Strichartz estimates and sub-Riemannian geometry
Lecture 3

Davide Barilari,
Dipartimento di Matematica “Tullio Levi-Civita”,
Università degli Studi di Padova

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Strichartz estimate in the Heisenberg group

Let $u_0$ in $S(\mathbb{H}^d)$ be radial and consider the Cauchy problem

\[
\begin{cases}
i \partial_t u - \Delta_\mathbb{H} u = 0 \\
u|_{t=0} = u_0.
\end{cases}
\]

Taking the partial Fourier transform with respect to the variable $w$

\[
\begin{cases}
i \frac{d}{dt} \mathcal{F}_\mathbb{H}(u)(t, n, m, \lambda) = -4|\lambda|(2|m| + d) \mathcal{F}_\mathbb{H}(u)(t, n, m, \lambda) \\
\mathcal{F}_\mathbb{H}(u)|_{t=0} = \mathcal{F}_\mathbb{H} u_0.
\end{cases}
\]

Notice that if we set $|m| = 0$ we see the “transport” part

\[
\mathcal{F}_\mathbb{H}(u)(t, n, 0, \lambda) = e^{4it|\lambda|(2|n| + d)} \mathcal{F}_\mathbb{H}(u_0)(|n|, |n|, \lambda) \delta_{n, m}.
\]
Applying the inverse Fourier formula

\[ u(t, z, s) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{H}^d} \mathcal{W}(\hat{x}, z, s) e^{4it|\lambda|(2|m|+d)} \mathcal{F}_{\mathbb{H}}(u_0)(|n|, |n|, \lambda) \delta_{n,m} d\hat{x}. \]

Re-expressed as the inverse Fourier transform in \( \mathbb{R} \times \mathbb{H}^d \) of \( \mathcal{F}_{\mathbb{H}}(u_0) \) \( d\Sigma \),

\[ \Sigma \overset{\text{def}}{=} \left\{ (\alpha, \hat{x}) = (\alpha, (n, n, \lambda)) \in \mathbb{R} \times \mathbb{H}^d \mid \alpha = 4|\lambda|(2|n| + d) \right\} . \]

endow \( \Sigma \) with the measure \( d\Sigma \) induced by the projection \( \mathbb{R} \times \mathbb{H}^d \to \mathbb{H}^d \)

\[ \int_{\mathbb{H}^d} \Phi(\alpha, \hat{x}) d\Sigma(\alpha, \hat{x}) = \int_{\mathbb{H}^d} \Phi(4|\lambda|(2|m| + d), \hat{x}) d\hat{x}, \]

**Theorem (Bahouri, DB, Gallagher, '21)**

*If* \( 1 \leq q \leq p \leq 2 \), *then for* \( f \) *radial*

\[ \| \mathcal{F}_{\mathbb{R} \times \mathbb{H}^d}(f) \|_{\mathcal{L}^2(d\Sigma)} \leq C_{p,q} \| f \|_{\ell^1_{s,L^q_t,L^p_z}}, \quad (1) \]
Using dual inequality, assuming that $\mathcal{F}_\mathbb{H} u_0$ is localized in the unit ball

For any $2 \leq p \leq q \leq \infty$

$$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \|\mathcal{F}_\mathbb{H} u_0\|_{L^2(\mathbb{H}^d)} = C \|u_0\|_{L^2(\mathbb{H}^d)},$$

- If $u_0$ is frequency localized in the ball $\mathcal{B}_\Lambda$,

  $$u_\Lambda(t, z, s) = u(\Lambda^{-2} t, \Lambda^{-1} z, \Lambda^{-2} s), \quad u_{0,\Lambda}(z, s) = u_0(\Lambda^{-1} z, \Lambda^{-2} s)$$

  we have

  $$\|u_\Lambda\|_{L^\infty_s L^q_t L^p_z} = \Lambda^{\frac{2}{q} + \frac{2d}{p}} \|u\|_{L^\infty_s L^q_t L^p_z}, \quad \|u_{0,\Lambda}\|_{L^2(\mathbb{H}^d)} = \Lambda^\frac{Q}{2} \|u_0\|_{L^2(\mathbb{H}^d)},$$

- we infer for $\sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}$

  $$\|u\|_{L^\infty_s L^q_t L^p_z} \leq C \Lambda^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}} \|u_0\|_{L^2(\mathbb{H}^d)} \leq C \|u_0\|_{H^\sigma(\mathbb{H}^d)}.$$
The inhomogeneous case

- Denoting by \( (\mathcal{U}(t))_{t \in \mathbb{R}} \) the solution operator of the Schrödinger equation on the Heisenberg group,
- \( (\mathcal{U}(t))_{t \in \mathbb{R}} \) is a one-parameter group of unitary operators on \( L^2(\mathbb{H}^d) \).
- the solution to the inhomogeneous equation

\[
\begin{cases}
    i \partial_t u - \Delta_{\mathbb{H}} u = f \\
    u|_{t=0} = 0,
\end{cases}
\]

writes

\[
 u(t, \cdot) = -i \int_0^t \mathcal{U}(t - t') f(t', \cdot) dt',
\]

(2)

It is enough to check that it satisfies, for all admissible pairs \((p, q)\),

\[
\|u\| \leq \|f\| \quad \text{with} \quad \sigma = \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}.
\]

