Strichartz estimates and sub-Riemannian geometry
Lecture 2

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Chapter 3: The Fourier restriction problem

Back to Schrödinger

Given a solution $u(t,x)$ of the classical Schrödinger equation (S) in $\mathbb{R}^n$

$$\begin{cases} i\partial_t u - \Delta u = 0 \\ u_{|t=0} = u_0 \,, \end{cases}$$

the Fourier transform $\widehat{u}(t,\xi)$ with respect to the spatial variable x satisfies

$$i\partial_t \widehat{u}(t,\xi) = -|\xi|^2 \widehat{u}(t,\xi), \qquad \widehat{u}(0,\xi) = \widehat{u}_0(\xi). \tag{1}$$

Solving the corresponding ODE and taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi \,. \tag{2}$$

Another viewpoint

One can also interpreted as the inverse Fourier transform of a data on the paraboloid $\widehat{S}$ in the space of frequencies

$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \widehat{u}_0(\xi) d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z) d\sigma(z)$$

where $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$, defined as

$$\widehat{S} \stackrel{\text{def}}{=} \left\{ (\alpha, \xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2 \right\}.$$

where $y = (t,x)$ and $z = (\alpha, \xi)$

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} = \|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^{n+1})}$$

Geometric interpretation

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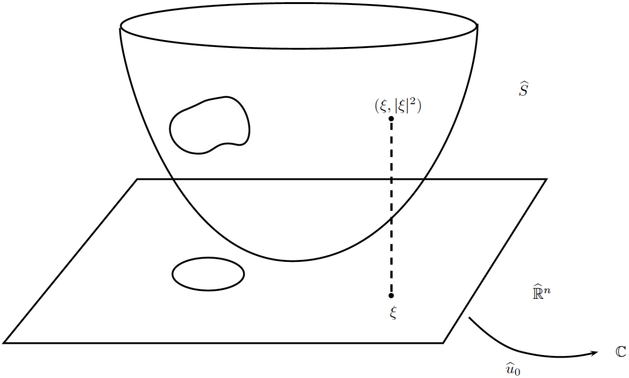
- Let us endow $\widehat{S}$ with the measure $d\sigma = d\xi$.

→ $d\sigma$ is not the intrinsic surface measure of $\widehat{S}$, which is $d\mu = \sqrt{1 + 2|\xi|}d\xi$.



The original approach of Strichartz, 1977

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RESTRICTIONS OF FOURIER TRANSFORMS TO QUADRATIC SURFACES AND DECAY OF SOLUTIONS OF WAVE EQUATIONS

ROBERT S. STRICHARTZ

§1. Introduction

Let S be a subset of $\mathbb{R}^n$ and $d\mu$ a positive measure supported on S and of temperate growth at infinity. We consider the following two problems:

Problem A. For which values of p, $1 \le p < 2$, is it true that $f \in L^p(\mathbb{R}^n)$ implies $\hat{f}$ has a well-defined restriction to S in $L^2(d\mu)$ with

$$\left(\int |\hat{f}|^2 d\mu\right)^{1/2} \le c_p \|f\|_p? \tag{1.1}$$

Problem B. For which values of q, $2 < q \le \infty$, is it true that the tempered distribution $F d\mu$ for each $F \in L^2(d\mu)$ has Fourier transform in $L^q(\mathbb{R}^n)$ with

$$\|(F d\mu)\hat{}\,\|_q \le c_q \left(\int |F|^2 d\mu\right)^{1/2}? \tag{1.2}$$

Strichartz says

A simple duality argument shows these two problems are completely equivalent if p and q are dual indices, $(1/p) + (1/q) = 1$. Interest in Problem A when S is a sphere stems from the work of C. Fefferman [3], and in this case the answer is known (see [11]). Interest in Problem B was recently signalled by I. Segal [6] who studied the special case $S = \{(x, y) \in \mathbb{R}^2 : y^2 - x^2 = 1\}$ and gave the interpretation of the answer as a space-time decay for solutions of the Klein-Gordon equation with finite relativistic-invariant norm.

