Strichartz estimates and sub-Riemannian geometry
Lecture 1

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Plan of the course

Lecture 1
- Strichartz estimates and dispersion: the Euclidean case
- Some sub-Riemannian geometry

Lecture 2
- Fourier restriction problem
- Strichartz estimates and dispersion: the Heisenberg case

Lecture 3
- Kirillov theory for Nilpotent groups
- Applications to some specific Carnot groups

Lecture 4
- The Engel group and the quartic oscillator
- Some comments on higher step Carnot groups
A part is based on joint works with

- Hajer Bahouri (LJLL, CNRS & Sorbonne Univ)
- Isabelle Gallagher (DMA, École Normale Supérieure)
- Matthieu Léautaud (IMO, Univ. Paris Saclay)

→ Main references:


**BBGM-23** H.Bahouri, D.Barilari, I.Gallagher, M.Léautaud *Spectral summability for the quartic oscillator with applications to the Engel group*, JST, 2023
Outline

1. Strichartz estimates and dispersion: the Euclidean case

2. Some sub-Riemannian geometry
Chapter 1: Strichartz estimates and dispersion
The Schrödinger equation

\[
\begin{align*}
    i \partial_t u - \Delta u &= 0 \\
    u|_{t=0} &= u_0,
\end{align*}
\]

we focus on

- dispersive estimates
- Strichartz estimates
- applications to NLS
- what happens for subelliptic laplacians? (very broad question)
- Riemannian $\rightarrow$ sub-Riemannian (non isotropic diffusions)
The Schrödinger equation on $\mathbb{R}^n$

The Schrödinger equation on $\mathbb{R}^n$

$$\begin{align*}
&\begin{cases}
i \partial_t u - \Delta u = 0 \\
u|_{t=0} = u_0,
\end{cases}
\end{align*}$$

From the explicit expression of the solution, using Fourier analysis:

$$u(t, \cdot) = \frac{e^{i|\cdot|^2}}{(4\pi it)^{n/2}} * u_0.$$ 

one obtains the basic dispersive estimate (for $t \neq 0$)

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi |t|)^{n/2}} \|u_0\|_{L^1(\mathbb{R}^n)} \quad (1)$$
Given a solution $u(t, x)$ of the classical Schrödinger equation $(S)$ in $\mathbb{R}^n$

$$
\begin{align*}
\left\{ \begin{array}{l}
    i \partial_t u - \Delta u = 0 \\
    u|_{t=0} = u_0,
\end{array} \right.
\end{align*}
$$

the Fourier transform $\hat{u}(t, \xi)$ with respect to the spatial variable $x$

$$
    i \partial_t \hat{u}(t, \xi) = -|\xi|^2 \hat{u}(t, \xi), \quad \hat{u}(0, \xi) = \hat{u_0}(\xi).
$$

(2)

Solving the corresponding ODE and taking the inverse Fourier transform

$$
    u(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^2)} \hat{u_0}(\xi) d\xi.
$$

(3)

This is the inverse Fourier of a product hence we get the convolution

$$
    u(t, \cdot) = \frac{e^{i \frac{|\cdot|^2}{4t}}}{(4\pi it)^{\frac{n}{2}}} \ast u_0.
$$
Given a solution \( u(t, x) \) of the classical Schrödinger equation \((S)\) in \( \mathbb{R}^n \)

\[
\begin{dcases}
  i\partial_t u - \Delta u = 0 \\
  u|_{t=0} = u_0 ,
\end{dcases}
\]

the Fourier transform \( \hat{u}(t, \xi) \) with respect to the spatial variable \( x \) satisfies

\[
i\partial_t \hat{u}(t, \xi) = -|\xi|^2 \hat{u}(t, \xi), \quad \hat{u}(0, \xi) = \hat{u}_0(\xi). \tag{4}
\]

Solving the corresponding ODE and taking the inverse Fourier transform

\[
u(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi)} e^{it|\xi|^2} \hat{u}_0(\xi) d\xi . \tag{5}
\]