(3)
By formula of the solution, we have for all $s \in \mathbb{R}$,

$$\| u(t, \cdot, s) \|_{L^p_z} \leq \int_{\mathbb{R}} \| U(t) U(-t') f(t', \cdot, s) \|_{L^p_z} dt'.$$

Therefore, still for all $s$,

$$\| u(\cdot, \cdot, s) \|_{L^q_t L^p_z} \leq \int_{\mathbb{R}} \| U(\cdot) U(-t') f(t', \cdot, s) \|_{L^q_t L^p_z} dt'.$$

Let us first assume that, for all $t$, the source term $f(t, \cdot)$ is frequency localized in in the unit ball $B_1$

if $g$ is frequency localized in a unit ball, then for all $2 \leq p \leq q \leq \infty$

$$\| U(t) g \|_{L^\infty_s L^q_t L^p_z} \lesssim \| g \|_{L^2(\mathbb{H}^d)}.$$ (4)
By formula of the solution, we have for all \( s \in \mathbb{R} \),

\[
\| u(t, \cdot, s) \|_{L^p_2} \leq \int_{\mathbb{R}} \| U(t)U(-t')f(t', \cdot, s) \|_{L^p_2} dt'.
\]

Therefore, still for all \( s \),

\[
\| u(\cdot, \cdot, s) \|_{L^q_t L^p_2} \leq \int_{\mathbb{R}} \| U(\cdot)U(-t')f(t', \cdot, s) \|_{L^q_t L^p_2} dt'.
\]

Let us first assume that, for all \( t \), the source term \( f(t, \cdot) \) is frequency localized in in the unit ball \( B_1 \)

if \( g \) is frequency localized in a unit ball, then for all \( 2 \leq p \leq q \leq \infty \)

\[
\| U(t)g \|_{L^\infty_t L^q_2} \lesssim \| g \|_{L^2(\mathbb{H}^d)}. \tag{5}
\]
Using homog Strichartz, we deduce that

$$\|u\|_{L^\infty_t L^q_s L^p_Y} \leq \int_{\mathbb{R}} \|\mathcal{U}(-t')f(t', \cdot)\|_{L^2(\mathbb{H}^d)} dt'.$$

Since $\mathcal{U}(-t')$ is unitary on $L^2(\mathbb{H}^d)$, we readily gather that

$$\|u\|_{L^\infty_t L^q_s L^p_Y} \leq \int_{\mathbb{R}} \|f(t', \cdot)\|_{L^2(\mathbb{H}^d)} dt'. \quad (6)$$

Now if for all $t$, $f(t, \cdot)$ is frequency localized in a ball of size $\Lambda$, then setting

$$f_\Lambda(t, \cdot) \overset{\text{def}}{=} \Lambda^{-2} f(\Lambda^{-2} t, \cdot) \circ \delta_{\Lambda^{-1}}$$

we find that on the one hand, $f_\Lambda(t, \cdot)$ is frequency localized in a unit ball for all $t$, and on the other hand that the solution to the Cauchy problem

$$\begin{cases} 
i \partial_t u_\Lambda - \Delta_{\mathbb{H}} u_\Lambda = f_\Lambda \\
u|_{t=0} = 0, \end{cases}$$

writes $u_\Lambda(t, w) = u(\Lambda^{-2} t, \cdot) \circ \delta_{\Lambda^{-1}}$. 
Now by scale invariance, we have

\[ \int_{\mathbb{R}} \| f_\Lambda(t', \cdot) \|_{L^2(\mathbb{H}^d)} dt' = \Lambda^{Q/2} \int_{\mathbb{R}} \| f(t', \cdot) \|_{L^2(\mathbb{H}^d)} dt' \]

and

\[ \| u_\Lambda \|_{L^\infty_t L^q_t L^p_Y} = \Lambda^{2q + \frac{2d}{p}} \| u \|_{L^\infty_t L^q_t L^p_Y}. \]

Consequently, we get

\[ \| u \|_{L^\infty_t L^q_t L^p_Y} \leq C \int_{\mathbb{R}} \Lambda^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}} \| f(t', \cdot) \|_{L^2(\mathbb{H}^d)} dt'. \]

Since \( \frac{Q}{2} - \frac{2}{q} - \frac{2d}{p} \geq 0 \), we have

\[ \Lambda^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}} \| f(t', \cdot) \|_{L^2(\mathbb{H}^d)} \lesssim \| f(t', \cdot) \|_{H^{\frac{Q}{2} - \frac{2}{q} - \frac{2d}{p}}(\mathbb{H}^d)}, \]

and then integrate in \( t \) to conclude
The statement

**Theorem (Bahouri, DB, Gallagher, ’21)**

*If* $1 \leq q \leq p \leq 2$, *then for* $f$ radial

$$
\| \mathcal{F}_{\mathbb{R} \times \mathbb{H}^d}(f) \Sigma \|_{L^2(d\Sigma)} \leq C_{p,q} \| f \|_{L^1_{s} L^q_{t} L^p_{z}},
$$

(7)

and its dual version

**Example**

*for any* $2 \leq p' \leq q' \leq \infty$, *there holds*

$$
\| \mathcal{F}_{\mathbb{R} \times \mathbb{H}^d}^{-1}(\theta \Sigma_{loc}) \|_{L^\infty_{s} L^{q'}_{t} L^{p'}_{\gamma}} \leq \| \theta \Sigma_{loc} \|_{L^2(d\Sigma_{loc})},
$$

