

Lecture notes on Differential Geometry

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Prelude

Differential Geometry is a vast subject, whose very first goal is to introduce instruments to develop differential and integral calculus on *manifolds*, i.e., smooth spaces which are not necessarily $\mathbb{R}^n$, or embedded in some Euclidean space.

Indeed when starting to study the subject, one often has already encountered smooth subsets of $\mathbb{R}^n$, called *submanifolds*, such as surfaces of $\mathbb{R}^3$ like the sphere, the torus, etc. To say what *smooth* means in this context one uses the notion of smooth function in $\mathbb{R}^n$. In this approach, the regularity properties of the object are *extrinsic*, i.e., related with the ambient space. In geometry it is preferable to have an approach to objects that permits to work with them using *intrinsic* properties, i.e., not referring to (or independent from) external objects or structures.

The goal of the course is to introduce the language of Differential Geometry. For this reason, a major part consists in building correct notions and acquire the right flexibility in order to work with them. A recurrent paradigm in Differential Geometry is the duality between the two viewpoints: *intrinsic* vs *in coordinates*. Several instances of this question are already present in linear algebra courses, comparing presentation of abstract vector spaces and their coordinate version when choosing a basis. For instance one might define what an eigenvalue for a square matrix is and then discover that similar matrices have the same eigenvalues. This might seem at first a beautiful coincidence or only the result of a surprisingly short proof. However, when one understands that the two matrices represent the same endomorphism in different coordinates, the proof is even no more needed! Indeed, the notion of an eigenvalue of an endomorphism *is independent on any choice of basis*, the statement being a coordinate-invariance property, it *must be true!*.

*La géométrie n'est pas vraie, elle est avantageuse.*¹

La Science et l'Hypothèse, 1902

Henri Poincaré, 1854 – 1912

These lecture notes presents some of the material taught by the author in the Master Degree of Mathematics at Università degli Studi di Padova, in the course of Differential Geometry. The reader is invited to have a look to classical books of Differential Geometry [Lee13, Laf15, Boo86, KN96b, KN96a, War83] and Riemannian Geometry. [dC92, GHL04, Pet16, Lee18, Cha06]

Padova, Academic years 2020—2025

¹Transl. Geometry is not true, it is advantageous.

Chapter 1

Smooth manifolds and smooth maps

*La notion générale de variété est assez difficile à définir avec précision.*¹

Leçons sur la Géométrie des espaces de Riemann, 1946

Élie Cartan, 1869 – 1951

The basic idea to build the definition of abstract manifold, is the one of a space that “looks locally like $\mathbb{R}^n$ ”. A naive approach might be dangerous (cf. É. Cartan). In what follows we formalize this idea.

1.1 Topological and smooth manifolds

Definition 1.1. Let M be a topological space, $U \subset M$ open. Let $\varphi : U \rightarrow \varphi(U) = V \subset \mathbb{R}^n$ be an homeomorphism onto an open set V of $\mathbb{R}^n$. The pair (U, φ) is called a *chart*. The inverse $\varphi^{-1} : V \rightarrow U$ is a *local parametrization*.

A chart (U, φ) gives local coordinates to points of U . Namely, for $q \in U \subset M$ we assign n coordinates to it, i.e., $\varphi(q) = (x_1, \dots, x_n) \in \mathbb{R}^n$. Here n is the number of scalar information needed to identify a point on the space M when looking in a (small) region U .

We stress that there is no regularity on the space M up to now. Moreover, notice that the number n is attached to a single chart and might a priori depend on the chart itself.

Remark 1.2. If we have two charts $\varphi_1 : U_1 \rightarrow \varphi(U_1) = V_1 \subset \mathbb{R}^{n_1}$ and $\varphi_2 : U_2 \rightarrow \varphi(U_2) = V_2 \subset \mathbb{R}^{n_2}$, with $U_1 \cap U_2 \neq \emptyset$, we can consider

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

which is an homeomorphism from an open set of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. By a topological argument, it follows that $n_1 = n_2$ (cf. Appendix of this chapter).

Definition 1.3. A *topological manifold* is a topological space together with an *atlas*, i.e., a family $\{(U_i, \varphi_i)\}_{i \in I}$ of charts such that M is covered by the open sets, $M = \cup_{i \in I} U_i$.

By Remark 1.2, when M is connected we have a well-defined notion of dimension of a topological manifold. For simplicity we develop the theory for connected spaces and we will say *M is a n -dimensional topological manifold* when charts take values in $\mathbb{R}^n$.

¹Transl. The general notion of manifold is quite hard to define precisely.

Remark 1.4. As a consequence of the definition, a topological manifold is always locally compact, locally connected and locally path connected (prove this as an exercise!). But this does not impose topological properties at the global level.

Exercise 1.5. Consider the “line with two origins”, that is the topological space given by the quotient $\mathbb{R} \times \{0, 1\}$ with the equivalence relation

$$(x, 0) \sim (x, 1), \quad x \neq 0.$$

It is not difficult to see that every point admits a neighbourhood which is homeomorphic to an open interval. Following Definition 1.3, this is a topological manifold of dimension 1. Nevertheless, the topology of this space is not locally Euclidean, since it is not a Hausdorff space.

Sometimes in the literature, the definition of topological manifold directly include further (global) topological assumptions, which we will always require from now on.

Standing assumptions. From now on we will always assume for a topological manifold M :

(C) M is connected

(H) M is Hausdorff (points are separated)

(SC) M is second countable (existence of a countable basis for the topology of M)

Assumption (C) could be replaced by asking that all connected components of a topological manifold have the same dimension. Since the theory is local, there is no restriction to assume connectedness.

Assumption (H) permits to avoid the example of the “line with two origins”, which is locally Euclidean but not Hausdorff.

Assumption (SC) is related (but not equivalent) to paracompactness, which is needed for partition of unity, for more details cf. Section 1.4.

Smooth manifolds and smooth functions

Let M be a n -dimensional topological manifold. Since M is a topological space, we have a well-defined notion of continuous functions $f : M \rightarrow \mathbb{R}$. Notice that, defining²

$$\widehat{f}_i := f \circ \varphi_i^{-1} : \varphi_i(U_i) \subset \mathbb{R}^n \rightarrow \mathbb{R},$$

we build a collection $\{\widehat{f}_i\}_{i \in I}$ of functions from (open subsets of) $\mathbb{R}^n$ to $\mathbb{R}$ and f is continuous if and only if each $\widehat{f}_i$ is continuous (we leave to the reader to check this fact).

Remark 1.6. We stress that $\widehat{f}_i$ is just f “read in coordinates” on the open set U_i .

The following definition permits to have a similar property for a smooth function,.

Definition 1.7. A *differentiable manifold of class C^k and dimension n* is a n -dimensional topological manifold M together with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ of charts such that all transition functions $\varphi_j \circ \varphi_i^{-1}$ are of class C^k .

²technically one should write $f|_{U_i} \circ \varphi_i^{-1}$

Remark 1.8. Notice that here k may take value in $\{0, 1, 2, \dots, \infty, \omega\}$ where ω stands for analytic. In what follows we focus on C^∞ case and smooth means always C^∞ , unless specified.

With some abuse of notation, we say that a $\mathcal{A}$ is a *smooth atlas* for M , and we often say that M is a smooth manifold, without referring to the pair $(M, \mathcal{A})$.

A smooth atlas is what defines on M a so-called *differentiable structure* or *smooth structure*.

Definition 1.9. Let M be a n -dimensional smooth manifold defined by the atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$. We say that a continuous function $f : M \rightarrow \mathbb{R}$ is smooth if for every $i \in I$ the following function is smooth

$$\widehat{f}_i := f \circ \varphi_i^{-1} : \varphi_i(U_i) \subset \mathbb{R}^n \rightarrow \mathbb{R}.$$

The previous definition is well posed. Indeed, if $U_i \cap U_j \neq \emptyset$, then we have that the *transition function* or *change of charts*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j).$$

is smooth, so that if $\widehat{f}_j$ is C^∞ then we have that

$$\widehat{f}_i = \widehat{f}_j \circ \varphi_j \circ \varphi_i^{-1} \tag{1.1}$$

is smooth as well. (Pay attention: the equality (1.1) is valid, but on which domain?)

The definition of smoothness is local. It is a good idea to get convinced of the following fact.

Exercise 1.10. Let M be a n -dimensional smooth manifold and $\{(U_i, \varphi_i)\}_{i \in I}$ the smooth atlas. Prove that $f : M \rightarrow \mathbb{R}$ is smooth if and only if for every point x there exists $j = j(x) \in I$ such that $x \in U_j$ and $\widehat{f}_j := f \circ \varphi_j^{-1}$ is smooth.

Remark 1.11. It turns out that introducing a smooth structure on M is equivalent to establish which continuous functions from M to $\mathbb{R}$ are smooth (C^∞). Cf. Remark ??.

Equivalent smooth structures

Two smooth atlases $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ and $\mathcal{A}' = \{(U'_j, \varphi'_j)\}_{j \in J}$ are said *compatible* if every change of chart between them is smooth, i.e., for every i, j such that $U_i \cap U'_j \neq \emptyset$ we have that the change of charts

$$\varphi'_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U'_j) \rightarrow \varphi'_j(U_i \cap U'_j)$$

is smooth. This is equivalent to say that the union $\mathcal{A} \cup \mathcal{A}'$ is still a smooth atlas.

We say that a chart is compatible with a smooth atlas $\mathcal{A}$ if adding the chart to the atlas $\mathcal{A}$ one gets a smooth atlas. In this case we say that the chart is a *smooth chart* (for the atlas $\mathcal{A}$).

Notice that if $\mathcal{A}$ and $\mathcal{A}'$ are compatible, they define the same set of smooth functions.

Example 1.12. We give some basic examples, which helps a bit to think about these notions:

1. The Euclidean space $\mathbb{R}^n$ has its canonical differentiable structure covered by a single chart (U, φ) , with $U = \mathbb{R}^n$ and $\varphi(x) = x$, the identity map. With this differentiable structure smooth functions with respect to the smooth structure are just the usual C^∞ functions in $\mathbb{R}^n$, i.e., those for which partial derivatives of every order are continuous.

2. On $M = \mathbb{R}$ we can define a differentiable structure with a single chart $(\mathbb{R}, \psi)$, where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\psi(x) = x^3$. Notice that since there is only one chart, there is nothing to check on the change of charts. In this case the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x$ in coordinates reads

$$\hat{f} = f \circ \psi^{-1}(x) = f(\sqrt[3]{x}) = \sqrt[3]{x},$$

which is not smooth. This differentiable structure is not compatible with the standard one: they define different smooth functions.

3. Now consider on $M = \mathbb{R}$ the atlas $\{(I_{y,r}, \varphi_{y,r})\}_{y \in \mathbb{R}, r > 0}$ where $I_{y,r} = (y - r, y + r)$ is the open interval and $\varphi_{y,r}(x) = x$ for every $x \in I_{y,r}$. We leave to the reader to check that this smooth structure is compatible with the standard structure (use the locality of the notion of smoothness).

A smooth atlas $\mathcal{A}$ is called *maximal* if it is not contained in a strictly larger smooth atlas, i.e., for every atlas $\mathcal{A}'$ compatible with $\mathcal{A}$ and such that $\mathcal{A} \subset \mathcal{A}'$, then $\mathcal{A} = \mathcal{A}'$.

Proposition 1.13. *Let M be a topological manifold. Then*

- (i) *every smooth atlas for M is contained in a unique maximal smooth atlas*
- (ii) *two smooth atlases for M determine the same maximal smooth atlas if and only if they are compatible*

Proof. For the reader interested in more details, we refer to [Lee13]. □

Remark 1.14. The previous results states that without loss of generality, one can assume that a smooth atlas is maximal. In some proofs this is useful and we will freely use this property.

Following the definition, to prove that a set S is a smooth manifold, one should first define a topological structure, and then introduce a smooth structure. Here is a *construction lemma* for smooth manifolds which permits to do both steps together.

Proposition 1.15 (construction lemma). *Let M be a set and $\{U_\alpha\}_{\alpha \in A}$ a collection of subsets together with $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ injective maps such that*

- 1. $\varphi_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$,
- 2. $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$
- 3. $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a diffeomorphism
- 4. M is covered by countably many U_α
- 5. if $x \neq y$ either they belong to the same U_α or to two disjoint ones

Then there exists a unique smooth manifold structure such that (U_α, ϕ_α) are charts.

Proof. Consider the topology generated by $\varphi_\alpha^{-1}(V)$, with V open set in $\mathbb{R}^n$. (Exercise: prove that this is a topology!). Then check that all other conditions included (H) and (SC) are satisfied by construction. For the reader interested in more details, we refer to [Lee13]. □

1.2 Fundamental examples

Let us start with some fundamental observations:

- Every n -dimensional vector space V can be endowed with a canonical structure of smooth n -dimensional manifold. Indeed if one choose a basis $e_1, \dots, e_n$ of V then one can build an atlas with a single chart $\varphi : V \rightarrow \mathbb{R}^n$ mapping v to its coordinates in the chosen basis:

$$\varphi(v) = (x_1, \dots, x_n), \quad \text{if} \quad v = \sum_{i=1}^n x_i e_i.$$

- Every open set U of a n -dim manifold M is also an n -dim manifold. Indeed if $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ is a smooth atlas for M then $\mathcal{A}' = \{(V_i, \psi_i)\}_{i \in I}$ where $V_i = U_i \cap U$ and $\psi_i = \varphi_i|_U$ is a smooth atlas for U .

Exercise 1.16. Prove that if M is an m -dimensional smooth manifold and N is n -dimensional smooth manifold, then $M \times N$ has a natural structure of $(m + n)$ -dimensional manifold. (Here natural means we can build a canonical atlas for $M \times N$ starting from the two atlas of M and N)

Spheres

We want to prove that the unit sphere S^n has the structure of smooth n -dimensional manifold. Let us consider

$$S^n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \|x\|^2 = 1\} \subset \mathbb{R}^{n+1}$$

with the topology induced by $\mathbb{R}^{n+1}$. We will build an atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ with two charts where

$$U_N = S^n \setminus \{N\}, \quad U_S = S^n \setminus \{S\},$$

with N and S the north and south pole respectively. The maps φ_N, φ_S are stereographic projections. For instance let us construct φ_N : given $P = (x_1, \dots, x_{n+1})$ in U_N we consider $\varphi_N(P)$ as the intersection of the segment $\overline{PN}$ with the hyperplane $H = \{x_{n+1} = 0\}$. Identifying H with $\mathbb{R}^n$ (just by removing the last component of the vector) we get a map

$$\varphi_N : U_N \rightarrow \mathbb{R}^n, \quad \varphi_N(P) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n),$$

which is well defined since on U_N we have $x_{n+1} \neq 1$. Similarly one can define and compute

$$\varphi_S : U_S \rightarrow \mathbb{R}^n, \quad \varphi_S(P) = \frac{1}{1 + x_{n+1}}(x_1, \dots, x_n).$$

A computation shows that the inverse has the following form

$$\varphi_N^{-1} : \mathbb{R}^n \rightarrow U_N, \quad \varphi_N^{-1}(u) = \frac{1}{\|u\|^2 + 1}(2u_1, \dots, 2u_n, \|u\|^2 - 1),$$

and similar results hold for φ_S^{-1} . We invite the reader to check the details and to compute that

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad y \mapsto \frac{y}{\|y\|^2}.$$

Notice that with the above identifications $\varphi_N(U_N \cap U_S) = \varphi_S(U_N \cap U_S) = \mathbb{R}^n \setminus \{0\}$.

Remark 1.17. The atlas $\mathcal{A}$ has two charts. This is the minimum. Indeed if we could build an atlas with one chart S^n would be homeomorphic to an open set of $\mathbb{R}^n$, which is not possible since S^n is compact.

Exercise 1.18. Consider the function $f : S^n \rightarrow \mathbb{R}$ given by $f(x) = x_{n+1}$ where $x = (x_1, \dots, x_{n+1}) \in S^n$. Notice that f is the restriction to the set S^n of a smooth function defined on $\mathbb{R}^{n+1}$ to the sphere. Show that

$$f \circ \varphi_N^{-1}(u) = \frac{\|u\|^2 - 1}{\|u\|^2 + 1}$$

and prove that f is smooth with respect to the smooth structure of S^n .

Exercise 1.19. Consider the atlas $\mathcal{A}' = \{(U_i^\pm, \varphi_i^\pm)\}_{i=1, \dots, n+1}$ defined as follows

$$U_i^+ = S^n \cap \{x_i > 0\}, \quad U_i^- = S^n \cap \{x_i < 0\}$$

with charts

$$\varphi_i^+(x_1, \dots, x_{n+1}) = \varphi_i^-(x_1, \dots, x_{n+1}) = (x_1, \dots, \widehat{x_i}, \dots, x_{n+1})$$

Show that $\mathcal{A}$ and $\mathcal{A}'$ are compatible, i.e., every change of charts between elements of the two atlases is smooth.

Projective spaces

The usual definition of $\mathbb{P}^n(\mathbb{R})$ is as the quotient (endowed with quotient topology):

$$\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

with the equivalence relation $x \sim y$ in $\mathbb{R}^{n+1} \setminus \{0\}$ if $x = \lambda y$ for some $\lambda \neq 0$. Denoting

$$x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$$

we denote

$$[x] = [x_0, \dots, x_n] \in \mathbb{P}^n(\mathbb{R})$$

the equivalence class. We define $U_i = \{[x] \in \mathbb{P}^n(\mathbb{R}) : x_i \neq 0\}$ for $i = 0, \dots, n$. It is easy to see that these are well defined and that

$$\mathbb{P}^n(\mathbb{R}) = \bigcup_{i=0}^n U_i.$$

We set $\varphi_i : U_i \rightarrow \mathbb{R}^n$ as follows

$$\varphi_i([x]) = \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \widehat{\frac{x_i}{x_i}}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) \in \mathbb{R}^n$$

where the “hat” stands for having removed that component (so it remains a n -vector). Geometrically $\varphi_i([x])$ are the coordinates of the intersection of the line represented by $[x]$ with the hyperplane $\{x_i = 1\}$ (after removing the i -th coordinate equal to 1). The map φ_i is an homeomorphism onto $\mathbb{R}^n$ with inverse

$$\varphi_i^{-1} : \mathbb{R}^n \rightarrow U_i, \quad y \mapsto [y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n]$$

One can check that (we set for instance³ $j < i$) the change of charts is smooth

$$\varphi_i \circ \varphi_j^{-1} : \mathbb{R}^n \setminus \{y_i = 0\} \rightarrow \mathbb{R}^n \setminus \{y_j = 0\} \rightarrow y \mapsto \left(\frac{y_0}{y_i}, \dots, \frac{y_{j-1}}{y_i}, \frac{1}{y_i}, \frac{y_{j+1}}{y_i}, \dots, \frac{y_{i-1}}{y_i}, \frac{\widehat{y_i}}{y_i}, \frac{y_{i+1}}{y_i}, \dots, \frac{y_n}{y_i} \right)$$

where we should notice that $\mathbb{R}^n \setminus \{y_i = 0\} = \varphi_j(U_i \cap U_j)$, and similarly for the other one. It follows that $\mathbb{P}^n(\mathbb{R})$ is a differentiable manifold of dimension n with this atlas.

The set of affine lines in the plane

Consider the set $AL(\mathbb{R}^2)$ of affine lines in $\mathbb{R}^2$. We identify a line ℓ with its equation $ax + by = c$ (notice that we can multiply both sides by a non zero constant, by describing the same line) and we define the sets

$$U_a = \{\ell \in AL(\mathbb{R}^2) \mid a \neq 0\}, \quad U_b = \{\ell \in AL(\mathbb{R}^2) \mid b \neq 0\},$$

and the charts $\varphi_i : U_i \rightarrow \mathbb{R}^2$ defined for $i = a, b$ by

$$\varphi_a(\ell) = \left(-\frac{b}{a}, \frac{c}{a} \right), \quad \varphi_b(\ell) = \left(-\frac{a}{b}, \frac{c}{b} \right),$$

These definitions are well-posed, i.e., independent on the coefficients a, b, c representing the line. It is not difficult to check that if (u, v) denote the coordinate on $\mathbb{R}^2$ then

$$\varphi_b(U_a \cap U_b) = \{(u, v) \in \mathbb{R}^2 \mid u \neq 0\} = \varphi_a(U_a \cap U_b)$$

and on this set

$$\varphi_b \circ \varphi_a^{-1} : \{(u, v) \in \mathbb{R}^2 \mid u \neq 0\} \rightarrow \{(u, v) \in \mathbb{R}^2 \mid u \neq 0\}, \quad (u, v) \mapsto \left(\frac{1}{u}, -\frac{v}{u} \right)$$

which is smooth on its domain. The atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i=a,b}$ gives the structure of smooth manifold to $AL(\mathbb{R}^2)$.

