Strichartz estimates and sub-Riemannian geometry
Lecture 4

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Chapter 6: Spectral summability of quartic oscillators & Engel group

The Engel group

This is the nilpotent Lie group of dimension 4 with a basis of the Lie algebra satisfying

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4$$

In particular we can consider the smooth functions $h_1, h_2, h_3, h_4 : \mathfrak{g}^* \to \mathbb{R}$. To find a basis of the Poisson vector fields it is enough to write down $\vec{h}_i$ for every $i = 1, 2, \ldots, 5$. Using our formulas

$$\vec{h}_1 = h_3 \partial_{h_2} + h_4 \partial_{h_3}, \qquad \vec{h}_2 = -h_3 \partial_{h_1}$$

$$\vec{h}_3 = -h_4 \partial_{h_1}$$

while h_4 is a casimir since the corresponding vector field X_0 is in the center. There is a second casimir.

$$f = \frac{1}{2} h_3^2 - h_2 h_4$$

Coadjoint orbits

All coadjoint orbits are contained in the level sets

$$\begin{cases} h_4 = \lambda, \\ \frac{1}{2}h_3^2 - h_4 h_2 = \nu \end{cases} \tag{1}$$

Note that $\{f, h_j\} = 0$ for $j \geq 2$ (the only non zero commutators must contain X_1) and

$$\{f, h_1\} = \{h_3, h_1\}h_3 - \{h_2, h_1\}h_4 = -h_4 h_3 + h_3 h_4 = 0$$

Combining this and the Poisson vector fields we have the orbits

(i) if $\lambda = \nu = 0$ then every point $(h_1, h_2, 0, 0)$ is an orbit

(ii) if $\lambda = 0$ and $\nu \neq 0$ then orbits are planes $h_4 = 0$, $h_3 = \pm\sqrt{2\nu}$

(iii) if $\lambda \neq 0$ then the orbit coincides with the set defined by the equations above

New representations

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Fix $\eta = (0, -\nu/\lambda, 0, \lambda)$ then we have a choice of maximal subalgebra

$$\mathfrak{h} = \mathrm{span}\{X_2, X_3, X_4\}, \qquad [\mathfrak{h}, \mathfrak{h}] = 0.$$

and the corresponding 1-dim representation

$$\mathcal{X}_{\nu,\lambda}(e^{x_2 X_2 + x_3 X_3 + x_4 X_4}) = e^{i\left(-\frac{\nu}{\lambda} x_2 + \lambda x_4\right)}.$$

We write points on G as

$$g = e^{x_2 X_2 + x_3 X_3 + x_4 X_4} e^{x_1 X_1}.$$

We take a complement $K = \exp(\mathbb{R} X_1)$ and we solve the Master equation

$$e^{\theta X_1} e^{x_2 X_2 + x_3 X_3 + x_4 X_4} e^{x_1 X_1} = \tag{2}$$

$$= e^{x_2 X_2 + (x_3 + \theta x_2) X_3 + (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) X_4} e^{(\theta + x_1) X_1} \tag{3}$$

We deduce that

$$\mathcal{R}_{\nu,\lambda} f(e^{\theta X_1}) = \mathcal{X}_{\nu,\lambda}(e^{x_2 X_2 + (x_3+\theta x_2) X_3 + (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) X_4}) f(e^{(\theta + x_1) X_1})$$

that is in the notation $\widetilde{f}(\theta) = f(e^{\theta X_1})$

$$\mathcal{R}_{\nu,\lambda} \widetilde{f}(\theta) = \exp\left[i \left(-\frac{\nu}{\lambda} x_2 + \lambda (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) \right) \right] \widetilde{f}(\theta + x_1)$$

Differentiating with respect to the x_i at zero we get also the representation of the Lie algebra

$$\begin{aligned} X_1 \widetilde{f} &= \frac{d}{dt} \widetilde{f}, \\ X_2 \widetilde{f} &= i \left(\frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right) \widetilde{f}, \\ X_3 \widetilde{f} &= i \lambda \theta \widetilde{f}, \\ X_4 \widetilde{f} &= i \lambda \widetilde{f} \end{aligned}$$

notice $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$.

