Strichartz estimates and sub-Riemannian geometry
Lecture 1

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Plan of the course

Lecture 1

- Strichartz estimates and dispersion: the Euclidean case
- Some sub-Riemannian geometry

Lecture 2

- Fourier restriction problem
- Strichartz estimates and dispersion: the Heisenberg case

Lecture 3

- Kirillov theory for Nilpotent groups
- Applications to some specific Carnot groups

Lecture 4

- The Engel group and the quartic oscillator
- Some comments on higher step Carnot groups

Joint work with

A part is based on joint works with

- Hajer Bahouri (LJLL, CNRS & Sorbonne Univ)
- Isabelle Gallagher (DMA, École Normale Supérieure)
- Matthieu Léautaud (IMO, Univ. Paris Saclay)

→ Main references:

BBG-21 H.Bahouri, D.Barilari, I.Gallagher, *Strichartz estimates and Fourier restriction theorems in the Heisenberg group*, JFAA, 2021

BBGM-23 H.Bahouri, D.Barilari, I.Gallagher, M.Léautaud *Spectral summability for the quartic oscillator with applications to the Engel group*, JST, 2023

Outline

1 Strichartz estimates and dispersion: the Euclidean case

2 Some sub-Riemannian geometry

Chapter 1: Strichartz estimates and dispersion

Motivation

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The Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \, , \end{cases}$$

we focus on

- dispersive estimates
- Strichartz estimates
- applications to NLS
- what happens for subelliptic laplacians? (very broad question)
- Riemannian $\rightarrow$ sub-Riemannian (non isotropic diffusions)

The Schrödinger equation on $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

From the explicit expression of the solution, using Fourier analysis:

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0 \,.$$

one obtains the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}} \|u_0\|_{L^1(\mathbb{R}^n)} \qquad (1)$$

Given a solution $u(t,x)$ of the classical Schrödinger equation (S) in $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

the Fourier transform $\widehat{u}(t,\xi)$ with respect to the spatial variable x

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi), \qquad \widehat{u}(0,\xi) = \widehat{u_0}(\xi). \tag{2}$$

Solving the corresponding ODE and taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u_0}(\xi) d\xi\,. \tag{3}$$

This is the inverse Fourier of a product hence we get the convolution

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0\,.$$

Given a solution $u(t,x)$ of the classical Schrödinger equation (S) in $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

the Fourier transform $\widehat{u}(t,\xi)$ with respect to the spatial variable x satisfies

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi), \qquad \widehat{u}(0,\xi) = \widehat{u}_0(\xi). \tag{4}$$

Solving the corresponding ODE and taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi)} e^{it|\xi|^2} \widehat{u}_0(\xi) d\xi\,. \tag{5}$$

This is the inverse Fourier of a product hence we get the convolution

$$u(t,\cdot) = \frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{n}{2}}} \star u_0\,.$$

and use Young inequality

The TT^* argument

Once one has the basic dispersive estimate (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}}}\|u_0\|_{L^1(\mathbb{R}^n)} \tag{6}$$

together with the conservation of the L^2 norm ($\rightarrow \widehat{u}(t,\xi) = e^{it|\xi|^2}\widehat{u}_0(\xi)$)

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \tag{7}$$

one can obtain interpolating estimates in L^p spaces

Fixed time interpolation

Interpolating the previous estimates one immediately has

$$\|u(t,\cdot)\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{(4\pi|t|)^{\frac{n}{2}(1-\frac{2}{p})}} \|u_0\|_{L^{p'}(\mathbb{R}^n)} \tag{8}$$

but we are rather interested in time-space estimates. Something like

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{9}$$

for suitable p, q.

