Strichartz estimates and sub-Riemannian geometry
Lecture 3

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Let u_0 in $\mathcal{S}(\mathbb{H}^d)$ be radial and consider the Cauchy problem

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u_{|t=0} = u_0 \,. \end{cases}$$

Taking the partial Fourier transform with respect to the variable w

$$\begin{cases} i\frac{d}{dt}\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = -4|\lambda|(2|m|+d)\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) \\ \mathcal{F}_{\mathbb{H}}(u)_{|t=0} = \mathcal{F}_{\mathbb{H}} u_0 \,. \end{cases}$$

$$\mathcal{F}_{\mathbb{H}}(u)(t,n,m,\lambda) = e^{4it|\lambda|(2|m|+d)}\mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\,.$$

→ Notice that if we set $|m| = 0$ we see the "transport" part

$$\mathcal{F}_{\mathbb{H}}(u)(t,0,0,\lambda) = e^{4it|\lambda|d}\mathcal{F}_{\mathbb{H}}(u_0)(0,0,\lambda)\,.$$

Applying the inverse Fourier formula

$$u(t,z,s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\widehat{\mathbb{H}}^d} \mathcal{W}(\widehat{x},z,s)\, e^{4it|\lambda|(2|m|+d)}\, \mathcal{F}_{\mathbb{H}}(u_0)(|n|,|n|,\lambda)\delta_{n,m}\, d\widehat{x}\,.$$

Re-expressed as the inverse Fourier transform in $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d$ of $\mathcal{F}_{\mathbb{H}}(u_0)\, d\Sigma$,

$$\Sigma \stackrel{\text{def}}{=} \Big\{(\alpha,\widehat{x}) = \big(\alpha,(n,n,\lambda)\big) \in \widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \,/\, \alpha = 4|\lambda|(2|n|+d)\Big\}\,.$$

endow Σ with the measure $d\Sigma$ induced by the projection $\widehat{\mathbb{R}} \times \widehat{\mathbb{H}}^d \to \widehat{\mathbb{H}}^d$

$$\int_{\widehat{\mathbb{D}}} \Phi(\alpha,\widehat{x})\, d\Sigma(\alpha,\widehat{x}) = \int_{\widehat{\mathbb{H}}^d} \Phi(4|\lambda|(2|m|+d),\widehat{x})\, d\widehat{x},$$

Theorem (Bahouri, DB, Gallagher, '21)

If $1 \le q \le p \le 2$, then for f radial

$$\|\mathcal{F}_{\mathbb{R}\times\mathbb{H}^d}(f)|_{\Sigma}\|_{L^2(d\Sigma)} \le C_{p,q} \|f\|_{L^1_s L^q_t L^p_z}\,, \qquad (1)$$

Using dual inequality, assuming that $\mathcal{F}_{\mathbb{H}} u_0$ is localized in the unit ball

For any $2 \leq p \leq q \leq \infty$

$$\|u\|_{L_s^\infty L_t^q L_z^p} \leq C \|\mathcal{F}_{\mathbb{H}} u_0\|_{L^2(\widehat{\mathbb{H}}^d)} = C \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- If u_0 is frequency localized in the ball $\mathcal{B}_\Lambda$,

$$u_\Lambda(t, z, s) = u(\Lambda^{-2} t, \Lambda^{-1} z, \Lambda^{-2} s), \qquad u_{0,\Lambda}(z, s) = u_0(\Lambda^{-1} z, \Lambda^{-2} s)$$

- we have

$$\|u_\Lambda\|_{L_s^\infty L_t^q L_z^p} = \Lambda^{\frac{2}{q} + \frac{2d}{p}} \|u\|_{L_s^\infty L_t^q L_z^p}, \qquad \|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^{\frac{Q}{2}} \|u_0\|_{L^2(\mathbb{H}^d)} ,$$

- we infer for $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$

$$\|u\|_{L_s^\infty L_t^q L_z^p} \leq C \Lambda^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)} \leq C \|u_0\|_{H^\sigma(\mathbb{H}^d)} .$$

The inhomogeneous case

- Denoting by $(\mathcal{U}(t))_{t\in\mathbb{R}}$ the solution operator of the Schrödinger equation on the Heisenberg group,
- $(\mathcal{U}(t))_{t\in\mathbb{R}}$ is a one-parameter group of unitary operators on $L^2(\mathbb{H}^d)$.
- the solution to the inhomogeneous equation

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = f \\ u_{|t=0} = 0\,, \end{cases}$$

writes

$$u(t,\cdot) = -i\int_0^t \mathcal{U}(t-t')f(t',\cdot)dt', \tag{2}$$

It is enough to check that it satisfies, for all admissible pairs (p,q) ,

$$\|u\|_{L^\infty_s L^q_t L^p_z} \lesssim \|f\|_{L^1_t H^\sigma(\mathbb{H}^d)} \tag{3}$$

with $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$.