Remark 1.20. We did not introduce topology on the set $AL(\mathbb{R}^2)$. One possibility is to identify it as a subset of the projective space

$$AL(\mathbb{R}^2) = \mathbb{P}^2(\mathbb{R}) \setminus [0, 0, 1]$$

noting that the triple (a, b, c) describe the same line as $(\lambda a, \lambda b, \lambda c)$ for $\lambda \neq 0$, and that the triples $(0, 0, c)$ do not define any line. Another possibility is to use Proposition 1.15, which we leave to the reader. (Pay attention: the atlas above does not satisfy condition 5. How should one adapt that?)

Remark 1.21. The charts corresponds to writing the line $ax + by = c$ in the “explicit” form

$$y = mx + p, \quad x = ny + q.$$

Indeed if the line is neither horizontal nor vertical (which is the intersection of the charts) we have

$$(m, p) = \left(-\frac{b}{a}, \frac{c}{a} \right), \quad (n, q) = \left(-\frac{a}{b}, \frac{c}{b} \right)$$

and the change of chart is the expression of (m, p) in terms of (n, q) .

Exercise 1.22. Let o be the origin of $\mathbb{R}^2$, for every affine line $\ell \in AL(\mathbb{R}^2)$ define $f(\ell) := \text{dist}(o, \ell)$, where dist denotes the Euclidean distance in $\mathbb{R}^2$ from a point to a line. Prove that the function f is not C^∞ with respect to the smooth structure of $AL(\mathbb{R}^2)$.

³the other case is similar but pay attention to indices if you write it!

Level sets

Let $U \subset \mathbb{R}^n$ be open and $F : U \rightarrow \mathbb{R}^m$ be a smooth map. We say that $x \in U$ is a *critical point* if $DF(x)$ is not surjective, x is a *regular point* otherwise.

A point $y \in \mathbb{R}^m$ is said to be a *regular value* for F if every $x \in F^{-1}(y)$ is a regular point.

Theorem 1.23. *Let $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map such that $y_0 \in \mathbb{R}^m$ is a regular value for F . Then $F^{-1}(y_0)$ is a smooth manifold of dimension $n - m$.*

Proof. It is not restrictive to assume $y_0 = 0$. Fix $x_0 \in U \cap F^{-1}(0)$. Since $\text{rank}(DF(x_0)) = m$ up to reordering variables we can split the space as $x = (x', x'') \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ in such a way that the $m \times m$ block $\partial F / \partial x''$ is invertible. Then applying the classical implicit function theorem there exists a neighborhood of $x_0 = (x'_0, x''_0)$, which we can take of the form $U' \times U''$ with $U' \subset \mathbb{R}^{n-m}$, $U'' \subset \mathbb{R}^m$, and a smooth function $f : U' \rightarrow U''$ such that $x = (x', x'')$ is a solution of

$$F(x) = 0, \quad x \in U' \times U'',$$

if and only if it satisfies

$$x'' = f(x'), \quad x' \in U'.$$

Then defining $\psi : U' \subset \mathbb{R}^{n-m} \rightarrow \mathbb{R}^n$ as $\psi(x') = (x', f(x'))$ we have proved that

$$(U' \times U'') \cap F^{-1}(0) = \psi(U')$$

with ψ invertible onto its image. Denoting $\varphi = \psi^{-1}$ we have that φ defines a chart on the open set $(U' \times U'') \cap F^{-1}(0)$.

We leave to the reader the check that the change of two such charts is smooth. (Notice that the operation of renaming variables (used at the very beginning of the proof) corresponds to apply an invertible linear transformation to the space.) $\square$

We will see how Theorem 1.23 generalizes to manifolds later.

Exercise 1.24. Characterize all $c \in \mathbb{R}$ for which the following subset of $\mathbb{R}^2$ is a smooth manifold

$$x^3 + xy + y^3 = c. \tag{1.2}$$

Exercise 1.25. Discuss whether the following subset of $\mathbb{R}^3$ is a smooth manifold

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x^2 + y^2 - x = 0 \end{cases} \tag{1.3}$$

This subset of $\mathbb{R}^3$ is also known as “Viviani’s window” (but check that on the web only after having tried to solve the exercise by yourself).

Examples in the space of matrices

The space $M_n(\mathbb{R})$ of square matrices with real entries can be endowed with a structure of smooth manifold of dimension n^2 , since it is a vector space. Let us consider some subset of it and show that they are actually smooth manifolds.

- the space $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is an open set within $M_n(\mathbb{R})$, since $\det$ is a continuous function, so it is a smooth manifold of dimension n^2 .
- the space $SL(n)$ is a smooth manifold of dimension $n^2 - 1$. Indeed we have

$$SL(n) = \{A \in GL_n(\mathbb{R}) \mid \det(A) = 1\} = \det^{-1}(1).$$

If we prove that $F := \det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}$ is smooth and the differential is surjective at every point, we are done. It is smooth because it is a polynomial map. Moreover we have

$$DF(I)H = \text{trace}(H) \quad (1.4)$$

Equation (1.4) can be proved by linearity of the differential and applying to a basis. This implies

$$\det(I + H) = 1 + \text{trace}(H) + o(\|H\|). \quad (1.5)$$

Using the properties of the determinant, for $A \in GL_n(\mathbb{R})$ and $H \in M_n(\mathbb{R})$

$$\begin{aligned} \det(A + H) &= \det(A) \det(I + A^{-1}H) \\ &= \det(A) + \det(A) \text{trace}(A^{-1}H) + o(\|H\|) \end{aligned} \quad (1.6)$$

where we used the identity (1.5) and the fact that given an invertible matrix A we have that $o(\|A^{-1}H\|) = o(\|H\|)$. This in turns gives for $A \in GL_n(\mathbb{R})$

$$DF(A)H = \text{trace}(A^{-1}H)$$

Notice that that $DF(A) : M_n(\mathbb{R}) \rightarrow \mathbb{R}$ is surjective as a linear map for every A (this means for every A there exists H such that $\text{trace}(A^{-1}H) \neq 0$, and it is enough to choose $H = A$).

- the space $O_n(\mathbb{R})$ is a smooth manifold of dimension $\frac{1}{2}n(n-1)$. Recall that

$$O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^T A = I\} = F^{-1}(I)$$

where $F : M_n(\mathbb{R}) \rightarrow \text{Sym}_n(\mathbb{R})$ is given by $F(A) = A^T A$. We have restricted the target space in order for F to have surjective differential. In fact, it is easy to see that

$$DF(A)H = H^T A + A^T H$$

where $DF(A) : M_n(\mathbb{R}) \rightarrow \text{Sym}_n(\mathbb{R})$. To prove surjective we have to show that for every $S \in \text{Sym}_n(\mathbb{R})$ we can find H such that $DF(A)H = S$. It is enough to choose $H = \frac{1}{2}AS$. Notice that

$$\dim O_n(\mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Exercise 1.26. What happens in the above argument if one starts with the space of special orthogonal matrices $SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^T A = I, \det = 1\} = O_n(\mathbb{R}) \cap SL_n(\mathbb{R})$?

1.3 Smooth maps

We start by defining what is a smooth map $F : M \rightarrow N$ between two smooth manifolds. We say that (U, φ) is a chart around a point q_0 if U is a neighborhood of q_0 (sometimes, without loss of generality, we can also assume $\varphi(q_0) = 0$).

Definition 1.27. Let M, N be two smooth manifolds and $F : M \rightarrow N$ be continuous. We say that F is smooth at a point $q_0 \in M$ if there exist charts (U, φ) around q_0 and (V, ψ) around $F(q_0)$ such that the following map is of class C^∞

$$\widehat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \rightarrow \psi(V) \subset \mathbb{R}^n.$$

Exercise 1.28. Show that in the previous definition one can replace “if there exist charts (U, φ) around q_0 and (V, ψ) around $F(q_0)$ ” with “for every chart (U, φ) around q_0 and (V, ψ) around $F(q_0)$ ”.

Definition 1.29. Let M, N be two smooth manifolds. A continuous map $F : M \rightarrow N$ is a *smooth diffeomorphism* if F is bijective with F and F^{-1} are both of class C^∞ .

We say that F is a *smooth local diffeomorphism* around $q_0 \in M$ if there exists U neighborhood of q_0 such that $F(U)$ is open in N and $F|_U : U \rightarrow F(U)$ is a diffeomorphism.

Remark 1.30. As before we focus on smooth maps (i.e., of class C^∞) but one can define C^k maps or C^k diffeomorphisms as well.

Remark 1.31. We stress that a diffeomorphism is more than a smooth homeomorphism. The inverse should also be smooth. Recall that a continuous injective map is a homeomorphism onto its image by Theorem 1.61.

The function $x \mapsto x^3$ is an example of a map which is a smooth homeomorphism but not a diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ (when both are endowed with the standard structure, cf. Example 1.34).

Exercise 1.32. Prove that if $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth then $G \circ F$ is smooth. If F and G are diffeomorphisms then $G \circ F$ is also a diffeomorphism. This implies that *being smoothly diffeomorphic* defines an equivalence relation between smooth manifolds.

Example 1.33. 1. The (open) unit ball B is diffeomorphic to $\mathbb{R}^n$ through (write explicitly the inverse of F)

$$F : B \rightarrow \mathbb{R}^n, \quad F(x) = \frac{x}{\sqrt{1 - \|x\|^2}}.$$

2. A trivial but important observation for what comes later is that if (U, φ) is a smooth chart on a smooth manifold then $\varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism when regarded as a map from the smooth manifold U and the open subset $\varphi(U)$ in $\mathbb{R}^n$ with the standard structure.

3. The two smooth manifolds S^1 and $SO(2)$ endowed with their standard smooth structures (defined as in the previous section) are diffeomorphic through the bijective smooth map

$$F : S^1 \rightarrow SO(2), \quad (x, y) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

where $x^2 + y^2 = 1$. It is left to the reader to check that both F and F^{-1} are smooth.

Exercise 1.34. Recall that $(\mathbb{R}, \varphi)$ with the ordinary smooth structure given by the identity chart $\varphi(x) = x$, and $(\mathbb{R}, \psi)$ with the chart $\psi : \mathbb{R} \rightarrow \mathbb{R}$ given by $\psi(x) = x^3$, are not defining the same smooth structure.

Nevertheless these two structures are equivalent up to diffeomorphism. Consider $F : (\mathbb{R}, \varphi) \rightarrow (\mathbb{R}, \psi)$ the map $F(x) = \sqrt[3]{x}$ which is an homeomorphism. Its coordinate presentation $\hat{F} = \psi \circ F \circ \varphi^{-1}(x) = x$ so it is smooth (and the same is true for the inverse).

Remark 1.35. We state here some general fact without proofs.

- If a smooth manifold M has dimension $\dim M \leq 3$, then all smooth structures on M are equivalent up to diffeomorphisms.
- There exists topological manifolds which admit no smooth structure. In particular Michael Freedman in 1982 found an example of a 4-dimensional topological manifold (called E_8 manifold) which can be proved to admit no smooth structure (using Rokhlin's theorem, or Donaldson's theorem).
- On $\mathbb{R}^n$, for $n \neq 4$, all smooth structures are equivalent up to diffeomorphisms. $\mathbb{R}^4$ admits an uncountable number of smooth structures that are non equivalent up to diffeomorphisms. These are called *fake* $\mathbb{R}^4$'s. (DeMichelis Freedman, 1982)
- John Milnor first proves the existence of a smooth structure on S^7 which is not diffeomorphic to the standard one (1956). There are actually 28 non equivalent (1963). These smooth structures are also called *exotic spheres*.

Extension “to infinity” for polynomials

We prove that a complex polynomial, as a map $P : \mathbb{C} \rightarrow \mathbb{C}$, can be naturally extended to a smooth map on the sphere S^2 .

Lemma 1.36. Consider the map $P : \mathbb{C} \rightarrow \mathbb{C}$, $P(z) = a_n z^n + \dots + a_1 z + a_0$, with $a_n \neq 0$ thought as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then set $\bar{P} : S^2 \rightarrow S^2$ defined by

$$\bar{P}(x) = \begin{cases} \varphi_N^{-1} \circ P \circ \varphi_N(x), & x \neq N \\ N, & x = N \end{cases}$$

Then the function $\bar{P}$ is smooth with respect to the C^∞ structure on S^2 given by the standard atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$.

Proof. It is immediate to check that the map $\bar{P}$ is smooth at every point of S^2 different from N . To check that it is smooth at N , since $\bar{P}(N) = N$, we can use chart (U_S, φ_S) both in domain and codomain. We have to check that the map

$$\varphi_S \circ \bar{P} \circ \varphi_S^{-1} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

is smooth at $z = 0$ (since $N = \varphi_S^{-1}(0)$). This is equivalent to show that

$$(\varphi_S \circ \varphi_N^{-1}) \circ P \circ (\varphi_N \circ \varphi_S^{-1}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

is smooth at $z = 0$. We have already computed $\varphi_S \circ \varphi_N^{-1}(z) = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$ (we use the identification in dimension 2) which coincides with its inverse so that the smoothness of the map become the smoothness at $z = 0$ of the map

$$z \mapsto \frac{1}{\overline{P(\bar{z}^{-1})}}$$

Now we have $P(\bar{z}^{-1}) = a_n \bar{z}^{-n} + \dots + a_1 \bar{z}^{-1} + a_0$, so that $\overline{P(\bar{z}^{-1})} = \bar{a}_n z^{-n} + \dots + \bar{a}_1 z^{-1} + \bar{a}_0$ and

$$\frac{1}{\overline{P(\bar{z}^{-1})}} = \frac{1}{\bar{a}_n z^{-n} + \dots + \bar{a}_1 z^{-1} + \bar{a}_0} = \frac{z^n}{\bar{a}_n + \dots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n}$$

which is smooth at $z = 0$ since $\bar{a}_n \neq 0$. □

Lie groups

Another key notion in geometry is the one of Lie group. This is both a group and a manifold and the group law is well-behaved with respect to the smooth structure.

Definition 1.37. We say that G is a *Lie group* if G is a smooth manifold and a group where the maps multiplication and inverse

$$m : G \times G \rightarrow G, \quad m(x, y) = xy$$

$$i : G \rightarrow G, \quad i(x) = x^{-1}$$

are smooth with respect to the differentiable structure.

Indeed one can check that it is enough to verify that the map $G \times G \rightarrow G$ that $(x, y) \mapsto xy^{-1}$ is smooth. Then we can define right and left translations

$$R_g : G \rightarrow G, \quad R_g(h) = hg$$

$$L_g : G \rightarrow G, \quad L_g(h) = gh$$

which are diffeomorphisms since they are smooth (composition of injection in a product and multiplication) with inverses $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$.

→ examples of Lie groups: $S^1, T^n, GL_n(\mathbb{R}), SO_n(\mathbb{R}), SU_n(\mathbb{C})$.

Coverings

Another class of smooth maps that are local diffeomorphisms are smooth coverings.

Definition 1.38. We say that $\pi : \widetilde{M} \rightarrow M$ is a *smooth covering* if π is surjective, of class C^∞ , and for every $q \in M$ has a connected neighborhood U such that for every connected component $\widetilde{U}$ of $\pi^{-1}(U)$ we have $\pi|_{\widetilde{U}} : \widetilde{U} \rightarrow U$ is a diffeomorphism.

Another way to say that: for every $q \in M$ has a connected neighborhood U such that

$$\pi^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$$

where V_α are open set diffeomorphic to U through π . A covering is said to be *universal covering* if $\widetilde{M}$ is simply connected. The universal covering is unique up to diffeomorphism.

Remark 1.39. Notice that the set A depends on q . If the cardinality of A is finite, then it is locally constant (prove it!), so that if M is connected and the cardinality is finite at one point, then it is everywhere constant. In this case we say that the covering has $m = \text{card}(A)$ sheets.

Example 1.40. example $\mathbb{R} \rightarrow S^1$ with $t \mapsto e^{2\pi it}$. Infinitely many sheets

Example 1.41. example $S^n \rightarrow \mathbb{P}^n$ with $x \mapsto [x]$. Two sheets. For $n = 3$ this has a Lie group incarnation: $SU(2) \rightarrow SO(3)$. Consider $S^3 \subset \mathbb{R}^4 \simeq \mathbb{C}^2$ such that $(u, v) \in \mathbb{C}^2$ is in S^3 if $|u|^2 + |v|^2 = 1$. Then

$$S^3 \rightarrow SU(2), \quad (u, v) \mapsto \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

is a bijective map, which is indeed a diffeomorphism with respect to the smooth structures.

The fact that $SO(3) \simeq P^3(\mathbb{R})$ is also diffeomorphic to $P^3(\mathbb{R})$ is interesting. Every matrix A in $SO(3)$ different from the identity represents a rotation around some axis $\mathbb{R}v$ and of some angle $\theta \in [-\pi, \pi]$. Then one can build the map $SO(3) \rightarrow P^3(\mathbb{R})$ given by $A \mapsto \theta v$

Another way to see that S^3 is a group, is to identify it to unit quaternions

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the imaginary elements.

Example 1.42. The covering $\pi : S^1 \rightarrow S^1$ where $z \mapsto z^n$ is a covering with n sheets.

Proper maps

We begin with some definitions

Definition 1.43. A continuous map $f : M \rightarrow N$ between smooth manifold is proper if $f^{-1}(K)$ is compact in M for any K compact in N .

Definition 1.44. A continous (resp. smooth) map $f : M \rightarrow N$ between smooth manifold is a *continuous (resp. smooth) covering map* if for every $y \in N$, there exists an open neighborhood V of y , such that $f^{-1}(V)$ is a union of disjoint open sets in M , each of which is mapped homeomorphically (resp. diffeomorphically) onto V .

Proposition 1.45. Any proper continuous map $f : M \rightarrow N$ between smooth manifold is closed, i.e., $f(C)$ is closed in N for every closed set $C \subset M$.

Proof. Take a sequence y_n in $f(C)$ with $y_n \rightarrow y$. Then $y_n = f(x_n)$ and $x_n \in C$. For n large enough, we can assume $y_n \in K$ where K compact neighborhood of y . So that $x_i \in f^{-1}(K)$ which is compact. hence $x_n \rightarrow x$ for some x (up to subsequences) and f continuous so that $f(x) = y$. $f(C)$ is closed. $\square$

Theorem 1.46. Let M, N be smooth connected manifold and $f : M \rightarrow N$ be a proper local diffeomorphism. Then f is a smooth covering map.

Proof. Since f is a local diffeomorphism, it is open. Since f is proper, it is closed. Hence $f(M)$ is open and closed in N and, by connectedness, f is surjective. Fix $y \in N$. Since f is a local diffeomorphism, each point of $f^{-1}(y)$ has a neighborhood on which f is injective, so $f^{-1}(y)$ is a

discrete set. Since the singleton $\{y\}$ is compact and f is proper, then $f^{-1}(y)$ is compact, hence finite. Set $f^{-1}(y) = \{x_1, \dots, x_k\}$. Fix U_i a neighborhood of x_i where f is a diffeomorphism. It is not restrictive to suppose that $U_i \cap U_j = \emptyset$ for $i \neq j$. Set $V = \bigcap_{i=1}^k f(U_i)$. Since each $f(U_i)$ is a neighborhood of y , V is a neighborhood of y also. By replacing V with the connected component of $V \setminus f(M \setminus \bigcup_i U_i)$ (which is open since f is closed) containing y , we can moreover assume that V is connected and $f^{-1}(V) \subset \bigcup_i U_i$. Hence if one set $\bar{U}_i := U_i \cap f^{-1}(V)$ one can check that $f^{-1}(V) = \bigcup_i \bar{U}_i$, disjoint union of its connected components, and that $f : \bar{U}_i \rightarrow V$ is a diffeomorphism, as desired. $\square$

Once it is known that the map f is a covering map, to show that it is injective one should prove that it is a 1-sheet covering, i.e., the preimage of each point is a single point. The following corollary provides a criterium.