In this paper we give a complete solution when S is a quadratic surface given by

$$S = \{x \in \mathbb{R}^n : R(x) = r\} \tag{1.3}$$

where $R(x)$ is a polynomial of degree two with real coefficients and r is a real constant. To avoid triviality we assume R is not a function of fewer than n variables, so that aside from isolated points S is a $n - 1$-dimensional C^∞ manifold. There is a canonical measure $d\mu$ associated to the function R (not intrinsic to the surface S, however) given by

$$d\mu = \frac{dx_1 \cdots dx_{n-1}}{|\partial R/\partial x_n|} \tag{1.4}$$

Fourier restriction

A lot of contributors: Stein, Fefferman, Tomas, etc.

Problem: Can we restrict Fourier transform of L^p functions to subsets ?

- f in $L^1(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ continuous $\rightarrow$ OK.
- f in $L^2(\mathbb{R}^n)$ implies $\mathcal{F}(f)$ in $L^2(\widehat{\mathbb{R}}^n)$ $\rightarrow$ arbitrary on a zero meas set $\widehat{S}$ of $\widehat{\mathbb{R}}^n$.

- what happens for $1 < p < 2$?
- it depends on the surface!
- if the surface is "flat" we cannot do a lot

First observation

→ The Fourier transform of a L^p function, for any $p > 1$, cannot be restricted to hyperplanes.

- This f belongs to $L^p(\mathbb{R}^n)$, for all $p > 1$

$$f(x) = \frac{e^{-|x'|^2}}{1 + |x_1|} \qquad x = (x_1, x') \in \mathbb{R}^n, \tag{3}$$

- its Fourier transform does not admit a restriction on $\widehat{S} = \{\xi_1 = 0\}$.

$$\widehat{f}(0, \xi') = \int_{\mathbb{R}^n} e^{-ix' \cdot \xi'} \frac{e^{-|x'|^2}}{1 + |x_1|} dx_1 dx'$$

→ what happens for different surfaces?

The statement

Tomas and Stein

One can restrict the Fourier transform of $L^p(\mathbb{R}^n)$ functions, for $p > 1$ (close to 1), to hypersurfaces $\widehat{S}$ that are "sufficiently curved", (main example: the sphere).

Let us state more formally the questions

Problem: given a hypersurface $\widehat{S} \subset \widehat{\mathbb{R}}^n$ endowed with a smooth measure $d\sigma$, the restriction problem asks for which pairs (p, q) an inequality of the form

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^q(\widehat{S},d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{4}$$

holds for all f in $\mathcal{S}(\mathbb{R}^n)$.

Dual approach

- The operator R_S is continuous from $L^p(\mathbb{R}^n)$ to $L^q(\widehat{S}, d\sigma)$?

$$R_S f = \mathcal{F}(f)|_{\widehat{S}}$$

$\rightarrow$ not completely settled in its general form

from now on

we focus on the case $q = 2$

- the adjoint operator R_S^* is continuous from $L^2(\widehat{S}, d\sigma)$ to $L^{p'}(\mathbb{R}^n)$?

$$R_S^* g = \mathcal{F}^{-1}(g d\sigma)$$

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(\widehat{S}, d\sigma)} \qquad (5)$$

A basic counterexample shows that the range of p for which the estimate holds cannot be the entire interval $1 \leq p \leq 2$;

Example (Knapp)

Let $\widehat{S}$ be the $(n-1)$-dimensional sphere in $\widehat{\mathbb{R}}^n$ endowed with the standard measure $d\mu$. The estimate can hold only if $p \leq \dfrac{2n+2}{n+3} = 2 - \dfrac{4}{n+3}$.

- Consider the equivalent formulation of the estimate

$$\|\widehat{g\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(S^{n-1})} \tag{6}$$

- Let $\delta > 0$ and let g_δ be the characteristic function "spherical cap"

$$\widehat{C}_\delta = \left\{x \in \widehat{S} : |x \cdot e_n| < \delta\right\}.$$

Proof of Knapp, I

- We consider the equivalent formulation of estimate

$$\|\widehat{g\sigma}\|_{L^{p'}(\mathbb{R}^n)} \leq C\|g\|_{L^2(S^{n-1})} \tag{7}$$

- Let $\delta > 0$ be small and let g_δ be the characteristic function on C_δ.
- $|C_\delta| \sim \delta^{n-1}$. This implies $\|g_\delta\|_{L^2(S^{n-1})} \sim \delta^{(n-1)/2}$.
- If $x \in \mathbb{R}^n$ is orthogonal to the vertical direction

$$|\widehat{g_\delta\sigma}(x)| = \left|\int_{S^{n-1}} e^{ix\cdot\xi} g_\delta(\xi) d\sigma(\xi)\right| = \left|\int_{C_\delta} e^{ix\cdot\xi} d\sigma(\xi)\right| \sim |C_\delta| \sim \delta^{n-1}.$$