For the free Schrödinger one has the following estimate

Strichartz estimate

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C_{p,q} \|u_0\|_{L^2(\mathbb{R}^n)} \,, \tag{10}$$

where (p, q) satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \qquad q \geq 2,\ (n, q, p) \neq (2, 2, \infty)$$

$\rightarrow$ the necessity can be obtained by rescaling

The rescaling argument

Assume the following holds for every $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \le C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{11}$$

Give a solution $u = u(t,x)$ with $u(0,\cdot) = u_0$ then

- also, $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution
- with initial datum $u_{0,\lambda}(x) = u(0,\lambda x) = u_0(\lambda x)$

Let us compute the two sides for u_λ

- $\|u_\lambda\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} = \lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}$.
- $\|u_{0,\lambda}\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}$

One gets

$$\lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \le C\lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{12}$$

which forces the equality

The rescaling argument

Assume the following holds for every $u_0 \in L^2(\mathbb{R}^n)$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{13}$$

Give a solution $u = u(t,x)$ with $u(0,\cdot) = u_0$ then

- also, $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution
- with initial datum $u_{0,\lambda}(x) = u(0,\lambda x) = u_0(\lambda x)$

Let us compute the two sides for u_λ

- $\|u_\lambda\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} = \lambda^{\frac{2}{q}+\frac{n}{p}}\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))}$.
- $\|u_{0,\lambda}\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}}\|u_0\|_{L^2(\mathbb{R}^n)}$

One gets

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\lambda^{\frac{n}{2}-\frac{2}{q}-\frac{n}{p}}\|u_0\|_{L^2(\mathbb{R}^n)}\,, \tag{14}$$

which forces the equality

For the free Schrödinger one has the following estimate

Strichartz estimate

For initial data $u_0 \in L^2(\mathbb{R}^n)$ we have the following

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C_{p,q}\|u_0\|_{H^\sigma(\mathbb{R}^n)}\,, \tag{15}$$

where (p,q) satisfies the admissibility condition

$$\frac{2}{q} + \frac{n}{p} \leq \frac{n}{2}, \qquad q \geq 2,\ (n,q,p) \neq (2,2,\infty)$$

→ the necessity can be obtained by rescaling

→ here $\sigma = \frac{n}{2} - \frac{2}{q} - \frac{n}{p}$

Strichartz estimates

The Schrödinger equation on $\mathbb{R}^n$ with right hand side $f = f(t,x)$

$$\begin{cases} i\partial_t u - \Delta u = f \\ u_{|t=0} = u_0 \,, \end{cases}$$

→ by Duhamel formula

$$u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} f(s) ds$$

or also for $U(t) = e^{it\Delta}$ (notice $U^*(s) = e^{-is\Delta}$)

$$u(t) = U(t)u_0 + \int_0^t U(t)U^*(s)f(s)ds$$

The TT^* argument, part II

dispersion implies Strichartz

If $(U(t))_{t\in\mathbb{R}}$ is a bounded family of continuous linear operators in L^2 and

$$\|U(t)U^*(t')f\|_{L^\infty} \le \frac{C}{|t-t'|^\sigma}\|f\|_{L^1}$$

then for any $(q,r)\in[2,\infty]^2$ such that

$$\frac{1}{q}+\frac{\sigma}{r}=\frac{\sigma}{2}, \qquad (q,r,\sigma)\neq(2,\infty,1)$$

one has

$$\|U(t)u_0\|_{L^qL^r} \le C\|u_0\|_{L^2}$$

$$\int_{s<t}\|U(t)U^*(s)f(s)\|_{L^qL^r} \le C\|f\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

here $(\tilde{q},\tilde{r})$ is also an admissible pair.

The TT^* argument, part II

Let us replace σ with the euclidean dispersion exponent

If $(U(t))_{t\in\mathbb{R}}$ is a bounded family of continuous linear operators in L^2 and

$$\|U(t)U^*(t')f\|_{L^\infty} \le \frac{C}{|t-t'|^{n/2}}\|f\|_{L^1}$$

then for any $(q,r)\in[2,\infty]^2$ such that

$$\frac{1}{q}+\frac{n}{2r}=\frac{n}{4}, \qquad (q,r,n)\neq(2,\infty,2)$$

one has

$$\|U(t)u_0\|_{L^qL^r} \le C\|u_0\|_{L^2}$$

$$\int_{s<t}\|U(t)U^*(s)f(s)\|_{L^qL^r} \le C\|f\|_{L^{\tilde q'}L^{\tilde r'}}$$