Corollary 1.47. *Under the previous assumptions, if N is simply connected, then f is a diffeomorphism.*

Proof. It is enough to show that the map f is injective. Let $x_1 \neq x_2$ in M such that $f(x_1) = f(x_2)$. Take a continuous curve $\alpha : [0, 1] \rightarrow M$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_1$ homotopic to a point. Its image $\gamma := f \circ \alpha : [0, 1] \rightarrow N$ is a closed loop in N such that $\gamma(0) = \gamma(1) = y$. Since N is simply connected there exists a continuous map

$$\Gamma : [0, 1] \times [0, 1] \rightarrow N$$

such that $\Gamma(0, t) = y$ and $\Gamma(1, t) = \gamma(t)$. For s sufficiently closed to 0 the curve $\gamma_s(t) = \Gamma(s, t)$ stays in the set V where f is a covering hence $f^{-1}(\gamma_s)$ is the union of k closed loops and it should be homotopic to a point. This gives a contradiction. $\square$

1.4 Bump functions, partition of unity and paracompactness

In this section we discuss partition of unity and its relation with paracompactness. For what described in this section is really important to assume the standing assumptions in Section 1.1, in particular the assumption (SC).

We start with the following auxiliary lemma.

Lemma 1.48. *There exists a C^∞ function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$(i) \quad 0 \leq \Phi \leq 1$$

$$(ii) \quad \Phi \equiv 1 \text{ on } B(0, 1/2) = \{x \in \mathbb{R}^n \mid \|x\| < 1/2\},$$

$$(iii) \quad \Phi \equiv 0 \text{ on } B(0, 1) = \{x \in \mathbb{R}^n \mid \|x\| > 1\}.$$

Proof. We start by recalling that the following function $h : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^∞

$$h(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ e^{-1/t}, & \text{if } t > 0. \end{cases}$$

It is enough then to define $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ as follows

$$\Phi(x) = \frac{h(1 - \|x\|^2)}{h(1 - \|x\|^2) + h(\|x\|^2 - 1/4)},$$

It is easily seen that Φ is C^∞ and satisfies the properties above. $\square$

The function Φ is what is called a *bump function*. To build bump functions on smooth manifolds we first need the following observation.

Proposition 1.49. *Every topological manifold admits a countable cover of precompact coordinate balls.*

Proof. Assume M is covered by one chart $\varphi : M \rightarrow \varphi(M) = V \subset \mathbb{R}^n$. Consider the set $\mathcal{B}$ of balls B in $\mathbb{R}^n$ with rational center and rational radius, whose closure is contained in V . This is a countable basis for the topology of V . Then consider the topology on M generated by the open set $\varphi^{-1}(B)$ for $B \in \mathcal{B}$. The general case follows using (SC) assumption. (Details left to the reader). $\square$

Proposition 1.50. *Let $K \subset V \subset M$ with K compact and V open subset of a smooth manifold M . There exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $f \equiv 1$ on K and $f \equiv 0$ on $M \setminus V$.*

Proof. Use an open cover with values in precompact balls. More precisely for every q in K take a chart $\varphi_q : U_q \rightarrow B(0, 2)$ with $\overline{U}_q \subset V$. By compactness take a finite number (relabelled $\varphi_i : U_i \rightarrow B(0, 2)$ for $i = 1, \dots, N$) of such charts such that

$$K \subset \bigcup_{i=1}^N \varphi_i^{-1}(B(0, 1/2)) \subset \bigcup_{i=1}^N U_i \subset V$$

Then set

$$f_i : U_i \rightarrow \mathbb{R}, \quad f_i(q) = \begin{cases} \Phi(\varphi_i(q)), & q \in U_i, \\ 0, & q \notin U_i. \end{cases}$$

Notice that f_i is smooth since $\varphi_i^{-1}(B_{1/2}) \subset U_i$. We define $f : M \rightarrow \mathbb{R}$ as

$$f(q) = 1 - \prod_{i=1}^N (1 - f_i(q))$$

which is clearly smooth. Moreover if $q \in K$ then at least one f_j is 1, hence $f(q) = 1$. If $q \notin V$ then all f_j are 0 hence $f(q) = 0$. $\square$

Of course one can exchange zero with one and reduce the compact to a point.

Corollary 1.51. *For every $q \in M$ and U open neighborhood of q , there exists a smooth function such that $f(q) = 0$ and $f \equiv 1$ outside U .*

Another such a property. Here by f smooth on a compact set means f coincides with the restriction to K of a smooth function defined on some open neighborhood of K .

Proposition 1.52. *Let $K \subset V \subset M$ with K compact and V open subset of a smooth manifold M . If $g : K \rightarrow \mathbb{R}^N$ is smooth then there exists a smooth extension $\hat{g} : M \rightarrow \mathbb{R}^N$ with $\text{supp } \hat{g} \subset V$.*

Recall that for $f \in C^\infty(M, \mathbb{R}^N)$ we denote $\text{supp } f = \overline{\{x \in M \mid f(x) \neq 0\}}$.

Proof. Let U be the neigh of K where there exists the extension so that $g = \tilde{g}|_K$ for $\tilde{g} : U \rightarrow \mathbb{R}^N$. Set $W = U \cap V$ which is still a neighborhood of K and apply the proposition to the pair $K \subset W$, i.e., find a smooth function f such that $f \equiv 1$ on K and $f \equiv 0$ outside W . Then consider:

$$\hat{g}(q) = \begin{cases} f(q)\tilde{g}(q), & q \in W, \\ 0, & q \notin W. \end{cases}$$

which satisfies the requirements. $\square$

Paracompactness and partition of unity

Given an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M we say that an open cover $\mathcal{V} = \{V_j\}_{j \in J}$ is a *refinement* of $\mathcal{U}$ if for every $j \in J$ one has $V_j \subset U_i$ for some $i \in I$.

An open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M is *locally finite* if each point $q \in M$ has a neighborhood V that $V \cap U_i \neq \emptyset$ only for a finite number of $i \in I$.

Definition 1.53. A smooth manifold is said to be *paracompact* if from every open cover we can extract a refinement which is locally finite.

Proposition 1.54. *Every smooth manifold is paracompact.*

Proof. For a complete proof we refer to [Lee13]. Here we only prove a key step which is given in the following Lemma. $\square$

Lemma 1.55. *Every topological manifold has a countable and locally finite open cover by precompact sets.*

Sketch of the proof. Start from $\{B_i\}_{i \in \mathbb{N}}$ countable open cover by precompact balls. Make it locally finite as follows: set $U_1 = B_1$ and then define iteratively U_j as follows: if U_j is defined then set m_j such that

$$\overline{U_j} \subset B_1 \cup \dots \cup B_{m_j}$$

which is finite by compactness. Set $U_{j+1} = B_1 \cup \dots \cup B_{m_j}$. Notice that necessarily $m_{j+1} > m_j$ in this construction. Then define $V_j = U_{j+2} \setminus U_j$. This is locally finite. $\square$

We stress that here the second countable assumption (SC) we added is crucial! This proves the existence of the partition of unity.

Definition 1.56. A *partition of unity* is a family of smooth functions $\{\psi_i\}_{i \in I}$ such that

- (i) $0 \leq \psi_i \leq 1$
- (ii) $\{\text{supp } \psi_i\}_{i \in I}$ is a locally finite cover of M
- (iii) $\sum_{i \in I} \psi_i = 1$

A partition of unity $\{\psi_i\}_{i \in I}$ is subordinated to an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ if $\text{supp } \psi_i \subset U_i$ for every $i \in I$. Notice that the sum in (iii) is finite at every point by condition (ii).

Theorem 1.57. *Every open cover $\mathcal{U} = \{U_i\}_{i \in I}$ admits a partition of unity $\{\psi_i\}_{i \in I}$ which is subordinated to $\mathcal{U}$.*

Proof. See [Lee13]. □

Notice that the initial open cover in the previous theorem is not necessarily locally finite. Thanks to partition of unity it is very easy to prove the existence of bump function even for closed sets (not only compact as in Proposition 1.50).

Proposition 1.58. *Let $C \subset V \subset M$ with C closed and V open subset of a smooth manifold M . There exists a smooth function $a : M \rightarrow \mathbb{R}$ such that $f \equiv 1$ on C and $f \equiv 0$ on $M \setminus V$.*

Proof. Let us consider the open cover $U_1 = V$ and $U_2 = M \setminus C$. Let ψ_1, ψ_2 a partition of unity subordinated to the open cover. We have $\psi_2 = 0$ on C i.e., outside $U_2 = M \setminus C$. Then $\psi_1 = 1$ on C and has support inside $U_1 = V$, i.e., satisfies the requirements. □

Similarly, but with little more work, one proves the corresponding extension lemma.

Proposition 1.59. *Let $C \subset V \subset M$ with C closed and V open subset of a smooth manifold M . If $f : C \rightarrow \mathbb{R}^N$ is smooth then there exists a smooth extension $\hat{f} : M \rightarrow \mathbb{R}^N$ with $\text{supp } \hat{f} \subset V$.*

Proof. See [Lee13]. □

To end the section let us prove another application: every noncompact manifold can be approximated via a family of compact subsets. Recall that an *exhaustion function* for a smooth manifold is a $f \in C^\infty(M)$ such that the sublevel set $\{f \leq c\}$ is compact for every $c \in \mathbb{R}$.

Proposition 1.60. *Every smooth manifold admit a smooth exhaustion function.*

Proof. Since the sublevel sets $\{f \leq c\}$ are closed and monotone with respect to $c \in \mathbb{R}$, it is enough to prove that $\{f \leq N\}$ is compact for every $N \in \mathbb{N}$.

Consider a countable open cover by precompact open sets $\{V_j\}_{j \in \mathbb{N}}$ and a partition of unity $\{\psi_j\}_{j \in \mathbb{N}}$ subordinated to it. Define

$$f(q) := \sum_{j=1}^{\infty} j\psi_j(q)$$

Notice that the sum is finite at every point since $\{\psi_j\}_{j \in \mathbb{N}}$ is a partition of unity (the supports define a locally finite cover). Fix $N \in \mathbb{N}$ and notice that if $q \notin \cup_{j=1}^N \overline{V}_j$ then $\psi_j(q) = 0$ for $1 \leq j \leq N$. We can estimate

$$f(q) = \sum_{j=N+1}^{\infty} j\psi_j(q) > N \sum_{j=N+1}^{\infty} \psi_j(q) = N \sum_{j=1}^{\infty} \psi_j(q) = N$$

It follows that the sublevel set satisfies for every $N \in \mathbb{N}$.

$$\{f \leq N\} \subset \cup_{j=1}^N \overline{V}_j$$

which is a closed set contained in a compact set, hence compact. □

1.5 Appendix: Brower invariance of domain

The following theorem is purely topological.

Theorem 1.61 (invariance of domain, Brower 1912). *Let $U \subset \mathbb{R}^n$ be open and $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ injective and continuous. Then $f(U)$ is open and f is a homeomorphism between U and $f(U)$.*

The proof of this theorem is not trivial.⁴ It has the following

Corollary 1.62. *A non-empty open subset of $\mathbb{R}^n$ can be homeomorphic to a non-empty open subset of $\mathbb{R}^m$ only if $n = m$.*

Proof. It is not restrictive to assume $m < n$ and consider a homeomorphism $h : U \subset \mathbb{R}^n \rightarrow V = h(U) \subset \mathbb{R}^m$. Then composing h with $i : \mathbb{R}^m \rightarrow \mathbb{R}^n$ the canonical injection. We have that $i \circ h$ is continuous and injective but it is not an open map since $i \circ h(U) = i(V) \subset \mathbb{R}^m \times \{0\}$. This is in contradiction with Theorem 1.61. $\square$

This corollary is intuitively obvious, but note that topological intuition is not always rigorous. For instance, it is intuitively plausible that there should be no continuous surjection from $\mathbb{R}^m$ to $\mathbb{R}^n$ for $n > m$, but such surjections always exist, thanks to variants of the Peano curve construction.

1.6 More examples

Grassmannians

Let us consider the Grassmannian of k planes in a n -dim vector space V .

$$G_k(V) = \{W \subset V \mid \dim(W) = k\}.$$

We want to show this is a $k(n - k)$ dimensional manifold. The idea behind the construction is that k -dimensional subspaces can be described as the graph of linear maps $\mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$, hence parametrized by $k(n - k)$ matrices.

Charts are build as follows: for every $U \subset V$, with $\dim U = n - k$, we consider

$$U^\natural = \{W \in G_k(V) \mid W \oplus U = V\} = \{W \in G_k(V) \mid W \cap U = \{0\}\}.$$

Of course $G_k(V)$ is covered by such open sets

$$G_k(V) = \bigcup_U U^\natural,$$

since every k -dimensional set is transversal to some $(n - k)$ -dim one. To define charts let us fix an element $Z \in U^\natural$, which will play the role of the origin of the chart. We recall the following lemma of linear algebra.

Lemma 1.63. *Every $W \in U^\natural$ is the graph of a unique linear map $A_W : Z \rightarrow U$.*

⁴It is strongly related to Brower fixed point theorem. For an interesting discussion on this topic one can see for instance the Terence Tao's blog <https://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/>

Choosing basis for Z and U we can easily build a *bijective* map

$$\varphi_U : U^\perp \rightarrow M_{k,n-k}(\mathbb{R}) \simeq \mathbb{R}^{k(n-k)}, \quad \varphi_U(W) = A_W.$$

Notice that $\varphi_U(Z) = 0$ (i.e., the subspace Z we have chosen is the origin in the coordinate chart).

Exercise 1.64. Let U_1, U_2 be two $(n - k)$ dimensional subspaces and fix Z_i in such a way that $Z_i \oplus U_i = V$ for $i = 1, 2$. Compute

$$\varphi_{U_1} \circ \varphi_{U_2}^{-1} : \mathbb{R}^{k(n-k)} \rightarrow \mathbb{R}^{k(n-k)}$$

and prove it is a diffeomorphism. Use the construction lemma (Proposition 1.15) to complete the proof that this gives $G_k(V)$ the structure of a manifold.

Chapter 2

Tangent space and differentials

The formal definitions in the preceding section do not help much to an understanding of what the notion of a vector really is.

“The mathematical notion of a vector”¹, 1923

Sir Arthur Stanley Eddington, 1882 – 1944

For submanifolds of $\mathbb{R}^n$ the tangent space is naturally defined as a vector space of the $\mathbb{R}^n$ itself, for smooth manifolds we cannot use the ambient space and we have to define an abstract notion. There are several approaches to define tangent space: historically one of the first is the characterisation we will see in Exercise 2.13, which is by the way a non evident definition of (tangent) vector at a first sight (cf. Sir Arthur Stanley Eddington).

2.1 Tangent space

A natural approach to tangent space is through (equivalence classes of) smooth curves. This has some drawbacks in particular when proving that the tangent space is a vector space and prove that the differential is linear.

Here we'll use the approach through derivations (defined on $C^\infty(M)$ and not on germs, exploiting bump functions). The reader is invited to have a look to the different presentations one can find in different books.

Definition 2.1. Let M be a smooth manifold. A *tangent vector* v at a point q in M is a linear map $v : C^\infty(M) \rightarrow \mathbb{R}$ which is a derivation, i.e., v satisfies for $f, g \in C^\infty(M)$

$$v(fg) = v(f)g(q) + f(q)v(g)$$

The tangent space T_qM is the set of all tangent vectors at q .

Notice that T_qM has the natural structure of vector space. This is not related to any external structure, which has an advantage: no verification is needed. Notice also that, at the moment, it is not at all clear what is the dimension of T_qM .

Lemma 2.2. *We have $v(c) = 0$ for every constant function c , moreover we have that $v(fg) = 0$ whenever $f(q) = g(q) = 0$.*

¹a section in the book “The Mathematical Theory of Relativity” by the same author

Proof. The second is trivial by definition of derivation. The first proof is also easy since

$$v(1) = v(1 \cdot 1) = 2v(1).$$

Hence $v(1) = 0$ and by linearity $v(c) = cv(1) = 0$. $\square$

Let us prove locality of tangent vectors. (Here it is crucial to work in the C^∞ category and not in the analytic one)

Proposition 2.3. *Let f and g agree on a neighborhood U of a point q . Then $v(f) = v(g)$ for every $v \in T_q M$.*

Proof. It is enough to show that if $h = 0$ on a neighborhood U of q then $v(h) = 0$. Indeed by linearity then we obtain the statement by choosing $h = f - g$.

Let ψ be a bump function such that $\psi = 0$ at q and $\psi \equiv 1$ on $\text{supp } h$, whose existence is guaranteed by Proposition 1.58 (choose $C = \text{supp } h$ and $V = M \setminus \{q\}$, clearly V open and $C \subset V$). Then $h = \psi h$ by construction (indeed, where h is non zero, then ψ is 1) and we have

$$v(h) = v(\psi h) = v(\psi)h(q) + \psi(q)v(h) = 0.$$

$\square$

2.2 Differential

Next we move to the definition of differential. This is a linear map between vector spaces. If one thinks in terms of smooth curves the following definition is natural, otherwise it is a little bit abstract.

Definition 2.4. Let $F : M \rightarrow N$ be a smooth map between smooth manifolds and $q \in M$. We define the *differential* $F_* : T_q M \rightarrow T_{F(q)} N$ as follows: for every $v \in T_q M$ we define $F_* v \in T_{F(q)} N$

$$(F_* v)(g) = v(g \circ F)$$

Easy properties of the differential follow from this definition:

Proposition 2.5. *Let $F : M \rightarrow N$ be a smooth map between smooth manifolds and $q \in M$*

- (i) $F_* : T_q M \rightarrow T_{F(q)} N$ is a linear map
- (ii) $(G \circ F)_* = G_* \circ F_*$
- (iii) $(\text{id}_M)_* = \text{id}_{T_q M}$
- (iv) if F is a diffeomorphism then F_* is an isomorphism

Proof. The proofs are easy: (i) is linearity of derivations : for every f

$$F_*(v + w)(f) = (v + w)(f \circ F) = v(f \circ F) + w(f \circ F) = F_*(v)(f) + F_*(w)(f)$$

(ii) is the definition again

$$((G \circ F)_* v)f = v(f \circ (G \circ F)) = v((f \circ G) \circ F) = (F_* v)(f \circ G) = (G_* F_* v)f$$

(iii) follows by definition. (iv) follows by $G = F^{-1}$ and (iii). $\square$

Combine the locality property with the previous Proposition 2.5 to prove.

Exercise 2.6. Let $F : M \rightarrow N$ is a local diffeo around $q \in M$. Then $F_* : T_q M \rightarrow T_{F(q)} N$ is an isomorphism. In particular show that if $U \subset M$ is open and $i : U \rightarrow M$ is the inclusion then $i_* : T_q U \rightarrow T_q M$ is an isomorphism.

Consider now a chart (U, φ) on M , let us write $\varphi(q) = (x_1, \dots, x_n)$. Sometimes we also write $(U, \{x_i\})$ to denote the chart. This defines n derivations associated with the charts² which we will denote

$$\left. \frac{\partial}{\partial x_1} \right|_q, \dots, \left. \frac{\partial}{\partial x_n} \right|_q$$

acting as follows: for any $f : U \rightarrow \mathbb{R}$ we set

$$\left. \frac{\partial}{\partial x_j} \right|_q f := \left. \frac{\partial(f \circ \varphi^{-1})}{\partial x_j} \right|_{\varphi(q)} = \left. \frac{\partial \hat{f}}{\partial x_j} \right|_{\varphi(q)} \quad (2.1)$$

where $\hat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f and the derivatives on the right are in $\mathbb{R}^n$.

Remark 2.7. This notation will make confusion at first. Do not be scared, it is a good idea to keep this notation because when you will be used to that you will just identify the two objects!