- By formula of the solution, we have for all $s \in \mathbb{R}$,

$$\|u(t,\cdot,s)\|_{L^p_z} \leq \int_{\mathbb{R}} \|\mathcal{U}(t)\mathcal{U}(-t')f(t',\cdot,s)\|_{L^p_z} dt'.$$

- Therefore, still for all s,

$$\|u(\cdot,\cdot,s)\|_{L^q_t L^p_z} \leq \int_{\mathbb{R}} \|\mathcal{U}(\cdot)\mathcal{U}(-t')f(t',\cdot,s)\|_{L^q_t L^p_z} dt'.$$

- Let us first assume that, for all t, the source term $f(t,\cdot)$ is frequency localized in in the unit ball $\mathcal{B}_1$
- if g is frequency localized in a unit ball, then for all $2 \leq p \leq q \leq \infty$

$$\|\mathcal{U}(t)g\|_{L^\infty_s L^q_t L^p_z} \lesssim \|g\|_{L^2(\mathbb{H}^d)}. \tag{4}$$

- By formula of the solution, we have for all $s \in \mathbb{R}$,

$$\|u(t,\cdot,s)\|_{L^p_z} \leq \int_{\mathbb{R}} \|\mathcal{U}(t)\mathcal{U}(-t')f(t',\cdot,s)\|_{L^p_z} dt'.$$

- Therefore, still for all s,

$$\|u(\cdot,\cdot,s)\|_{L^q_t L^p_z} \leq \int_{\mathbb{R}} \|\mathcal{U}(\cdot)\mathcal{U}(-t')f(t',\cdot,s)\|_{L^q_t L^p_z} dt'.$$

- Let us first assume that, for all t, the source term $f(t,\cdot)$ is frequency localized in in the unit ball $\mathcal{B}_1$
- if g is frequency localized in a unit ball, then for all $2 \leq p \leq q \leq \infty$

$$\|\mathcal{U}(t)g\|_{L^\infty_s L^q_t L^p_z} \lesssim \|g\|_{L^2(\mathbb{H}^d)}\,. \tag{5}$$

- Using homog Strichartz, we deduce that

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq \int_{\mathbb{R}} \|\mathcal{U}(-t')f(t',\cdot)\|_{L^2(\mathbb{H}^d)} dt'.$$

- Since $\mathcal{U}(-t')$ is unitary on $L^2(\mathbb{H}^d)$, we readily gather that

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq \int_{\mathbb{R}} \|f(t',\cdot)\|_{L^2(\mathbb{H}^d)} dt'. \tag{6}$$

- Now if for all t, $f(t,\cdot)$ is frequency localized in a ball of size Λ, then setting

$$f_\Lambda(t,\cdot) \stackrel{\text{def}}{=} \Lambda^{-2} f(\Lambda^{-2}t,\cdot) \circ \delta_{\Lambda^{-1}}$$

- we find that on the one hand, $f_\Lambda(t,\cdot)$ is frequency localized in a unit ball for all t, and on the other hand that the solution to the Cauchy problem

$$\begin{cases} i\partial_t u_\Lambda - \Delta_{\mathbb{H}} u_\Lambda = f_\Lambda \\ u_{|t=0} = 0\,, \end{cases}$$

writes $u_\Lambda(t,w) = u(\Lambda^{-2}t,\cdot) \circ \delta_{\Lambda^{-1}}$.

Now by scale invariance, we have

$$\int_{\mathbb{R}} \|f_\Lambda(t',\cdot)\|_{L^2(\mathbb{H}^d)} dt' = \Lambda^{\frac{Q}{2}} \int_{\mathbb{R}} \|f(t',\cdot)\|_{L^2(\mathbb{H}^d)} dt'$$

and

$$\|u_\Lambda\|_{L^\infty_s L^q_t L^p_Y} = \Lambda^{\frac{2}{q}+\frac{2d}{p}} \|u\|_{L^\infty_s L^q_t L^p_Y} .$$

Consequently, we get

$$\|u\|_{L^\infty_s L^q_t L^p_Y} \leq C \int_{\mathbb{R}} \Lambda^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}} \|f(t',\cdot)\|_{L^2(\mathbb{H}^d)} dt' .$$

Since $\dfrac{Q}{2} - \dfrac{2}{q} - \dfrac{2d}{p} \geq 0$, we have

$$\Lambda^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}} \|f(t',\cdot)\|_{L^2(\mathbb{H}^d)} \lesssim \|f(t',\cdot)\|_{H^{\frac{Q}{2}-\frac{2}{q}-\frac{2d}{p}}(\mathbb{H}^d)} ,$$

and then integrate in t to conclude

The statement

Theorem (Bahouri, DB, Gallagher, '21)

If $1 \leq q \leq p \leq 2$, *then for* f *radial*

$$\|\mathcal{F}_{\mathbb{R}\times\mathbb{H}^d}(f)|_{\Sigma}\|_{L^2(d\Sigma)} \leq C_{p,q}\|f\|_{L^1_s L^q_t L^p_z}\,, \qquad (7)$$

and its dual version

Example

for any $2 \leq p' \leq q' \leq \infty$, there holds

$$\|\mathcal{F}^{-1}_{\widehat{\mathbb{R}}\times\widehat{\mathbb{H}}^d}(\theta_{|\Sigma_{\rm loc}})\|_{L^\infty_s L^{q'}_t L^{p'}_Y} \leq \|\theta_{|\Sigma_{\rm loc}}\|_{L^2(d\Sigma_{\rm loc})}\,, \qquad (8)$$