This is the inverse Fourier of a product hence we get the convolution

\[
u(t, \cdot) = \frac{e^{i \frac{|\cdot|^2}{4t}}}{(4\pi it)^{n/2}} \ast u_0 .
\]

and use Young inequality
The $TT^*$ argument

Once one has the basic dispersive estimate (for $t \neq 0$)

$$
\| u(t, \cdot) \|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi |t|)^{n/2}} \| u_0 \|_{L^1(\mathbb{R}^n)}
$$

(6)

together with the conservation of the $L^2$ norm ($\rightarrow \hat{u}(t, \xi) = e^{it|\xi|^2} \hat{u}_0(\xi)$)

$$
\| u(t, \cdot) \|_{L^2(\mathbb{R}^n)} = \| u_0 \|_{L^2(\mathbb{R}^n)}
$$

(7)

one can obtain interpolating estimates in $L^p$ spaces.
Interpolating the previous estimates one immediately has

\[ \| u(t, \cdot) \|_{L^p(\mathbb{R}^n)} \leq \frac{1}{(4\pi |t|)^n (1 - \frac{2}{p})} \| u_0 \|_{L^p'(\mathbb{R}^n)} \]  

but we are rather interested in time-space estimates. Something like

\[ \| u \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C \| u_0 \|_{L^2(\mathbb{R}^n)} , \]  

for suitable \( p, q \).
Strichartz estimates

For the free Schrödinger one has the following estimate

**Strichartz estimate**

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C_{p,q} \|u_0\|_{L^2(\mathbb{R}^n)},$$

(10)

where $(p, q)$ satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad q \geq 2, \quad (n, q, p) \neq (2, 2, \infty)$$

→ the necessity can be obtained by rescaling
The rescaling argument

Assume the following holds for every \( u_0 \in L^2(\mathbb{R}^n) \)

\[
\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q} \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{11}
\]

Give a solution \( u = u(t,x) \) with \( u(0,\cdot) = u_0 \) then

- also, \( u_\lambda(t,x) = u(\lambda^2 t, \lambda x) \) is a solution
- with initial datum \( u_{0,\lambda}(x) = u(0, \lambda x) = u_0(\lambda x) \)

Let us compute the two sides for \( u_\lambda \)

- \( \|u_\lambda\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} = \lambda^{\frac{n}{q} + \frac{n}{p}} \|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \).
- \( \|u_{0,\lambda}\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}} \|u_0\|_{L^2(\mathbb{R}^n)} \)

One gets

\[
\lambda^{\frac{n}{q} + \frac{n}{p}} \|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C \lambda^{\frac{n}{2}} \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{12}
\]

which forces the equality
The rescaling argument

Assume the following holds for every \( u_0 \in L^2(\mathbb{R}^n) \)

\[
\| u \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C_{p,q} \| u_0 \|_{L^2(\mathbb{R}^n)},
\]  
\[\text{(13)}\]

Give a solution \( u = u(t, x) \) with \( u(0, \cdot) = u_0 \) then

- also, \( u_\lambda(t, x) = u(\lambda^2 t, \lambda x) \) is a solution
- with initial datum \( u_{0, \lambda}(x) = u(0, \lambda x) = u_0(\lambda x) \)

Let us compute the two sides for \( u_\lambda \)

- \( \| u_\lambda \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} = \lambda^{\frac{2}{q} + \frac{n}{p}} \| u \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))}. \)
- \( \| u_{0, \lambda} \|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}} \| u_0 \|_{L^2(\mathbb{R}^n)} \)

One gets

\[
\| u \|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C \lambda^{\frac{n}{2} - \frac{2}{q} - \frac{n}{p}} \| u_0 \|_{L^2(\mathbb{R}^n)},
\]  
\[\text{(14)}\]

which forces the equality
For the free Schrödinger one has the following estimate

**Strichartz estimate**

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C_{p,q} \|u_0\|_{H^\sigma(\mathbb{R}^n)},$$

(15)

where $(p, q)$ satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} \leq \frac{n}{2}, \quad q \geq 2, \quad (n, q, p) \neq (2, 2, \infty)$$

→ the necessity can be obtained by rescaling

→ here $\sigma = \frac{n}{2} - \frac{2}{q} - \frac{n}{p}$
The Schrödinger equation on $\mathbb{R}^n$ with right hand side $f = f(t, x)$

$$
\begin{aligned}
\begin{cases}
  i\partial_t u - \Delta u &= f \\
  u|_{t=0} &= u_0,
\end{cases}
\end{aligned}
$$