(8)
The completion of the frequency set

- The frequency set $\widehat{\mathbb{H}}^d$ comes with a measure

$$\int_{\widehat{\mathbb{H}}^d} \theta(\hat{x}) \, d\hat{x} \overset{\text{def}}{=} \int_{\mathbb{R}} \sum_{(n,m) \in \mathbb{N}^2} \theta(n, m, \lambda)|\lambda|^d \, d\lambda.$$  

- endowed with a distance

$$d(\hat{x}, \hat{x}') \overset{\text{def}}{=} |\lambda(n+m) - \lambda'(n'+m')|_{\ell^1} + |(n-m) - (n'-m')|_{\ell^1} + d|\lambda - \lambda'|,$$

- $(\widehat{\mathbb{H}}^d, d)$ it is not complete $\rightarrow$ build the metric completion $\widehat{\mathbb{H}}^d$

Some advantages of [Bahouri, Chemin, Danchin]

- definition of $\mathcal{S}(\widehat{\mathbb{H}}^d)$,
- interpretation smoothness $\leftrightarrow$ decay
- $\rightarrow$ give a meaning to the unit sphere $\mathcal{S}_{\widehat{\mathbb{H}}^d}$ of $\widehat{\mathbb{H}}^d$. 
Recall that for $\theta$ being the Fourier transform of a radial function

$$
\int_{\mathbb{H}^d} \theta(\hat{x}) d\hat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d d\lambda.
$$

For spherical measures (on sphere of radius $R$) we want

$$
\int_{\mathbb{H}^d} \theta(\hat{x}) d\hat{x} = \int_{0}^{\infty} \left( \int_{S_{\mathbb{H}^d}^R} \theta(\hat{x}) d\sigma_R(\hat{x}) \right) dR
$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$
\int_{S_{\mathbb{H}^d}^R} \theta(\hat{x}) d\sigma_R(\hat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2R^{2d+1}}{(2|n| + d)^{d+1}} \left( \sum_{\pm} \theta(n, n, \frac{\pm R^2}{2|n| + d}) \right)
$$
On the surface measure, $R = 1$

Recall that for $\theta$ Fourier transform of radial function

$$\int_{\mathbb{H}^d} \theta(\hat{x}) d\hat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda)|\lambda|^d d\lambda.$$  

For spherical measures (on sphere of radius $R$) we want

$$\int_{\mathbb{H}^d} \theta(\hat{x}) d\hat{x} = \int_0^\infty \left( \int_{S_{\mathbb{H}^d}} \theta(\hat{x}) d\sigma_R(\hat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{S_{\mathbb{H}^d}} \theta(\hat{x}) d\sigma_1(\hat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2}{(2|n| + d)^{d+1}} \left( \sum_{\pm} \theta(n, n, \frac{\pm 1}{2|n| + d}) \right)$$
The result of Müller

- D.Müller [Annals of Math, 1990]: works in terms of spectral decomposition

\[ L = \int_0^{\infty} \lambda dE(\lambda), \quad Pf = f \ast G \]

- proves the estimate ("restriction for the sphere"): if \( 1 \leq p \leq 2 \)

\[ \left[ \sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \left( \sum_{\pm} \left| \mathcal{F}_H(f)(n, n, \frac{\pm 1}{2|n| + d}) \right|^2 \right) \right]^{\frac{1}{2}} \leq C_p \| f \|_{L_1^1 L_p^2} \]

- can be reinterpreted as follows: If \( 1 \leq p \leq 2 \), then for radial \( f \)

\[ \| \mathcal{F}_H(f)|_{S_{\mathbb{H}^d}} \|_{L^2(S_{\mathbb{H}^d})} \leq C_p \| f \|_{L_1^1 L_p^2} , \quad (9) \]

- valid on the full interval: for \( p \in [1, 2] \)
- crucial: the anisotropic norm \( L_1^1 L_p^2 \) (\( r = 1 \) is necessary in vertical)
- false for \( p > 2 \)
Fourier transform of the surface measure

Up to a measure zero set on \( \mathbb{H}^d \)

\[
\mathcal{S}_{\mathbb{H}^d} = \left\{ (n, n, \lambda) \in \mathbb{H}^d / (2|n| + d)|\lambda| = 1 \right\}
\]

By definition, the tempered distribution \( G = \mathcal{F}_{\mathbb{H}}^{-1}(d\sigma_{\mathcal{S}_{\mathbb{H}^d}}) \)

**Lemma**

\( G \) is the bounded function on \( \mathbb{H}^d \) defined by

\[
G(z, s) = \frac{2^d}{\pi^{d+1}} \sum_{n \in \mathbb{N}^d} \frac{1}{(2|n| + d)^{d+1}} \cos \left( \frac{s}{2|n| + d} \right) \mathcal{W} \left( n, n, 1, \frac{z}{\sqrt{2|n| + d}} \right)
\]

(10)

For the sphere of radius \( R^{1/2} \) we have the homogeneity property:

\[
G_R(z, s) \overset{\text{def}}{=} R^d (G \circ \delta_{\sqrt{R}})(z, s).
\]