The completion of the frequency set

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- The frequency set $\widetilde{\mathbb{H}}^d$ comes with a measure

$$\int_{\widetilde{\mathbb{H}}^d} \theta(\widehat{x})\, d\widehat{x} \stackrel{\text{def}}{=} \int_{\mathbb{R}} \sum_{(n,m)\in\mathbb{N}^{2d}} \theta(n,m,\lambda)|\lambda|^d\, d\lambda\,.$$

- endowed with a distance

$$d(\widehat{x},\widehat{x}') \stackrel{\text{def}}{=} \big|\lambda(n+m)-\lambda'(n'+m')\big|_{\ell^1}+\big|(n-m)-(n'-m')|_{\ell^1}+d|\lambda-\lambda'|\,,$$

- $(\widetilde{\mathbb{H}}^d, d)$ it is not complete [→] build the metric completion $\widehat{\mathbb{H}}^d$

Some advantages of [Bahouri, Chemin, Danchin]

- definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$,
- interpretation smoothness $\leftrightarrow$ decay

→ give a meaning to the unit sphere $\mathbb{S}_{\widehat{\mathbb{H}}^d}$ of $\widehat{\mathbb{H}}^d$.

Recall that for θ being the Fourier transform of a radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x})d\widehat{x} = \int_{\mathbb{R}} \sum_{n\in\mathbb{N}^d} \theta(n,n,\lambda)|\lambda|^d \, d\lambda \,.$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x})d\widehat{x} = \int_0^\infty \left(\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x})d\sigma_R(\widehat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n|+d)|\lambda|$)

$$\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x})d\sigma_R(\widehat{x}) = \sum_{n\in\mathbb{N}^d} \frac{2R^{2d+1}}{(2|n|+d)^{d+1}} \Big(\sum_{\pm} \theta(n,n,\frac{\pm R^2}{2|n|+d}) \Big)$$

On the surface measure, $R = 1$

Recall that for θ Fourier transform of radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d \, d\lambda \, .$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_0^\infty \left(\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{\mathbb{S}_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_1(\widehat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \theta(n, n, \frac{\pm 1}{2|n| + d}) \Big)$$

The result of Müller

- D.Müller [Annals of Math, 1990]: works in terms of spectral decomposition

$$L = \int_0^\infty \lambda dE(\lambda), \qquad \mathcal{P}f = f * G$$

- proves the estimate ("restriction for the sphere"): if $1 \leq p \leq 2$

$$\Big[\sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \Big| \mathcal{F}_{\mathbb{H}}(f)(n, n, \frac{\pm 1}{2|n| + d}) \Big|^2 \Big) \Big]^{\frac{1}{2}} \leq C_p \| f \|_{L^1_s L^p_z}$$

- can be reinterpreted as follows: If $1 \leq p \leq 2$, then for radial f

$$\| \mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}} \|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C_p \| f \|_{L^1_s L^p_z} , \tag{9}$$

→ valid on the full interval: for $p \in [1, 2]$

→ crucial: the anisotropic norm $L^1_s L^p_z$ ($r = 1$ is necessary in vertical)

- false for $p > 2$

Fourier transform of the surface measure

Up to a measure zero set on $\hat{\mathbb{H}}^d$

$$\mathbb{S}_{\widehat{\mathbb{H}}^d} = \Big\{(n, n, \lambda) \in \widehat{\mathbb{H}}^d \,/\, (2|n| + d)|\lambda| = 1\Big\}$$

By definition, the tempered distribution $G = \mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathbb{S}_{\widehat{\mathbb{H}}^d}})$

Lemma

G is the bounded function on $\mathbb{H}^d$ defined by

$$G(z,s) = \frac{2^d}{\pi^{d+1}} \sum_{n\in\mathbb{N}^d} \frac{1}{(2|n|+d)^{d+1}} \cos\Big(\frac{s}{2|n|+d}\Big) \mathcal{W}\Big(n, n, 1, \frac{z}{\sqrt{2|n|+d}}\Big) \tag{10}$$

For the sphere of radius $R^{1/2}$ we have the homogeneity property:

$$G_R(z,s) \stackrel{\text{def}}{=} R^d (G \circ \delta_{\sqrt{R}})(z,s)\,. \tag{11}$$

Measure on the paraboloid

Proceeding as for the restriction theorem on the sphere of $\widehat{\mathbb{H}}^d$, let us first compute

$$G_{\Sigma_{\rm loc}} \stackrel{\rm def}{=} \mathcal{F}^{-1}_{\hat{\mathbb{R}}\times\hat{\mathbb{H}}^d}(d\Sigma_{\rm loc})\,.$$

Lemma

With the above notation, $G_{\Sigma_{\rm loc}}$ is the bounded function on $\mathbb{R}\times\hat{\mathbb{H}}^d$ defined by

$$G_{\Sigma_{\rm loc}}(t,w) = 2\pi\int_0^\infty G_\alpha(w)\,e^{-it\,\alpha}\psi(\alpha)\,d\alpha\,, \tag{12}$$

where G_R is the inverse Fourier of the measure of sphere of radius $R^{1/2}$.

