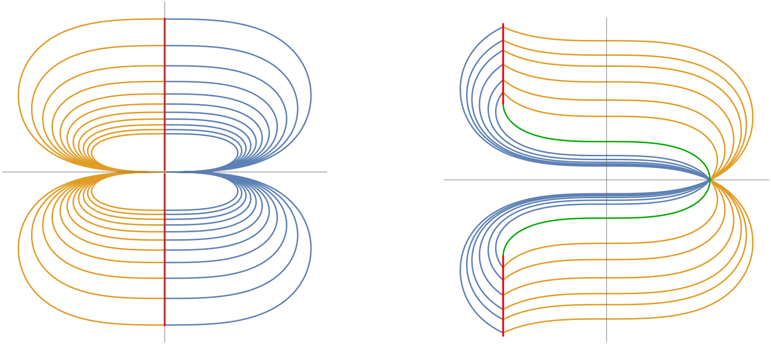
$$X_1 = \partial_x, \qquad X_2 = x\partial_y. \tag{12}$$

We equip the Grushin plane with the corresponding sub-Riemannian distance $\mathsf{d}_{\mathbb{G}_2}$ and the Lebesgue measure $\mathbb{R}^2$.

We set $\mathsf{G}_{\mathbb{G}_2} : \mathbb{G}_2 \times \mathbb{G}_2 \to [0, \infty)$:

$$\mathsf{G}_{\mathbb{G}_2}(q, q') = v, \qquad \forall (q, q') \notin \mathrm{Cut}(\mathbb{G}_2), \tag{13}$$

where Fix $q = (x, y) \in \mathbb{R}^2$ and $q' \notin \mathrm{Cut}(q)$. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the geodesic from q to q'. Let $\lambda = u\, dx + v\, dy \in T^*_{(x,y)}\mathbb{R}^2$ be its initial covector.



Geodesics and cut-locus (in red) of the Grushin plane starting from the origin and from $q = (1, 0)$. Displayed geodesics have initial covector $\lambda = u\, dx + v\, dy$, with $v = \pm\pi$ and different values of u.

- is a meek gauge function
- The Grushin plane is an ideal structure, it supports interpolation inequalities for densities $n = 2$
- distortion coefficient

$$\beta_t^{(\mathbb{G}_2}(q, q') = t\frac{(u^2 + tuv^2x + v^2x^2)\sin(tv) - tu^2v\cos(tv)}{(u^2 + uv^2x + v^2x^2)\sin(v) - u^2v\cos(v)}, \qquad \forall t \in [0, 1], \tag{14}$$

where $\lambda = u\,dx + v\,dy \in T^*_{(x,y)}\mathbb{R}^2$ is the initial covector of the geodesic joining q with q'.

- not in the standard form
- For all $(q, q') \notin \mathrm{Cut}(\mathbb{G}_2)$ it holds

$$\beta_t^{\mathbb{G}_2, \mathsf{d}_{\mathbb{G}_2}}(q, q') \geq \beta_t^{\mathbb{H}^1}(\mathsf{G}_{\mathbb{G}_2}(q, q')),$$

Theorem

The Grushin plane satisfies the $\mathsf{CD}(\beta^{\mathbb{H}^1}, 2)$ *condition.*