(11)
Measure on the paraboloid

Proceeding as for the restriction theorem on the sphere of $\mathbb{H}^d$, let us first compute

$$G_{\Sigma_{\text{loc}}} \overset{\text{def}}{=} \mathcal{F}^{-1}_{\mathbb{R} \times \hat{\mathbb{H}}^d}(d\Sigma_{\text{loc}}) .$$

**Lemma**

*With the above notation, $G_{\Sigma_{\text{loc}}}$ is the bounded function on $\mathbb{R} \times \hat{\mathbb{H}}^d$ defined by*

$$G_{\Sigma_{\text{loc}}} (t, w) = 2\pi \int_{0}^{\infty} G_{\alpha}(w) e^{-it\alpha} \psi(\alpha) \, d\alpha , \quad (12)$$

*where $G_{\mathbb{R}}$ is the inverse Fourier of the measure of sphere of radius $R^{1/2}$. This gives for all $f$ in $S_{\text{rad}}(\mathcal{D})$*

$$(R_{\Sigma_{\text{loc}}}^* R_{\Sigma_{\text{loc}}} f)(t, z, s) = \left(\frac{\pi}{2}\right)^d (G_{\Sigma_{\text{loc}}} * \check{f})(-t, -z, s) , \quad (13)$$
Consider the restriction operator

$$R_{\Sigma_{loc}} f = \mathcal{F}_{\mathbb{R} \times \mathbb{H}^d}(f)|_{\Sigma_{loc}}$$

Indeed applying the Hölder inequality, we deduce that

$$\| R_{\Sigma_{loc}} f \|_{L^2(\Sigma_{loc})}^2 \leq \| R_{\Sigma_{loc}}^* R_{\Sigma_{loc}} f \|_{L_{sL_{t}^{q'}L_{Y}^{p'}}} \| f \|_{L_{sL_{t}^{q}L_{Y}^{p}}}$$

$$\leq \| \tilde{f} * D G_{\Sigma_{loc}} \|_{L_{sL_{t}^{q'}L_{Y}^{p'}}} \| f \|_{L_{sL_{t}^{q}L_{Y}^{p}}} ,$$

Then as in the Euclidean case, we are reduced to proving that

$$R_{\Sigma_{loc}}^* R_{\Sigma_{loc}}$$

is bounded from $L_{sL_{t}^{q}L_{Y}^{p}}$ into $L_{sL_{t}^{q'}L_{Y}^{p'}}$. 
Proof for $1 \leq p < 2$ (non endpoint)

**Main lemma**

\[
\|f \ast G_{\Sigma_{\text{loc}}} \|_{L^\infty_s L_t^q L_z^{p'}} \lesssim \left\| \mathcal{F}_\mathbb{R}(f)(-\alpha, \cdot) \right\|_{L^{p'}_z L^1_s} \alpha^d \left(1 - \frac{2}{p'} \right) \psi(\alpha) \|_{L^q_\alpha}
\]

- Hölder estimate in $\alpha$ + Hausdorff-Young inequality: for any $a \geq 2$

\[
\|f \ast G_{\Sigma_{\text{loc}}} \|_{L^\infty_s L_t^q L_z^{p'}} \lesssim \| \mathcal{F}_\mathbb{R}(f) \|_{L^a_t L^{p'}_z L^1_s} \| \alpha^d \left(1 - \frac{2}{p'} \right) \psi(\alpha) \|_{L^b_\alpha}
\]

\[
\lesssim \| f \|_{L^a'_t L^{p'}_z L^1_s} \| \alpha^d \left(1 - \frac{2}{p'} \right) \psi(\alpha) \|_{L^b_\alpha(\mathbb{R})},
\]

where $a'$ is the conjugate exponent of $a$ and $\frac{1}{a} + \frac{1}{b} = \frac{1}{q}$.

- Finally for $a' = q$ and Minkowski’s inequality, we get for $q' \geq p' > 2$

\[
\|f \ast G_{\Sigma_{\text{loc}}} \|_{L^\infty_s L_t^q L_z^{p'}} \lesssim \| f \|_{L^1_t L^q_z L_z^p}
\]

→ endpoint $p = 2$: ad hoc argument
Chapter 5: Kirillov Theory for Nilpotent groups
Here $V$ is a vector space finite or infinite dimensional.

- Given a Lie group $G$ a representation of $G$ is a smooth homomorphism

$$R : G \rightarrow GL(V), \quad R(g_1 g_2) = R(g_1) R(g_2)$$

where in the left hand side we have the product in $G$ while in the right hand side the composition in $GL(V)$.

- A subspace $W$ of $V$ is an invariant subspace if $R(g) w \in W$ for all $g \in G$ and $w \in W$.

- The representation is said to be irreducible if the only invariant subspaces of $V$ are the zero space and $V$ itself.
if \( \mathcal{R} \) map into the group of unitary operators, we say \textit{unitary representation}.

The representation is said \textit{one-dimensional} if \( V \) has dimension 1.