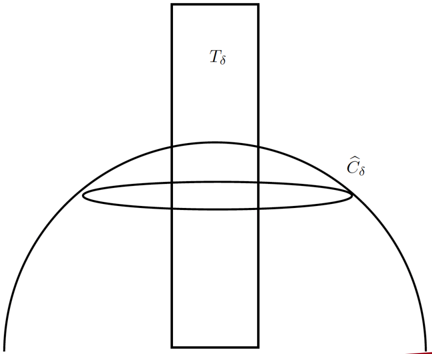
$$\|\widehat{g_\delta\sigma}\|_{L^{p'}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'}$$

Geometric interpretation

- let T_δ be the tube in the x space oriented orthogonally to the sphere

$$[-\delta^{-1}, \delta^{-1}] \times \ldots \times [-\delta^{-1}, \delta^{-1}] \times [-\delta^{-2}, \delta^{-2}]$$

- $|T_\delta| \sim \delta^{-n-1}$.



Proof of Knapp, II

- For x in T_δ and δ very small the quantity $x \cdot \xi$ is almost zero for $\xi \in C_\delta$.

$$\|\widehat{g_\delta \sigma}\|_{L^{p'}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'} \tag{8}$$

$$\geq \left(\int_{T_\delta} |\widehat{g\sigma}(x)|^{p'} dx\right)^{1/p'} \tag{9}$$

$$\sim \left(\int_{T_\delta} \delta^{(n-1)p'} dx\right)^{1/p'} \tag{10}$$

$$\sim \delta^{(n-1)} |T_\delta|^{1/p'} \sim \delta^{(n-1)} \delta^{(-n-1)/p'} \tag{11}$$

The estimate can hence be valid only if (the inequality is $\geq$ since $\delta \to 0$)

$$n - 1 - \frac{n+1}{p'} \geq \frac{n-1}{2}$$

which is the conclusion.

The above range is indeed the correct one for non vanishing curvature.

Theorem (Tomas-Stein, 1975)

Let $\widehat{S}$ be a smooth compact hypersurface in $\widehat{\mathbb{R}}^n$ with non vanishing Gaussian curvature at every point, and let $d\sigma$ be a smooth measure on $\widehat{S}$. Then

$$\|\mathcal{F}(f)|_{\widehat{S}}\|_{L^2(\widehat{S},d\sigma)} \le C_p \|f\|_{L^p(\mathbb{R}^n)} .$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$ and every $p \le (2n+2)/(n+3)$,

- A similar result is possible for surfaces with vanishing Gaussian curvature (that are not flat).
- In this case the range of p is smaller depending on the order of tangency of the surface to its tangent space.
- The assumption about compactness of $\widehat{S}$ can be removed by replacing $d\sigma$ with a compactly supported smooth measure.

Equivalent to the continuity from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ of the operator

$$R_S^* R_S f = f * \widehat{\sigma} \tag{12}$$

$$\|\mathcal{F}(f)|_{\widehat{S}}\|^2_{L^2(\widehat{S},d\sigma)} = \int (f * \widehat{\sigma}) f dx \leq \|f * \widehat{\sigma}\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}$$

Recall that the Fourier transform of the measure $d\sigma$ is a function given by

$$\widehat{\sigma}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} d\sigma(x) \tag{13}$$

Let S be a smooth compact hypersurfaces with non-zero Gaussian curvature at every point. Then

$$|\widehat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-\frac{n-1}{2}} \tag{14}$$

some comments

Let S be a smooth compact hypersurfaces with non-zero Gaussian curvature at every point. Then

$$|\widehat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}} \tag{15}$$

- only with decay one only gets $p \leq \frac{4n}{3n+1}$ (Fefferman, Stein)

$$n = 3, \qquad \widehat{\sigma}(\xi) = 2\frac{\sin(2\pi|x|)}{|x|}$$

- using a dyadic decomposition and real interpolation $p < \frac{2(n+1)}{n+3}$ (Tomas)
- with complex interpolation $p = \frac{2(n+1)}{n+3}$ (Stein)

From restriction to Strichartz estimates

The classical Schrödinger equation in $\mathbb{R}^n$: taking the inverse Fourier transform

$$u(t,x) = \int_{\widehat{\mathbb{R}}^n} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u}_0(\xi)d\xi\,. \tag{16}$$