In particular notice that

$$X_1 \widetilde{f} = \frac{d}{dt} \widetilde{f},$$
$$X_2 \widetilde{f} = i \left(\frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right) \widetilde{f},$$

Notice that the Laplacian is

$$X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{2} \theta^2 - \frac{\nu}{\lambda} \right)^2$$

This gives the basis of left-invariant vector fields

$$X_1 = \partial_{x_1}, \qquad X_2 = \partial_{x_2} + x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4}$$

$$X_3 = \partial_{x_3} + x_1 \partial_{x_4}, \qquad X_4 = \partial_{x_4}$$

The Engel group

$\mathbb{E} \sim \mathbb{R}^4$

$$X_1 := \partial_1\,, \quad X_2 := \partial_2 + x_1\partial_3 + \frac{x_1^2}{2}\partial_4\,, \quad X_3 := \partial_3 + x_1\partial_4\,, \quad X_4 := \partial_4\,.$$

Group law:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_1 y_2 \\ x_4 y_4 + x_1 y_3 + \frac{x_1^2}{2} y_2 \end{pmatrix}$$

Homogeneous dimension: $Q = \sum_j j \dim \mathfrak{g}_j = 7$

$$\delta_\varepsilon(x_1, x_2, x_3, x_4) = (\varepsilon x_1, \varepsilon x_2, \varepsilon^2 x_3, \varepsilon^3 x_4)$$

In general

$$\Delta := \sum_{X_j \in \mathfrak{g}_1} X_j^2$$

so on $\mathbb{H}$ and $\mathbb{E}$

$$\Delta = X_1^2 + X_2^2 .$$

Homogeneous and inhomogeneous Sobolev spaces are defined by

$$\|u\|_{\dot{H}^s} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2} , \quad \|f\|_{H^s} = \|(\mathrm{Id} - \Delta)^{\frac{s}{2}} u\|_{L^2} .$$

Questions :

- "Space of frequencies" for Fourier Analysis
- Summation formula
- Some applications

The Fourier transform on $\mathbb{E}$

For any integrable function u on $\mathbb{E}$

$$\forall(\nu,\lambda)\in\mathbb{R}\times\mathbb{R}^*\,,\quad \widehat{u}(\nu,\lambda):=\int_{\mathbb{E}} u(x)\mathcal{R}_x^{\nu,\lambda}dx\,,$$

- $\mathcal{R}^{\nu,\lambda}$ the group homomorphism between $\mathbb{E}$ and $\mathcal{U}(L^2(\mathbb{R}))$
- for all x in $\mathbb{E}$ and ϕ in $L^2(\mathbb{R})$, by

$$\mathcal{R}_x^{\nu,\lambda}\phi(\theta):=\exp\left(i\lambda x_4+i\lambda\theta x_3-i\frac{\nu}{\lambda}x_2+i\lambda\frac{\theta^2}{2}x_2\right)\phi(\theta+x_1)\,.$$

- λ is dual to the center X_4 (homogeneous of degree 3)
- ν is representing the operator (homogeneous of degree 4)

$$X_4X_2-\frac{1}{2}X_3^2$$

$$\widehat{-\Delta_{\mathbb{E}} u}(\nu,\lambda) = \widehat{u}(\nu,\lambda)\circ P_{\nu,\lambda}\,, \quad \text{with} \quad P_{\nu,\lambda} := -\frac{d^2}{d\theta^2} + \left(\lambda\frac{\theta^2}{2} - \frac{\nu}{\lambda}\right)^2 .$$

- $\mathrm{Sp}(P_{\nu,\lambda}) = \{E_m(\nu,\lambda), m\in\mathbb{N}\}$ not explicit!
- $\psi_m^{\nu,\lambda}$ the eigenfunctions of $P_{\nu,\lambda}$ associated with $E_m(\nu,\lambda)$.

Homogeneity reduces to the study

$$P_\mu := -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2$$

Setting $T_\alpha\varphi := \alpha^{\frac{1}{2}}\varphi(\alpha\,\cdot)$ and $\mu = \frac{\nu}{|\lambda|^{4/3}}$ then $P_{\nu,\lambda} = |\lambda|^{2/3}T_{|\lambda|^{1/3}}\mathrm{P}_\mu T_{|\lambda|^{-1/3}}$

$$E_m(\nu,\lambda) = |\lambda|^{2/3}\mathrm{E}_m(\mu) \quad \text{and} \quad \psi_m^{\nu,\lambda} = T_{|\lambda|^{1/3}}\varphi_m^\mu$$

The Lai-Robert, Colin de Verdière-Letrouit, Helffer, Helffer-Léautaud...