For \( V = \mathbb{C} \), a 1-dim representation of \( G \) will be a smooth homomorphism
\[
\chi : G \to U(\mathbb{C}) = S^1
\]

Let \( G \) nilpotent, \( \eta \in \mathfrak{g}^* \) and \( H \subset G \) be such that \( \eta([\mathfrak{h}, \mathfrak{h}]) = 0 \): we can define the one-dimensional representation
\[
\chi_\eta : H \to S^1 = U(\mathbb{C})
\]
\[
\chi_\eta(e^X) = e^{i \langle \eta, X \rangle}, \quad X \in \mathfrak{h}.
\]
where as usual \( \langle \eta, X \rangle \) denotes the duality product \( g^* \) and \( g \).
The Kirillov theory gives a way to describe all possible irreducible unitary representations of $G$ in terms of coadjoint orbits of the group.

An algorithm in four steps:

1. Fix an element $\eta \in \mathfrak{g}^*$.
2. Fix any maximal Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ s.t. $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$.
3. Consider the one-dimensional representation

\[ \mathcal{X}_{\eta, \mathfrak{h}} : H \rightarrow S^1 = U(\mathbb{C}) \]

\[ \mathcal{X}_{\eta, \mathfrak{h}}(e^X) = e^{i\langle \eta, X \rangle}, \quad X \in \mathfrak{h}. \]

where as usual $\langle \eta, X \rangle$ denotes the duality product $\mathfrak{g}^*$ and $\mathfrak{g}$.

4. Compute the induced representation $\mathcal{R}_{\eta, \mathfrak{h}} : G \rightarrow U(\mathcal{W})$.

→ a way to lift a representation to the group $G$. 


Coadjoint orbits

Given a Lie group $G$

- the conjugation map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$.
- the *adjoint action* of $G$ onto its Lie algebra

$$\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad \text{Ad}_g = (C_g)^*$$

- Notice that $\text{Ad} : G \to GL(\mathfrak{g})$ given by $g \mapsto \text{Ad}_g$ is a finite dimensional representation of $G$.
- This induces the so called *coadjoint action* dual of the above

$$\text{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle \text{Ad}_g^* \eta, \nu \rangle := \langle \eta, (\text{Ad}_{g^{-1}})^* \nu \rangle$$

- Notice that $\text{Ad}^*$ is indeed an action of $G$ on $\mathfrak{g}^*$. Given $\eta \in \mathfrak{g}^*$ the *coadjoint orbit* of $\eta$ is by definition the set

$$\mathcal{O}_\eta = \{ \text{Ad}_g^* \eta \mid g \in G \}.$$
The Kirillov theorem states the following:

**Theorem**

The map which assigns to \( \eta \in g^*/G \) to \( R_{\eta,\mathfrak{h}} \) in \( \hat{G} \) (where \( \mathfrak{h} \) is some maximal Lie subalgebra) is a bijection. More precisely:

(a) every irreducible unitary representation of a nilpotent Lie group \( G \) is of the form \( R_{\eta,\mathfrak{h}} \) for some \( \eta \) and \( \mathfrak{h} \)

(b) two representations \( R_{\eta,\mathfrak{h}} \) and \( R_{\eta',\mathfrak{h}'} \) are equivalent if and only if \( \eta \) and \( \eta' \) belong to the same orbit.

Here two irreducible unitary representations \( R_1 : G \to U(W_1) \) and \( R_2 : G \to U(W_2) \) are equivalent if there exists an isometry between the Hilbert spaces \( T : W_1 \to W_2 \) such that

\[
T \circ R_1(g) \circ T^{-1} = R_2(g), \quad \forall g \in G
\]
The Kirillov theory gives a way to describe all possible irreducible unitary representations of $G$ in terms of coadjoint orbits of the group.

An algorithm in four steps:

1. Fix an element $\eta \in \mathfrak{g}^*$ in every leaf.
2. Fix any maximal Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ s.t. $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$.
3. Consider the one-dimensional representation
   \[
   \chi_{\eta, \mathfrak{h}} : H \to S^1 = U(\mathbb{C})
   \]
   \[
   \chi_{\eta, \mathfrak{h}}(e^X) = e^{i\langle \eta, X \rangle}, \quad X \in \mathfrak{h}.
   \]
   where as usual $\langle \eta, X \rangle$ denotes the duality product $\mathfrak{g}^*$ and $\mathfrak{g}$.
4. Compute the induced representation $\mathcal{R}_{\eta, \mathfrak{h}} : G \to U(W)$.

→ a way to lift a representation to the group $G$
Poisson structure on the dual $g^*$

Let $a, b : g^* \to \mathbb{R}$ be smooth functions.

- Poisson manifold with the bracket

$$\{a, b\}(\eta) = \langle \eta, [da, db]\rangle$$

- Given a smooth $a : g^* \to \mathbb{R}$ we can define its Poisson vector field by setting for every smooth $b : g^* \to \mathbb{R}$

$$\tilde{a}(b) = \{a, b\}$$

- The set of all Poisson vector at a point defines a distribution

$$D_\eta = \{\tilde{a}(\eta) \mid a \in C^\infty(g^*)\}$$

which has no constant rank (notice $D_0 = \{0\}$).
We can define also the *Poisson orbit* of $\eta \in g^*$ in the sense of dynamical systems as follows

$$O^P_\eta = \{ e^{t_1 a_1} \circ \ldots \circ e^{t_\ell a_\ell}(\eta) \mid \ell \in \mathbb{N}, t_i \in \mathbb{R}, a_i \in C^\infty(g^*) \}.$$ 

Notice that both $O^P_\eta$ and $O_\eta$ are subsets of $g^*$ containing $\eta$.