→ by Duhamel formula

$$
u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}f(s)ds
$$

or also for $U(t) = e^{it\Delta}$ (notice $U^*(s) = e^{-is\Delta}$)

$$
u(t) = U(t)u_0 + \int_0^t U(t)U^*(s)f(s)ds
$$
dispersion implies Strichartz

If \((U(t))_{t \in \mathbb{R}}\) is a bounded family of continuous linear operators in \(L^2\) and

\[
\|U(t)U^*(t')f\|_{L^\infty} \leq \frac{C}{|t - t'|^{\sigma}} \|f\|_{L^1}
\]

then for any \((q, r) \in [2, \infty]^2\) such that

\[
\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}, \quad (q, r, \sigma) \neq (2, \infty, 1)
\]

one has

\[
\|U(t)u_0\|_{L^q L^r} \leq C\|u_0\|_{L^2}
\]

and

\[
\int_{s < t} \|U(t)U^*(s)f(s)\|_{L^q L^r} \leq C\|f\|_{\tilde{L}^{\tilde{q}} \tilde{L}^{\tilde{r}}}
\]

here \((\tilde{q}, \tilde{r})\) is also an admissible pair.
Let us replace $\sigma$ with the euclidean dispersion exponent

If $(U(t))_{t \in \mathbb{R}}$ is a bounded family of continuous linear operators in $L^2$ and

$$\|U(t)U^*(t')f\|_{L^\infty} \leq \frac{C}{|t - t'|^{n/2}} \|f\|_{L^1}$$

then for any $(q, r) \in [2, \infty]^2$ such that

$$\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}, \quad (q, r, n) \neq (2, \infty, 2)$$

one has

$$\|U(t)u_0\|_{L^qL^r} \leq C\|u_0\|_{L^2}$$

$$\int_{s < t} \|U(t)U^*(s)f(s)\|_{L^qL^r} \leq C\|f\|_{L^\tilde{q}'L^\tilde{r}'}$$
Let us replace $\sigma$ with the euclidean dispersion exponent

If $(U(t))_{t \in \mathbb{R}}$ is a bounded family of continuous linear operators in $L^2$ and

$$\|U(t)U^*(t')f\|_{L^\infty} \leq \frac{C}{|t - t'|^{n/2}} \|f\|_{L^1}$$

then for any $(q, r) \in [2, \infty]^2$ such that

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad (q, r, n) \neq (2, \infty, 2)$$

one has

$$\|U(t)u_0\|_{L^qL^r} \leq C\|u_0\|_{L^2}$$

$$\int_{s < t} \|U(t)U^*(s)f(s)\|_{L^qL^r} \leq C\|f\|_{L^{q'}L^{r'}}$$
The dispersive inequality also yields the following Strichartz inequalities for the inhomogeneous Schrödinger equation $i\partial_t u - \Delta u = f$

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C \left( \|u_0\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^{q'}(\mathbb{R}, L^{p'}(\mathbb{R}^n))} \right), \quad (16)$$

- $(p, q)$ and $(p_1, q_1)$ satisfy the admissibility condition
- $a'$ the dual exponent of any $a \in [1, \infty]$.  
- crucial in the study of semilinear and quasilinear Schrödinger equations

An application: the cubic semilinear equation in $\mathbb{R}^2$

$$\begin{cases} 
    i\partial_t u - \Delta u = |u|^2 u \\
    u|_{t=0} = u_0, 
\end{cases}$$
Fixed point method

For sufficiently small datum \( u_0 \in L^2 \) the cubic equation in \( \mathbb{R}^2 \)

\[
\begin{cases}
  i\partial_t u - \Delta u = |u|^2 u \\
  u|_{t=0} = u_0,
\end{cases}
\]

has a solution in the space \( L^\infty_t L^2_x \cap L^3_t L^6_x \).