The frequency space on $\mathbb{E}$

Set $\widehat{x} := (n, m, \nu, \lambda) \in \widehat{\mathbb{E}} = \mathbb{N}^2 \times \mathbb{R} \times \mathbb{R}^*$, and

$$\begin{aligned}\mathcal{F}_{\mathbb{E}}(u)(n, m, \nu, \lambda) &:= \left(\widehat{u}(\lambda)\psi_m^{\nu,\lambda} | \psi_n^{\nu,\lambda}\right)_{L^2(\mathbb{R})} \\ &=: \int_{\mathbb{H}} \mathcal{W}(\widehat{x}, x) u(x) dx\end{aligned}$$

where

$$\mathcal{W}((n, m, \nu, \lambda), x) := e^{i(\lambda x_4 - \frac{\nu}{\lambda} x_2)} \int_{\mathbb{R}} e^{i\lambda(\theta x_3 + \frac{\theta^2}{2} x_2)} \psi_m^{\nu,\lambda}(\theta + x_1) \psi_n^{\nu,\lambda}(\theta) d\theta \,.$$

Then

$$\mathcal{F}_{\mathbb{E}}(-\Delta_{\mathbb{E}} u)(n, m, \nu, \lambda) = \underbrace{E_m(\nu, \lambda)}_{\text{frequency}} \mathcal{F}_{\mathbb{E}}(u)(n, m, \nu, \lambda) \,.$$

Theorem (Bahouri-DB-Gallagher-Léautaud 2023)

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathrm{E}_m(\mu)^\gamma} d\mu < \infty \iff \gamma > 2$$

Moreover assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{5}{2}} dr)$

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} \Phi\big(E_m(\nu,\lambda)\big)\, d\nu d\lambda = \mathrm{C} \int_0^\infty \Phi(r) r^{\frac{5}{2}}\, dr\,.$$

where

$$\mathrm{C} = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{3}{\mathrm{E}_m(\mu)^{\frac{7}{2}}} d\mu\,.$$

- it splits the contribution of the spectrum and the one of F
- it is a summability result for all the spectra

Spectral summability

Theorem (Bahouri-DB-Gallagher-Léautaud 2023)

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathsf{E}_m(\mu)^{\gamma}} d\mu < \infty \iff \gamma > 2$$

Moreover assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{Q-2}{2}} dr)$

$$\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} \Phi\big(E_m(\nu,\lambda)\big)\, d\nu d\lambda = \mathsf{C} \int_0^{\infty} \Phi(r) r^{\frac{Q-2}{2}}\, dr\,.$$

where

$$\mathsf{C} = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{3}{\mathsf{E}_m(\mu)^{\frac{Q}{2}}} d\mu\,.$$

- it splits the contribution of the spectrum and the one of F
- it is a summability result for all the spectra

Formula in simpler situations

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Analogue in Heisenberg $\mathbb{H}^d$

$$\sum_{m\in\mathbb{N}^d}\int_0^\infty \Phi\big(|\lambda|(2|m|+d)\big)|\lambda|^d d\lambda = \left(\sum_{m\in\mathbb{N}^d}\frac{2}{(2|m|+d)^{d+1}}\right)\int_0^\infty \Phi(r)r^d\,dr\,.$$

- notice the Plancherel measure in LHS and $d=(Q-2)/2$, $d+1=Q/2$.
- the convergence in this case is easy

Analogue in $\mathbb{R}^n$ would be the spherical coordinate formula

$$\int_0^\infty \Phi\big(|\xi|^2\big)d\xi = |S^{d-1}|\int_0^\infty \Phi(r)r^{\frac{n-2}{2}}\,dr\,.$$

On the surface measure in Heisenberg

Recall that for θ being the Fourier transform of a radial function

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}^d} \theta(n, n, \lambda) |\lambda|^d \, d\lambda \, .$$

For spherical measures (on sphere of radius R) we want

$$\int_{\widehat{\mathbb{H}}^d} \theta(\widehat{x}) d\widehat{x} = \int_0^\infty \left(\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) \right) dR$$

So we have (change of variable $R^2 = (2|n| + d)|\lambda|$)

$$\int_{\mathbb{S}^R_{\widehat{\mathbb{H}}^d}} \theta(\widehat{x}) d\sigma_R(\widehat{x}) = \sum_{n \in \mathbb{N}^d} \frac{2R^{2d+1}}{(2|n| + d)^{d+1}} \Big(\sum_{\pm} \theta(n, n, \frac{\pm R^2}{2|n| + d}) \Big)$$

Let G be a simply connected nilpotent Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ its dual.

Lemma (Kirillov lemma)

It exists in $\mathfrak{g}^*$

- *a G-invariant subset V (open in the Zariski topology),*
- *a linear submanifold Q of $\mathfrak{g}^*$*

such that all coadjoint orbits lying in V intersect Q at exactly one point.