This gives for all f in $\mathcal{S}_{\rm rad}(\mathcal{D})$

$$(R^*_{\Sigma_{\rm loc}}R_{\Sigma_{\rm loc}}f)(t,z,s) = \left(\frac{\pi}{2}\right)^d (G_{\Sigma_{\rm loc}}\star\check{f})(-t,-z,s)\,, \tag{13}$$

Reduction to the estimate on convolution

Consider the restriction operator

$$R_{\Sigma_{\mathrm{loc}}} f = \mathcal{F}_{\mathbb{R}\times\mathbb{H}^d}(f)_{|\Sigma_{\mathrm{loc}}}$$

Indeed applying the Hölder inequality, we deduce that

$$\begin{aligned}\|R_{\Sigma_{\mathrm{loc}}} f\|^2_{L^2(\Sigma_{\mathrm{loc}})} &\leq \|R^*_{\Sigma_{\mathrm{loc}}} R_{\Sigma_{\mathrm{loc}}} f\|_{L^\infty_s L^{q'}_t L^{p'}_Y} \|f\|_{L^1_s L^q_t L^p_Y} \\ &\leq \|\check{f} \star_{\mathcal{D}} G_{\Sigma_{\mathrm{loc}}}\|_{L^\infty_s L^{q'}_t L^{p'}_Y} \|f\|_{L^1_s L^q_t L^p_Y},\end{aligned}$$

Then as in the Euclidean case, we are reduced to proving that $R^*_{\Sigma_{\mathrm{loc}}} R_{\Sigma_{\mathrm{loc}}}$ is bounded from $L^1_s L^q_t L^p_z$ into $L^\infty_s L^{q'}_t L^{p'}_z$.

Proof for $1 \leq p < 2$ (non endpoint)

Main lemma

$$\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} \lesssim \left\| \|\mathcal{F}_{\mathbb{R}}(f)(-\alpha,\cdot)\|_{L_z^p L_s^1} \alpha^{d(1-\frac{2}{p'})}\ \psi(\alpha) \right\|_{L_\alpha^q}$$

- Hölder estimate in α + Hausdorff-Young inequality: for any $a \geq 2$

$$\begin{aligned}\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} &\lesssim \|\mathcal{F}_{\mathbb{R}}(f)\|_{L_\alpha^a L_z^p L_s^1} \|\alpha^{d(1-\frac{2}{p'})}\ \psi(\alpha)\|_{L_\alpha^b} \\ &\lesssim \|f\|_{L_t^{a'} L_z^p L_s^1} \|\alpha^{d(1-\frac{2}{p'})}\ \psi(\alpha)\|_{L_\alpha^b(\mathbb{R})},\end{aligned}$$

where a' is the conjugate exponent of a and $\frac{1}{a} + \frac{1}{b} = \frac{1}{q}$.

- Finally for $a' = q$ and Minkowski's inequality, we get for $q' \geq p' > 2$

$$\|f \star G_{\Sigma_{\mathrm{loc}}}\|_{L_s^\infty L_t^{q'} L_z^{p'}} \lesssim \|f\|_{L_s^1 L_t^q L_z^p}$$

→ endpoint $p = 2$: ad hoc argument

Chapter 5: Kirillov Theory for Nilpotent groups

Here V is a vector space finite or infinite dimensional.

- Given a Lie group G a representation of G is a smooth homomorphism

$$\mathcal{R} : G \to GL(V), \qquad \mathcal{R}(g_1 g_2) = \mathcal{R}(g_1)\mathcal{R}(g_2)$$

  where in the left hand side we have the product in G while in the right hand side the composition in $GL(V)$.
- A subspace W of V is an invariant subspace if $\mathcal{R}(g)w \in W$ for all $g \in G$ and $w \in W$.
- The representation is said to be *irreducible* if the only invariant subspaces of V are the zero space and V itself.

Important 1D unitary representations

- if $\mathcal{R}$ map into the group of unitary operators, we say *unitary* representation .
- The representation is said *one-dimensional* if V has dimension 1.
- For $V = \mathbb{C}$, a 1-dim representation of G will be a smooth homomorphism

$$\mathcal{X} : G \to U(\mathbb{C}) = S^1$$

- Let G nilpotent, $\eta \in \mathfrak{g}^*$ and $H \subset G$ be such that $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$: we can define the one-dimensional representation

$$\mathcal{X}_\eta : H \to S^1 = U(\mathbb{C})$$

$$\mathcal{X}_\eta(e^X) = e^{i\langle \eta, X\rangle}, \qquad X \in \mathfrak{h}.$$

where as usual $\langle \eta, X\rangle$ denotes the duality product $\mathfrak{g}^*$ and $\mathfrak{g}$.

Kirillov theory

The Kirillov theory gives a way to describe all possible irreducible unitary representations of G in terms of coadjoint orbits of the group.