The TT^* argument, part II

Let us replace σ with the euclidean dispersion exponent

If $(U(t))_{t\in\mathbb{R}}$ is a bounded family of continuous linear operators in L^2 and

$$\|U(t)U^*(t')f\|_{L^\infty} \leq \frac{C}{|t-t'|^{n/2}}\|f\|_{L^1}$$

then for any $(q,r)\in[2,\infty]^2$ such that

$$\frac{2}{q}+\frac{n}{r}=\frac{n}{2}, \qquad (q,r,n)\neq(2,\infty,2)$$

one has

$$\|U(t)u_0\|_{L^qL^r} \leq C\|u_0\|_{L^2}$$

$$\int_{s<t}\|U(t)U^*(s)f(s)\|_{L^qL^r} \leq C\|f\|_{L^{\tilde{q}'}L^{\tilde{r}'}}$$

Consequence

The dispersive inequality also yields the following Strichartz inequalities for the inhomogeneous Schrödinger equation $i\partial_t u - \Delta u = f$

$$\|u\|_{L^q(\mathbb{R},L^p(\mathbb{R}^n))} \leq C\left(\|u_0\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{p}'}(\mathbb{R}^n))}\right), \quad (16)$$

- (p, q) and (p_1, q_1) satisfy the admissibility condition
- a' the dual exponent of any $a \in [1, \infty]$.
- crucial in the study of semilinear and quasilinear Schrödinger equations

An application : the cubic semilinear equation in $\mathbb{R}^2$

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u \\ u_{|t=0} = u_0 \, , \end{cases}$$

Fixed point method

For sufficiently small datum $u_0 \in L^2$ the cubic equation in $\mathbb{R}^2$

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u \\ u_{|t=0} = u_0 \,, \end{cases}$$

has a solution in the space $L^\infty_t L^2_x \cap L^3_t L^6_x$.

u is a solution if and only if it is a fixed point of the map

$$u \mapsto F(u) = U(t)u_0 + Q(u)$$

where $U(t)u_0 = e^{it\Delta}u_0$ and

$$Q(u) = \int_0^t e^{i(t-s)\Delta}|u(s)|^2 u(s) ds$$

we have $$\|Q(u)\|_{L^3_t L^6_x} \leq \||u|^2 u\|_{L^1_t L^2_x} \leq \|u\|^3_{L^3_t L^6_x}$$

u is a solution if and only if it is a fixed point of the map

$$u \mapsto F(u) = U(t)u_0 + Q(u)$$

so that

$$\|F(u)\|_{L^3_t L^6_x} \leq \|U(t)u_0\|_{L^3_t L^6_x} + \|Q(u)\|_{L^3_t L^6_x}$$

and

$$\|F(u)\|_{L^3_t L^6_x} \leq C\|u_0\|_{L^2} + C\|u\|^3_{L^3_t L^6_x}$$

→ if $8C^2\|u_0\|^2_{L^2} \leq \frac{1}{2}$ then F sends $B(0, 2C\|u_0\|_{L^2})$ in $L^3_t L^6_x$ into itself.

→ if $8C^2\|u_0\|^2_{L^2} \leq \frac{1}{2}$ then F is a contraction (similar computations)

By Picard fixed point theorem we have existence and uniqueness.