Consider the paraboloid $\widehat{S}$ in the space of frequencies $\widehat{\mathbb{R}}^{n+1} = \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n$

$$\widehat{S} = \left\{(\alpha,\xi) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{R}}^n \mid \alpha = |\xi|^2\right\}.$$

- Given $\widehat{u}_0 : \widehat{\mathbb{R}}^n \to \mathbb{C}$ define $g : \widehat{S} \to \mathbb{C}$ as $g(|\xi|^2,\xi) = \widehat{u}_0(\xi)$. Then

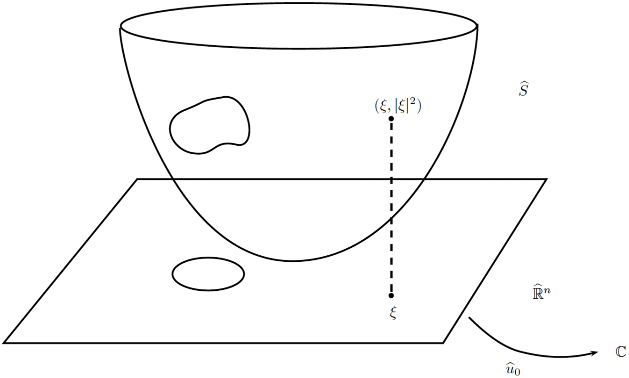
$$u(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u}_0(\xi)d\xi = \int_{\widehat{S}} e^{iy\cdot z} g(z)d\sigma(z)$$

where $y = (t,x)$ and $z = (\alpha,\xi)$.

Geometric interpretation

- Let us endow $\widehat{S}$ with the measure $d\sigma = d\xi$.

→ $d\sigma$ is not the intrinsic surface measure of $\widehat{S}$, which is $d\mu = \sqrt{1 + 2|\xi|}d\xi$.



The Fourier restriction theorem

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\widehat{\mathbb{R}}^{n+1})} \leq C_p \|g\|_{L^2(\widehat{S}, d\mu)}, \tag{17}$$

for all $g \in L^2(\widehat{S}, d\mu)$ and all $p' \geq 2(n+2)/n$.

By construction $\|g\|_{L^2(\widehat{S}, d\mu)} = \|\widehat{u}_0\|_{L^2(\widehat{\mathbb{R}}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}$

→ we stress that we apply the result in dimension $n+1$, i.e., in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$

Applying the statement to g related to a initial data u_0 such that $\widehat{u}_0$ is supported on a unit ball

$$\|u\|_{L^{p'}(\mathbb{R}^{n+1})} \leq C \|u_0\|_{L^2(\mathbb{R}^n)}, \tag{18}$$

for all $p' \geq 2(n+2)/n$.

A scaling argument and the density of spectrally localized functions in $L^2(\mathbb{R}^n)$, give the result for $p' = 2 + \frac{4}{n}$, and all $u_0 \in L^2(\mathbb{R}^n)$.

Some difficulties

1. Prove a Fourier restriction on the Heisenberg group

- a result of D.Müller $\rightarrow$ specific for the sphere
- what is the sphere? what about paraboloid?

2. We do not exactly need restriction theorems for $\mathbb{H}^d$

- we applied the result to a surface in the space $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$

$\rightarrow$ the paraboloid for the Schrödinger eq. (the cone for the wave equation).

- when dealing with equations defined on the Heisenberg group $\mathbb{H}^d$, one is naturally lead to consider surfaces in the space $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$, which is not related to $\mathbb{H}^{d'}$ for some d'.

Chapter 4: Strichartz estimates in the Heisenberg group

The result

A function ϕ on $\mathbb{H}^1$ is said to be *radial* if $\phi(x, y, z) = \phi(x^2 + y^2, z)$.

Theorem (Bahouri, DB, Gallagher, '21)

Given (p, q) *belonging to the admissible set*

$$\mathcal{A} = \left\{(p, q) \in [2, \infty]^2 \,/\, p \leq q \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} = \frac{Q}{2}\right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ *with radial data satisfies*

$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{L^2(\mathbb{H}^d)}\right).$$

- restrictive due to $p \leq q$. Indeed $p = q = 2$.
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

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$$\|u\|_{L^\infty_z L^q_t L^p_{x,y}} \leq C_{p,q,p_1,q_1}\left(\|u_0\|_{H^\sigma(\mathbb{H}^d)}\right).$$

- $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$ is the loss of derivatives, $\sigma = 0$ forces $p = q$
- we stress that $L^\infty_z \; L^q_t \; L^p_{x,y} \neq L^\infty_t \; L^q_z \; L^p_{x,y}$
- similar for inhomogeneous and wave

The result

A function ϕ on $\mathbb{H}^d$ is said to be *radial* if $\phi(z,s) = f(|z|,s)$.