Proposition 2.8. Let M be a smooth manifold and $q \in M$. If $(U, \{x_i\})$ is a chart near q , then the coordinate vectors

$$\left. \frac{\partial}{\partial x_1} \right|_q, \dots, \left. \frac{\partial}{\partial x_n} \right|_q$$

define a basis for $T_q M$. In particular $\dim T_q M = \dim M$.

Proof. To prove this, we notice that by definition we have

$$\left. \frac{\partial}{\partial x_j} \right|_q = \varphi_*^{-1} \left(\left. \frac{\partial}{\partial x_j} \right|_{\varphi(q)} \right)$$

where the derivative in the right hand side is the usual derivative with respect to the j -th variable in $\mathbb{R}^n$. Since φ_*^{-1} is an isomorphism (by construction φ is a smooth local diffeo), it is enough to prove that the family $\{\left. \frac{\partial}{\partial x_j} \right|_{\varphi(q)}\}$ is a basis for $T_{\varphi(q)} \mathbb{R}^n$, which is proved in the following lemma. $\square$

Lemma 2.9. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be smooth, then there exist smooth functions $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, n$ such that $g_i(0) = 0$ and

$$f(x) = f(0) + \sum_{i=1}^n \left. \frac{\partial f}{\partial x_i} \right|_0 x_i + \sum_{i=1}^n g_i(x) x_i.$$

²meaning that these *depends* on the choice of the charts

Proof. We have

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{ds} f(sx) ds = \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(sx) ds \\ &= \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i=1}^n x_i \int_0^1 \left(\frac{\partial f}{\partial x_i}(sx) - \frac{\partial f}{\partial x_i}(0) \right) ds. \end{aligned}$$

and it is sufficient to set $g_i(x) = \int_0^1 \left(\frac{\partial f}{\partial x_i}(sx) - \frac{\partial f}{\partial x_i}(0) \right) ds$. □

Lemma 2.10. *Let $x_0 \in \mathbb{R}^n$. Then $\frac{\partial}{\partial x_1}|_{x_0}, \dots, \frac{\partial}{\partial x_n}|_{x_0}$ is a basis for $T_{x_0}\mathbb{R}^n$.*

Proof. We have to prove that the space of derivations at x_0 in $\mathbb{R}^n$ is generated by $\frac{\partial}{\partial x_1}|_{x_0}, \dots, \frac{\partial}{\partial x_n}|_{x_0}$ and that these derivations are independent. It is enough to prove this statement for $x_0 = 0$.

(a). Assume that $v = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_0 = 0$. This means that for every smooth function $f \in C^\infty(\mathbb{R}^n)$

$$0 = vf = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} \Big|_0 f$$

By choosing $f = x_j$ we get $\alpha_j = 0$ for every $j = 1, \dots, n$, hence the derivations are independent.

(b). Now we want to show that every derivation v at 0 can be written as $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_0$ for some α_i . Let us choose $\alpha_j := v(x_j)$ and set $X_v = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_0$ for this choice. We show $v = X_v$. Using Lemma 2.9

$$f(x) = f(0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) x_i + \sum_{i=1}^n g_i(x) x_i,$$

for some smooth functions g_i such that $g_i(0) = 0$. Then

$$v(f) = 0 + \sum_{i=1}^n v(x_i) \frac{\partial f}{\partial x_i}(0) + 0 = \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(0) = X_v(f). \quad \square$$

The differential in coordinates

Let $F : M \rightarrow N$ be smooth, let us compute the representation of F_* in coordinates. Recall that if we have charts (U, φ) and (V, ψ) around q and $F(q)$ respectively then F in coordinates read

$$\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \rightarrow \psi(V) \subset \mathbb{R}^m$$

Denoting $(x_1, \dots, x_n)$ and $(y_1, \dots, y_m)$ the corresponding coordinates on M and N we want to understand what is the matrix $A = (a_{ij})$ such that

$$F_* \frac{\partial}{\partial x_i} \Big|_q = \sum_{j=1}^m a_{ij} \frac{\partial}{\partial y_j} \Big|_{F(q)}$$

It is not difficult: we have to sit down, apply to a function f the left hand side, apply definitions and see what happens:

$$\begin{aligned} \left(F_* \frac{\partial}{\partial x_i} \Big|_q \right) f &= \frac{\partial}{\partial x_i} \Big|_q (f \circ F) \\ &= \frac{\partial}{\partial x_i} \Big|_{\varphi(q)} [f \circ F \circ \varphi^{-1}] = \frac{\partial}{\partial x_i} \Big|_{\varphi(q)} [f \circ \psi^{-1} \circ \psi \circ F \circ \varphi^{-1}] \\ &= \frac{\partial}{\partial x_i} \Big|_{\varphi(q)} [\widehat{f} \circ \widehat{F}] \end{aligned}$$

hence applying the chain rule in $\mathbb{R}^n$ we get

$$\begin{aligned} \left(F_* \frac{\partial}{\partial x_i} \Big|_q \right) f &= \sum_{j=1}^m \frac{\partial \widehat{f}}{\partial y_j} \Big|_{\widehat{F}(\varphi(q))} \frac{\partial \widehat{F}_j}{\partial x_i} \Big|_{\varphi(q)} = \sum_{j=1}^m \frac{\partial \widehat{f}}{\partial y_j} \Big|_{\psi(F(q))} \frac{\partial \widehat{F}_j}{\partial x_i} \Big|_{\varphi(q)} \\ &= \sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i} \Big|_{\varphi(q)} \frac{\partial}{\partial y_j} \Big|_{F(q)} f \end{aligned}$$

where we have used $\psi \circ F = \widehat{F} \circ \varphi$ (pay attention to $\frac{\partial}{\partial x_i}$ when they act on $\mathbb{R}^n$ or M).

Proposition 2.11. *Let $F : M \rightarrow N$ be smooth map between manifolds and $q \in M$. Given charts (U, φ) and (V, ψ) around q and $F(q)$ respectively, then we have that*

$$F_* \frac{\partial}{\partial x_i} \Big|_q = \sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i} \Big|_q \frac{\partial}{\partial y_j} \Big|_{F(q)}$$

In particular the matrix representing the differential F_* in coordinates is the Jacobian matrix $D\widehat{F}(\varphi(q))$ of the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$

$$D\widehat{F} = \begin{pmatrix} \frac{\partial \widehat{F}_1}{\partial x_1} & \cdots & \frac{\partial \widehat{F}_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \widehat{F}_m}{\partial x_1} & \cdots & \frac{\partial \widehat{F}_m}{\partial x_n} \end{pmatrix}$$

Remark 2.12. There are several other common notations for the differential of a map at q

$$F_{*,q}, F'(q), DF(q), D_q F, d_q F, dF(q), T_q F$$

Exercise 2.13 (Change of coordinates for vectors). Given two coordinate sets $(U, \{x_i\})$ and $(U', \{x'_i\})$ with $q \in U \cap U'$ show that if $v \in T_q M$ writes as

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q = \sum_{j=1}^n v'_j \frac{\partial}{\partial x'_j} \Big|_q.$$

then

$$v'_j = \sum_{i=1}^n \frac{\partial x'_j}{\partial x_i} v_i.$$

(Here, if φ, φ' denotes the coordinate maps, the quantity $\frac{\partial x'_j}{\partial x_i}$ is the jacobian of $\varphi' \circ \varphi^{-1}$).

Use what said for $\mathbb{R}^n$ to prove the following.

Exercise 2.14. Let V be a vector space and $v \in V$. Prove that $T_v V$ is canonically isomorphic to V (i.e., the isomorphism is independent on the choice of a basis in V).

2.3 A bit of geometry: tangent vector and curves

Now that we have a differential, given a smooth curve $\gamma : I \rightarrow M$ defined on an open interval and $t_0 \in I$ we can define simply

$$\dot{\gamma}(t_0) := \gamma_* \left(\frac{d}{dt} \Big|_{t=t_0} \right) \in T_{\gamma(t_0)} M$$

where $d/dt|_{t=t_0}$ is the standard basis for the one-dimensional space $T_{t_0} I \simeq T_{t_0} \mathbb{R}$. By definition

$$\dot{\gamma}(t_0) f = \frac{d}{dt} \Big|_{t=t_0} (f \circ \gamma)$$

which is indeed coherent with the intuitive notion of differentiating along a curve.

Remark 2.15. Notice that if $\gamma(t) = \varphi^{-1}(\hat{\gamma}(t))$ for some chart (U, φ) centered at $q = \gamma(0)$ we have

$$\dot{\gamma}(t_0) f = \frac{d}{dt} \Big|_{t=t_0} (f \circ \gamma) = \frac{d}{dt} \Big|_{t=t_0} (\hat{f} \circ \hat{\gamma}) = \sum_{i=1}^n \frac{\partial \hat{f}}{\partial x_i} \Big|_0 \hat{\gamma}'_i(t_0) = \sum_{i=1}^n \hat{\gamma}'_i(t_0) \frac{\partial}{\partial x_i} \Big|_q f$$

This means that the coordinate components of the vector can be computed in any coordinate set!

Lemma 2.16. *Let $q \in M$. Every tangent vector in $T_q M$ is the tangent vector to a smooth curve.*

Proof. Take a smooth chart (U, φ) centered at q and write $v \in T_q M$ as $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q$. Then consider $\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n)$. This curve is smooth and is tangent to v at $t = 0$ by the above computation. $\square$

We can now formally prove the property which was at the basis of our motivation for the definition of differential.

Proposition 2.17. *Let $F : M \rightarrow N$ and let $v \in T_q M$ such that $v = \dot{\gamma}(t_0)$ for some $\gamma : I \rightarrow M$. Then we have*

$$F_*(v) = \frac{d}{dt} \Big|_{t=t_0} (F \circ \gamma).$$

Proof. The proof is just a collage of definitions: define the curve $\eta := F \circ \gamma$

$$\dot{\eta}(t_0) = \eta_* \left(\frac{d}{dt} \Big|_{t=t_0} \right) = (F \circ \gamma)_* \frac{d}{dt} \Big|_{t=t_0} = F_* \gamma_* \frac{d}{dt} \Big|_{t=t_0} = F_* \dot{\gamma}(t_0)$$

$\square$

This permits the geometric interpretation of the differential: given $v \in T_q M$ to compute $F_* v$ it is enough to compute the tangent vector to the curve $F \circ \gamma$ where γ is a curve tangent to v .

2.4 Examples

1. (Tangent space to a submanifold of $\mathbb{R}^n$.) Consider a submanifold S of $\mathbb{R}^n$ defined by

$$S = \{x \in \mathbb{R}^n \mid F(x) = y_0\},$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has surjective differential at every point of $S = F^{-1}(y_0)$, for $y_0 \in \mathbb{R}^m$. Then we have that for $x_0 \in S$

$$T_{x_0}S = \ker DF(x_0) = \{v \in \mathbb{R}^n \mid DF(x_0)v = 0\}.$$

Indeed for every curve $\gamma : I \rightarrow S$ such that $x_0 = \gamma(0)$ and $v = \dot{\gamma}(0)$ we have that $F(\gamma(t)) = y_0$ for every t and by differentiating at $t = 0$ we have

$$0 = \left. \frac{d}{dt} \right|_{t=0} F(\gamma(t)) = DF(\gamma(0))\dot{\gamma}(0) = DF(x_0)v.$$

2. (Tangent space to $SL(n)$ and $SO(n)$ at the identity.) Let us compute the tangent space to these matrix groups. Consider

$$SL(n) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\} = \det^{-1}(1).$$

We have proved that this is a $(n^2 - 1)$ -dimensional manifold. Using the previous fact we have that, denoting I the identity matrix,

$$\mathfrak{sl}(n) := T_I SL(n) = \ker D \det(I) = \{H \in M_n(\mathbb{R}) \mid \text{trace}(H) = 0\}.$$

Similarly one proves that for

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I\} = F^{-1}(I)$$

where $F : O(n) \rightarrow \text{Sym}(n)$ is given by $F(A) = A^T A$ we have

$$\mathfrak{so}(n) := T_I O(n) = \ker DF(I) = \{H \in M_n(\mathbb{R}) \mid H + H^T = 0\}.$$

where we used that $DF(A)H = A^T H + H^T A$.

3. (Tangent space on Lie groups.) If G is a Lie group then its tangent space to the identity $e \in G$ is a vector space which is called its *Lie algebra* and denoted by $\mathfrak{g}$, namely

$$\mathfrak{g} := T_e G$$

If $g \in G$ then the left translation $L_g : G \rightarrow G$ is a diffeomorphism hence $L_{g*} : T_e G \rightarrow T_g G$ is a diffeomorphism, hence one can recover the tangent space to every point just by applying the differential of the left translation

$$T_g G = L_{g*}(T_e G) = L_{g*}(\mathfrak{g}).$$

Similar relations are of course true also for right translations.

Exercise 2.18. Prove that if G is a Lie group of matrices, i.e., G is a subgroup of $GL_n(\mathbb{R})$ (or $GL_n(\mathbb{C})$) with the product of matrices as operation, then if $g \in G$ is a matrix then $T_g G = gT_e G$ in the sense that

$$T_g G = \{gv \mid v \in T_e G\},$$

where now g and v are two matrices and gv is their product in $GL_n(\mathbb{R})$.

Exercise 2.19 (Tangent space to the Grassmannian.). Let $G_k(V)$ be the Grassmannian of k -di subspaces of a n -dim vector space V . Given $W \in G_k(V)$, prove that

$$T_W G_k(V) \simeq \text{Hom}(W, V/W).$$

Chapter 3

Immersions, embeddings. Submanifolds

Geometry is the science of correct reasoning on incorrect figures
George Pólya, 1887 – 1985

In this chapter we discuss the notion of submanifold. The basic idea, thinking to a manifold as a set which “locally looks like $\mathbb{R}^n$ ” for some n , is that a submanifold should be a subset of a manifold which “locally looks like $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$ ” for some $0 \leq k \leq n$. In this context we also have a “bifurcation” of notions of submanifold, namely immersed and embedded submanifold.

To introduce these concepts, let us first start by some further considerations on smooth maps between manifolds.

3.1 Immersions, submersions, embeddings

Definition 3.1. Let $F : M \rightarrow N$ be smooth. The rank of F at q is the rank of the linear map $F_* : T_q M \rightarrow T_{F(q)} N$, i.e., the dimension of $\text{im } F_*$.

- (i) F is an *immersion* if F_* is injective at every point $q \in M$.
- (ii) F is a *submersion* if F_* is surjective at every point $q \in M$.
- (iii) F is an *embedding* if F is an immersion and $F : M \rightarrow F(M) \subset N$ is a homeomorphism onto its image endowed with the subspace topology of N .

Later we introduce *immersed* (resp. *embedded*) submanifolds. As we will see, these are exactly images $F(M)$ of a smooth manifold M under an injective immersion (resp. embedding) $F : M \rightarrow N$.

- Example 3.2.**
- 1. A local diffeomorphism $F : M \rightarrow N$ is both an immersion and a submersion.
 - 2. Projections from products $\pi_i : M_1 \times \dots \times M_n \rightarrow M_i$ are submersions. Injections in products $i_i : M_i \rightarrow M_1 \times \dots \times M_n$ are embeddings.
 - 3. A smooth curve $\gamma : I \rightarrow M$ is an immersion if and only if $\dot{\gamma}(t) \neq 0$ for every $t \in I$.

4. The curve $\gamma_1 : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\gamma(t) = (t^2, t^3)$ is injective but not an immersion since $\dot{\gamma}(t) = (2t, 3t^2)$ vanish at $t = 0$. Notice that the image of γ is contained in (in fact it coincides with) the set $\{(x, y) \in \mathbb{R}^2 \mid x^3 = y^2\}$.

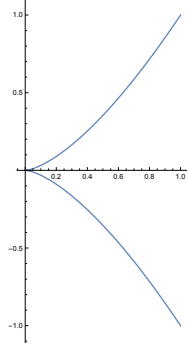


Figure 3.1: The curve γ_1

5. The curve $\gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $\gamma_2(t) = (t^3 - 4t, t^2 - 4)$ is an immersion but is not injective since $\dot{\gamma}_2(t) = (3t^2 - 4, 2t) \neq (0, 0)$ but $\gamma_2(2) = \gamma_2(-2)$. It is not an embedding (not injective hence not homeo). Notice that the image of γ_2 is contained in (in fact it coincides with) the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 = (y + 4)y^2\}$.

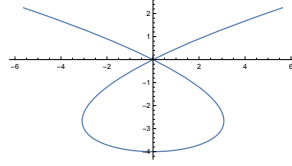


Figure 3.2: The curve γ_2

6. The curve $\gamma_3 : I =]-\pi/2, 3\pi/2[\rightarrow \mathbb{R}^2$ given by $\gamma_3(t) = (\sin(2t), \cos(t))$ is an injective immersion. But it is not an embedding since $\gamma_3(I) \cap B(0, r)$ is not homeomorphic to an interval for any $r > 0$! (make a picture!). Otherwise show that $\gamma_3(I)$ is closed and bounded in $\mathbb{R}^2$, hence compact, while I is not! One can observe that the image of the curve is contained in (in fact it coincides with) the set $\{(x, y) \in \mathbb{R}^2 \mid x^2 = 4y^2(1 - y^2)\}$.
7. Let $c \in \mathbb{R} \setminus \mathbb{Q}$. The curve $\gamma : \mathbb{R} \rightarrow S^1 \times S^1$ given by $\gamma(t) = (e^{2\pi it}, e^{2\pi ict})$ is an injective immersion. If one writes $S^1 = [0, 1]/\sim$ then $\gamma : \mathbb{R} \rightarrow [0, 1]^2/\sim$ becomes $\gamma(t) = (t, ct) \bmod 1$. The curve γ is not an embedding since the closure of $\gamma(\mathbb{R})$ is $S^1 \times S^1$, i.e., the curve γ has dense image. See Appendix of the chapter.

In general it is not easy to check whether an injective immersion is an embedding. This is a particular case when it is possible.

Proposition 3.3. *Let $F : M \rightarrow N$ be an injective immersion. If M is compact then F is an embedding.*

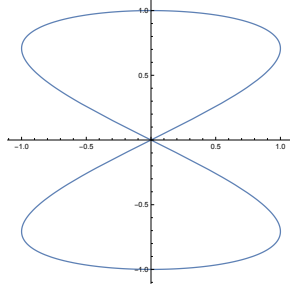


Figure 3.3: The curve γ_3

Proposition 3.3 follows from purely topological considerations: if $f : X \rightarrow Y$ is continuous and bijective from X compact topological space and Y Hausdorff topological space, then f is a homeomorphism. The reader is invited to check the details.

Exercise 3.4. Recall that $F : M \rightarrow N$ is proper if and only if $F^{-1}(K)$ is compact for every $K \subset N$ compact. Prove that if F is proper then F is a closed map (i.e., $F(C)$ is closed for any closed set $C \subset M$) and then show that the assumption M compact in Proposition 3.3 can be replaced by F proper.

3.2 The constant rank theorem

Our perspective is mainly local so we are now interested in the description of local properties of immersions or submersions. We first prove an important result which is a consequence of the inverse function theorem in $\mathbb{R}^n$.

Theorem 3.5. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a smooth map with constant rank equal to r in a neighborhood of $x_0 \in \mathbb{R}^n$. Then there exists*

- (i) *a local diffeomorphism $\varphi : U \rightarrow U_0$ from a neigh U of x_0 to a neigh U_0 of 0*
- (ii) *a local diffeomorphism $\psi : V \rightarrow V_0$ from a neigh V of $F(x_0)$ to a neigh V_0 of 0*

such that $\varphi(x_0) = 0$, $\psi(F(x_0)) = 0$ and

$$\psi \circ F|_U \circ \varphi^{-1} : U_0 \rightarrow V_0, \quad \psi \circ F|_U \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_r, 0, \dots, 0).$$

Proof. We split the space $(x, y) \in \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$ and $(u, v) \in \mathbb{R}^m = \mathbb{R}^r \times \mathbb{R}^{m-r}$ in such a way that we write

$$F(x, y) = (F_1(x, y), F_2(x, y))$$

with $F_1 : \mathbb{R}^r \times \mathbb{R}^{n-r} \rightarrow \mathbb{R}^r$ and $F_2 : \mathbb{R}^r \times \mathbb{R}^{n-r} \rightarrow \mathbb{R}^{m-r}$. It is not restrictive to assume that $x_0 = (0, 0)$ and $F(x_0) = (0, 0)$. In this notation, we have to prove the existence of ψ and φ such that $\psi \circ F \circ \varphi^{-1}(x, y) = (x, 0)$ for all (x, y) close to $(0, 0)$.