Chapter 2: Some sub-Riemannian geometry

Outline

1 Strichartz estimates and dispersion: the Euclidean case

2 Some sub-Riemannian geometry

The Heisenberg group $\mathbb{H}$

$\mathbb{H} \sim \mathbb{R}^3$

$$X_1 := \partial_1 - \frac{x_2}{2}\partial_3 \ , \quad X_2 := \partial_2 + \frac{x_1}{2}\partial_3 \, , \quad X_3 := \partial_3 \, .$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \end{pmatrix}$$

- We have $[X_1, X_2] = X_3$
- the distribution $D = \mathrm{span}\{X_1, X_2\}$ is bracket generating
- we can define a sub-Riemannian distance
- it is also left-invariant

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Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3^\varepsilon = \varepsilon\frac{\partial}{\partial z}$$

- $(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

→ The Riemannian Hamiltonian is degenerate for $\varepsilon \to 0$:

$$H_\varepsilon(p,x) = \frac{1}{2}\sum_{i,j=1}^{3} g_\varepsilon^{ij}(x) p_i p_j$$

- $g^{ij}(x) = \lim_{\varepsilon\to 0} g_\varepsilon^{ij}(x)$ is ≥ 0 but not invertible at any x
- it is like if the "inverse" $g_{ij}(x)$ has one eigenvalue $= +\infty$.

A limiting procedure: the Heisenberg group

Define on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \qquad X_3^\varepsilon = \varepsilon\frac{\partial}{\partial z}$$

- $(\mathbb{R}^3, g^\varepsilon)$ Riemannian structure with $\{X_1, X_2, X_3^\varepsilon\}$ o.n. frame.

As metric spaces $(\mathbb{R}^3, d^\varepsilon) \to (\mathbb{R}^3, d_{SR})$ (in the Gromov-Hausdorff sense)

- $D^\varepsilon = \text{span}\{X_1, X_2, X_3^\varepsilon\} \to D = \text{span}\{X_1, X_2\}$

The sequence of curvatures is unbounded from below:

- $\text{Ric}^\varepsilon(v) \to -\infty$ for all $v \in D$

Sub-Riemannian geometry

Sub-Riemannian structure

- M smooth, connected manifold
- $D \subseteq TM$ distribution of (non necessarily) constant rank
  - Hörmander condition: $\mathrm{Lie}(D)|_x = T_x M$ for all $x \in M$
- g smooth scalar product on D

Admissible curve: $\gamma : [0,1] \to M$ such that $\dot{\gamma}(t) \in D_{\gamma(t)}$

$$\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt$$

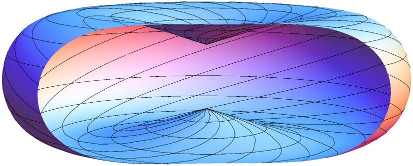
Sub-Riemannian distance: (or Carnot-Carathédory)

$$d_{SR}(x,y) = \inf\{\ell(\gamma) \mid \gamma \text{ admissible joining } x \text{ with } y\}$$

Regularity of d_{SR}

Assume M connected:

Chow-Rashevskii: $d_{SR} < +\infty$ and (M, d_{SR}) has the same topology of M



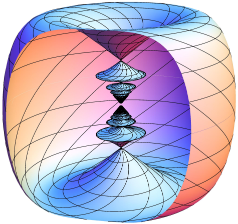
Features of general sub-Riemannian structures:

- $d_{SR}^2 : M \times M \to \mathbb{R}$ is *never* smooth on the diagonal
- geodesics are not parametrized by initial vector
- no Levi-Civita connection in general
- metric Hausdorff dimension $\dim_H(M) > \dim(M)$

Sub-Riemannian balls

Even simple “Riemannian” questions are not trivial in this geometry

- regularity of length-minimizers
- regularity of balls / cut locus ?
- what is curvature ?
- what is an intrinsic volume ?



The Heisenberg group $\mathbb{H}$

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$\mathbb{H} \sim \mathbb{R}^3$

$$X_1 := \partial_1 - \frac{x_2}{2}\partial_3 \ , \quad X_2 := \partial_2 + \frac{x_1}{2}\partial_3 \, , \quad X_3 := \partial_3 \, .$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2) \end{pmatrix}$$

The Haar measure is equal to the Lebesgue measure.