\( u \) is a solution if and only if it is a fixed point of the map

\[
u \mapsto F(u) = U(t)u_0 + Q(u)
\]

where \( U(t)u_0 = e^{it\Delta}u_0 \) and

\[
Q(u) = \int_0^t e^{i(t-s)\Delta}|u(s)|^2 u(s)ds
\]

we have

\[
\|Q(u)\|_{L^3_t L^6_x} \leq \|u\|^2 \|u\|_{L^1_t L^2_x} \leq \|u\|^3 \|L^3_t L^6_x
\]
A fixed point argument

$u$ is a solution if and only if it is a fixed point of the map

$$u \mapsto F(u) = U(t)u_0 + Q(u)$$

so that

$$\|F(u)\|_{L^3_t L^6_x} \leq \|U(t)u_0\|_{L^3_t L^6_x} + \|Q(u)\|_{L^3_t L^6_x}$$

and

$$\|F(u)\|_{L^3_t L^6_x} \leq C\|u_0\|_{L^2} + C\|u\|_{L^3_t L^6_x}^3$$

$\rightarrow$ if $8C^2\|u_0\|_{L^2}^2 \leq \frac{1}{2}$ then $F$ sends $B(0, 2C\|u_0\|_{L^2})$ in $L^3_t L^6_x$ into itself.

$\rightarrow$ if $8C^2\|u_0\|_{L^2}^2 \leq \frac{1}{2}$ then $F$ is a contraction (similar computations)

By Picard fixed point theorem we have existence and uniqueness.
Chapter 2: Some sub-Riemannian geometry
Outline

1. Strichartz estimates and dispersion: the Euclidean case

2. Some sub-Riemannian geometry
The Heisenberg group \( \mathbb{H} \)

\( \mathbb{H} \sim \mathbb{R}^3 \)

\[ X_1 := \partial_1 - \frac{x_2}{2} \partial_3 \, , \quad X_2 := \partial_2 + \frac{x_1}{2} \partial_3 \, , \quad X_3 := \partial_3 \, . \]

Group law:

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 
\end{pmatrix} \cdot \begin{pmatrix}
  y_1 \\
  y_2 \\
  y_3 
\end{pmatrix} = \begin{pmatrix}
  x_1 + y_1 \\
  x_2 + y_2 \\
  x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) 
\end{pmatrix}
\]

- We have \([X_1, X_2] = X_3\)
- the distribution \( D = \text{span}\{X_1, X_2\} \) is bracket generating
- we can define a sub-Riemannian distance
- it is also left-invariant
A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z}$$

$(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

$\rightarrow$ The Riemannian Hamiltonian is degenerate for $\varepsilon \to 0$:  

$$H_\varepsilon(p, x) = \frac{1}{2} \sum_{i,j=1}^{3} g_\varepsilon^{ij}(x)p_i p_j$$

$g^{ij}(x) = \lim_{\varepsilon \to 0} g_\varepsilon^{ij}(x)$ is $\geq 0$ but not invertible at any $x$

it is like if the “inverse” $g_{ij}(x)$ has one eigenvalue $= +\infty$. 

A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

\[X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3^\varepsilon = \varepsilon \frac{\partial}{\partial z}\]

\[\text{span}\{X_1, X_2, X_3^\varepsilon\} \rightarrow \text{span}\{X_1, X_2\}\]

\(\mathbb{R}^3, g^\varepsilon\) Riemannian structure with \(\{X_1, X_2, X_3^\varepsilon\}\) o.n. frame.

As metric spaces \((\mathbb{R}^3, d^\varepsilon) \rightarrow (\mathbb{R}^3, d_{\text{SR}})\) (in the Gromov-Hausdorff sense)

\[D^\varepsilon = \text{span}\{X_1, X_2, X_3^\varepsilon\} \rightarrow D = \text{span}\{X_1, X_2\}\]

The sequence of curvatures is unbounded from below:

\[\text{Ric}^\varepsilon(\nu) \rightarrow -\infty\text{ for all }\nu \in D\]
Sub-Riemannian geometry

Sub-Riemannian structure

- $M$ smooth, connected manifold
- $D \subseteq TM$ distribution of (non necessarily) constant rank
  - Hörmander condition: $\text{Lie}(D)|_x = T_x M$ for all $x \in M$
- $g$ smooth scalar product on $D$

Admissible curve: $\gamma : [0, 1] \rightarrow M$ such that $\dot{\gamma}(t) \in D_{\gamma(t)}$