We can assume that $\text{rank } DF(x, y) = r$ at every point $(x, y) \in \mathbb{R}^n$ and (up to reordering variables) that the $r \times r$ matrix $D_x F_1(0, 0)$ invertible. Let us set

$$\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \varphi(x, y) = (F_1(x, y), y)$$

By construction $D\varphi(0,0)$ is invertible, hence φ is a local diffeo on a neighborhood U of $(0,0)$ (since we assume that $x_0 = (0,0)$ in the proof it is not restrictive to take $U = U_0$). It is a direct check to see that the composition $F \circ \varphi^{-1}$ has the form

$$F \circ \varphi^{-1}(x, y) = (x, \tilde{F}_2(x, y))$$

for a suitable map $\tilde{F}_2 : \mathbb{R}^r \times \mathbb{R}^{n-r} \rightarrow \mathbb{R}^{m-r}$. Since a local diffeomorphism does not change the rank, it holds $\text{rank } D(F \circ \varphi^{-1})(x, y) = r$ on U . Writing down explicitly $D(F \circ \varphi^{-1})(x, y)$, one easily see that this implies that $\tilde{F}_2$ does not depend on y , i.e.,

$$F \circ \varphi^{-1}(x, y) = (x, \tilde{F}_2(x)).$$

By setting $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ as $\psi(u, v) = (u, v - \tilde{F}_2(u))$ (notice that x and u both belong to $\mathbb{R}^r$, the space where $\tilde{F}_2$ is defined) then

$$\psi \circ F \circ \varphi^{-1}(x, y) = \psi(x, \tilde{F}_2(x)) = (x, \tilde{F}_2(x) - \tilde{F}_2(x)) = (x, 0)$$

which is equivalent to the statement. $\square$

Remark 3.6. The statement is local. In particular it can be applied to a smooth map defined on an open set $\Omega \subset \mathbb{R}^n$.

Remark 3.7. Notice the following two particular cases of Theorem 3.5: if F is an immersion then $r = n$ and there exists local diffeomorphisms φ, ψ such that

$$\psi \circ F|_U \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0). \quad (3.1)$$

If F is a submersion then $r = m$ and there exist local diffeomorphisms φ, ψ such that

$$\psi \circ F|_U \circ \varphi^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_m). \quad (3.2)$$

Notice that (3.1) is the canonical linear immersion of $\mathbb{R}^n$ into $\mathbb{R}^n \times \mathbb{R}^{m-n}$ (for $n < m$), and (3.2) is the canonical linear projection of $\mathbb{R}^{n-m} \times \mathbb{R}^m$ into $\mathbb{R}^m$ (for $n > m$).

We get the following theorem for manifolds just by applying the previous discussion.

Theorem 3.8 (Rank theorem for manifolds). *Let $F : M \rightarrow N$ be smooth map between manifolds of constant rank r in a neighborhood of a point q . Then there exist a chart (U, φ) around q a chart (V, ψ) around $F(q)$ such that the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1}$ of F is given by*

$$\hat{F}(x_1, \dots, x_n) = (x_1, \dots, x_r, 0, \dots, 0).$$

Proof. Let us consider coordinates (U, φ) centered at q and (V, ψ) centered at $F(q)$. Then $\hat{F} := \psi \circ F|_U \circ \varphi^{-1} : \phi(U) \subset \mathbb{R}^n \rightarrow \psi(V) \subset \mathbb{R}^m$ by definition is a smooth map such that $\text{rank}(D\hat{F}(0,0)) = r$. Applying Theorem 3.5 to $\hat{F}$ there exists local diffeo φ', ψ' such that

$$\psi' \circ \hat{F} \circ (\varphi')^{-1}(x_1, \dots, x_n) = (x_1, \dots, x_r, 0, \dots, 0).$$

Writing

$$\psi' \circ \hat{F} \circ (\varphi')^{-1} = \psi' \circ \psi \circ F|_U \circ \varphi^{-1} \circ (\varphi')^{-1}$$

and using as charts $\Psi := \psi' \circ \psi$ and $\Phi = \varphi' \circ \varphi$ the statement is proved. $\square$

A series of consequences descend from this result. We can say that immersions are locally injective, submersions are locally surjective, where the word “locally” should be properly understood. The following is a more formal statement and the reader is invited to check the details.

Proposition 3.9. *If $F : M \rightarrow N$ is an immersion then every point $q \in M$ has a neighborhood such that $F|_U : U \rightarrow N$ is an embedding. If $F : M \rightarrow N$ is a submersion then F is an open map.*

Combining the above considerations, also the following corollary is immediate.

Corollary 3.10 (Inverse function theorem for manifolds). *Let $F : M \rightarrow N$ be smooth and $q \in M$ such that $F_* : T_q M \rightarrow T_{F(q)} N$ is an isomorphism. Then F is a local diffeomorphism at q .*

The following result also holds (but it works only for constant rank maps!)

Corollary 3.11. *Let $F : M \rightarrow N$ be smooth map of constant rank r between manifolds. Then*

- (i) *if F is injective then it is an immersion.*
- (ii) *if F is surjective then it is a submersion.*

The reader is invited to deduce (i) from the rank theorem. The proof of (ii) requires the definition of a “set of measure zero” on a manifold, it is not difficult but at the moment it is omitted.

3.3 An application: the fundamental theorem of algebra

The goal is to give a proof of the following well-known theorem.

Theorem 3.12. *Let $P(z) = a_n z^n + \dots + a_1 z + a_0$ be a nonconstant polynomial with complex coefficients. Then there exists $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.*

Recall that a polynomial map $P : \mathbb{C} \rightarrow \mathbb{C}$ can be extended to a smooth map on the sphere S^2 .

Lemma 3.13. *Consider the map $P : \mathbb{C} \rightarrow \mathbb{C}$, $P(z) = a_n z^n + \dots + a_1 z + a_0$, with $a_n \neq 0$ thought as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Then set $\bar{P} : S^2 \rightarrow S^2$ (endowed by the standard atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$) defined by*

$$\bar{P}(x) = \begin{cases} \varphi_N^{-1} \circ P \circ \varphi_N(x), & x \neq N \\ N, & x = N \end{cases}$$

Then the function $\bar{P}$ smooth with respect to the C^∞ structure on S^2 .

Exercise 3.14. Consider the map $P : \mathbb{C} \rightarrow \mathbb{C}$ given by $P(z) = a_n z^n + \dots + a_1 z + a_0$, with $a_n \neq 0$ thought as a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that the differential $DP(z_0) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ when regarded as a linear map $\mathbb{C} \rightarrow \mathbb{C}$ is the multiplication by the complex number $P'(z_0)$.

For the proof we will also use the following fact.

Lemma 3.15. *Let $F : M \rightarrow N$ be a smooth map between smooth manifolds of the same dimension, with M compact. Let y_0 be a regular value of F . Then*

(i) $F^{-1}(y_0)$ is finite;

(ii) there exists an open set V containing y_0 such that: $\forall y \in V$, $\text{card } F^{-1}(y) = \text{card } F^{-1}(y_0)$

Proof. If $F^{-1}(y_0) = \emptyset$ it suffices to prove (ii). But $F(M)$ is the image of a compact subset M under a continuous map, so it is a compact subset and is therefore closed in N . Then take $V = N \setminus F(M)$.

If $F^{-1}(y_0) \neq \emptyset$ we first note that it is a compact subset of M , as it is a closed subset of a compact set. Let $x \in F^{-1}(y_0)$. By hypothesis, x is a regular point of F : there exists an open set U_x containing x such that $F|_{U_x}$ is a diffeomorphism between U_x and $F(U_x)$. In particular, x is the only point of U_x where $F(x) = y$. The family $\{U_x\}_{x \in F^{-1}(y_0)}$ is an open cover of $F^{-1}(y_0)$ and we can extract a finite subcover $\{U_{x_i}\}_{i=1, \dots, N}$, which shows that $F^{-1}(y_0)$ is a finite subset with N elements (more concisely we can say the topological space $F^{-1}(y_0)$ is both compact and discrete, and therefore finite).

To prove the existence of V : assume by contradiction there exists a sequence $y_n \rightarrow y_0$ with $\text{card } F^{-1}(y_n) > N$. Consider a sequence x'_n not belonging to the union $\{U_{x_i}\}_{i=1, \dots, N}$ such that $F(x'_n) = y_n$. Since M is compact, up to subsequence $x'_n \rightarrow \bar{x}$. By continuity $\bar{x}$ not belonging to the union $\{U_{x_i}\}_{i=1, \dots, N}$ but $F(\bar{x}) = y_0$ contradiction. $\square$

Proof of Theorem 3.12. Let $P(z) = a_n z^n + \dots + a_1 z + a_0$ be a nonconstant polynomial with complex coefficients. We can associate an “extension to infinity” to P , denoted here by $F := \bar{P}$, defined in Lemma 3.13.

We show that F has a finite number of non-regular values. On $S^2 \setminus \{N\}$, F has the same number of non-regular values as P (we composed on the right and left by diffeomorphisms given by charts). By Exercise 3.14, the differential of F at a point z of $S^2 \setminus \{N\}$ is precisely the multiplication by the complex number $P'(z)$, seen as a linear map of $\mathbb{R}^2$ in $\mathbb{R}^2$. The non-regular points of P are those $z \in \mathbb{C}$ such that $P'(z) = 0$. Since P is non constant, there are only finitely many of these, and so P has only a finite number of non-regular values. The same holds for F seen as a map of S^2 in S^2 (if necessary, we add the north pole). Let C denote the set of these values. Since C is finite, $S^2 \setminus C$ is connected. By Lemma 3.15, the function $y \rightarrow \text{card } F^{-1}(y)$ is locally constant on $S^2 \setminus C$, therefore constant by connectedness.

This constant is nonzero. If not, F , and moreover P have only singular values, which says that P' is identically zero and therefore P is constant, contrary to the hypothesis. All points in $S^2 \setminus C$, and therefore all the points in S^2 are values taken by F . Returning to the definition of F , we deduce that all of the points of $\mathbb{C}$ are values taken by P . $\square$

3.4 Submanifolds

We start with the following observation: an injective immersion F defines a smooth structure on the image of F . The proof is left to the reader.

Lemma 3.16. *Let $F : M \rightarrow N$ be an injective immersion. Given a smooth atlas $\{(U_i, \varphi_i)\}_{i \in \mathbb{N}}$ of M prove that $\{(F(U_i), \varphi_i \circ F|_{U_i}^{-1})\}_{i \in \mathbb{N}}$ is a smooth atlas for $F(M)$.*

Notice that with this smooth structure $F : M \rightarrow F(M)$ is a diffeomorphism.

Definition 3.17. An *immersed submanifold* of a smooth manifold M is a subset $S \subset M$ such that S is endowed with a smooth manifold structure such that the canonical injection $i : S \rightarrow M$

is a smooth immersion. The dimension of an immersed submanifold S is the dimension of S as a smooth manifold.

More or less by construction (we leave the details to the reader) we have the following property.

Proposition 3.18. *Immersed submanifolds are precisely images of smooth injective immersions.*

Proof. If S is immersed submanifold then S is the image of the canonical injection $i : S \rightarrow M$, which is a smooth injective immersion by definition.

We have to prove that given $F : M \rightarrow N$ an injective immersion, the canonical inclusion $i : F(M) \rightarrow N$ is an immersion, where $F(M)$ is endowed with the smooth structure of Lemma 3.16. But this is the composition of the map $F^{-1}|_{F(M)} : F(M) \rightarrow M$ which is a smooth diffeomorphism and $F : M \rightarrow N$ which is an injective immersion. $\square$

A question is then whether the structure of manifold $F(M)$ has from the injective immersion “agrees” with the smooth structure $F(M)$ might inherit from N .

Another simple but crucial observation before moving to the following definition.

Lemma 3.19. *Let $F : M \rightarrow N$ be an embedding. For every point $F(q)$ in $F(M)$ there exists a chart (V, ψ) of N with V neighborhood of $F(q)$ such that*

$$\psi(V \cap F(M)) = \psi(V) \cap (\mathbb{R}^k \times \{0\})$$

Proof. Let $U \subset M$ be an open set containing q . Since F embedding then (crucial!) we have that $F(U) = V \cap F(M)$ for some V open in N .

Since F is an immersion there exists charts (U, φ) and (V, ψ) such that

$$\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V), \quad \widehat{F}(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$$

In particular we have $\psi(F(U)) = \psi(V \cap F(M)) = \psi(V) \cap (\mathbb{R}^k \times \{0\})$. $\square$

Remark 3.20. If the inclusion $i : S \rightarrow M$ is an embedding, then renaming the chart ψ as φ and applying the above consideration: for every $q \in S$ there exist a chart φ satisfying

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$$

This is the existence of a “slice chart” for S at every point.

Definition 3.21. An *embedded submanifold* of a smooth manifold M is a subset $S \subset M$ such that S is endowed with a smooth manifold structure such that the canonical injection $i : S \rightarrow M$ is a smooth embedding.

Similarly as in Proposition 3.18 we have the following.

Proposition 3.22. *Embedded submanifolds are precisely images of smooth embeddings.*

Proof. Let $S \subset M$ be an embedded submanifold in such a way that $i : S \rightarrow M$ is an embedding. This means that $A \subset S$ is open then $A = U \cap S$ where U is open in M .

Take $q \in S$. Then there exists neighborhood of q such that the inclusion $i : U \cap S \rightarrow U$ writes in coordinates as $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$. More precisely we can take a chart $\varphi : U \rightarrow \mathbb{R}^n$ to write

$$\widehat{i} = \varphi \circ i \circ \varphi^{-1} : \varphi(U \cap S) \rightarrow \varphi(U), \quad \widehat{i}(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$$

In particular we have $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$. $\square$

Remark 3.23. (again on the difference: immersed vs embedded.) It is important to stress that given $S \subset M$ an immersed submanifold Proposition 3.9 guarantees that for every $q \in M$ there exists a neighborhood V of q in S such that V is an embedded submanifold. But in general it is not true that there exists a neighborhood U of M such that $U \cap S$ is an embedded submanifold.

An easy consequence of the existence of slice charts is the following fact.

Proposition 3.24. *Let $f : M \rightarrow \mathbb{R}$ be smooth and let S be an embedded submanifold of M . Then the restriction $f|_S : S \rightarrow \mathbb{R}$ is smooth.*

We can also characterize the tangent space to embedded submanifolds.

Proposition 3.25. *Let $S \subset M$ be an embedded submanifold. Let $i : S \rightarrow M$ the inclusion and $q \in S$. Then*

$$i_*(T_q S) = \{v \in T_q M \mid vf = 0, \forall f \in C^\infty(M), f|_S = 0\}. \quad (3.3)$$

Identifying elements of $T_q S$ with $i_*(T_q S) \subset T_q M$ we can see $T_q S$ as a vector subspace of $T_q M$.

Proof. Let us first show the inclusion $\subseteq$ in (3.3). Let $w \in T_q S$ let us prove that $v = i_* w$ satisfies $vf = (i_* w)f = 0$ for every $f \in C^\infty(M), f|_S = 0$. Indeed

$$(i_* w)f = w(f \circ i) = w(0) = 0.$$

Conversely let $v \in T_q M$ such that $vf = 0$ for all $f \in C^\infty(M), f|_S = 0$. Then we want to show that $v = i_* w$ for some $w \in T_q S$. It is easy to show that the set

$$W = \{v \in T_q M \mid vf = 0, \forall f \in C^\infty(M), f|_S = 0\}$$

is a vector subspace of $T_q M$. Let $q \in S$ and choose coordinates (U, φ) such that $\varphi(S \cap U) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$. Then we have

$$\varphi(S \cap U) = \{(x_1, \dots, x_n) \in \varphi(U) \mid x_{k+1} = \dots = x_n = 0\}.$$

which can be expressed by saying that locally in coordinates

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \dots = x_n = 0\}.$$

which more formally means: $f|_S = f(x_1, \dots, x_k, 0, \dots, 0)$ and W is spanned by $\partial/\partial x_j$ for $j = 1, \dots, k$ and one sees in coordinates that $\dim W = k = \dim S$. Hence $W = T_q S$. $\square$

This characterization is interesting from an abstract viewpoint, but not so convenient for computations. Indeed what one does is to show that embedded submanifold can be always locally described as regular level set of some map.

How to compute the tangent space to an immersion If $F : M \rightarrow N$ is an immersion then locally we can look at it as $F : \mathbb{R}^k \rightarrow \mathbb{R}^n$ assume $F(0) = x_0$ and let $\Sigma = F(U)$ so that

$$T_{x_0}\Sigma = F_*T_0\mathbb{R}^k$$

this is generated by the vectors $\{F_*\partial_{u_i}\}_{i=1,\dots,k}$. But we have

$$F_*\frac{\partial}{\partial u_i} = \frac{\partial F}{\partial u_i} \quad (3.4)$$

Indeed

$$\left(F_*\frac{\partial}{\partial u_i}\right)g = \frac{\partial}{\partial u_i}(g \circ F) = \sum_{j=1}^m \frac{\partial g}{\partial y_j} \frac{\partial F_j}{\partial u_i} = \left(\sum_{j=1}^m \frac{\partial F_j}{\partial u_i} \frac{\partial}{\partial y_j}\right)g$$

which means exactly (3.4) as a vector in $\mathbb{R}^n$

Level sets

The following results can be obtained simply by corresponding results in $\mathbb{R}^n$ and using charts, as in the proof of Theorem 3.8.

Proposition 3.26. *Let $F : M \rightarrow N$ be a smooth map of constant rank equal to k . Then each level set $F^{-1}(q)$, for $q \in N$, is a smooth submanifold of M of dimension $n - k$.*

We have this corollary, which is the previous one with $k = \dim N$.

Corollary 3.27. *Let $F : M \rightarrow N$ be a smooth submersion. Then each level set $F^{-1}(q)$, for $q \in N$, is a smooth submanifold of M of dimension $\dim M - \dim N$.*

Notice that this result can be strengthened considerably, since actually we only need to check the assumption on the level set. Following the notation already introduced.

Definition 3.28. Let $F : M \rightarrow N$ be smooth. We say that $q \in M$ is *critical point* if F_*,q is not surjective, q is a *regular point* otherwise. A point $y \in N$ is said *regular value* if every $q \in F^{-1}(y)$ is a regular point.

Proposition 3.29. *Let $F : M \rightarrow N$ be smooth. Assume $q \in N$ is a regular value for F , then $F^{-1}(q)$ is a smooth submanifold of M of dimension $\dim M - \dim N$.*

Proposition 3.30. *Let $S \subset M$ be an embedded submanifold of dimension k and $q \in S$. There exists $U \subset M$ neighborhood of q and $\Phi : U \rightarrow \mathbb{R}^{n-k}$ such that $S \cap U$ is a regular level set $U \cap \Phi^{-1}(y)$ for some regular value y in N*

Proof. Work as in the proof of Lemma 3.19 and subsequent Remark. For every $q \in S \subset M$ of dimension k , there exists a chart (U, φ) such that $\varphi(S \cap U) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Consider the map $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$ which forgets the first k variables. Then $\pi \circ \varphi : U \rightarrow \mathbb{R}^{n-k}$ and

$$\pi \circ \varphi(S \cap U) = \pi\left(\varphi(U) \cap (\mathbb{R}^k \times \{0\})\right) = \{0\}.$$

The map π is a submersion and φ a local diffeomorphism, so that $\Phi := \pi \circ \varphi$ is a submersion. $\square$

We say that $\Phi : U \rightarrow N$ is a *local map defining S* near $q \in S$ if $S \cap U$ is a regular level set $U \cap \Phi^{-1}(y)$ for some regular value y in N . By Proposition 3.30 there always exists local maps defining S .