Convolution product $f \star g(x) := \int_{\mathbb{H}} f(x \cdot y^{-1}) g(y) \, dy \, .$

Homogeneous dimension
$Q = \sum_j j \dim \mathfrak{g}_j = 4 \, , \quad \left| B_{\mathbb{H}}(x, r) \right| = r^Q \left| B_{\mathbb{H}}(0, 1) \right|$

Laplacian in Heisenberg

- the horizontal vector fields X and Y are defined by

$$X = \partial_x - \frac{y}{2}\partial_z, \qquad Y = \partial_y + \frac{x}{2}\partial_z\,.$$

- The horizontal gradient

$$\nabla_{\mathbb{H}} u = (Xu)X + (Yu)Y.$$

- Complex notations $Z = X + iY$ and $\bar{Z} = X - iY$

$$\Delta_{\mathbb{H}} u = (X^2 + Y^2)u = Z\overline{Z} - i\partial_z,$$

Remark (on Shrödinger equation in $\mathbb{H}$)

$$i\partial_t u - \Delta_{\mathbb{H}} u = 0 \quad \Leftrightarrow \quad i(\partial_t + \partial_z)u = Z\overline{Z}u$$

No dispersion in Heisenberg

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The linear Schrödinger equations on $\mathbb{H}$ associated with the sublaplacian

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = u_0 \,, \end{cases}$$

Theorem (Bahouri-Gérard-Xu 2000)

There exists a function u_0 in the Schwartz class $\mathcal{S}(\mathbb{H})$ such that the solution to the free Schrödinger equation satisfies

$$u(t, x_1, x_2, x_3) = u_0(x_1, x_2, x_3 + t)\,.$$

In particular for all $1 \leq p \leq \infty$

$$\|u(t,\cdot)\|_{L^p(\mathbb{H}^d)} = \|u_0\|_{L^p(\mathbb{H}^d)}$$

$\rightarrow$ no dispersion

The Lie algebra $\mathfrak{g}$ of a Carnot (stratified Lie) group of step r admits the following stratification

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i] \,.$$

A sub-Riemannian structure is given by a scalar product on $\mathfrak{g}_1$

Heisenberg group $\mathbb{H}$ (step 2)

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \,, \quad \overbrace{X_1, X_2}^{\mathfrak{g}_1} , \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}$$

Engel group $\mathbb{E}$ (step 3)

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \,, \quad \overbrace{X_1, X_2}^{\mathfrak{g}_1} , \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2} , \quad \overbrace{X_4 = [X_1, X_3]}^{\mathfrak{g}_3}$$

Higher codimensions

The situation for dispersion on general step 2 is different

Theorem (Bahouri-Fermanian-Gallagher 2016)

Let G be a step 2 stratified Lie group with

- *center of dimension p*
- *radical index k.*
- *non-degeneracy assumption (*) holds.*

If $u_0 \in L^1(G)$ is spectrally localized in a ring, then

$$\|u(t,\cdot)\|_{L^\infty(G)} \leq \frac{C}{|t|^{\frac{k}{2}}\left(1+|t|^{\frac{p-1}{2}}\right)}\|u_0\|_{L^1(G)}$$

In Heisenberg $k = 0$ and $p = 1$!

In $L^2 = L^2(\mathbb{R}^2, dxdy)$, consider the action of the Baouendi-Grushin operator

$$\Delta_G = \partial_x^2 + x^2\partial_y^2. \tag{17}$$

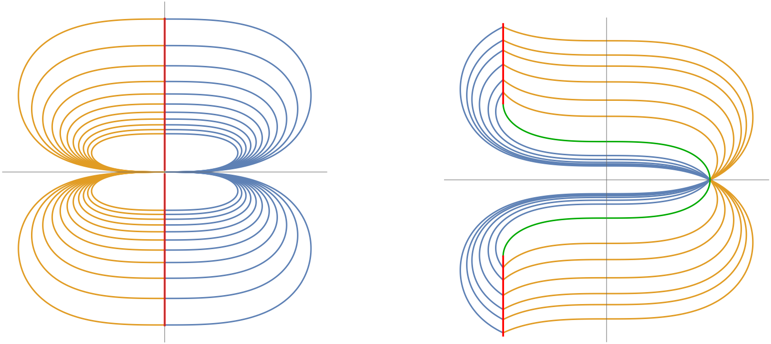
This operator is the Laplacian of the sub-Riemannian structure on $\mathbb{R}^2$ defined by