Elements of $\mathfrak{g} = \mathfrak{g}^{**}$ = linear functions on $\mathfrak{g}^*$.
We choose a basis of $\mathfrak{g}$ by

$$X_1, \dots, X_m, Y_{m+1}, \dots, Y_n$$

such that

- $Y_{m+1}, \dots, Y_n$ will be constant on Q,
- $X_1, \dots, X_m$ as coordinates on Q,

A formula for that

For every point $\eta_X \in Q$ with coordinates $X = (X_1, \ldots, X_m)$ we consider a skew-symmetric matrix A of size $n - m$ with elements

$$B_{ij}(X) = \langle \eta_X, [Y_i, Y_j] \rangle, \qquad i, j = m+1, \ldots, n$$

Theorem

The Plancherel measure is

$$\mu = \sqrt{\det B(X_1, \ldots, X_m)} dX_1 \wedge \ldots \wedge dX_m$$

where $dX_1, \ldots, dX_m, dY_{m+1}, \ldots, dY_n$ *is the dual basis.*

Case of the Heisenberg and Engel → at the blackboard.

It relies on a refined analysis of the spectrum of P_μ: recall

$$\mathsf{P}_\mu = -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2, \quad \mu \in \mathbb{R}$$

This operator appears also in different contexts:

- in quantum mechanics;
- in the study of Schrödinger operators with magnetic fields

It is defined on the domain

$$D(\mathsf{P}_\mu) = \left\{ u \in L^2(\mathbb{R}), \quad -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2 u \in L^2(\mathbb{R}) \right\}, \tag{4}$$

and that its spectrum consists in countably many real eigenvalues $\{\mathsf{E}_m(\mu)\}_{m\in\mathbb{N}}$ of multiplicity 1 and satisfying

$$0 < \mathsf{E}_0(\mu) < \mathsf{E}_1(\mu) < \cdots < \mathsf{E}_m(\mu) < \mathsf{E}_{m+1}(\mu) \to +\infty .$$

On the summability of the spectrum

It relies on a refined analysis of the spectrum of P_μ: recall

$$\mathsf{P}_\mu = -\frac{d^2}{d\theta^2} + \left(\frac{\theta^2}{2} - \mu\right)^2, \quad \mu \in \mathbb{R}$$

The behavior of the potential depends on the sign of the parameter μ:

- It admits a single well when $\mu < 0$
- It admits a double well when $\mu > 0$.
- need combination of microlocal and semiclassical analysis along with known spectral results.

Another observation for later

- it is the square of a polynomial of degree 2 (with no 1st order term)

Discuss (in terms of the parameter γ) convergence of

$$\mathcal{J}_\gamma = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathsf{E}_k(\mu)^\gamma} d\mu = \int_{\mathbb{R}\times\mathbb{N}} \frac{1}{\mathsf{E}_k(\mu)^\gamma} d\mu d\delta(k)\,,$$

where $d\delta(k)$ is the counting measure on $\mathbb{N}$.

- three main regimes to be considered in the analysis of the eigenvalues $\mathsf{E}_k(\mu)$.
- In each of these regimes, we will use a semiclassical reformulation

1. $|\mu| \lesssim 1$ or $|\mu| \ll \sqrt{\mathsf{E}_k(\mu)}$ (classical and perturbative classical regime) that is, μ bounded or going to $\pm\infty$ not too fast,
2. $\mu \to -\infty$ and $\mathsf{E}_k(\mu) \lesssim \mu^2$ (Semiclassical Harmonic oscillator/single well regime),
3. $\mu \to +\infty$ and $\mathsf{E}_k(\mu) \lesssim \mu^2$ (Semiclassical double well regime).

We shall then split $\mathcal{J}_\gamma$ accordingly, for some $\varepsilon > 0$ (small) and $\mu_0 > 0$ (large) as

$$\mathcal{J}_\gamma = \mathcal{J}_\gamma^-(\varepsilon, \mu_0) + \mathcal{J}_\gamma^0(\varepsilon, \mu_0) + \mathcal{J}_\gamma^+(\varepsilon, \mu_0), \quad \text{with} \tag{5}$$

$$\mathcal{J}_\gamma^\bullet(\varepsilon, \mu_0) \stackrel{\text{def}}{=} \int_{\mathcal{E}^\bullet(\varepsilon,\mu_0)} \frac{d\mu d\delta(k)}{\mathsf{E}_k(\mu)^\gamma} \tag{6}$$

$$\mathcal{E}^0(\varepsilon, \mu_0) \stackrel{\text{def}}{=} \{(\mu, k) \in \mathbb{R} \times \mathbb{N}, |\mu| \leq \mu_0 \text{ or } |\mu|^2 \leq \varepsilon^2 \mathsf{E}_k(\mu)\},$$

$$\mathcal{E}^-(\varepsilon, \mu_0) \stackrel{\text{def}}{=} \{(\mu, k) \in \mathbb{R} \times \mathbb{N}, \mu \leq -\mu_0 \text{ and } |\mu|^2 \geq \varepsilon^2 \mathsf{E}_k(\mu)\},$$

$$\mathcal{E}^+(\varepsilon, \mu_0) \stackrel{\text{def}}{=} \{(\mu, k) \in \mathbb{R} \times \mathbb{N}, \mu \geq \mu_0 \text{ and } |\mu|^2 \geq \varepsilon^2 \mathsf{E}_k(\mu)\}.$$

- Note that the (necessary and sufficient) condition $\gamma > 2$ for having $\mathcal{J}_\gamma < \infty$, as stated in Theorem, comes from the third (double well) region

Recover known results

As for instance some Sobolev embeddings. Remember here $Q = 7$.