An algorithm in four steps:

1. Fix an element $\eta \in \mathfrak{g}^*$.
2. Fix any maximal Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ s.t. $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$.
3. Consider the one-dimensional representation

$$\mathcal{X}_{\eta,\mathfrak{h}} : H \to S^1 = U(\mathbb{C})$$

$$\mathcal{X}_{\eta,\mathfrak{h}}(e^X) = e^{i\langle \eta, X \rangle}, \qquad X \in \mathfrak{h}.$$

   where as usual $\langle \eta, X \rangle$ denotes the duality product $\mathfrak{g}^*$ and $\mathfrak{g}$.

4. Compute the induced representation $\mathcal{R}_{\eta,\mathfrak{h}} : G \to U(W)$.

$\to$ a way to lift a representation to the group G

Coadjoint orbits

Given a Lie group G

- the conjugation map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$.
- the *adjoint action* of G onto its Lie algebra

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{Ad}_g = (C_g)_*$$

- Notice that $\mathrm{Ad} : G \to GL(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$ is a finite dimensional representation of G.
- This induces the so called *coadjoint action* dual of the above

$$\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*, \qquad \langle \mathrm{Ad}^*_g \eta, v \rangle := \langle \eta, (\mathrm{Ad}_{g^{-1}})_* v \rangle$$

- Notice that Ad^* is indeed an action of G on $\mathfrak{g}^*$. Given $\eta \in \mathfrak{g}^*$ the *coadjoint orbit* of η is by definition the set

$$\mathcal{O}_\eta = \{\mathrm{Ad}^*_g \eta \mid g \in G\}.$$

Kirillov theorem

The Kirillov theorem states the following:

Theorem

The map which assigns to $\eta \in \mathfrak{g}^*/G$ to $\mathcal{R}_{\eta,\mathfrak{h}}$ in $\widehat{G}$ (where $\mathfrak{h}$ is some maximal Lie subalgebra) is a bijection. More precisely:

(a) *every irreducible unitary representation of a nilpotent Lie group G is of the form $\mathcal{R}_{\eta,\mathfrak{h}}$ for some η and H*

(b) *two representations $\mathcal{R}_{\eta,\mathfrak{h}}$ and $\mathcal{R}_{\eta',\mathfrak{h}'}$ are equivalent if and only if η and η' belong to the same orbit.*

Here two irreducible unitary representations $R_1 : G \to U(W_1)$ and $R_2 : G \to U(W_2)$ are equivalent if there exists an isometry between the Hilbert spaces $T : W_1 \to W_2$ such that

$$T \circ R_1(g) \circ T^{-1} = R_2(g), \qquad \forall g \in G$$

Kirillov theory

The Kirillov theory gives a way to describe all possible irreducible unitary representations of G in terms of coadjoint orbits of the group.

An algorithm in four steps:

1. Fix an element $\eta \in \mathfrak{g}^*$ in every leaf
2. Fix any maximal Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ s.t. $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$.
3. Consider the one-dimensional representation

$$\mathcal{X}_{\eta,\mathfrak{h}} : H \to S^1 = U(\mathbb{C})$$

$$\mathcal{X}_{\eta,\mathfrak{h}}(e^X) = e^{i\langle \eta, X\rangle}, \qquad X \in \mathfrak{h}.$$

   where as usual $\langle \eta, X\rangle$ denotes the duality product $\mathfrak{g}^*$ and $\mathfrak{g}$.

4. Compute the induced representation $\mathcal{R}_{\eta,\mathfrak{h}} : G \to U(W)$.

$\rightarrow$ a way to lift a representation to the group G

Poisson structure on the dual $\mathfrak{g}^*$

Let $a, b : \mathfrak{g}^* \to \mathbb{R}$ be smooth functions.

- Poisson manifold with the bracket

$$\{a, b\}(\eta) = \langle \eta, [da, db] \rangle$$

- Given a smooth $a : \mathfrak{g}^* \to \mathbb{R}$ we can define its *Poisson vector field* by setting for every smooth $b : \mathfrak{g}^* \to \mathbb{R}$

$$\vec{a}(b) = \{a, b\}$$

- The set of all Poisson vector at a point defines a distribution

$$D_\eta = \{\vec{a}(\eta) \mid a \in C^\infty(\mathfrak{g}^*)\}$$

which has no constant rank (notice $D_0 = \{0\}$).

Poisson orbit

We can define also the *Poisson orbit* of $\eta \in \mathfrak{g}^*$ in the sense of dynamical systems as follows

$$\mathcal{O}_\eta^P = \{e^{t_1 \vec{a}_1} \circ \ldots \circ e^{t_\ell \vec{a}_\ell}(\eta) \mid \ell \in \mathbb{N}, t_i \in \mathbb{R}, a_i \in C^\infty(\mathfrak{g}^*)\}.$$

Notice that both $\mathcal{O}_\eta^P$ and $\mathcal{O}_\eta$ are subsets of $\mathfrak{g}^*$ containing η.

Proposition

For every $\eta \in \mathfrak{g}^*$ we have the equality $\mathcal{O}_\eta^P = \mathcal{O}_\eta$. Each orbit is an even dimensional symplectic manifold.