$$X = \partial_x, \quad Y = x\partial_y. \tag{18}$$

meaning that $\Delta_G = X^2 + Y^2$. Consider the associated Schrödinger equation

$$i\partial_t u(x, y, t) + \Delta_G u(x, y, t) = 0, \quad u(0, \cdot) = u_0. \tag{19}$$

Geodesics of the Grushin plane



Geodesics of the Grushin plane starting from the origin and from $(1, 0)$.

No dispersion

The associated Schrödinger equation

$$i\partial_t u(x,y,t) + \Delta_{BG} u(x,y,t) = 0, \quad u(t=0) = u_0. \tag{20}$$

is also nondispersive.

there exist initial data u_0 for which the solution u satisfies

$$\|u(t)\|_{L^p} = \|u_0\|_{L^p} \quad \forall t \in \mathbb{R}, \quad p \geq 1. \tag{21}$$

This phenomenon is due to a transport behaviour of Δ_{BG} in the vertical direction. Let us show this fact.

Baouendi-Grushin operator in Fourier

For any $u \in L^2$, write

$$u(x, y) = \int_{\mathbb{R}} e^{i\lambda y} \widehat{u}(x, \lambda) d\lambda,$$

where $\widehat{u}(x, \lambda)$ is the Fourier transform of u w.r.t. the y-variable.

$$\Delta_G u = \int_{\mathbb{R}} e^{i\lambda y} (\partial_x^2 - x^2\lambda^2) \widehat{u}(x, \lambda) d\lambda =: \int_{\mathbb{R}} e^{i\lambda y} \widehat{\Delta_G}(\lambda) \widehat{u}(x, \lambda) d\lambda,$$

where we defined the Hermite operator

$$\widehat{\Delta_G}(\lambda) = \partial_x^2 - x^2\lambda^2$$

for which we know eigenvalues and eigenfunctions.

Let $h_n(x)$ be the n^{th} Hermite function, which satisfies the ODE

$$\frac{d^2}{dx^2}h_n(x) - x^2 h_n(x) = -(2n+1)h_n(x),$$

then $h_n^\lambda(x) := h_n(\sqrt{|\lambda|}x)$ satisfies

$$\frac{d^2}{dx^2}h_n^\lambda(x) - x^2\lambda^2 h_n^\lambda(x) = -(2n+1)|\lambda| h_n^\lambda(x).$$

We can then write for any $\lambda \neq 0$

$$\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} \widehat{u}_n(\lambda) h_n^\lambda(x), \tag{22}$$

and obtain

$$\widehat{\Delta_G}(\lambda)\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} -(2n+1)|\lambda|\widehat{u}_n(\lambda)h_n^\lambda(x).$$

Let $h_n(x)$ be the n^{th} Hermite function, which satisfies the ODE

$$\frac{d^2}{dx^2}h_n(x) - x^2 h_n(x) = -(2n+1)h_n(x),$$

then $h_n^\lambda(x) := h_n(\sqrt{|\lambda|}x)$ satisfies

$$\frac{d^2}{dx^2}h_n^\lambda(x) - x^2\lambda^2 h_n^\lambda(x) = -(2n+1)|\lambda| h_n^\lambda(x).$$

We can then write for any $\lambda \neq 0$

$$\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} \widehat{u}_n(\lambda) h_n^\lambda(x), \tag{23}$$

and obtain

$$\widehat{\Delta_G}(\lambda)\widehat{u}(x,\lambda) = \sum_{n\in\mathbb{N}} -(2n+1)|\lambda| \widehat{u}_n(\lambda) h_n^\lambda(x).$$