Proposition

For $s > Q/2$, then $H^s(\mathbb{E})$ embeds in $L^\infty(\mathbb{E})$.

Recall that

$$\|u\|^2_{H^s(\mathbb{E})} := \int_{\widehat{E}} |\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})|^2 (1 + E_m(\nu, \lambda))^s \, d\widehat{x}$$

Start from the inversion formula

$$u(x) = (2\pi)^{-3} \int_{\widehat{E}} \mathcal{W}(\widehat{x}, x^{-1}) \mathcal{F}_{\mathbb{E}}(u)(\widehat{x}) \, d\widehat{x}$$

so that

$$|u(x)| \leq \int_{\widehat{E}} |\mathcal{W}(\widehat{x}, x)| |\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})| \, d\widehat{x}$$

Sobolev embeddings

Multiplying/dividing $(1+E_m(\nu,\lambda))^{s/2}$ and using Cauchy-Schwartz

$$|u(x)| \leq \|u\|_{H^s} \left(\int_{\widehat{E}} |\mathcal{W}(\widehat{x}, x^{-1})|^2 (1+E_m(\nu,\lambda))^{-s} \, d\widehat{x} \right)^{1/2}$$

Since $\sum_{n\in\mathbb{N}} |\mathcal{W}(\widehat{x}, x^{-1})|^2 = 1$ due to the fact that representation are unitary it remains to estimate

$$\left(\sum_{m\in\mathbb{N}} \int_{\mathbb{R}\times\mathbb{R}^*} (1+E_m(\nu,\lambda))^{-s} d\lambda d\nu \right)^{1/2}$$

which thanks to the summation formula is finite for $s > Q/2$

$$\leq \left(\int_0^\infty (1+r)^{-s} r^{\frac{Q-2}{2}} \, dr \right) \left(\sum_{m\in\mathbb{N}} \int_{\mathbb{R}} \frac{1}{\mathrm{E}_m(\mu)^{\frac{Q}{2}}} d\mu \right)$$

An application

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We are interested in the assumptions on Φ giving,

$$\Phi(-\Delta_{\mathbb{E}})u = u \star k_\Phi\,, \quad \text{for all } u \in \mathcal{S}(\mathbb{E})\,, \tag{7}$$

Theorem (BBGL, 23)

Assume $\Phi \in L^1(\mathbb{R}_+, r^{\frac{5}{2}}dr)$. *Then*

- *For any* $u \in \mathcal{S}(\mathbb{E})$, *then* $\Phi(-\Delta_{\mathbb{E}}) : \mathcal{S} \to L^\infty$ *is well-defined by*

$$\Phi(-\Delta_{\mathbb{E}})u \stackrel{\text{def}}{=} \mathcal{F}_{\mathbb{E}}^{-1}\Big(\Phi\big(E_m(\nu,\lambda)\big)\mathcal{F}_{\mathbb{E}}(u)(\widehat{x})\Big)\,.$$

- *Moreover, there is* k_Φ *in* $\mathcal{S}'(\mathbb{E})$ *such that* $\Phi(-\Delta_{\mathbb{E}})u = u \star k_\Phi$ *and we have the continuous map*

$$\begin{aligned} L^1(\mathbb{R}_+, r^{\frac{5}{2}}dr) &\longrightarrow \mathcal{S}'(\mathbb{E}) \\ \Phi &\longmapsto k_\Phi \end{aligned}$$

- Indeed k_Φ belongs to $C^0 \cap L^\infty(\mathbb{E})$ and there holds

$$\|k_\Phi\|_{L^\infty(\mathbb{E})} \leq (2\pi)^{-3}\mathrm{C}\int_0^\infty r^{5/2}|\Phi(r)|dr \quad \text{and}$$
$$k_\Phi(0) = (2\pi)^{-3}\mathrm{C}\int_0^\infty r^{5/2}\Phi(r)dr\,,$$

where

$$\mathrm{C} \stackrel{\text{def}}{=} \sum_{m\in\mathbb{N}}\int_{\mathbb{R}} \frac{3}{\mathrm{E}_m(\mu)^{\frac{7}{2}}}d\mu < \infty\,.$$