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$$

Sub-Riemannian distance: (or Carnot-Carathédory)

$$d_{SR}(x, y) = \inf \{ \ell(\gamma) \mid \gamma \text{ admissible joining } x \text{ with } y \}$$
Regularity of $d_{SR}$

Assume $M$ connected:

**Chow-Rashevskii:** $d_{SR} < +\infty$ and $(M, d_{SR})$ has the same topology of $M$

Features of general sub-Riemannian structures:

- $d_{SR}^2 : M \times M \rightarrow \mathbb{R}$ is *never* smooth on the diagonal
- geodesics are not parametrized by initial vector
- no Levi-Civita connection in general
- metric Hausdorff dimension $\dim_H(M) > \dim(M)$
Sub-Riemannian balls

Even simple “Riemannian” questions are not trivial in this geometry

- regularity of length-minimizers
- regularity of balls / cut locus
- what is curvature
- what is an intrinsic volume
The Heisenberg group $\mathbb{H}$

$\mathbb{H} \sim \mathbb{R}^3$

$x_1 := \partial_1 - \frac{x_2}{2} \partial_3$, $x_2 := \partial_2 + \frac{x_1}{2} \partial_3$, $x_3 := \partial_3$.

Group law:

$$\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right) \cdot \left(\begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\right) = \left(\begin{array}{c} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \end{array}\right)$$

The Haar measure is equal to the Lebesgue measure.

Convolution product $f \ast g(x) := \int_{\mathbb{H}} f(x \cdot y^{-1})g(y) \, dy$.

Homogeneous dimension

$Q = \sum_j j \dim g_j = 4$, $|B_\mathbb{H}(x, r)| = r^Q |B_\mathbb{H}(0, 1)|$
the horizontal vector fields $X$ and $Y$ are defined by

$$X = \partial_x - \frac{y}{2} \partial_z, \quad Y = \partial_y + \frac{x}{2} \partial_z.$$ 

The horizontal gradient

$$\nabla_\mathbb{H} u = (Xu)X + (Yu)Y.$$ 

Complex notations $Z = X + iY$ and $\bar{Z} = X - iY$

$$\Delta_\mathbb{H} u = (X^2 + Y^2)u = ZZ - i\partial_z,$$

**Remark (on Shrödinger equation in $\mathbb{H}$)**

$$i\partial_t u - \Delta_\mathbb{H} u = 0 \iff i(\partial_t + \partial_z) u = ZZ u$$
No dispersion in Heisenberg

The linear Schrödinger equations on $\mathbb{H}$ associated with the sublaplacian

$$\begin{cases} 
  i \partial_t u - \Delta_{\mathbb{H}} u = f \\
  u|_{t=0} = u_0 ,
\end{cases}$$

**Theorem (Bahouri-Gérard-Xu 2000)**

There exists a function $u_0$ in the Schwartz class $\mathcal{S}(\mathbb{H})$ such that the solution to the free Schrödinger equation satisfies

$$u(t, x_1, x_2, x_3) = u_0(x_1, x_2, x_3 + t).$$

In particular for all $1 \leq p \leq \infty$

$$\|u(t, \cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

→ no dispersion
The Lie algebra $\mathfrak{g}$ of a Carnot (stratified Lie) group of step $r$ admits the following stratification

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i].$$

A sub-Riemannian structure is given by a scalar product on $\mathfrak{g}_1$

**Heisenberg group $\mathbb{H}$ (step 2)**

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \underbrace{\mathfrak{g}_1}_{X_1, X_2}, \quad \underbrace{\mathfrak{g}_2}_{X_3 = [X_1, X_2]}$$

**Engel group $\mathbb{E}$ (step 3)**

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \quad \underbrace{\mathfrak{g}_1}_{X_1, X_2}, \quad \underbrace{\mathfrak{g}_2}_{X_3 = [X_1, X_2]}, \quad \underbrace{\mathfrak{g}_3}_{X_4 = [X_1, X_3]}$$
Higher codimensions

The situation for dispersion on general step 2 is different

**Theorem (Bahouri-Fermanian-Gallagher 2016)**

Let $G$ be a step 2 stratified Lie group with

- center of dimension $p$
- radical index $k$.
- non-degeneracy assumption (*) holds.