Proposition 3.31. *Let $S \subset M$ be an embedded submanifold. If $\Phi : U \rightarrow N$ is any local map defining S near $q \in S$ then*

$$T_q S = \ker \Phi_* : T_q M \rightarrow T_{\Phi(q)} N$$

Proof. Identifying as before $T_q S$ with $i_* T_q S \subset T_q M$ under the inclusion map $i : S \rightarrow M$. Notice that $\Phi \circ i$ is a constant map on $S \cap U$, by definition. Hence $\Phi_* \circ i_* = 0$. Then $\text{im } i_* \subset \ker \Phi_*$. On the other hand a dimensional count says that

$$\dim \ker \Phi_* = \dim T_q M - \dim T_{\Phi(q)} N = \dim T_q S = \dim \text{im } i_*. \quad \square$$

Example 3.32 (Matrices of fixed rank). Recall that the set $M_{m,n}^k(\mathbb{R})$ of matrices of size $m \times n$ and of rank k is an open set of $M_{m,n}(\mathbb{R})$ when $k = \min\{m, n\}$ hence a submanifold of dimension mn , or of codimension 0.

The goal of this exercise is to prove that for $0 < k < \min\{m, n\}$ the set $M_{m,n}^k(\mathbb{R})$ is an embedded submanifold of $M_{m,n}(\mathbb{R})$ of codimension $(m - k)(n - k)$.

Let us consider the open subset of $M_{m,n}(\mathbb{R})$ given by

$$U = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \det(A) \neq 0 \right\}$$

where the upper left A is of size $k \times k$ and the other ones accordingly to the fact that M is of size $m \times n$. Clearly matrices in U have rank $\geq r$ so that we are interested in describing $U \cap M_{m,n}^k(\mathbb{R})$. Following the proof of the constant rank theorem: we consider the inverse of the map $\varphi(x, y) = (Ax + By, y)$. This means $x' = Ax + By$ and $y' = y$. Inverting, then $y = y'$ and $x = A^{-1}(x' - By')$. Then consider

$$P = \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I \end{pmatrix}$$

and note that

$$MP = \begin{pmatrix} I & 0 \\ CA^{-1} & D - CA^{-1}B \end{pmatrix}$$

Since M has rank k the same is true for MP being P invertible, hence $D - CA^{-1}B$ should be the zero matrix. This suggests to consider the map

$$\Phi : U \subset M_{m,n}(\mathbb{R}) \rightarrow M_{m-k,n-k}(\mathbb{R}), \quad \Phi(M) = D - CA^{-1}B$$

and to observe that $U \cap M_{m,n}^k(\mathbb{R}) = \Phi^{-1}(0)$. The map Φ is a submersion if and only if for every matrix $X \in M_{m-k,n-k}(\mathbb{R})$ there exists a curve $\gamma(t) \in U$ such that $\gamma(0) = M_0$ and $(\Phi \circ \gamma)'(0) = X$. This is easily done by taking

$$\gamma(t) = \begin{pmatrix} A & B \\ C & D + tX \end{pmatrix}$$

Adjusting the proof for every minor (i.e., applying linear invertible maps to the open set U), one gets that $M_{m,n}^k(\mathbb{R})$ is a smooth submanifold of dimension

$$\dim M_{m,n}^k(\mathbb{R}) = nm - (n - k)(m - k).$$

An example to understand Let us consider S^2 (as a subset of $\mathbb{R}^3$) and consider the map $F : S^2 \rightarrow \mathbb{R}$ given by

$$F(x, y, z) = x$$

Find all regular values y of F and show that for every such y the set $F^{-1}(y)$ is a circle. Notice that there are two critical points in S^2 .

Exercise 3.33. Prove that the map $G : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$G(\varphi, \theta) = ((2 + \cos \phi) \cos \theta, (2 + \cos \phi) \sin \theta, \sin \phi)$$

is an immersion. Show that $G(\mathbb{R}^2)$ is contained in the set

$$T = \{(x, y, z) \mid ((x^2 + y^2)^{1/2} - 2)^2 + z^2 - 1 = 0\}$$

Indeed $G(\mathbb{R}^2) = T$. Let $\Phi(x, y, z) = ((x^2 + y^2)^{1/2} - 2)^2 + z^2 - 1$. Then

$$T = \{(x, y, z) \mid ((x^2 + y^2)^{1/2} - 2)^2 + z^2 - 1 = 0\} = \Phi^{-1}(0)$$

and it is easy to see that $D\Phi(x, y, z)$ has maximal rank if $(x, y) \neq (0, 0)$, which is true on T . Hence T is an embedded 2-dimensional submanifold.

Let $F : T \rightarrow \mathbb{R}$ defined by $F(x, y, z) = x$ for every $(x, y, z) \in T$. Prove that F is smooth. Find the critical points of F .

Exercise 3.34. Consider the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $F(x, y) = x^3 + 3xy + y^3$. For which values of $c \in \mathbb{R}$ the set $\{(x, y) \in \mathbb{R}^2 \mid F(x, y) = c\}$ is a smooth embedded submanifold?

3.5 Transversality

This is intended as an introduction to the basic concepts of transversality. Transversality is a description of how two objects intersect. It can be thought of as the “opposite” of tangency. It extends the natural definition of transversality of vector subspaces: if W_1, W_2 are vector subspaces of V then they are said to be *transversal* if $W_1 + W_2 = V$. Notice that two subspaces cannot be transversal if

$$\dim W_1 + \dim W_2 < \dim V$$

in particular this notion depends on the ambient space V and not only on the subspaces W_1, W_2 . If two vector subspaces W_1, W_2 are transversal then by the Grassmann formula we have

$$\dim(W_1 \cap W_2) = m_1 + m_2 - n$$

denoting $n = \dim V$ and $m_i = \dim W_i$ for $i = 1, 2$.

We expect then that if S_1, S_2 are two embedded submanifold of M which intersect “transversally” then the intersection $S_1 \cap S_2$ is an embedded submanifold of M of dimension $m_1 + m_2 - n$, where $n = \dim M$ and $m_i = \dim S_i$ for $i = 1, 2$. Let us formalize this.

Definition 3.35. Let S_1, S_2 be two embedded submanifold of M . We say that S_1, S_2 are *transversal*, and write $S_1 \pitchfork S_2$ if

$$T_q S_1 + T_q S_2 = T_q M, \quad \forall q \in S_1 \cap S_2.$$

We can define also the notion of a map transverse to a submanifold in the target space.

Definition 3.36. Let $F : M \rightarrow N$ be a smooth map and $S \subset N$ be an embedded submanifold of the target space. We say that F is *transversal* to S , and write $F \pitchfork S$ if

$$F_*(T_q M) + T_{F(q)} S = T_{F(q)} N, \quad \forall q \in F^{-1}(S).$$

Given an embedded submanifold S of N we denote $\text{codim}_N S := \dim N - \dim S$. We can generalize the level set theorem as follows.

Theorem 3.37. Let $F : M \rightarrow N$ be a smooth map and $S \subset N$ be an embedded submanifold of the target space. Assume that F is transversal to S . Then $F^{-1}(S)$ is an embedded submanifold of M and $\text{codim}_M F^{-1}(S) = \text{codim}_N S$.

Proof. We will construct a map $G : M \rightarrow \mathbb{R}^d$ for a suitable d which is a submersion, and such that $G^{-1}(0) = F^{-1}(S)$. Since S is an embedded submanifold of N , thanks to Proposition ? there exists a submersion $\Phi : N \rightarrow \mathbb{R}^{m-s}$ such that $\Phi^{-1}(0) = S$, where $s = \dim S$ and $m = \dim N$. Consider then the composition $G := \Phi \circ F : M \rightarrow \mathbb{R}^{m-s}$. For every $q \in F^{-1}(S)$, we have

$$G_{*,q} = \Phi_{*,F(q)} \circ F_{*,q} : T_q M \rightarrow T_{F(q)} N \rightarrow \mathbb{R}^{m-s}.$$

We know that $T_{F(q)} S = \ker \Phi_{*,F(q)}$ and due to the transversality condition we have

$$F_*(T_q M) + T_{F(q)} S = T_{F(q)} N$$

which implies in particular that $G_{*,q}$ is surjective. □

We have the following corollary.

Corollary 3.38. Let S_1, S_2 be two embedded submanifold of M , with $s_i = \dim S_i$ and $n = \dim M$. Assume that S_1, S_2 are transversal, then $S_1 \cap S_2$ is a smooth embedded submanifold and

$$\dim(S_1 \cap S_2) = s_1 + s_2 - n$$

3.6 Appendix

Proposition 3.39. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The curve $\gamma : \mathbb{R} \rightarrow S^1 \times S^1$ given by $\gamma(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$ has dense image.

Proof. If we identify $S^1 \times S^1$ with $[0, 1]^2 / \sim$ with the identification of the boundary then γ is rewritten as

$$\gamma(t) = (t, \alpha t) \pmod{1}.$$

It is not difficult to see that γ has dense image in $[0, 1]^2 / \sim$ if and only if the following sequence is dense in $[0, 1]$

$$(\alpha_n)_{n \in \mathbb{N}} \subset [0, 1], \quad \alpha_n := n\alpha \pmod{1}.$$

Let us prove this fact: first of all we notice that $\alpha_n \neq \alpha_m$ for $n \neq m$ otherwise we would have $(n - m)\alpha = k$ for some integer k , which is a contradiction with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Moreover we notice that for every $n > m$ we have $\alpha_n - \alpha_m = \alpha_{n-m}$.

Let us show that 0 is an accumulation point for the sequence: for every N there exists $n \in \mathbb{N}$ such that $|\alpha_n| < \frac{1}{N}$. Indeed consider the first $N+1$ element of the sequence $\alpha_1, \dots, \alpha_{N+1}$. By pigeonhole principle there exists two integers n, m such that $\alpha_n, \alpha_m \in [\frac{k}{N}, \frac{k+1}{N}]$ for the same $k \in \{0, \dots, N-1\}$. Then the element $x := \alpha_{n-m} = \alpha_n - \alpha_m$ belongs to $[0, \frac{1}{N}]$.

Then one can notice that for arbitrary $p \in \mathbb{N}$ we have $p\alpha_{n-m} = \alpha_{p(n-m)}$ and then we can find $p = p(k)$ such that $p\alpha_{n-m}$ belongs in $[\frac{k}{N}, \frac{k+1}{N}]$ (we leave this last check to the reader). $\square$

Chapter 4

Tangent bundle and vector fields

A smooth assignment of a tangent vector to each point of a manifold is called a vector field. This seems very clear, up to the moment you start to ask yourself: what does it mean *smooth* here?

These kind of situations puzzle the modern mathematician which then feel more comfortable with the following neat definition:

A smooth vector field is a smooth section of the tangent bundle.

The next pages are devoted to give a meaning to the last sentence.

4.1 The tangent bundle

The tangent bundle is defined as the disjoint union

$$TM = \bigcup_{q \in M} T_q M$$

endowed with the natural projection

$$\pi : TM \rightarrow M, \quad \pi(v) = q, \text{ if } v \in T_q M$$

Proposition 4.1. *TM has the natural structure of smooth manifold with $\dim TM = 2 \dim M$.*

Proof. Consider charts $\{(U_i, \varphi_i)\}$ on M . Then $\pi^{-1}(U_i)$ is an open set of TM and the reunion of these sets is an open cover of TM . We define coordinates

$$\bar{\varphi}_i : \pi^{-1}(U_i) \rightarrow \varphi_i(U_i) \times \mathbb{R}^n$$

as follows

$$\bar{\varphi}_i(v) = (x_1, \dots, x_n, v_1, \dots, v_n)$$

if $\pi(v) = q$ with $\varphi_i(q) = (x_1, \dots, x_n)$, and moreover

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q$$

Notice that if $U_i \cap U_j \neq \emptyset$ then $\pi^{-1}(U_i) \cap \pi^{-1}(U_j) \neq \emptyset$ and

$$\bar{\varphi}_i \circ \bar{\varphi}_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \rightarrow \varphi_i(U_i \cap U_j) \times \mathbb{R}^n$$

where $\bar{\varphi}_i \circ \bar{\varphi}_j^{-1}(x, v) = (x', v')$ if (cf. with Exercise 2.13)

$$x' = \varphi_i \circ \varphi_j^{-1}(x), \quad v' = \frac{\partial(\varphi_i \circ \varphi_j^{-1})}{\partial x} v \quad (4.1)$$

which completes the proof. $\square$

Notice that the projection $\pi : TM \rightarrow M$ is smooth in the atlas defined above since in coordinates it is just the linear projection $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto (x_1, \dots, x_n)$.

We can treat the collection of the differentials at different points as a single map $F_* : TM \rightarrow TN$. The previous computations shows that.

Corollary 4.2. *Let $F : M \rightarrow N$ be smooth. Then $F_* : TM \rightarrow TN$ is smooth.*

Proof. Consider coordinates as above. The map $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto (x'_1, \dots, x'_n, v'_1, \dots, v'_n)$ can be written as follows:

$$x' = F(x), \quad v' = DF(x)v \quad (4.2)$$

where $F(x) = \varphi_i \circ \varphi_j^{-1}(x)$. To show that is is smooth in coordinates

If in coordinates $F : M \rightarrow N$ is smooth and $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^m Y_j \frac{\partial}{\partial y_j}$ then the equality $F_*X = Y$ means

$$\sum_{j=1}^m Y_j \frac{\partial g}{\partial y_j} = \sum_{i=1}^n X_i \frac{\partial(g \circ F)}{\partial x_i} = \sum_{j=1}^m \sum_{i=1}^n X_i \frac{\partial g}{\partial y_j} \frac{\partial F_j}{\partial x_i}$$

so that the statement follows from formulas

$$F_* \left(\sum_{i=1}^n X_i \frac{\partial}{\partial x_i} \right) = \sum_{j=1}^m \left(\sum_{i=1}^n X_i \frac{\partial F_j}{\partial x_i} \right) \frac{\partial}{\partial y_j}$$

$\square$

Example 4.3 (Tangent bundle TS^1). We prove that the tangent bundle TS^1 to the circle S^1 is diffeomorphic to $S^1 \times \mathbb{R}$. Let us identify

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

and first notice that $T_1 S^1 = i\mathbb{R}$. This is a simple consequence of the following computation: let $\gamma(t) = e^{i\theta(t)}$ for some smooth function θ with $\theta(0) = 0$. Then $\dot{\gamma}(0) = i\dot{\theta}(0) \in i\mathbb{R}$.

Given $v \in T_z S^1$ we notice that $z^{-1}v \in i\mathbb{R}$. Indeed let $\gamma(t) = e^{i\theta(t)}$ for some smooth function θ with $\theta(0) = \theta_0$ with $z = \gamma(0)$ and $v = \dot{\gamma}(0)$. Then

$$v = \dot{\gamma}(0) = i\dot{\theta}(0)e^{i\theta(0)} = i\dot{\theta}(0)z, \quad \Rightarrow \quad z^{-1}v \in i\mathbb{R}.$$

It follows that we can define the map (identifying $i\mathbb{R} \simeq \mathbb{R}$)

$$\Psi : TS^1 \rightarrow S^1 \times \mathbb{R}, \quad v \mapsto (z, z^{-1}v)$$

where $z = \pi(v)$. The verification that Ψ is a diffeomorphism is left to the reader.

Notice that S^1 is a Lie group. Where did we use the Lie group structure here?

4.2 Vector bundles

We start by the definition of vector bundle.

Definition 4.4. Let M be a smooth n -dimensional manifold. A *smooth vector bundle of rank k over M* is a smooth manifold E of dimension $n + k$ together with a smooth surjective map $\pi : E \rightarrow M$ such that

- (i) for each $q \in M$ the set $E_q := \pi^{-1}(q)$ is a vector space of dimension k
- (ii) for every $q \in M$ there exists a neighborhood $U \subset M$ and a diffeomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ such that¹ $\pi = \text{pr}_1 \circ \Phi$ and $\Phi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a linear isomorphism.

We say that E is the total space, M the base of the vector bundle, and π the projection. The diffeomorphism $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ is called *local trivialization*. If one can choose $U = M$ then E is diffeomorphic to $M \times \mathbb{R}^k$ and we say that the vector bundle is *trivial*.

Technically one should write that a vector bundle is a triple (E, M, π) . For simplicity we will also say that $\pi : E \rightarrow M$ (or even $E \rightarrow M$) is a vector bundle.

Lemma 4.5. *Let M be a smooth manifold. Then TM is a vector bundle of rank equal to $\dim M$.*

Proof. Given any smooth chart (U, φ) for M with coordinates $\{x_i\}$ we set $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ as

$$\Phi \left(\sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q \right) = (q, v_1, \dots, v_n)$$

We notice that $(\varphi \times \text{id}) \circ \Phi = \bar{\varphi}$ where $\varphi \times \text{id} : U \times \mathbb{R}^n \rightarrow \varphi(U) \times \mathbb{R}^n$ is defined as $(q, v) \mapsto (\varphi(q), v)$. Since both $\varphi \times \text{id}$ and $\bar{\varphi}$ are diffeomorphisms, it follows that Φ is a diffeomorphism as well. We have build local trivializations. The reader is invited to check all others requirements. $\square$

If a bundle is not trivial then we need more than one local trivialization.

Proposition 4.6. *Let $\pi : E \rightarrow M$ be a smooth vector bundle of rank k and let $\Phi_1 : \pi^{-1}(U_1) \rightarrow U_1 \times \mathbb{R}^k$ and $\Phi_2 : \pi^{-1}(U_2) \rightarrow U_2 \times \mathbb{R}^k$ be two local trivialization with $U_1 \cap U_2 \neq \emptyset$. Then*

$$\Phi_2 \circ \Phi_1^{-1} : (U_1 \cap U_2) \times \mathbb{R}^k \rightarrow (U_1 \cap U_2) \times \mathbb{R}^k$$

writes as $\Phi_2 \circ \Phi_1^{-1}(q, v) = (q, \tau(q)v)$ where $\tau : (U_1 \cap U_2) \rightarrow GL_k(\mathbb{R})$ is smooth.

Proof. Note that by construction $\text{pr}_1 \circ \Phi_2 \circ \Phi_1^{-1} = \text{pr}_1$ hence $\Phi_2 \circ \Phi_1^{-1}(q, v) = (q, \sigma(q, v))$ with $\sigma : (U_1 \cap U_2) \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ smooth. For a fixed q , the map $v \mapsto \sigma(q, v)$ is linear. Hence $\sigma(q, v) = \tau(q)v$ for some matrix $\tau(q)$. We have just to prove that $q \mapsto \tau(q)$ is smooth. But the coordinates $\tau(q) = (\tau_{ij}(q))$ satisfy $\tau_{ij}(q) = \text{pr}_i(\sigma(q, e_j))$ hence they are smooth². $\square$

The map τ is called *transition function*. As for smooth manifolds, we have a construction lemma for vector bundles.

¹given a product $X \times Y$, we denote $\text{pr}_1 : X \times Y \rightarrow X$ the projection onto the first factor.

²here $\text{pr}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the projection onto the i -th factor $\text{pr}_i(x_1, \dots, x_n) = x_i$

Proposition 4.7. *Let M be smooth manifold. Assume for every $q \in M$ we have a k -dimensional vector space E_q and define*

$$E = \bigcup_{q \in M} E_q, \quad \pi : E \rightarrow M$$

where π is the natural projection. Then assume that we have

- (i) an open cover $\{U_i\}_{i \in \mathbb{N}}$ of M
- (ii) diffeomorphisms $\Phi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{R}^k$ that are linear isomorphisms on fibers

such that

- (a) for every $U_i \cap U_j \neq \emptyset$ there exists $\tau_{ij} : U_i \cap U_j \rightarrow GL_k(\mathbb{R})$ such that

$$\Phi_j \circ \Phi_i^{-1}(q, v) = (q, \tau_{ij}(q)v).$$

Then there exists a unique smooth structure on E such that π is smooth, E is a smooth vector bundle of rank k over M , and $\{\Phi_i\}_{i \in \mathbb{N}}$ is a local trivialization. ***cocycle property***

Given a smooth vector bundle $E \rightarrow M$, a (local) *section* of E is a map $\sigma : U \subset M \rightarrow E$ such that $\pi \circ \sigma = \text{id}_U$. If U can be chosen equal to M , then σ is a *global section*.