- Finally $k_\Phi \in L^2(\mathbb{E})$ if and only if $\Phi \in L^2(\mathbb{R}_+, r^{5/2}dr)$ and there holds

$$\|k_\Phi\|^2_{L^2(\mathbb{E})} = (2\pi)^{-3}\mathrm{C}\int_0^\infty r^{5/2}|\Phi(r)|^2dr\,.$$

Chapter 7: Higher steps groups: some observations and comments

The Goursat group in dim 5

This is the nilpotent Lie group of dimension 5 with a basis of the Lie algebra satisfying

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_1, X_4] = X_5$$

In particular we can consider the (linear) smooth functions $h_1, h_2, h_3, h_4, h_5 : \mathfrak{g}^* \to \mathbb{R}$. To find a basis of the Poisson vector fields it is enough to write down $\vec{h}_i$ for every $i = 1, 2, \ldots, 5$. Using our formulas

$$\vec{h}_1 = h_3 \partial_{h_2} + h_4 \partial_{h_3} + h_5 \partial_{h_4}, \qquad \vec{h}_2 = -h_3 \partial_{h_1}$$

$$\vec{h}_3 = -h_4 \partial_{h_1}, \qquad \vec{h}_4 = -h_5 \partial_{h_1}$$

while h_5 is a casimir since the corresponding vector field X_5 is in the center. There is a second casimir similar to Engel.

Lemma

The function $f = \frac{1}{2} h_4^2 - h_3 h_5$ is a casimir.

Notice that $\{f, h_j\} = 0$ for $j \geq 2$ since the only non zero commutators between the vector fields must contain X_1 and

$$\{f, h_1\} = \{h_4, h_1\}h_4 - \{h_3, h_1\}h_5 = -h_5 h_4 + h_4 h_5 = 0$$

There is a third casimir. It is necessary since the dimension of the leaves should be even, hence in this case is $2 = 5 - 3$.

Lemma

This function is a casimir

$$f = h_2 h_5^2 + \frac{1}{3} h_4^3 - h_3 h_4 h_5$$

All coadjoint orbits are contained in the level sets

$$\begin{cases} h_5 = \lambda, \\ \frac{1}{2} h_4^2 - h_5 h_3 = \nu \\ h_2 h_5^2 + \frac{1}{3} h_4^3 - h_3 h_4 h_5 = \mu \end{cases} \tag{8}$$

The Poisson orbits are NOT NEEDED

$\rightarrow$ It is enough to fix one point.
On the orbit we take $\eta = (0, \mu/\lambda^2, -\nu/\lambda, 0, \lambda)$ then we have a choice of maximal subalgebra

$$\mathfrak{h} = \mathrm{span}\{X_2, X_3, X_4, X_5\}, \qquad [\mathfrak{h}, \mathfrak{h}] = 0.$$

and the corresponding 1-dim representation

$$\mathcal{X}_{\nu,\lambda}(e^{x_2 X_2 + x_3 X_3 + x_4 X_4 + x_5 X_5}) = \exp i \left(\frac{\mu}{\lambda^2} x_2 - \frac{\nu}{\lambda} x_3 + \lambda x_5 \right).$$

We write points on G as

$$g = e^{x_2 X_2 + x_3 X_3 + x_4 X_4 + x_5 X_5} e^{x_1 X_1}.$$

We take a complement $K = \exp(\mathbb{R}X_1)$ and we solve the Master equation

$$e^{\theta X_1} e^{x_2 X_2 + x_3 X_3 + x_4 X_4 + x_5 X_5} e^{x_1 X_1} = \tag{9}$$

$$= e^{x_2 X_2 + (x_3 + \theta x_2) X_3 + (x_4 + \theta x_3 + \frac{\theta^2}{2} x_2) X_4 + (x_5 + \theta x_4 + \frac{\theta^2}{2} x_3 + \frac{\theta^3}{6} x_2) X_5} e^{(\theta + x_1) X_1}$$

We deduce that in the notation $\widetilde{f}(\theta) = f(e^{\theta X_1})$

$$\mathcal{R}_{\mu,\nu,\lambda}\widetilde{f}(\theta) = \exp\left[i\left(\frac{\mu}{\lambda^2}x_2 - \frac{\nu}{\lambda}(x_3 + \theta x_2) + \lambda(x_5 + \theta x_4 + \frac{\theta^2}{2}x_3 + \frac{\theta^3}{6}x_2)\right)\right] \cdot$$

$$\cdot\ \widetilde{f}(\theta + x_1)$$

Differentiating with respect to the x_i at zero we get also the representation of the Lie algebra

$$X_1\widetilde{f} = \frac{d}{d\theta}\widetilde{f},$$
$$X_2\widetilde{f} = i\left(\frac{\mu}{\lambda^2} - \frac{\nu}{\lambda}\theta + \frac{\lambda}{6}\theta^3\right)\widetilde{f},$$
$$X_3\widetilde{f} = i\left(-\frac{\nu}{\lambda} + \frac{\lambda}{2}\theta^2\right)\widetilde{f},$$
$$X_4\widetilde{f} = i\lambda\theta\widetilde{f}, \qquad X_5\widetilde{f} = i\lambda\widetilde{f}$$

notice $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ $[X_1, X_4] = X_5$.