It is enough to use as a_i the linear on fibers function associated to a basis

$$h_i(p, x) = p \cdot X_i(x)$$

Computation of coadjoint orbits

Fix a basis of the Lie algebra $X_1, \ldots, X_n$ such that

$$[X_i, X_j] = c_{ij}^k X_k$$

for some constants c_{ij}^k. Define the corresponding coordinates on the fibers of T^*G given by

$$h_i(p, x) = p \cdot X_i(x)$$

These can be thought as smooth functions on $\mathfrak{g}^*$ and satisfy

$$\{h_i, h_j\} = c_{ij}^k h_k.$$

We recall that a *casimir* is a smooth function $f \in C^\infty(\mathfrak{g}^*)$ such that

$$\{a, f\} = 0, \qquad \forall\, a \in C^\infty(\mathfrak{g}^*)$$

If we write $f = f(h_1, \ldots, h_n)$ to check that f is a casimir it is enough to check that

$$\{f, h_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial h_i}\{h_i, h_j\} = \sum_{i,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k = 0, \qquad j = 1, \ldots, n$$

that means

$$\sum_{i=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k = 0, \qquad j, k = 1, \ldots, n$$

The Poisson vector field associated to a function f is

$$\vec{f} = \sum_{i,j,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k \frac{\partial}{\partial h_j}$$

The Poisson vector field associated to a casimir is the zero vector field.

Casimir

If we write $f = f(h_1, \dots, h_n)$ to check that f is a casimir it is enough to check that

$$\{f, h_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial h_i}\{h_i, h_j\} = \sum_{i,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^{k} h_k = 0, \qquad j = 1, \dots, n$$

that means

$$\sum_{i=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^{k} = 0, \qquad j, k = 1, \dots, n$$

The Poisson vector field associated to a function f is

$$\vec{h}_i = \sum_{i,j,k=1}^{n} c_{ij}^{k} h_k \frac{\partial}{\partial h_j}$$

The Poisson vector field associated to a casimir is the zero vector field.

The Heisenberg group

Let us go back to the main example, the Heisenberg group.

$$[X, Y] = Z$$

- relabel $(X, Y, Z) = (X_1, X_2, X_0)$
- Consider $h_1, h_2, h_0 : \mathfrak{g}^* \to \mathbb{R}$
- write down $\vec{h}_i$ for every $i = 1, 2, 0.$

$$\vec{h}_1 = h_0 \partial_{h_2}, \qquad \vec{h}_2 = -h_0 \partial_{h_1}$$

- h_0 is a casimir: the corresponding vector field X_0 is in the center.

Hence we have the coadjoint orbits.

- if $h_0 = 0$ then every point $(h_1, h_2, 0)$ is an orbit
- if $h_0 \neq 0$ then every plane $h_0 = \lambda$ is an orbit

To compute the representations.

- If we take $\eta = (h_1, h_2, 0) \in \mathfrak{g}^*$ then we can take $\mathfrak{h} = \mathfrak{g}$ since $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}X_0$ and the corresponding character

$$\mathcal{X}_\eta(g) = e^{i(h_1 x + h_2 y)}$$

  where $g = e^{xX+yY+zZ}$. Notice that since we can take $\mathfrak{h} = \mathfrak{g}$ there is "nothing to induce", so these are representation of the abelian $\mathbb{R}^2$.

- If we take $\eta = (0, 0, h_0) \in \mathfrak{g}^*$ with $\lambda \neq 0$ as representative of the orbit. We can take $\mathfrak{h} = \mathrm{span}\{Y, Z\}$ since $[\mathfrak{h}, \mathfrak{h}] = 0$ and it is maximal

$$\mathcal{X}_\eta(g) = e^{i\lambda z}$$

  what to do then?

we have to understand the induced representations!

Induced representations

Let G be a nilpotent Lie group and H be a subgroup.

- Given a representation $\mathcal{X} : H \to U(V)$ we want to build a representation $\mathcal{R} : G \to U(W)$ that is *induced* by $\mathcal{X}$.
- We first build the Hilbert space W. Consider the set of functions $f : G \to V$ such that

$$f(hg) = \mathcal{X}(h)f(g) \tag{14}$$

- Notice that this means that

$$\mathcal{X}(h)f = f \circ L_h$$

- For such a function, since $\mathcal{X}$ is unitary, we have that $\|f(hg)\|$ is independent on h and hence the norm of $\|f(Hg)\|$ is well-defined, where Hg denotes the left coset of g in $H\backslash G$.

Very abstract!

- We require that

$$\int_{H\backslash G} \|f(Hg)\|^2 d\mu < \infty \tag{15}$$

  where $d\mu$ is a right invariant measure on $H\backslash G$.

- Then we set

$$W = \{f : G \to V \mid f \text{ satisfies (14)-(15)}\}$$

- Once we have set the space W we can define $\mathcal{R} : G \to U(W)$ as follows

$$\mathcal{R}(g)f = f \circ R_g, \qquad \text{i.e., } (\mathcal{R}(g)f)(g') = f(g'g)$$

  where the R_g is the right translation.

- One can check that $\mathcal{R}$ is unitary and strongly continuous.