Remark 4.8. Recall that given a vector bundle there exists always a particular section, which is the *zero global section*. This is the section $\zeta : M \rightarrow E$ defined as $\zeta(q) = 0_q$ where $0_q \in E_q$ is the origin of the vector space.

Remark 4.9. The space of smooth functions $C^\infty(M)$ can be identified with the space of smooth sections of the trivial vector bundle $M \times \mathbb{R}$ of rank 1.

Definition 4.10. We say that a family of sections $\sigma_1, \dots, \sigma_r : U \rightarrow E$ are independent if $\{\sigma_i(q)\}_{i=1, \dots, r}$ are linearly independent as vectors in E_q for every $q \in U$. When $r = \text{rank}(E)$ then we say that $\{\sigma_1, \dots, \sigma_r\}$ is a local frame.

If $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ is a local trivialization we can always define a local frame on U as follows

$$\sigma_i(q) = \Phi^{-1}(q, e_i)$$

where e_i is the canonical basis of $\mathbb{R}^k$. Conversely if $\{\sigma_1, \dots, \sigma_k\}$ is a local frame on U then we can build a local trivialization as follows

$$\Psi : U \times \mathbb{R}^k \rightarrow \pi^{-1}(U), \quad \Psi(q, v) = \sum_{i=1}^k v_i \sigma_i(q).$$

We have proved the following fact.

Corollary 4.11. *Every local frame for a smooth vector bundle is associated to a local trivialization. A smooth vector bundle is trivial if and only if it admits a smooth global frame.*

We observe that a local chart (U, φ) on M together with a local frame $\{\sigma_1, \dots, \sigma_k\}$ gives coordinates to E

$$\bar{\varphi} \left(\sum_{i=1}^k v_i \sigma_i(q) \right) = (\varphi(q), v_1, \dots, v_k) = (x_1(q), \dots, x_n(q), v_1, \dots, v_k).$$

Remark 4.12. We observe that TS^1 is trivial, while the Möbius band is not.

4.3 Vector fields

Definition 4.13. A *smooth vector field* on an open set $U \subset M$ is a smooth section of the tangent bundle TM , i.e., a smooth map $X : U \rightarrow TM$ such that $\pi \circ X = \text{id}_U$.

The value at a point q of a vector field X is denoted $X(q)$ or X_q and is an element of T_qM . The set of all smooth vector fields in M is denoted $\text{Vec}(M)$.

Given coordinate open set (U, φ) with coordinates $\{x_i\}$, we notice that for every $q \in U$ we have a basis of the tangent space

$$\left. \frac{\partial}{\partial x_1} \right|_q, \dots, \left. \frac{\partial}{\partial x_n} \right|_q$$

and a smooth vector field X on U is decomposed along the canonical basis

$$X(q) = \sum_{i=1}^n X_i(q) \left. \frac{\partial}{\partial x_i} \right|_q$$

where $X_i : U \subset M \rightarrow \mathbb{R}$ are functions defined on U .

Lemma 4.14. X is smooth in U if and only if X_i are smooth functions.

Proof. Taking the standard chart on $\pi^{-1}(U) \subset TM$ we have

$$\widehat{\varphi}(X(q)) = (x_1, \dots, x_n, \widehat{X}_1(x), \dots, \widehat{X}_n(x))$$

where $\widehat{X}_i = X_i \circ \varphi^{-1}$ is the coordinate representation of X_i . □

We can use bump functions to prove that

Exercise 4.15. Let $q \in M$ and $v \in T_qM$. Then there exists X in $\text{Vec}(M)$ such that $X(q) = v$.

Given $f \in C^\infty(M)$ and $X \in \text{Vec}(M)$ we notice that fX is a new smooth vector field such that $(fX)(q) = f(q)X(q)$. $\text{Vec}(M)$ is a $C^\infty(M)$ -module, i.e., a module over the ring $C^\infty(M)$.

Another way to “couple” vector fields and functions is to use X to differentiate smooth functions: for $f \in C^\infty(M)$ we can set

$$Xf(q) = X|_q f$$

where in the right hand side the vector differentiate the function.

Corollary 4.16. X is smooth if and only if Xf is smooth for every $f \in C^\infty(M)$.

Proof. Let

$$X(q) = \sum_{i=1}^n X_i(q) \left. \frac{\partial}{\partial x_i} \right|_q$$

If we have a chart (U, φ) and $q = \varphi^{-1}(x)$ we have

$$\widehat{Xf}(x) = Xf(\varphi^{-1}(x)) = X_{\varphi^{-1}(x)} f = \sum_{i=1}^n X_i(\varphi^{-1}(x)) \left. \frac{\partial}{\partial x_i} \right|_{\varphi^{-1}(x)} f = \sum_{i=1}^n \widehat{X}_i(x) \left. \frac{\partial f}{\partial x_i} \right|_x$$

Then if f is smooth clearly Xf is smooth. If we know that Xf is smooth for every f smooth, take $f = x_j$ and you get that $X_j(x)$ smooth. □

Remark 4.17. We have proved exactly that the identity of vector fields

$$X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$$

implies the identity of functions (in coordinates)

$$Xf = \sum_{i=1}^n X_i \frac{\partial f}{\partial x_i}$$

where we have removed the “hat” in the notation. In what follows we keep this spirit.

A vector field hence induces a linear map $X : C^\infty(M) \rightarrow C^\infty(M)$ which satisfies

$$X(fg) = f \cdot Xg + g \cdot Xf$$

where $\cdot$ here is usual multiplication of functions. Every vector field is a derivation of the algebra $C^\infty(M)$. Indeed one can prove the converse: every derivation of the algebra $C^\infty(M)$ is a vector field!

Example 4.18. Graphical example of TS^1 and a vector field on it.

Vector fields and smooth maps

If $F : M \rightarrow N$ is a smooth map and X is a vector field on M then for every $q \in M$ we can define F_*X_q which is an element of $T_{F(q)}N$. This in general does not define a vector field on N . (Think for instance the case when F is not injective)

If $X \in \text{Vec}(M)$ and $Y \in \text{Vec}(N)$ we say that Y is F -related to X if $F_*X_q = Y_{F(q)}$ for every $q \in M$.

Exercise 4.19. Prove that $F : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $F(t) = (\cos t, \sin t)$ then $\partial/\partial t$ is F -related to $-y \partial/\partial x + x \partial/\partial y$.

Lemma 4.20. Let $F : M \rightarrow N$ be a smooth map, $X \in \text{Vec}(M)$, $Y \in \text{Vec}(N)$. Then Y is F -related to X if for every $g \in C^\infty(N)$

$$Yg \circ F = X(g \circ F).$$

When F is a diffeomorphism then F_*X is a well defined smooth vector field on N , called the *push-forward* of X via F , and satisfies the following identity

$$(F_*X)g = X(g \circ F) \circ F^{-1}$$

Notice that, at the level of vectors, we have $(F_*X)_q g = X_{F^{-1}(q)}(g \circ F)$.

$$F_* \left(\sum_{i=1}^n X_i \frac{\partial}{\partial x_i} \right) = \sum_{j=1}^n \left(\sum_{i=1}^n X_i \frac{\partial F_j}{\partial x_i} \right) \frac{\partial}{\partial y_j}$$

Exercise 4.21. Let $F : \mathbb{R}_x \rightarrow \mathbb{R}_y^+$ $F(x) = e^x$ then $F_*(\partial/\partial x) = y \partial/\partial y$

4.4 Lie brackets

If one thinks at a vector field as a first order differential operator, one might ask if (or when) a Schwartz-like formula holds: give two vector fields X, Y in $\text{Vec}(M)$, when is it true that $XYf = YXf$ for every $f \in C^\infty(M)$?

Remark 4.22. Notice that in general the map $D : C^\infty(M) \rightarrow C^\infty(M)$ defined by $D = XY$ is not a derivation. A simple check is with $M = \mathbb{R}^2$, $X = \partial/\partial x$, $Y = \partial/\partial y$ and $f(x, y) = x$, $g(x, y) = y$.

Exercise 4.23. Let X, Y in $\text{Vec}(M)$, prove that the operator $D : C^\infty(M) \rightarrow C^\infty(M)$ defined by $D = XY - YX$ is a derivation of $C^\infty(M)$ hence corresponds to a vector field.

The first think we can do is to define the *Lie bracket*

Definition 4.24. We define the *Lie bracket* between X, Y in $\text{Vec}(M)$ as the vector field corresponding to the derivation $[X, Y] := XY - YX$

Lemma 4.25. Let $(U, \{x_i\})$ be a coordinate set and X, Y in $\text{Vec}(M)$. Then if on U we have

$$X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n Y_i(x) \frac{\partial}{\partial x_i}$$

we have

$$[X, Y] = \sum_{i,j=1}^n \left(X_i(x) \frac{\partial Y_j}{\partial x_i}(x) - Y_i(x) \frac{\partial X_j}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j}$$

The proof is left as an exercise.

Exercise 4.26. Compute the Lie bracket $[X, Y]$ between the two vector fields in $\mathbb{R}^3$ with coordinates (x, y, z)

$$X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$$

and

$$X = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}, \quad Y = -\sin z \frac{\partial}{\partial x} + \cos z \frac{\partial}{\partial y}$$

The Lie bracket is clearly a bilinear skew-symmetric form on $\text{Vec}(M)$ (as $\mathbb{R}$ -vector space). It enjoys also more interesting properties related to the fact that $\text{Vec}(M)$ is a module over $C^\infty(M)$ (recall that $C^\infty(M)$ is an associative $\mathbb{R}$ -algebra).³

Proposition 4.27. The Lie bracket satisfies for X, Y, Z in $\text{Vec}(M)$ and f in $C^\infty(M)$

$$(i) \quad [X, Y] = -[Y, X]$$

$$(ii) \quad [X + Y, Z] = [X, Z] + [Y, Z]$$

³recall that an *associative algebra* A is endowed with two compatible operations addition, multiplication (assumed to be associative), and a scalar multiplication by elements in some field K . The addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a vector space over K . A *commutative algebra* is an associative algebra that has a commutative multiplication, or, equivalently, an associative algebra that is also a commutative ring.

$$(iii) [fX, Y] = f[X, Y] - (Yf)X$$

$$(iv) [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Proof. (i) and (ii) are direct consequences of the definition.

(iii) It is enough to prove the statement by applying both sides to every smooth $g \in C^\infty(N)$ and then apply (i) and (ii) to get the general statement. For every $g \in C^\infty(M)$ we have

$$[fX, Y]g = f \cdot XYg - Y(f \cdot Xg) = f \cdot XYg - Yf \cdot Xg - f \cdot YXg = f[X, Y]g - Yf \cdot Xg$$

which proves the statement. We used a $f \cdot g$ as product between smooth functions f and g . (iv) The proof is a simple check applying the whole expansion to a smooth function f and expanding, it is left as an exercise. $\square$

Remark 4.28. Collecting the different properties we have also for every X, Y in $\text{Vec}(M)$ and every f, g in $C^\infty(M)$

$$[fX, gY] = fg[X, Y] - g(Yf)X + f(Xg)Y. \quad (4.3)$$

Lemma 4.29. *Let $F : M \rightarrow N$ be a diffeomorphism. Then for every X_1, X_2 in $\text{Vec}(M)$ we have*

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]. \quad (4.4)$$

Proof. This is a particular case of the more general fact: if F is smooth and X_1, X_2 are F -related with Y_1, Y_2 then $[X_1, X_2]$ is F -related to $[Y_1, Y_2]$. To prove this claim recall that by Lemma 4.20 we have

$$Y_i g \circ F = X_i(g \circ F), \quad i = 1, 2,$$

hence

$$X_i X_j(g \circ F) = X_i(Y_j g \circ F) = Y_i Y_j g \circ F, \quad i = 1, 2,$$

by linearity

$$[X_1, X_2](g \circ F) = [Y_1, Y_2]g \circ F$$

which proves the desired property. Then the statement follows when F is a diffeomorphism and $Y_i = F_*X_i$ for $i = 1, 2$. $\square$

Exercise 4.30. Prove that if $X(x) = Ax$ and $Y(x) = Bx$ are linear vector fields in $\mathbb{R}^n$ then $[X, Y](x) = [A, B]x$ where $[A, B] = AB - BA$ is the commutator of matrices.

Computations in coordinates

It is interesting to prove some identities in coordinates and see the difference between proofs with intrinsic language.

Exercise 4.31. Let us prove the following identity in coordinates

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2]. \quad (4.5)$$

We set, we omit the sum

$$X = X_i \frac{\partial}{\partial x_i}, \quad Y = Y_j \frac{\partial}{\partial x_j}$$

so that we have

$$[X, Y] = \left(X_i \frac{\partial Y_k}{\partial x_i} - Y_i \frac{\partial X_k}{\partial x_i} \right) \frac{\partial}{\partial x_k}$$

We also have

$$F_* X = \underbrace{X_i \frac{\partial F_\ell}{\partial x_i}}_{\tilde{X}_\ell} \frac{\partial}{\partial y_\ell}, \quad F_* Y = \underbrace{Y_j \frac{\partial F_\ell}{\partial x_j}}_{\tilde{Y}_\ell} \frac{\partial}{\partial y_\ell}$$

Then

$$F_*[X, Y] = \left(X_i \frac{\partial Y_k}{\partial x_i} - Y_i \frac{\partial X_k}{\partial x_i} \right) \frac{\partial F_\ell}{\partial x_k} \frac{\partial}{\partial y_\ell}$$

while

$$[F_* X, F_* Y] = \left(\tilde{X}_\ell \frac{\partial \tilde{Y}_k}{\partial y_\ell} - \tilde{Y}_\ell \frac{\partial \tilde{X}_k}{\partial y_\ell} \right) \frac{\partial}{\partial y_k}$$

expanding

$$[F_* X, F_* Y] = \left(X_i \frac{\partial F_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell} \left(Y_j \frac{\partial F_k}{\partial x_j} \right) - Y_j \frac{\partial F_\ell}{\partial x_j} \frac{\partial}{\partial y_\ell} \left(X_i \frac{\partial F_k}{\partial x_i} \right) \right) \frac{\partial}{\partial y_k}$$

so that we have two terms

$$\begin{aligned} [F_* X, F_* Y] &= \left(X_i Y_j \frac{\partial F_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell} \left(\frac{\partial F_k}{\partial x_j} \right) - X_i Y_j \frac{\partial F_\ell}{\partial x_j} \frac{\partial}{\partial y_\ell} \left(\frac{\partial F_k}{\partial x_i} \right) \right) \frac{\partial}{\partial y_k} \\ &\quad + \left(X_i \frac{\partial F_\ell}{\partial x_i} \frac{\partial Y_j}{\partial y_\ell} \frac{\partial F_k}{\partial x_j} - Y_j \frac{\partial F_\ell}{\partial x_j} \frac{\partial X_i}{\partial y_\ell} \frac{\partial F_k}{\partial x_i} \right) \frac{\partial}{\partial y_k} \end{aligned}$$

the first row is zero since we have a sum of the kind $a_{ij}v_i v_j$ with a_{ij} symmetric (why?)

$$[F_* X, F_* Y] = \left(X_i \underbrace{\frac{\partial F_\ell}{\partial x_i} \frac{\partial Y_j}{\partial y_\ell}}_{\tilde{X}_\ell \tilde{Y}_j} \frac{\partial F_k}{\partial x_j} - Y_j \underbrace{\frac{\partial F_\ell}{\partial x_j} \frac{\partial X_i}{\partial y_\ell}}_{\tilde{Y}_\ell \tilde{X}_i} \frac{\partial F_k}{\partial x_i} \right) \frac{\partial}{\partial y_k}$$

now notice that

$$\frac{\partial F_\ell}{\partial x_i} \frac{\partial Y_j}{\partial y_\ell} = \frac{\partial Y_j}{\partial x_i}, \quad \frac{\partial F_\ell}{\partial x_j} \frac{\partial X_i}{\partial y_\ell} = \frac{\partial X_i}{\partial x_j}$$

so that

$$\begin{aligned} [F_* X, F_* Y] &= \left(X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial F_k}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \frac{\partial F_k}{\partial x_i} \right) \frac{\partial}{\partial y_k} \\ &= \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_j \frac{\partial X_i}{\partial x_j} \right) \frac{\partial F_k}{\partial x_j} \frac{\partial}{\partial y_k} \\ &= F_*[X, Y] \end{aligned}$$

Exercise 4.32. Prove the following identity in coordinates

$$[fX, gY] = fg[X, Y] - g(Yf)X + f(Xg)Y. \quad (4.6)$$

Chapter 5

Integral curves and flows

*Knowing what is big and what is small is more important
than being able to solve differential equations*
Stanislaw Ulam (1909–1984)

In this chapter we discuss integral curves and flows of vector fields, thanks to which we can give a more geometric interpretation of the Lie bracket.

5.1 Integral curves and flows

Definition 5.1. Let M be a smooth manifold and $X \in \text{Vec}(M)$. An *integral curve* of the vector field X is a smooth curve $\gamma : J \rightarrow M$, where $J \subset \mathbb{R}$ is an open interval, such that

$$\dot{\gamma}(t) = X(\gamma(t)), \quad \forall t \in J. \quad (5.1)$$

Take a vector field X defined on M and a chart (U, φ) . Write X in coordinates, i.e., consider the vector field φ_*X

$$\widehat{X} = \varphi_*X = \sum_{i=1}^n \widehat{X}_i(x) \frac{\partial}{\partial x_i}$$

Consider a solution $x(t)$ to the ODE associated to $\widehat{X}$ in $\mathbb{R}^n$

$$\dot{x} = \widehat{X}(x)$$

defined on some open interval J containing 0. Recall that this means for every $t \in J$ the curve $x(t)$ satisfies in $\mathbb{R}^n$ the system of autonomous differential equations

$$\dot{x}_i(t) = \widehat{X}_i(x_1(t), \dots, x_n(t)), \quad i = 1, \dots, n$$

Then it is easy to see that $\gamma(t) = \varphi^{-1}(x(t))$ is an integral curve of X . Indeed

$$\dot{\gamma}(t) = \varphi_*^{-1}(\dot{x}(t)) = \varphi_*^{-1}(\widehat{X}(x(t))) = (\varphi_*^{-1}\widehat{X})(\varphi^{-1}(x(t))) = X(\gamma(t)).$$

A consequence of the classical existence uniqueness theorem for solution of ODEs ensures that, for every initial condition, there exists a unique integral curve of a smooth vector field, defined on some open interval.

Theorem 5.2. Let $X \in \text{Vec}(M)$ and fix $t_0 \in \mathbb{R}$, $q_0 \in M$. Then there exists a unique integral curve $\gamma : I \rightarrow M$ of X such that $\gamma(t_0) = q_0$ defined on some maximal open interval I containing t_0 .

Remark 5.3. Notice that if $\gamma : I \rightarrow M$ is an integral curve then $\gamma_c : I_c \rightarrow M$ defined by $\gamma_c(t) = \gamma(t - c)$ is also an integral curve (defined for $t \in I_c := I + c$).

Hence we can always shift initial time and consider integral curves $\gamma : I \rightarrow M$ where I is an open interval containing 0 and $\gamma(0) = q_0$. Notice that this is a consequence of the fact that the vector field is autonomous, i.e., the right hand side of the corresponding ODE does not depend explicitly on t .

Since vector fields under consideration are smooth, the corresponding ODEs in $\mathbb{R}^n$ have smooth coefficients. Thus we have not only existence and uniqueness of solutions but also smoothness with respect to initial data. This is translated into the following result.

Theorem 5.4. Let $X \in \text{Vec}(M)$. There exists an open set $\mathcal{U} \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and a map $\Phi^X : \mathcal{U} \rightarrow M$ of class C^∞ such that

- (a) for $q \in M$ the set $\mathcal{I}^q = \{t \in \mathbb{R} \mid (t, q) \in \mathcal{U}\}$ open neighborhood of 0,
- (b) for $q \in M$ the curve $\gamma^q : \mathcal{I}^q \rightarrow M$ given by $\gamma^q(t) = \Phi^X(t, q)$ integral curve of X , $\gamma^q(0) = q$,
- (c) for every $s \in \mathcal{I}^q$ and $t \in \mathcal{I}^{\Phi^X(s, q)}$ we have $t + s \in \mathcal{I}^q$ and

$$\Phi^X(t, \Phi^X(s, q)) = \Phi^X(t + s, q)$$

We denote by $\Phi_t^X := \Phi^X(t, \cdot)$ the flow of X at time t , which is a map defined on the open set $\mathcal{U}^t = \{q \in M \mid (t, q) \in \mathcal{U}\}$. Then the property (c) is rewritten as $\Phi_t^X \circ \Phi_s^X = \Phi_{t+s}^X$.