If $u_0 \in L^1(G)$ is spectrally localized in a ring, then

$$\|u(t, \cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{\frac{k}{2}}(1 + |t|^{\frac{p-1}{2}})} \|u_0\|_{L^1(G)}$$

In Heisenberg $k = 0$ and $p = 1$!
In $L^2 = L^2(\mathbb{R}^2, dx\,dy)$, consider the action of the Baouendi-Grushin operator

$$\Delta_G = \partial_x^2 + x^2 \partial_y^2. \quad (17)$$

This operator is the Laplacian of the sub-Riemannian structure on $\mathbb{R}^2$ defined by

$$X = \partial_x, \quad Y = x \partial_y. \quad (18)$$

meaning that $\Delta_G = X^2 + Y^2$. Consider the associated Schrödinger equation

$$i\partial_t u(x, y, t) + \Delta_G u(x, y, t) = 0, \quad u(0, \cdot) = u_0. \quad (19)$$
Geodesics of the Grushin plane starting from the origin and from (1, 0).
The associated Schrödinger equation

\[ i \partial_t u(x, y, t) + \Delta_{BG} u(x, y, t) = 0, \quad u(t = 0) = u_0. \quad (20) \]

is also nondispersive.

There exist initial data \( u_0 \) for which the solution \( u \) satisfies

\[ \|u(t)\|_{L^p} = \|u_0\|_{L^p} \quad \forall t \in \mathbb{R}, \quad p \geq 1. \quad (21) \]

This phenomenon is due to a transport behaviour of \( \Delta_{BG} \) in the vertical direction. Let us show this fact.
For any $u \in L^2$, write

$$u(x, y) = \int_{\mathbb{R}} e^{i\lambda y} \hat{u}(x, \lambda) d\lambda,$$

where $\hat{u}(x, \lambda)$ is the Fourier transform of $u$ w.r.t. the $y$-variable.

$$\Delta_G u = \int_{\mathbb{R}} e^{i\lambda y} (\partial^2_x - x^2 \lambda^2) \hat{u}(x, \lambda) d\lambda =: \int_{\mathbb{R}} e^{i\lambda y} \hat{\Delta}_G(\lambda) \hat{u}(x, \lambda) d\lambda,$$

where we defined the Hermite operator

$$\hat{\Delta}_G(\lambda) = \partial^2_x - x^2 \lambda^2$$

for which we know eigenvalues and eigenfunctions.
Let \( h_n(x) \) be the \( n^{th} \) Hermite function, which satisfies the ODE

\[
\frac{d^2}{dx^2} h_n(x) - x^2 h_n(x) = -(2n + 1) h_n(x),
\]

then \( h_n^\lambda(x) := h_n(\sqrt{|\lambda|} x) \) satisfies

\[
\frac{d^2}{dx^2} h_n^\lambda(x) - x^2 \lambda^2 h_n^\lambda(x) = -(2n + 1) |\lambda| h_n^\lambda(x).
\]

We can then write for any \( \lambda \neq 0 \)

\[
\hat{u}(x, \lambda) = \sum_{n \in \mathbb{N}} \hat{u}_n(\lambda) h_n^\lambda(x),
\]

and obtain

\[
\Delta^G_\lambda(x, \lambda) = \sum_{n \in \mathbb{N}} -(2n + 1) |\lambda| \hat{u}_n(\lambda) h_n^\lambda(x).
\]
Let \( h_n(x) \) be the \( n^{th} \) Hermite function, which satisfies the ODE
\[
\frac{d^2}{dx^2} h_n(x) - x^2 h_n(x) = -(2n + 1) h_n(x),
\]
then \( h_n^\lambda(x) := h_n(\sqrt{\lambda}|x) \) satisfies
\[
\frac{d^2}{dx^2} h_n^\lambda(x) - x^2 \lambda^2 h_n^\lambda(x) = -(2n + 1)|\lambda| h_n^\lambda(x).
\]
We can then write for any \( \lambda \neq 0 \)
\[
\hat{u}(x, \lambda) = \sum_{n \in \mathbb{N}} \hat{u}_n(\lambda) h_n^\lambda(x), \quad (23)
\]
and obtain
\[
\widehat{\Delta_G}(\lambda)\hat{u}(x, \lambda) = \sum_{n \in \mathbb{N}} -(2n + 1)|\lambda| \hat{u}_n(\lambda) h_n^\lambda(x).
\]
Summing up, by writing