Similar comments to Engel case

Observation

Notice that the Laplacian is

$$X_1^2 + X_2^2 = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{6}\theta^3 - \frac{\nu}{\lambda}\theta + \frac{\mu}{\lambda^2}\right)^2$$

- it is a polynomial of degree = 2(step-1)
- it does not has term on degree step-2
- it is arbitrary!
- oscillator with polynomial potential!

Homogeneity

Replacing $\theta \mapsto \alpha\theta$ (rescaling the functions) gives

$$\frac{1}{\alpha^2}\frac{d^2}{d\theta^2} - \left(\frac{\lambda}{6}\alpha^3\theta^3 - \frac{\nu}{\lambda}\alpha\theta + \frac{\mu}{\lambda^2}\right)^2 = \frac{1}{\alpha^2}\left[\frac{d^2}{d\theta^2} - \left(\frac{\lambda}{6}\alpha^4\theta^3 - \frac{\nu}{\lambda}\alpha^2\theta + \alpha\frac{\mu}{\lambda^2}\right)^2\right]$$

which is

$$\frac{1}{\alpha^2}\left[\frac{d^2}{d\theta^2} - \left(\frac{\alpha^4\lambda}{6}\theta^3 - \frac{\alpha^6\nu}{\alpha^4\lambda}\theta + \frac{\alpha^9\mu}{(\alpha^4\lambda)^2}\right)^2\right]$$

- one recovers the good homogeneity μ, ν, λ of degree $9, 6, 4$.
- These numbers also follows from the polynomial structure of the casimir
- assigning weights $(1, 1, 2, 3, 4)$ to the coordinates $(h_1, \ldots, h_5)$.

More formally

More formally as in the Engel group

$$P_{\alpha^9\mu,\alpha^6\nu,\alpha^4\lambda} = \alpha^2 T_\alpha P_{\mu,\nu,\lambda} T_\alpha^{-1}$$

Normalizing $\alpha^{-4} = \lambda$ we have

$$\frac{d^2}{d\theta^2} - \left(\frac{\theta^3}{6} - \lambda^{-6/4}\nu\theta + \lambda^{-9/4}\mu\right)^2$$

and renaming $a = \lambda^{-6/4}\nu$ and $b = \lambda^{-9/4}\mu$ we "reduce" the study to the following family

$$\frac{d^2}{d\theta^2} - \left(\frac{\theta^3}{6} - a\theta + b\right)^2$$

An algebraic observation

The normalized potential at step s satisfies

$$\frac{d^2}{d\theta^2} - (V_s(\theta))^2$$

with at each s being the primitive $V_{s+1} = \int V_s d\theta$

$$V_2 = \theta$$

$$V_3 = \frac{\theta^2}{2} + a$$

$$V_4 = \frac{\theta^3}{6} + a\theta + b$$

A summation formula on the eigenvalue of the operator ?

Generalization of summability

- The next case would be

$$-\frac{d^2}{d\theta^2} + \left(\frac{\lambda}{6}\theta^3 - \frac{\nu_2}{\lambda}\theta + \frac{\nu_3}{\lambda^2}\right)^2$$

  with ν_3, ν_2, λ homogeneous of degree $9, 6, 4$ respectively.
- Denoting $E_m(\nu_2, \nu_3, \lambda)$ the corresponding eigenvalues we are asking for which γ

$$\sum_{m\in\mathbb{N}} \int \frac{1}{E_m(\nu_2, \nu_3, 1)^\gamma} d\nu < \infty$$

- I do not know!
- relation with the measure of the unit sphere?