Crucial for computations!

- We have a natural projection $\pi : G \to H\backslash G$.
- Given any section $s : H\backslash G \to G$ (this means that $\pi \circ s = \mathrm{id}$ on $H\backslash G$) we can consider the image of the section $K = s(H\backslash G)$ and try to write elements of G as products $H \cdot K$.
- Write $g'g = hk$ we can split

$$(R(g)f)(g') = f(g'g) = f(hk) = \mathcal{X}(h)f(k) \tag{16}$$

Crucial step: solve the *Master equation*

$$g'g = h \cdot k$$

- it is enough to solve the Master equation for $g' \in K$ (use the last equality in (16) and f is a equivariant function)

$$K \cdot G = H \cdot K$$

Back to Heisenberg

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To compute the representations.

- If we take $\eta = (h_1, h_2, 0) \in \mathfrak{g}^*$ then we can take $\mathfrak{h} = \mathfrak{g}$ since $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}X_0$ and the corresponding character

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  where $g = e^{xX+yY+zZ}$. Notice that since we can take $\mathfrak{h} = \mathfrak{g}$ there is "nothing to induce", so these are representation of the abelian $\mathbb{R}^2$.

- If we take $\eta = (0, 0, h_0) \in \mathfrak{g}^*$ with $\lambda \neq 0$ as representative of the orbit. We can take $\mathfrak{h} = \mathrm{span}\{Y, Z\}$ since $[\mathfrak{h}, \mathfrak{h}] = 0$ and it is maximal

$$\mathcal{X}_\eta(g) = e^{i\lambda z}$$

  what to do then?

The induced representation in this case works as follows: we can take as complement $K = e^{\mathbb{R}X}$ and then try to write the elements as product $H \cdot K$ as follows. Let us take $k = e^{\theta X}$ in K and $g = e^{yY+zZ}e^{xX}$ general element (it is convient to use these coordinates). We have

$$(\mathcal{X}_\eta(g)f)(k) = f(kg)$$

and we have to write

$$e^{\theta X}e^{yY+zZ}e^{xX}$$

as an element of H times an element of K. We have

$$e^{\theta X}e^{yY+zZ}e^{xX} = e^{yY+(z+\theta y)Z}e^{(\theta+x)X}$$

so that

$$\begin{aligned}(\mathcal{R}_\eta(g)f)(k) &= f(e^{yY+(z+\theta y)Z}e^{(\theta+x)X}) &(17)\\ &= \mathcal{X}_\eta(e^{yY+(z+\theta y)Z})f(e^{(\theta+x)X}) &(18)\end{aligned}$$

Last step

Writing explicitly the character and $\widetilde{f}(\theta) = f(e^{\theta X})$ as a function on $L^2(\mathbb{R})$ instead of $L^2(K)$ we have

$$(\mathcal{R}_\eta(g)\widetilde{f})(\theta) = e^{i\lambda(z+\theta y)}\widetilde{f}(\theta + x) \tag{19}$$

One can recognise the representation of the Lie algebra which are skew-adjoint operators on the same space of functions

$$X_1\widetilde{f} = \frac{d}{dt}\widetilde{f}, \qquad X_2\widetilde{f} = i\lambda\theta\widetilde{f}, \qquad X_0\widetilde{f} = i\lambda\widetilde{f}$$

which indeed satisfy $[X_1, X_2] = X_0$.

$$\Delta = X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \lambda^2\theta^2$$

The trick for the Master equation

This is related to CHB formula

Lemma

Assume that the Lie algebra generated by A, B is nipotent. Then we have that $e^A e^B e^{-A} = e^{C(A,B)}$ where

$$C(A,B) = e^{\mathrm{ad}(A)} B = \sum_{k=0}^{\infty} \frac{\mathrm{ad}^k(A)}{k!} B = B + [A,B] + \frac{1}{2}[A,[A,B]] + \dots$$

Notice that the sum is finite due to nilpotency assumption.

In Heisenberg

$$e^{\theta X} e^{yY+zZ} e^{xX} = e^{yY+zZ+[\theta X, yY+zZ]} e^{(\theta+x)X}$$

since

$$e^{\theta X} e^{yY+zZ} e^{xX} = e^{yY+(z+\theta y)Z} e^{(\theta+x)X}$$

An observation on the coordinates

The Heisenberg group $\mathfrak{g} = \mathrm{span}\{X, Y, Z\}$ with the only non trivial commutator

$$[X, Y] = Z$$

Elements of $G = \exp(\mathfrak{g})$ can be also written as follows $g = e^{yY} e^{zZ} e^{xX} = e^{yY+zZ} e^{xX}$. This means that we identify

$$(x, y, z) = e^{yY+zZ} e^{xX}$$

With this coordinate representation of G we have the group law

$$\begin{aligned}(x, y, z) \cdot (x', y', z') &= e^{yY+zZ} e^{xX} e^{y'Y+z'Z} e^{x'X} \\ &= e^{(y+y')Y+(z+z'+xy')Z} e^{(x+x')X} \\ &= (x + x', y + y', z + z' + xy')\end{aligned}$$

using the same trick

The Engel group

This is the nilpotent Lie group of dimension 4 with a basis of the Lie algebra satisfying