Summing up, by writing

$$u(x,y) = \int_{\mathbb{R}} e^{i\lambda y} \left(\sum_{n\in\mathbb{N}} h_n^\lambda(x) \widehat{u}_n(\lambda) \right) d\lambda, \tag{24}$$

we obtain

$$\Delta_{BG} u(x,y) = \int_{\mathbb{R}} |\lambda| e^{i\lambda y} \left(\sum_{n\in\mathbb{N}} -(2n+1) h_n^\lambda(x) \widehat{u}_n(\lambda) \right) d\lambda.$$

Suppose now that the initial datum u_0 is supported only on the Hermite mode $n = \widetilde{n}$ (and on positive Fourier modes $\lambda \geq 0$), that is,

$$u_0(x,y) = \int_0^\infty e^{i\lambda y} h_{\widetilde{n}}^\lambda(x) u_{0,\widetilde{n}}(\lambda) d\lambda, \tag{25}$$

then we realize that

$$\Delta_{BG} u_0 = i(2\widetilde{n}+1)\partial_y u_0,$$

- If the initial datum f is supported only on the Hermite mode $n = \widetilde{n}$ and on positive Fourier modes $\lambda \geq 0$

$$\Delta_{BG} u_0 = i(2\widetilde{n} + 1)\partial_y u_0,$$

$\rightarrow$ a transport equation in the vertical direction y with velocity $2\widetilde{n} + 1$. The solution u of (20) associated to such an initial datum $u_0 \in V_{\widetilde{n},+}$ is thus given by

$$u(x, y, t) = u_0(x, y - (2\widetilde{n} + 1)t), \quad \forall t \in \mathbb{R}. \tag{26}$$

- Analogously, if the initial datum u_0 is supported only on the Hermite mode $n = \widetilde{n}$ and on negative Fourier modes $\lambda \leq 0$

Since $\|h_n^\lambda\|_{L^2(\mathbb{R},dx)} \sim \lambda^{-1/2}$, equality (24) holds in $L^2(\mathbb{R}^2, dxdy)$ iff

$$\sum_{n\in\mathbb{N}} \int_{\mathbb{R}} |\widehat{u}_n(\lambda)|^2 \lambda^{-1/2} d\lambda < \infty.$$

In $L^2 = L^2(\mathbb{R}^3, dxdy_1dy_2)$, consider the action of the Baouendi-Grushin operator

$$\Delta_{BG2} = \partial_x^2 + x^2(\partial_{y_1}^2 + \partial_{y_2}^2) =: \partial_x^2 + x^2\Delta_y, \qquad (27)$$

where we have defined the vertical Laplacian

$$\Delta_y = \partial_{y_1}^2 + \partial_{y_2}^2.$$

Consider the associated Schrödinger equation

$$i\partial_t u(x,y,t) + \Delta_{BG2}u(x,y,t) = 0, \quad u(t=0) = u_0. \qquad (28)$$

Arguing as before, for any $u \in L^2$, write

$$u(x, y_1, y_2) = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \widehat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2,$$

where $\widehat{u}(x, \lambda_1, \lambda_2)$ is the Fourier transform of u w.r.t. the y_1 and y_2 variables. We have that

$$\begin{aligned} \Delta_{BG2} u &= \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} (\partial_x^2 - x^2(\lambda_1^2 + \lambda_2^2)) \widehat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\ &=: \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \widehat{\Delta}(\lambda_1, \lambda_2) \widehat{u}(x, \lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \end{aligned}$$

where we defined the Hermite operator

$$\widehat{\Delta}(\lambda_1, \lambda_2) = \partial_x^2 - x^2(\lambda_1^2 + \lambda_2^2).$$

Let $h_n(x)$ be the n^{th} Hermite function, and define $h_n^{\lambda_1,\lambda_2}(x) := h_n((\lambda_1^2+\lambda_2^2)^{1/4}x)$ which satysfies

$$\frac{d^2}{dx^2}h_n^{\lambda_1,\lambda_2}(x) - x^2(\lambda_1^2+\lambda_2^2)h_n^{\lambda_1,\lambda_2}(x) = -(2n+1)\sqrt{\lambda_1^2+\lambda_2^2}\,h_n^{\lambda_1,\lambda_2}(x).$$