\[ u(x, y) = \int_{\mathbb{R}} e^{i\lambda y} \left( \sum_{n \in \mathbb{N}} h_n^\lambda(x) \hat{u}_n(\lambda) \right) d\lambda, \]  

we obtain

\[ \Delta_{BG} u(x, y) = \int_{\mathbb{R}} |\lambda| e^{i\lambda y} \left( \sum_{n \in \mathbb{N}} - (2n + 1) h_n^\lambda(x) \hat{u}_n(\lambda) \right) d\lambda. \]

Suppose now that the initial datum \( u_0 \) is supported only on the Hermite mode \( n = \tilde{n} \) (and on positive Fourier modes \( \lambda \geq 0 \)), that is,

\[ u_0(x, y) = \int_{0}^{\infty} e^{i\lambda y} h_\tilde{n}^\lambda(x) u_{0, \tilde{n}}(\lambda) d\lambda, \]  

then we realize that

\[ \Delta_{BG} u_0 = i(2\tilde{n} + 1) \partial_y u_0, \]
If the initial datum \( f \) is supported only on the Hermite mode \( n = \tilde{n} \) and on positive Fourier modes \( \lambda \geq 0 \)

\[ \Delta_{BG} u_0 = i(2\tilde{n} + 1) \partial_y u_0, \]

\[ \rightarrow \text{a transport equation in the vertical direction } y \text{ with velocity } 2\tilde{n} + 1. \]

The solution \( u \) of (20) associated to such an initial datum \( u_0 \in V_{\tilde{n},+} \) is thus given by

\[ u(x, y, t) = u_0(x, y - (2\tilde{n} + 1)t), \quad \forall t \in \mathbb{R}. \] (26)

Analogously, if the initial datum \( u_0 \) is supported only on the Hermite mode \( n = \tilde{n} \) and on negative Fourier modes \( \lambda \leq 0 \)

Since \( \| h_n^\lambda \|_{L^2(\mathbb{R},dx)} \sim \lambda^{-1/2} \), equality (24) holds in \( L^2(\mathbb{R}^2, dx dy) \) iff

\[ \sum_{n \in \mathbb{N}} \int_{\mathbb{R}} |\hat{u}_n(\lambda)|^2 \lambda^{-1/2} d\lambda < \infty. \]
In $L^2 = L^2(\mathbb{R}^3, dx dy_1 dy_2)$, consider the action of the Baouendi-Grushin operator

$$\Delta_{BG2} = \partial_x^2 + x^2(\partial_{y_1}^2 + \partial_{y_2}^2) =: \partial_x^2 + x^2 \Delta_y,$$

where we have defined the vertical Laplacian

$$\Delta_y = \partial_{y_1}^2 + \partial_{y_2}^2.$$

Consider the associated Schrödinger equation

$$i \partial_t u(x, y, t) + \Delta_{BG2} u(x, y, t) = 0, \quad u(t = 0) = u_0.$$
Arguing as before, for any \( u \in L^2 \), write

\[
u(x, y_1, y_2) = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \hat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2,
\]

where \( \hat{u}(x, \lambda_1, \lambda_2) \) is the Fourier transform of \( u \) w.r.t. the \( y_1 \) and \( y_2 \) variables. We have that

\[
\Delta_{BG2} u = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} (\partial_x^2 - x^2(\lambda_1^2 + \lambda_2^2)) \hat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2
\]

\[
=: \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \hat{\Delta}(\lambda_1, \lambda_2) \hat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2,
\]

where we defined the Hermite operator

\[
\hat{\Delta}(\lambda_1, \lambda_2) = \partial_x^2 - x^2(\lambda_1^2 + \lambda_2^2).
\]
Let $h_n(x)$ be the $n^{th}$ Hermite function, and define $h_{n}^{\lambda_1,\lambda_2}(x) := h_n((\lambda_1^2 + \lambda_2^2)^{1/4}x)$ which satisfies

$$
\frac{d^2}{dx^2} h_{n}^{\lambda_1,\lambda_2}(x) - x^2(\lambda_1^2 + \lambda_2^2) h_{n}^{\lambda_1,\lambda_2}(x) = -(2n + 1) \sqrt{\lambda_1^2 + \lambda_2^2} h_{n}^{\lambda_1,\lambda_2}(x).
$$