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4$$

In particular we can consider the smooth functions $h_1, h_2, h_3, h_4 : \mathfrak{g}^* \to \mathbb{R}$. To find a basis of the Poisson vector fields it is enough to write down $\vec{h}_i$ for every $i = 1, 2, \ldots, 5$. Using our formulas

$$\vec{h}_1 = h_3 \partial_{h_2} + h_4 \partial_{h_3}, \qquad \vec{h}_2 = -h_3 \partial_{h_1}$$

$$\vec{h}_3 = -h_4 \partial_{h_1}$$

while h_4 is a casimir since the corresponding vector field X_0 is in the center. There is a second casimir.

$$f = \frac{1}{2} h_3^2 - h_2 h_4$$

Coadjoint orbits

All coadjoint orbits are contained in the level sets

$$\begin{cases} h_4 = \lambda, \\ \frac{1}{2}h_3^2 - \lambda h_2 = \nu \end{cases} \tag{20}$$

Note that $\{f, h_j\} = 0$ for $j \geq 2$ (the only non zero commutators must contain X_1) and

$$\{f, h_1\} = \{h_3, h_1\}h_3 - \{h_2, h_1\}h_4 = -h_4 h_3 + h_3 h_4 = 0$$

Combining this and the Poisson vector fields we have the orbits

(i) if $\lambda = \nu = 0$ then every point $(h_1, h_2, 0, 0)$ is an orbit
(ii) if $\lambda = 0$ and $\nu \neq 0$ then orbits are planes $h_4 = 0,\ h_3 = \pm\sqrt{2\nu}$
(iii) if $\lambda \neq 0$ then the orbit coincides with the set defined by the equations above

New representations

Fix $\eta = (0, -\nu/\lambda, 0, \lambda)$ then we have a choice of maximal subalgebra

$$\mathfrak{h} = \mathrm{span}\{X_2, X_3, X_4\}, \qquad [\mathfrak{h}, \mathfrak{h}] = 0.$$

and the corresponding 1-dim representation

$$\mathcal{X}_{\nu,\lambda}(e^{x_2 X_2 + x_3 X_3 + x_4 X_4}) = e^{i\left(-\frac{\nu}{\lambda} x_2 + \lambda x_4\right)}.$$

We write points on G as

$$g = e^{x_2 X_2 + x_3 X_3 + x_4 X_4} e^{x_1 X_1}.$$

We take a complement $K = \exp(\mathbb{R} X_1)$ and we solve the Master equation

$$e^{\theta X_1} e^{x_2 X_2 + x_3 X_3 + x_4 X_4} e^{x_1 X_1} = \tag{21}$$

$$= e^{x_2 X_2 + (x_3 + \theta x_2) X_3 + (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) X_4} e^{(\theta + x_1) X_1} \tag{22}$$

We deduce that

$$\mathcal{R}_{\nu,\lambda} f(e^{\theta X_1}) = \mathcal{X}_{\nu,\lambda}(e^{x_2 X_2 + (x_3+\theta x_2)X_3 + (x_4+\theta x_3+\frac{\theta^2}{2}x_2)X_4}) f(e^{(\theta+x_1)X_1})$$

that is in the notation $\widetilde{f}(\theta) = f(e^{\theta X_1})$

$$\mathcal{R}_{\nu,\lambda}\widetilde{f}(\theta) = \exp\left[i\left(-\frac{\nu}{\lambda}x_2 + \lambda(x_4 + \theta x_3 + \frac{\theta^2}{2}x_2) \right)\right] \widetilde{f}(\theta + x_1)$$

Differentiating with respect to the x_i at zero we get also the representation of the Lie algebra

$$\begin{aligned} X_1\widetilde{f} &= \frac{d}{dt}\widetilde{f}, \\ X_2\widetilde{f} &= i\left(\frac{\lambda}{2}\theta^2 - \frac{\nu}{\lambda}\right)\widetilde{f}, \\ X_3\widetilde{f} &= i\lambda\theta\widetilde{f}, \\ X_4\widetilde{f} &= i\lambda\widetilde{f} \end{aligned}$$

notice $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$.

In particular notice that

$$X_1 \widetilde{f} = \frac{d}{dt} \widetilde{f},$$
$$X_2 \widetilde{f} = i \left(\frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right) \widetilde{f},$$

Notice that the Laplacian is

$$X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right)^2$$

This gives the basis of left-invariant vector fields

$$X_1 = \partial_{x_1}, \qquad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4}$$

$$X_3 = \partial_{x_3} + x_1 \partial_{x_4}, \qquad X_4 = \partial_{x_4}$$

Final comments for today?

Observation

Notice that the Laplacian is

$$X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{2}\theta^2 - \frac{\nu}{\lambda}\right)^2$$

- it is the square of a polynomial of degree $=$ 2(step-1)
- polynomial which does not has term on degree step-2
- it is arbitrary!
- oscillator with polynomial potential!
- what is the spectrum?
- summability property and relation with the Plancherel formula
- proof in the case of the Engel group, remark in higher steps