The summation formula

We denote $\mathrm{E}_m(\mu,\nu,\lambda)$ is the m-th eigenvalue of $\mathrm{P}_{\mu,\nu,\lambda}$, I assume they exists but I do not know :)
By scaling we have (check the signs) setting $\nu' = \frac{\nu}{\lambda^{6/4}}$ $\mu' = \frac{\mu}{\lambda^{9/4}}$

$$P_{\mu,\nu,\lambda} = |\lambda|^{1/2} T_{|\lambda|^{1/4}} \mathrm{P}_{\mu',\nu',1} T_{|\lambda|^{-1/4}}, \tag{11}$$

$$E_m(\mu,\nu,\lambda) = |\lambda|^{1/2} E_m(\mu',\nu',1), \tag{12}$$

Then we compute

$$\sum_{m\in\mathbb{N}} \int F\big(E_m(\mu,\nu,\lambda)\big)\lambda^{-2} d\lambda d\nu d\mu = \sum_{m\in\mathbb{N}} \int F\left(\lambda^{1/2} E_m\left(\frac{\mu}{\lambda^{9/4}}, \frac{\nu}{\lambda^{6/4}}, 1\right)\right)\lambda^{-2} d\lambda$$

so setting $\nu' = \frac{\nu}{\lambda^{6/4}}$ $\mu' = \frac{\mu}{\lambda^{9/4}}$ (for fixed λ)

$$= \sum_{m\in\mathbb{N}} \int F\left(\lambda^{1/2} E_m\left(\mu',\nu',1\right)\right)\lambda^{\frac{7}{4}} d\lambda d\nu' d\mu'$$

and then setting $r = \lambda^{1/2} E_m(\mu', \nu', 1)$ (for fixed μ', ν') we find

$$dr = \lambda^{-1/2} E_m(\mu', \nu', 1)\, d\lambda$$

so that

$$\lambda^{\frac{7}{4}} d\lambda = \lambda^{\frac{9}{4}} \lambda^{-\frac{1}{2}} d\lambda = \left(\frac{r}{E_m}\right)^{9/2} \frac{dr}{E_m}$$

and

$$\sum_{m \in \mathbb{N}} \int F\big(E_m(\mu, \nu, \lambda)\big) \lambda^{-2} d\lambda d\nu d\mu =$$

$$= \left(\int r^{9/2} F(r) dr\right) \sum_{m \in \mathbb{N}} \int \frac{1}{E_m(\mu, \nu, 1)^{11/2}} d\mu d\nu\,.$$

The filiform/Goursat group in general dimension

This is the nilpotent Lie group of dimension n and step $s = n - 1$ with a basis of the Lie algebra satisfying

$$[X_1, X_i] = X_{i+1}, \qquad i = 2, \ldots, n$$

Example: Filiform/Goursat group (step 4)

$$\overbrace{X_1, X_2}^{\mathfrak{g}_1}, \quad \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}, \quad \overbrace{X_4 = [X_1, X_3]}^{\mathfrak{g}_3}, \quad \overbrace{X_5 = [X_1, X_4]}^{\mathfrak{g}_4}$$

- dimension increase each time by 1
- s step, then $n = s + 1$ dimension
- it is always rank 2
- it is always the same vector field of $\mathfrak{g}_1$ generating the new direction

Generalization of summability

Generalization (only for this class of groups at the moment) as follows :

- the set of parameters will be $s-1=n-2$ dimensional : (ν, λ)
- $\nu = (\nu_2, \ldots, \nu_{s-1})$ a set of $s-2$ parameters
- the Plancherel measure as $f(\lambda) d\lambda d\nu$,
- $Q = 1 + s(s+1)/2$ be the homogeneous dimension

$$\sum_{m\in\mathbb{N}} \int \Phi\big(E_m(\nu,\lambda)\big) f(\lambda) d\lambda d\nu = c_n \left(\int r^{(Q-2)/2} \Phi(r) dr\right) \left(\sum_{m\in\mathbb{N}} \int \frac{1}{E_m(\nu,1)^{Q/2}} d\nu\right) \tag{13}$$

where $E_m(\nu, 1)$ is the family of eigenvalue of a 1D oscillator of the form

$$-\frac{d^2}{d\theta^2} + \big(V_s(\nu;\theta)\big)^2$$

with $V_s(\nu; \cdot)$ polynomial of degree $s-1$ with no term of degree $s-2$

Formula for the V_s

Better to show in dim $n+2$ (or step s, with $s=n+1$)

$$V_s(\nu;\cdot) = \frac{d^2}{d\theta^2} - \left(\frac{\lambda}{n!}\theta^n + \sum_{k=2}^{n}(-1)^{k-1}\frac{\nu_k}{(k-2)!\lambda^{k-1}}\frac{\theta^{n-k}}{n-k!}\right)^2$$

- λ is the dual variable to the center
- the ν represents the casimirs

$$\frac{1}{k}X_2^k + \sum_{\ell=1}^{k-1}(-1)^\ell\frac{(k-2)!}{(k-\ell-1)!}X_1^\ell X_2^{k-\ell-1}X_{\ell+2}$$

- explicit homogeneity

A reference or... an advertisement

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