Theorem (Bahouri, DB, Gallagher, '21)

Given (p,q) and (p_1,q_1) belonging to the admissible set

$$\mathcal{A} = \left\{ (p,q) \in [2,\infty]^2 \, / \, q \leq p \quad \textit{and} \quad \frac{2}{q} + \frac{2d}{p} \leq \frac{Q}{2} \right\},$$

the solution to the Schrödinger equation $(S_{\mathbb{H}})$ with radial data satisfies

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C_{p,q,p_1,q_1} \left(\|u_0\|_{H^\sigma(\mathbb{H}^d)} + \|f\|_{L^1_t H^\sigma(\mathbb{H}^d)} \right).$$

The Fourier transform on $\mathbb{H}$

It is defined using irreducible unitary representations : for any integrable function u on $\mathbb{H}$ (Kirillov theory)

$$\forall \lambda \in \mathbb{R}^*, \quad \widehat{u}(\lambda) := \int_{\mathbb{H}} u(x) \mathcal{R}_x^\lambda dx,$$

with $\mathcal{R}^\lambda$ the group homomorphism between $\mathbb{H}$ and the unitary group $\mathcal{U}(L^2(\mathbb{R}))$ of $L^2(\mathbb{R})$ given for all x in $\mathbb{H}$ and ϕ in $L^2(\mathbb{R})$, by

$$\mathcal{R}_x^\lambda \phi(\theta) := \exp\Big(i\lambda x_3 + i\lambda\theta x_2\Big)\phi(\theta + x_1).$$

Then $\widehat{u}(\lambda)$ is a family of bounded operators on $L^2(\mathbb{R})$, with many properties similar to $\mathbb{R}^d$: $\underbrace{\text{inversion formula}}_{Trace}$, $\underbrace{\text{Fourier-Plancherel identity}}_{Hilbert-Schmidt}$

The Fourier transform of the sublaplacian on $\mathbb{H}$

The sub-Laplacian

$$\Delta_{\mathbb{H}} = X_1^2 + X_2^2$$

There holds

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda) = \widehat{u}(\lambda) \circ P_\lambda\,, \quad \text{with} \quad P_\lambda := -\frac{d^2}{d\theta^2} + \lambda^2\theta^2\,.$$

The spectrum of the **rescaled harmonic oscillator** is

$$\mathrm{Sp}(P_\lambda) = \big\{|\lambda|(2m+1)\,,\, m \in \mathbb{N}\big\}$$

and the eigenfunctions are the Hermite functions ψ_m^λ. So for all $m \in \mathbb{N}$,

$$\widehat{-\Delta_{\mathbb{H}} u}(\lambda)\psi_m^\lambda = E_m(\lambda)\widehat{u}(\lambda)\psi_m^\lambda\,.$$

The frequency space on $\mathbb{H}$

Set $\widehat{x} := (n, m, \lambda) \in \widehat{\mathbb{H}} = \mathbb{N}^2 \times \mathbb{R}^*$, and

$$\mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda) := (\widehat{u}(\lambda)\psi_m^\lambda | \psi_n^\lambda)_{L^2(\mathbb{R})}$$
$$= \int_{\mathbb{H}} \mathcal{W}(\widehat{x}, x) u(x) dx$$

where $\mathcal{W}(\widehat{x}, x) := e^{i\lambda x_3} e^{-|\lambda|(x_1^2+x_2^2)} \underbrace{L_m(2|\lambda|(x_1^2 + x_2^2))}_{\text{Laguerre polynomial}}$.