**Proposition**

For every $\eta \in g^*$ we have the equality $O^P_\eta = O_\eta$. Each orbit is an *even dimensional* symplectic manifold.

It is enough to use as $a_i$ the linear on fibers function associated to a basis $h_i(p, x) = p \cdot X_i(x)$.
Fix a basis of the Lie algebra $X_1, \ldots, X_n$ such that

$$[X_i, X_j] = c_{ij}^k X_k$$

for some constants $c_{ij}^k$. Define the corresponding coordinates on the fibers of $T^*G$ given by

$$h_i(p, x) = p \cdot X_i(x)$$

These can be thought as smooth functions on $g^*$ and satisfy

$$\{h_i, h_j\} = c_{ij}^k h_k.$$

We recall that a casimir is a smooth function $f \in C^\infty(g^*)$ such that

$$\{a, f\} = 0, \quad \forall a \in C^\infty(g^*)$$
If we write \( f = f(h_1, \ldots, h_n) \) to check that \( f \) is a casimir it is enough to check that

\[
\{f, h_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial h_i} \{h_i, h_j\} = \sum_{i,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k = 0, \quad j = 1, \ldots, n
\]

that means

\[
\sum_{i=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k = 0, \quad j, k = 1, \ldots, n
\]

The Poisson vector field associated to a function \( f \) is

\[
\vec{f} = \sum_{i,j,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k \frac{\partial}{\partial h_j}
\]

The Poisson vector field associated to a casimir is the zero vector field.
If we write $f = f(h_1, \ldots, h_n)$ to check that $f$ is a casimir it is enough to check that

$$\{f, h_j\} = \sum_{i=1}^{n} \frac{\partial f}{\partial h_i} \{h_i, h_j\} = \sum_{i,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k = 0, \quad j = 1, \ldots, n$$

that means

$$\sum_{i=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k = 0, \quad j, k = 1, \ldots, n$$

The Poisson vector field associated to a function $f$ is

$$\vec{h}_i = \sum_{i,j,k=1}^{n} c_{ij}^k h_k \frac{\partial}{\partial h_j}$$

The Poisson vector field associated to a casimir is the zero vector field.
Let us go back to the main example, the Heisenberg group.

\[ [X, Y] = Z \]

- relabel \((X, Y, Z) = (X_1, X_2, X_0)\)
- Consider \(h_1, h_2, h_0 : \mathfrak{g}^* \to \mathbb{R}\)
- write down \(\vec{h}_i\) for every \(i = 1, 2, 0\).

\[
\begin{align*}
\vec{h}_1 &= h_0 \partial_{h_2}, \\
\vec{h}_2 &= -h_0 \partial_{h_1}
\end{align*}
\]

- \(h_0\) is a casimir: the corresponding vector field \(X_0\) is in the center.

Hence we have the coadjoint orbits.
- if \(h_0 = 0\) then every point \((h_1, h_2, 0)\) is an orbit
- if \(h_0 \neq 0\) then every plane \(h_0 = \lambda\) is an orbit
To compute the representations.

- If we take $\eta = (h_1, h_2, 0) \in g^*$ then we can take $\mathfrak{h} = g$ since $[g, g] = \mathbb{R}X_0$ and the corresponding character

  $$\chi_\eta(g) = e^{i(h_1x + h_2y)}$$

  where $g = e^{xX + yY + zZ}$. Notice that since we can take $\mathfrak{h} = g$ there is “nothing to induce”, so these are representation of the abelian $\mathbb{R}^2$.

- If we take $\eta = (0, 0, h_0) \in g^*$ with $\lambda \neq 0$ as representative of the orbit. We can take $\mathfrak{h} = \text{span}\{Y, Z\}$ since $[\mathfrak{h}, \mathfrak{h}] = 0$ and it is maximal

  $$\chi_\eta(g) = e^{i\lambda z}$$

  what to do then?

  we have to understand the **induced representations**!
Induced representations

Let $G$ be a nilpotent Lie group and $H$ be a subgroup.

- Given a representation $\mathcal{X} : H \to U(V)$ we want to build a representation $\mathcal{R} : G \to U(W)$ that is induced by $\mathcal{X}$.
- We first build the Hilbert space $W$. Consider the set of functions $f : G \to V$ such that

$$f(hg) = \mathcal{X}(h)f(g) \quad (14)$$

- Notice that this means that

$$\mathcal{X}(h)f = f \circ L_h$$

- For such a function, since $\mathcal{X}$ is unitary, we have that $\|f(hg)\|$ is independent on $h$ and hence the norm of $\|f(Hg)\|$ is well-defined, where $Hg$ denotes the left coset of $g$ in $H \backslash G$. 

We require that
\[ \int_{H \setminus G} \|f(Hg)\|^2 d\mu < \infty \]  
(15)

where \( d\mu \) is a right invariant measure on \( H \setminus G \).

Then we set

\[ W = \{ f : G \to V \mid f \text{ satisfies (14)-(15)} \} \]

Once we have set the space \( W \) we can define \( \mathcal{R} : G \to U(W) \) as follows

\[ \mathcal{R}(g)f = f \circ R_g, \text{ i.e., } (\mathcal{R}(g)f)(g') = f(g'g) \]

where the \( R_g \) is the right translation.