We can then write for any $\lambda_1 \neq 0, \lambda_2 \neq 0$,

$$
\hat{u}(x, \lambda_1, \lambda_2) = \sum_{n \in \mathbb{N}} \hat{u}_n(\lambda_1, \lambda_2) h_{n}^{\lambda_1,\lambda_2}(x), \quad (29)
$$

and obtain

$$
\hat{\Delta}(\lambda_1, \lambda_2)\hat{u}(x, \lambda_1, \lambda_2) = \sum_{n \in \mathbb{N}} -(2n + 1) \sqrt{\lambda_1^2 + \lambda_2^2} \hat{u}_n(\lambda) h_{n}^{\lambda_1,\lambda_2}(x).
$$

Summing up

$$
u(x, y_1, y_2) = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \left( \sum_{n \in \mathbb{N}} h_{n}^{\lambda_1,\lambda_2}(x) \hat{u}_n(\lambda_1, \lambda_2) \right) d\lambda_1 d\lambda_2, \quad (30)$$
We obtain

$$\Delta_{BG2} u(x, y_1, y_2) = \int_{\mathbb{R}^2_y} \sqrt{\lambda_1^2 + \lambda_2^2} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \times$$

$$\times \left( \sum_{n \in \mathbb{N}} -(2n + 1) h_n^{\lambda_1, \lambda_2}(x) \hat{u}_n(\lambda_1, \lambda_2) \right) d\lambda_1 d\lambda_2. \quad (31)$$

We immediately remark the appearance of $\sqrt{\lambda_1^2 + \lambda_2^2}$, which is the symbol associated with the operator $\sqrt{-\Delta_y}$.

Suppose now that the initial datum $f$ is supported only on the Hermite mode $n = \tilde{n}$, that is,

$$f(x, y_1, y_2) = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} h_{\tilde{n}}^{\lambda_1, \lambda_2}(x) f_{\tilde{n}}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \quad (32)$$
then we realize that

- $\Delta_{BG2} f = (2\tilde{n} + 1)\sqrt{-\Delta_y} f$
- $(i\partial_t + \Delta_{BG2}) f = 0 \iff (i\partial_t + (2\tilde{n} + 1)\sqrt{-\Delta_y}) f = 0$.

By multiplying the last equation with $(i\partial_t - (2\tilde{n} + 1)\sqrt{-\Delta_y})$,

$$ (i\partial_t + \Delta_{BG2}) f = 0 \Rightarrow (-\partial_t^2 + (2\tilde{n} + 1)^2\Delta_y) f = 0, $$

a solution to (28) with initial datum belonging to the space of functions $V_{\tilde{n}}$ defined by (32) is also a solution to the wave equation in the vertical direction $y_1, y_2$ with velocity $2\tilde{n} + 1$.

Assume now that the Fourier transform in the $y_1, y_2$ variables of the initial datum $f \in V_{\tilde{n}} \cap L^1(\mathbb{R}^3)$ is supported in an annulus.

Thanks to the dispersive estimates enjoyed by the wave equation in $\mathbb{R}^d$, we obtain that there exists a constant $C$ such that the solution $u$ to (28) satisfies

$$ \|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{1/2}} \|u_0\|_{L^1(\mathbb{R}^3)}. $$
The wave equation on $\mathbb{R}^n$:

\[
(W) \quad \left\{ \begin{array}{l}
\frac{\partial^2}{\partial t^2} u - \Delta u = 0 \\
(u, \partial_t u)|_{t=0} = (u_0, u_1),
\end{array} \right.
\]

The classical dispersive estimate writes (for $t \neq 0$)

\[
\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|t|^{n-1/2}} \left( \|u_0\|_{L^1(\mathbb{R}^n)} + \|u_1\|_{L^1(\mathbb{R}^n)} \right).
\]

→ oscillatory integrals and stationary phase theorem.