Then

$$\mathcal{F}_{\mathbb{H}}(-\Delta_{\mathbb{H}} u)(n, m, \lambda) = \underbrace{E_m(\lambda)}_{\text{frequency}} \mathcal{F}_{\mathbb{H}}(u)(n, m, \lambda) .$$

Bahouri, Chemin, Danchin

Some formulas

Inversion and Fourier-Plancherel formulae

$$f(\widehat{x}) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widetilde{\mathbb{H}}^d} \mathcal{W}(\widehat{x}, x)\mathcal{F}_{\mathbb{H}} f(\widehat{x})\, d\widehat{x}$$

and

$$(\mathcal{F}_{\mathbb{H}} f | \mathcal{F}_{\mathbb{H}} g)_{L^2(\widetilde{\mathbb{H}}^d)} = \frac{\pi^{d+1}}{2^{d-1}} (f|g)_{L^2(\mathbb{H}^d)}\,,$$

Action of the Laplacian

$$\mathcal{F}_{\mathbb{H}}(\Delta_{\mathbb{H}} f)(\widehat{x}) = -4|\lambda|(2|m| + d)\mathcal{F}_{\mathbb{H}}(f)(\widehat{x})\,.$$

Radial functions $f(z,s) = f(|z|, s)$

$$\mathcal{F}_{\mathbb{H}}(f)(n,m,\lambda) = \mathcal{F}_{\mathbb{H}}(f)(n,m,\lambda)\delta_{n,m} = \mathcal{F}_{\mathbb{H}}(f)(|n|,|n|,\lambda)\delta_{n,m}\,.$$

Convolution for radial functions

$$\mathcal{F}_{\mathbb{H}}(f \star g)(\ell,\ell,\lambda) = \mathcal{F}_{\mathbb{H}} f(\ell,\ell,\lambda)\mathcal{F}_{\mathbb{H}} g(\ell,\ell,\lambda)\,.$$

Strichartz estimate in the Heisenberg group

Let u_0 in $\mathcal{S}(\mathbb{H}^d)$ be radial and consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \, . \end{cases}$$

Taking the partial Fourier transform with respect to the variable w

$$\begin{cases} i\frac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} = \mathcal{F}_{\mathbb{H}} u_0 \, . \end{cases}$$

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m} \, .$$

→ Notice that if we set $|m| = 0$ we see the "transport" part

$$\mathcal{F}_{\mathbb{H}}(u)(t,0,0,\lambda) = e^{4it|\lambda|d}\mathcal{F}_{\mathbb{H}}(u_0)(0,0,\lambda) \, .$$

Applying the inverse Fourier formula

$$u(t,z,s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} \mathcal{W}(\widehat{x}, z, s)\, e^{4it|\lambda|(2|m|+d)}\, \mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\, d\widehat{x}\,.$$

Re-expressed as the inverse Fourier transform in $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$ of $\mathcal{F}_{\mathbb{H}}(u_0)\, d\Sigma$,

$$\Sigma \stackrel{\text{def}}{=} \Big\{(\alpha,\widehat{x}) = \big(\alpha,(n,n,\lambda)\big) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \,/\, \alpha = 4|\lambda|(2|n|+d)\Big\}\,.$$

endow Σ with the measure $d\Sigma$ induced by the projection $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha,\widehat{x})\, d\Sigma(\alpha,\widehat{x}) = \int_{\widehat{\mathbb{H}}^d} \Phi(4|\lambda|(2|m|+d),\widehat{x})\, d\widehat{x},$$

Theorem (Bahouri, DB, Gallagher, '19)

If $1 \leq q \leq p \leq 2$, then for f radial

$$\|\mathcal{F}_{\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d}(f)|_\Sigma\|_{L^2(d\Sigma)} \leq C_{p,q}\|f\|_{L^1_s L^q_t L^p_z}\,, \qquad (19)$$

Using dual inequality, assuming that $F_{\mathbb{H}} u_0$ is localized in the unit ball

For any $2 \leq p \leq q \leq \infty$

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \|\mathcal{F}_{\mathbb{H}} u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- If u_0 is frequency localized in the ball $\mathcal{B}_\Lambda$,

$$u_\Lambda(t, z, s) = u(\Lambda^{-2} t, \Lambda^{-1} z, \Lambda^{-2} s), \qquad u_{0,\Lambda}(z, s) = u_0(\Lambda^{-1} z, \Lambda^{-2} s)$$

- we have

$$\|u_\Lambda\|_{L^\infty_s L^q_t L^p_z} = \Lambda^{\frac{2}{q} + \frac{2d}{p}} \|u\|_{L^\infty_s L^q_t L^p_z}, \qquad \|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{\frac{Q}{2}} \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- we infer

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \Lambda^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)} .$$