One can check that \( \mathcal{R} \) is unitary and strongly continuous.
Crucial for computations!

- We have a natural projection $\pi : G \to H\backslash G$.
- Given any section $s : H\backslash G \to G$ (this means that $\pi \circ s = \text{id}$ on $H\backslash G$) we can consider the image of the section $K = s(H\backslash G)$ and try to write elements of $G$ as products $H \cdot K$.
- Write $g'g = hk$ we can split

\[(R(g)f)(g') = f(g'g) = f(hk) = \chi(h)f(k)\] (16)

Crucial step: solve the Master equation

\[g'g = h \cdot k\]

- It is enough to solve the Master equation for $g' \in K$ (use the last equality in (16) and $f$ is a equivariant function)

\[K \cdot G = H \cdot K\]
To compute the representations.

- If we take \( \eta = (h_1, h_2, 0) \in \mathfrak{g}^* \) then we can take \( \mathfrak{h} = \mathfrak{g} \) since \([\mathfrak{g}, \mathfrak{g}] = \mathbb{R}X_0 \) and the corresponding character

\[
\chi_{\eta}(g) = e^{i(h_1x + h_2y)}
\]

where \( g = e^{xX+yY+zZ} \). Notice that since we can take \( \mathfrak{h} = \mathfrak{g} \) there is “nothing to induce”, so these are representation of the abelian \( \mathbb{R}^2 \).

- If we take \( \eta = (0, 0, h_0) \in \mathfrak{g}^* \) with \( \lambda \neq 0 \) as representative of the orbit. We can take \( \mathfrak{h} = \text{span}\{Y, Z\} \) since \([\mathfrak{h}, \mathfrak{h}] = 0 \) and it is maximal

\[
\chi_{\eta}(g) = e^{i\lambda z}
\]

what to do then?
The induced representation in this case works as follows: we can take as complement $K = e^{\mathbb{R}X}$ and then try to write the elements as product $H \cdot K$ as follows. Let us take $k = e^{\theta X}$ in $K$ and $g = e^{yY+zZ}e^{xX}$ general element (it is convenient to use these coordinates). We have

$$(\mathcal{X}_\eta(g) f)(k) = f(kg)$$

and we have to write

$$e^{\theta X}e^{yY+zZ}e^{xX}$$

as an element of $H$ times an element of $K$. We have

$$e^{\theta X}e^{yY+zZ}e^{xX} = e^{yY+(z+\theta y)Z}e^{(\theta + x)X}$$

so that

$$(\mathcal{R}_\eta(g)f)(k) = f(e^{yY+(z+\theta y)Z}e^{(\theta + x)X})$$

$$= \mathcal{X}_\eta(e^{yY+(z+\theta y)Z})f(e^{(\theta + x)X})$$

(17)
Last step

Writing explicitly the character and \( \tilde{\mathcal{f}}(\theta) = f(e^{\theta X}) \) as a function on \( L^2(\mathbb{R}) \) instead of \( L^2(K) \) we have

\[
(\mathcal{R}_\eta(g)\tilde{f})(\theta) = e^{i\lambda(z+\theta y)}\tilde{f}(\theta + x)
\]  \hspace{1cm} \text{(19)}

One can recognise the representation of the Lie algebra which are skew-adjoint operators on the same space of functions

\[
X_1 \tilde{f} = \frac{d}{dt} \tilde{f}, \quad X_2 \tilde{f} = i\lambda \theta \tilde{f}, \quad X_0 \tilde{f} = i\lambda \tilde{f}
\]

which indeed satisfy \( [X_1, X_2] = X_0 \).

\[
\Delta = X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \lambda^2 \theta^2
\]
The trick for the Master equation

This is related to CHB formula

**Lemma**

Assume that the Lie algebra generated by $A, B$ is nipotent. Then we have that $e^A e^B e^{-A} = e^{C(A,B)}$ where

$$C(A, B) = e^{\text{ad}(A)} B = \sum_{k=0}^{\infty} \frac{\text{ad}^k(A)}{k!} B = B + [A, B] + \frac{1}{2} [A, [A, B]] + \ldots$$

Notice that the sum is finite due to nilpotency assumption.

In Heisenberg

$$e^{\theta X} e^{y Y + z Z} e^{x X} = e^{y Y + z Z + [\theta X, y Y + z Z]} e^{(\theta + x)X}$$

since

$$e^{\theta X} e^{y Y + z Z} e^{x X} = e^{y Y + (z + \theta y)Z} e^{(\theta + x)X}$$
An observation on the coordinates

The Heisenberg group $g = \text{span}\{X, Y, Z\}$ with the only non trivial commutator

$$[X, Y] = Z$$

Elements of $G = \exp(g)$ can be also written as follows

$$g = e^{yY} e^{zZ} e^{xX} = e^{yY+zZ} e^{xX}.$$ This means that we identify

$$(x, y, z) = e^{yY+zZ} e^{xX}$$

With this coordinate representation of $G$ we have the group law

$$(x, y, z) \cdot (x', y', z') = e^{yY+zZ} e^{xX} e^{y'Y+z'Z} e^{x'X}$$

$$= e^{(y+y')Y+(z+z'+xy')Z} e^{(x+x')X}$$

$$= (x + x', y + y', z + z' + xy')$$

using the same trick
The Engel group

This is the nilpotent Lie group of dimension 4 with a basis of the Lie algebra satisfying

\[ [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4 \]