We can then write for any $\lambda_1 \neq 0, \lambda_2 \neq 0$,

$$\widehat{u}(x,\lambda_1,\lambda_2) = \sum_{n\in\mathbb{N}} \widehat{u}_n(\lambda_1,\lambda_2)h_n^{\lambda_1,\lambda_2}(x), \tag{29}$$

and obtain

$$\widehat{\Delta}(\lambda_1,\lambda_2)\widehat{u}(x,\lambda_1,\lambda_2) = \sum_{n\in\mathbb{N}} -(2n+1)\sqrt{\lambda_1^2+\lambda_2^2}\,\widehat{u}_n(\lambda)h_n^{\lambda_1,\lambda_2}(x).$$

Summing up

$$u(x,y_1,y_2) = \int_{\mathbb{R}_y^2} e^{i(\lambda_1 y_1+\lambda_2 y_2)}\left(\sum_{n\in\mathbb{N}} h_n^{\lambda_1,\lambda_2}(x)\widehat{u}_n(\lambda_1,\lambda_2)\right)d\lambda_1 d\lambda_2, \tag{30}$$

We obtain

$$\Delta_{BG2} u(x, y_1, y_2) = \int_{\mathbb{R}^2_y} \sqrt{\lambda_1^2 + \lambda_2^2} e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \times \\ \times \left(\sum_{n \in \mathbb{N}} -(2n+1) h_n^{\lambda_1, \lambda_2}(x) \widehat{u}_n(\lambda_1, \lambda_2) \right) d\lambda_1 d\lambda_2. \tag{31}$$

We immediately remark the appearance of $\sqrt{\lambda_1^2 + \lambda_2^2}$, which is the symbol associated with the operator $\sqrt{-\Delta_y}$.

Suppose now that the initial datum f is supported only on the Hermite mode $n = \widetilde{n}$, that is,

$$f(x, y_1, y_2) = \int_{\mathbb{R}^2_y} e^{i(\lambda_1 y_1 + \lambda_2 y_2} h_{\widetilde{n}}^{\lambda_1, \lambda_2}(x) f_{\widetilde{n}}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \tag{32}$$

then we realize that

- $\Delta_{BG2} f = (2\widetilde{n}+1)\sqrt{-\Delta_y} f$
- $(i\partial_t + \Delta_{BG2})f = 0 \Leftrightarrow (i\partial_t + (2\widetilde{n}+1)\sqrt{-\Delta_y})f = 0.$
- By multiplying the last equation with $(i\partial_t - (2\widetilde{n}+1)\sqrt{-\Delta_y})$,

$$(i\partial_t + \Delta_{BG2})f = 0 \Rightarrow (-\partial_t^2 + (2\widetilde{n}+1)^2\Delta_y)f = 0,$$

- a solution to (28) with initial datum belonging to the space of functions $V_{\widetilde{n}}$ defined by (32) is also a solution to the wave equation in the vertical direction y_1, y_2 with velocity $2\widetilde{n}+1$.
- Assume now that the Fourier transform in the y_1, y_2 variables of the initial datum $f \in V_{\widetilde{n}} \cap L^1(\mathbb{R}^3)$ is supported in an annulus.
- Thanks to the dispersive estimates enjoyed by the wave equation in $\mathbb{R}^d$, we obtain that there exists a constant C such that the solution u to (28) satisfies

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{1/2}}\|u_0\|_{L^1(\mathbb{R}^3)}.$$

The wave equation on $\mathbb{R}^n$

$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1)\,, \end{cases}$$

The classical dispersive estimate writes (for $t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|t|^{\frac{n-1}{2}}}\left(\|u_0\|_{L^1(\mathbb{R}^n)} + \|u_1\|_{L^1(\mathbb{R}^n)}\right).$$

$\rightarrow$ oscillatory integrals and stationary phase theorem.