In particular we can consider the smooth functions \( h_1, h_2, h_3, h_4 : g^* \to \mathbb{R} \). To find a basis of the Poisson vector fields it is enough to write down \( \mathbf{h}_i \) for every \( i = 1, 2, \ldots, 5 \). Using our formulas

\[ \mathbf{h}_1 = h_3 \partial_{h_2} + h_4 \partial_{h_3}, \quad \mathbf{h}_2 = -h_3 \partial_{h_1} \]

\[ \mathbf{h}_3 = -h_4 \partial_{h_1} \]

while \( h_4 \) is a casimir since the corresponding vector field \( X_0 \) is in the center. There is a second casimir.

\[ f = \frac{1}{2} h_3^2 - h_2 h_4 \]
Coadjoint orbits

All coadjoint orbits are contained in the level sets

\[
\begin{align*}
  h_4 &= \lambda, \\
  \frac{1}{2} h_3^2 - \lambda h_2 &= \nu
\end{align*}
\]  

(20)

Note that \( \{ f, h_j \} = 0 \) for \( j \geq 2 \) (the only non zero commutators must contain \( X_1 \)) and

\[
\{ f, h_1 \} = \{ h_3, h_1 \} h_3 - \{ h_2, h_1 \} h_4 = -h_4 h_3 + h_3 h_4 = 0
\]

Combining this and the Poisson vector fields we have the orbits

(i) If \( \lambda = \nu = 0 \) then every point \( (h_1, h_2, 0, 0) \) is an orbit

(ii) If \( \lambda = 0 \) and \( \nu \neq 0 \) then orbits are planes \( h_4 = 0, \ h_3 = \pm \sqrt{2\nu} \)

(iii) If \( \lambda \neq 0 \) then the orbit coincides with the set defined by the equations above
Fix $\eta = (0, -\nu/\lambda, 0, \lambda)$ then we have a choice of maximal subalgebra

$$\mathfrak{h} = \text{span}\{X_2, X_3, X_4\}, \quad [\mathfrak{h}, \mathfrak{h}] = 0.$$ 

and the corresponding 1-dim representation

$$\chi_{\nu,\lambda}(e^{x_2X_2+x_3X_3+x_4X_4}) = e^{i\left(-\frac{\nu}{\lambda}x_2 + \lambda x_4\right)}.$$

We write points on $G$ as

$$g = e^{x_2X_2+x_3X_3+x_4X_4}e^{x_1X_1}.$$

We take a complement $K = \exp(\mathbb{R}X_1)$ and we solve the Master equation

$$e^{\theta X_1} e^{x_2X_2+x_3X_3+x_4X_4} e^{x_1X_1} =$$

$$= e^{x_2X_2+(x_3+\theta x_2)X_3+(x_4+\theta x_3+\frac{\theta^2}{2}x_2)X_4} e^{(\theta+x_1)X_1} \quad (21)$$

$$= e^{x_2X_2+(x_3+\theta x_2)X_3+(x_4+\theta x_3+\frac{\theta^2}{2}x_2)X_4} e^{(\theta+x_1)X_1} \quad (22)$$
We deduce that

\[ \mathcal{R}_{\nu, \lambda} f(e^{\theta X_1}) = X_{\nu, \lambda}(e^{x_2 X_2 + (x_3 + \theta x_2) X_3 + (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) X_4}) f(e^{(\theta + x_1) X_1}) \]

that is in the notation \( \tilde{f}(\theta) = f(e^{\theta X_1}) \)

\[ \mathcal{R}_{\nu, \lambda} \tilde{f}(\theta) = \exp \left[ i \left( -\frac{\nu}{\lambda} x_2 + \lambda (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) \right) \right] \tilde{f}(\theta + x_1) \]

Differentiating with respect to the \( x_i \) at zero we get also the representation of the Lie algebra

\[ X_1 \tilde{f} = \frac{d}{dt} \tilde{f}, \]
\[ X_2 \tilde{f} = i \left( \frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right) \tilde{f}, \]
\[ X_3 \tilde{f} = i \lambda \theta \tilde{f}, \]
\[ X_4 \tilde{f} = i \lambda \tilde{f} \]

notice \([X_1, X_2] = X_3\) and \([X_1, X_3] = X_4\).
In particular notice that

$$X_1 \tilde{f} = \frac{d}{dt} \tilde{f},$$

$$X_2 \tilde{f} = i \left( \frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right) \tilde{f},$$

Notice that the Laplacian is

$$X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left( \frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right)^2$$

This gives the basis of left-invariant vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4}$$

$$X_3 = \partial_{x_3} + x_1 \partial_{x_4}, \quad X_4 = \partial_{x_4}$$
Notice that the Laplacian is

\[ X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left( \frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right)^2 \]

- it is the square of a polynomial of degree = 2 (step-1)
- polynomial which does not have a term of degree step-2
- it is arbitrary!
- oscillator with polynomial potential!
- what is the spectrum?
- summability property and relation with the Plancherel formula
- proof in the case of the Engel group, remark in higher steps