Remark 5.5. It will be also convenient to use the exponential notation $\Phi_t^X = e^{tX}$, in such a way that the property (c) is written as

$$e^{tX} \circ e^{sX} = e^{(t+s)X}.$$

Notice that by definition for every $t \in \mathcal{I}^q$ we have¹

$$\frac{d}{dt} e^{tX}(q) = X(e^{tX}(q)), \quad X(q) = \left. \frac{d}{dt} e^{tX}(q) \right|_{t=0}.$$

Remark 5.6. When $X(x) = Ax$ is a linear vector field on $\mathbb{R}^n$, where A is a $n \times n$ matrix, the corresponding flow is the matrix exponential $\Phi_t^X(x) = e^{tA}x$.

Example 5.7. Let us compute the integral curves and flow of the vector field

$$X = x\partial_x + y\partial_y.$$

Given (x_0, y_0) in $\mathbb{R}^2$ the integral curve which passes through (x_0, y_0) at $t = 0$ is the solution

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \\ x(0) = x_0, y(0) = y_0 \end{cases}$$

The solution is easily computed $(x(t), y(t)) = (x_0 e^t, y_0 e^t)$. If $(x_0, y_0) = (0, 0)$ the vector field is zero at that point and the corresponding integral curve is constant.

¹notice the formula $\frac{d}{dt} e^{tX} = X e^{tX}$ (this should be compared with the more formal (5.9))

Appendix: on completeness

A vector field $X \in \text{Vec}(M)$ is called *complete* if, for every $q_0 \in M$, the maximal solution of the equation is defined on $I = \mathbb{R}$.

The classical theory of ODE ensures completeness of the vector field $X \in \text{Vec}(M)$ in the following cases:

- (i) M is a compact manifold,
- (ii) X has compact support in M ,
- (iii) $M = \mathbb{R}^n$ and X has sub-linear growth at infinity, i.e., there exists $C_1, C_2 > 0$ such that

$$\|X(x)\| \leq C_1\|x\| + C_2, \quad \forall x \in \mathbb{R}^n.$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

When we are interested in the behavior of the trajectories of a vector field $X \in \text{Vec}(M)$ in a compact subset K of M , the assumption of completeness is not restrictive.

Indeed consider an open neighborhood U of a compact K in M . There exists a smooth bump function $\psi : M \rightarrow \mathbb{R}$ that is identically 1 on K , and that vanishes out of U . Then the vector field ψX is complete, since it has compact support in M . Moreover, the vector fields X and ψX coincide on K , hence their integral curves coincide on K as well.

Exercise 5.8. Let $f : M \rightarrow \mathbb{R}$ be a smooth function and $X \in \text{Vec}(M)$ be a complete vector field. Denote by γ an integral curve of X and let $\varphi_f : \mathbb{R} \rightarrow \mathbb{R}$ be a solution of the differential equation $\dot{\varphi}(t) = f(\gamma(\varphi(t)))$. Prove that the curve defined by $\gamma_f(t) = \gamma(\varphi_f(t))$ is an integral curve of fX .

Pushforward and flows

If $F : M \rightarrow N$ is a diffeomorphism and $X \in \text{Vec}(M)$, then F_*X is the vector field whose integral curves are the image under F of integral curves of X . The vector field F_*X is also called the *pushforward of X through F* .

Lemma 5.9. Let $F : M \rightarrow N$ be a diffeomorphism, $X \in \text{Vec}(M)$. Then

$$e^{tF_*X} = F \circ e^{tX} \circ F^{-1}, \quad (5.2)$$

Proof. Given $q \in N$ consider the curve $\eta(t) = F \circ e^{tX} \circ F^{-1}(q)$, we want to prove that $\eta(t)$ is an integral curve of F_*X . Indeed

$$\begin{aligned} \dot{\eta}(t) &= \frac{d}{dt} F \circ e^{tX} \circ F^{-1}(q) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} F \circ e^{\varepsilon X} \circ e^{tX} \circ F^{-1}(q) = \\ &= F_*(X(e^{tX} \circ F^{-1}(q))) = (F_*X)(F \circ e^{tX} \circ F^{-1}(q)) = (F_*X)(\eta(t)) \end{aligned}$$

This proves that

$$F \circ e^{tX} \circ F^{-1}(q) = \eta(t) = e^{tF_*X}(q)$$

for every $q \in N$, which is the statement. □

Along the lines we also proved that for $q \in N$

$$(F_*X)(q) = \frac{d}{dt} \Big|_{t=0} e^{tF_*X}(q) = \frac{d}{dt} \Big|_{t=0} F \circ e^{tX} \circ F^{-1}(q), \quad (5.3)$$

5.2 Lie derivatives and Lie brackets

In this section we introduce the *Lie derivative* of Y in the direction of X .

Definition 5.10. Let $X, Y \in \text{Vec}(M)$. We define the *Lie derivative* of Y wrt X as the vector field

$$L_X Y := \frac{\partial}{\partial t} \Big|_{t=0} e_*^{-tX} Y \quad (5.4)$$

More precisely, this means that for every $q \in M$ we define

$$L_X Y(q) = \lim_{t \rightarrow 0} \frac{1}{t} \left(e_*^{-tX}(Y_{e^{tX}(q)}) - Y_q \right)$$

Notice that for every t the vector $e_*^{-tX}(Y_{e^{tX}(q)})$ belong to $T_q M$ hence the limit makes sense in $T_q M$.

Remark 5.11. The geometric meaning of the Lie bracket can be understood by writing explicitly

$$L_X Y|_q = \frac{\partial}{\partial t} \Big|_{t=0} e_*^{-tX} Y|_q = \frac{\partial}{\partial t} \Big|_{t=0} e_*^{-tX}(Y|_{e^{tX}(q)}) = \frac{\partial}{\partial s \partial t} \Big|_{t=s=0} e^{-tX} \circ e^{sY} \circ e^{tX}(q). \quad (5.5)$$

Remark 5.12. We have used that for $X, Y \in \text{Vec}(M)$ we can reinterpret the pushforward of Y with respect to X as follows:

$$(e_*^{tX} Y)|_q = e_*^{tX}(Y|_{e^{-tX}(q)}) = \frac{d}{ds} \Big|_{s=0} e^{tX} \circ e^{sY} \circ e^{-tX}(q). \quad (5.6)$$

where we used that

$$Y(e^{-tX}(q)) = \frac{d}{ds} \Big|_{s=0} e^{sY} \circ e^{-tX}(q)$$

and the definition of pushforward.

The main goal of the section is to prove the following result, which is not evident at a first glance.

$$L_X Y = [X, Y] \quad (5.7)$$

Before going into the proof of the theorem we need some preliminary observations.

Vector fields as operators on functions

The action of a vector field $X \in \text{Vec}(M)$ on the algebra $C^\infty(M)$ of smooth functions on M can be rewritten as follows

$$Xf(q) = \frac{d}{dt} \Big|_{t=0} f(e^{tX}(q)), \quad q \in M. \quad (5.8)$$

In other words Xf is the derivative of the function f along the integral curves of X .

Notice that given a point q there exists an open neighborhood $\mathcal{U}$ of $(0, q) \in \mathbb{R} \times M$ where the function $f(t, q) := f(e^{tX}(q))$ is defined and smooth.

The next statement makes more precise in which sense Xf is the first order term in the expansion of $f \circ e^{tX}$ with respect to t .

Lemma 5.13. Let us denote $f : \mathcal{U} \rightarrow \mathbb{R}$ the function $f(t, q) := f(e^{tX}(q))$. Then we can write

$$f(t, q) = f(q) + g(t, q)t$$

where $g : \mathcal{U} \rightarrow \mathbb{R}$ is smooth (both in q and t) and satisfies $g(0, q) = Xf(q)$.

Proof. Let us fix $(t, q) \in \mathcal{U}$ and consider $F : \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(s) := f(st, q)$. The fundamental theorem of calculus

$$F(1) - F(0) = \int_0^1 \frac{d}{ds} F(s) ds$$

is rewritten using that $\frac{d}{ds} F(s) = t \frac{\partial f}{\partial t}(st, q)$ as follows

$$f(t, q) = f(0, q) + t \int_0^1 \frac{\partial f}{\partial t}(st, q) ds$$

which proves the statement with $g(t, q) := \int_0^1 \frac{\partial f}{\partial t}(st, q) ds$ which is smooth. Clearly we have also $g(0, q) = \frac{\partial f}{\partial t}(0, q) = Xf(q)$. $\square$

Remark 5.14. The previous result intuitively says that $t \mapsto f_t = f \circ e^{tX}$ is “smooth” (but formally we should give a smooth structure to $C^\infty(M)$) and states that Xf represents the first order term in the expansion of f_t when $t \rightarrow 0$.

We can state this in the following manner: for $q \in M$ one has

$$f_t(q) = f(q) + t(Xf)(q) + r_1(t, q)$$

where $\sup_{q \in U} t^{-1}|r_1(t, q)| \rightarrow 0$ for $t \rightarrow 0$ (and the same is true also for every spatial derivative). We can write this as

$$f_t(q) = f(q) + t(Xf)(q) + o(t).$$

Since the remainder is locally uniform, we can interpret this as the identity of functions

$$f_t = f + t(Xf) + o(t)$$

Exercise 5.15. Let $f \in C^\infty(M)$ and $X \in \text{Vec}(M)$, and denote $f_t = f \circ e^{tX}$. Prove the following formulas²

$$\frac{d}{dt} f_t = Xf_t, \tag{5.9}$$

$$f_t = f + tXf + \frac{t^2}{2!}X^2f + \frac{t^3}{3!}X^3f + \dots + \frac{t^k}{k!}X^kf + o(t^k). \tag{5.10}$$

where as before $o(t^k)$ is a function $r_k(t, q)$ such that $\sup_{q \in U} t^{-k}|r_k(t, q)| \rightarrow 0$ for $t \rightarrow 0$.

²we can think to the formula

$$f \circ e^{tX} = \left(\text{Id} + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \dots + \frac{t^k}{k!}X^k + o(t^k) \right) f$$

Theorem 5.16. *The Lie derivative $L_X Y$ is a smooth vector field and, as derivations on functions, it satisfies*

$$L_X Y = [X, Y]. \quad (5.11)$$

Proof. We want to prove that for every $q \in M$

$$L_X Y(q) = \left. \frac{\partial}{\partial t} \right|_{t=0} (e_*^{-tX} Y)(q) = [X, Y](q).$$

Set $g = f \circ e^{-tX}$ for f smooth and we have ³

$$(e_*^{-tX} Y)f = Y(f \circ e^{-tX}) \circ e^{tX} = Yg \circ e^{tX} = Yg + t(XYg) + o(t)$$

Now use that $g = f - t(Xf) + o(t)$ and

$$\begin{aligned} Yg &= Yf - tYXf + o(t) \\ XYg &= XYf + o(1) \end{aligned}$$

hence collecting the results we have

$$(e_*^{-tX} Y)f = Yf + t[X, Y]f + o(t)$$

and the statement is proved. □

5.3 Lie brackets and commutativity of flows

Lemma 5.17. *Let $X, Y \in \text{Vec}(M)$ be complete. Then the two properties are equivalent*

$$(i) \quad [X, Y] = 0$$

$$(ii) \quad \text{for every } t \in \mathbb{R} \text{ we have } e_*^{-tX} Y = Y.$$

Proof. If $e_*^{-tX} Y = Y$ then by definition of Lie brackets clearly $[X, Y] = 0$.

Assume now that $[X, Y] = \left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Y = 0$ and we want to prove that $\left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Y = 0$ for all $t \in \mathbb{R}$. Indeed we have

$$\begin{aligned} \frac{d}{dt} e_*^{-tX} Y &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e_*^{-(t+\varepsilon)X} Y = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e_*^{-tX} e_*^{-\varepsilon X} Y \\ &= e_*^{-tX} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} e_*^{-\varepsilon X} Y = e_*^{-tX} [X, Y] = 0, \end{aligned}$$

It follows that $e_*^{-tX} Y$ does not depend on t , hence it coincides with its value at $t = 0$ which is $e_*^{-0X} Y = \text{id}_* Y = Y$. □

Remark 5.18. Notice that since $[X, Y] = -[Y, X]$, then $[X, Y] = 0$ is also equivalent to $e_*^{-tY} X = X$ for every $t \in \mathbb{R}$.

³Setting $h(t, q) := g(-t, q)$ we have $f(e^{-tX}(q)) = f(q) - h(t, q)t$, with h smooth and $h(0, q) = Xf(q)$.

To end this section, we show that the Lie bracket of two vector fields is zero (i.e., they commute as operator on functions) if and only if their flows commute. We state this for complete vector fields but indeed the result is local so we can apply the argument in the “Appendix: on completeness” and the result is indeed general.

Proposition 5.19. *Let $X, Y \in \text{Vec}(M)$ be complete. The following properties are equivalent:*

- (i) $[X, Y] = 0$,
- (ii) $e^{tX} \circ e^{sY} = e^{sY} \circ e^{tX}, \quad \forall t, s \in \mathbb{R}$.

Proof. (i) $\Rightarrow$ (ii). Fix $t \in \mathbb{R}$. Let us show that $\phi_s := e^{-tX} \circ e^{sY} \circ e^{tX}$ is the flow generated by Y . Indeed we have

$$\begin{aligned} \frac{\partial}{\partial s} \phi_s(q) &= \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} e^{-tX} \circ e^{(s+\varepsilon)Y} \circ e^{tX}(q) \\ &= \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} e^{-tX} \circ e^{\varepsilon Y} \circ e^{tX} \circ \underbrace{e^{-tX} \circ e^{sY} \circ e^{tX}}_{\phi_s}(q) \\ &= e_*^{-tX} Y \circ \phi_s(q) = Y \circ \phi_s(q) \end{aligned}$$

where in the last equality we used the previous lemma. Using uniqueness of the flow generated by a vector field we get

$$e^{-tX} \circ e^{sY} \circ e^{tX} = e^{sY}, \quad \forall t, s \in \mathbb{R},$$

which is equivalent to (ii).

(ii) $\Rightarrow$ (i). Using that $[X, Y] = L_X Y$ and the characterization (5.5) we have

$$\begin{aligned} [X, Y](q) &= \frac{\partial}{\partial s \partial t} \Big|_{t=s=0} e^{-tX} \circ e^{sY} \circ e^{tX}(q) = \frac{\partial}{\partial s \partial t} \Big|_{t=s=0} e^{-tX} \circ e^{tX} \circ e^{sY}(q) \\ &= \frac{\partial}{\partial s \partial t} \Big|_{t=s=0} e^{sY}(q) = 0. \end{aligned}$$

□

Exercise 5.20. Let $X, Y \in \text{Vec}(M)$ and $q \in M$. Consider the curve on M

$$\gamma(t) = e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(q).$$

Prove that for every $f \in C^\infty(M)$ we have

$$f(\gamma(t)) = f(q) + t^2[X, Y]f(q) + o(t^2)$$

This can be interpreted in the following way: the curve $t \mapsto \gamma(\sqrt{t})$ is differentiable at $t = 0$ (but not smooth!), and its tangent vector at $t = 0$ is $[X, Y](q)$.

Exercise 5.21 (Another proof of Jacobi identity). Prove that the Lie bracket satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (5.12)$$

by differentiating the identity $e_*^{tX}[Y, Z] = [e_*^{tX}Y, e_*^{tX}Z]$ with respect to t at $t = 0$.

Exercise 5.22. Let $X, Y \in \text{Vec}(M)$. Using the semigroup property of the flow, prove that

$$\frac{d}{dt} e_*^{-tX} Y = e_*^{-tX} [X, Y]. \quad (5.13)$$

Deduce the following formal series expansion

$$\begin{aligned} e_*^{-tX} Y &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad } X)^n Y \\ &= Y + t[X, Y] + \frac{t^2}{2} [X, [X, Y]] + \frac{t^3}{6} [X, [X, [X, Y]]] + \dots \end{aligned} \quad (5.14)$$

where we have introduced the notation $(\text{ad } X)Y := [X, Y]$.

5.4 Left-invariant vector fields on a Lie group

A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with an operation

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

that is bilinear, skew-symmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for every $X, Y, Z \in \mathfrak{g}$.

Remark 5.23. Given an associative algebra A with multiplication $(x, y) \mapsto xy$ denoted with juxtaposition, we can always give a Lie algebra structure to A by defining the Lie bracket

$$[x, y] = xy - yx$$

Indeed in this case the Jacobi identity is trivially satisfied.

The set of smooth vector fields $\text{Vec}(M)$ on a smooth manifold M is naturally a Lie algebra using as operation the Lie brackets of vector fields.

If M is a Lie group we can give the following definition.

Definition 5.24. Let G be a Lie group and X a vector field on G . We say that X is left-invariant if for every $g \in G$ we have $(L_g)_* X = X$, i.e., we have

$$(L_g)_* X_h = X_{gh}, \forall g, h \in G$$

Let us denote by $\text{Vec}^L(G)$ the subset of left-invariant vector fields on G .

Proposition 5.25. *The Lie bracket of left-invariant vector fields is left-invariant. Hence $\text{Vec}^L(G)$ is a Lie algebra.*

Proof. If X and Y are left-invariant then $(L_g)_* X = X$ and $(L_g)_* Y = Y$ for every $g \in G$. Hence

$$(L_g)_* [X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]$$

for every g , which proves that $[X, Y]$ is left-invariant. $\square$

Notice that a left-invariant vector field in particular satisfies the following property

$$X_g = (L_g)_* X_e, \quad \forall g \in G$$

where e is the identity of the group G . Indeed this property characterizes left-invariant vector fields since if this is true we have

$$(L_g)_* X_h = (L_g)_* (L_h)_* X_e = (L_g \circ L_h)_* X_e = (L_{gh})_* X_e = X_{gh}$$

This says that a left-invariant vector field is characterized by its value at one point, for instance the origin. Hence we have

Proposition 5.26. *The map $\varepsilon : \text{Vec}^L(G) \rightarrow T_e G$ given by $\varepsilon(X) = X_e$ is an isomorphism of vector spaces. In particular $\dim(\text{Vec}^L(G)) = \dim G$.*

The two spaces $\text{Vec}^L(G)$ and $T_e G$ are then identified and we denote by $\mathfrak{g}$ both the set of left-invariant vector fields of G or, equivalently, the tangent space to the identity.

Remark 5.27. Analogously we can introduce *right-invariant vector fields* $\text{Vec}^R(G)$. This space can be also identified through right translations with the tangent space to the identity but notice that in general a left-invariant vector field is not right-invariant.

It is a convention that the Lie algebra of a Lie group is the one of left-invariant vector fields.

Exercise 5.28. Consider in $M = \mathbb{R}^2 \times S^1$ with coordinates (x, y, θ) the two vector fields

$$X = \cos \theta \partial_x + \sin \theta \partial_y, \quad Y = \partial_\theta$$

1. Prove that $[X, Y] = \sin \theta \partial_x - \cos \theta \partial_y$ and give a geometrical interpretation of this fact in terms of the Exercise 5.20.
2. Prove that $X, Y, [X, Y]$ are linearly independent at every point of $\mathbb{R}^2 \times S^1$
3. Prove that the following is a group law in M

$$(x, y, \theta) \cdot (x', y', \theta') = (x + (\cos \theta)x' - (\sin \theta)y', y + (\sin \theta)x' + (\cos \theta)y', \theta + \theta')$$

and that M is a Lie group.

4. Prove that X, Y are left-invariant.

Chapter 6

Vector distributions: integrability vs non-integrability

*If a notion bears a personal name, then this name
is not the name of the discoverer.*

The Arnold Principle.
V.I. Arnold (1937–2010)

In this chapter, we discuss some results about integrability of vector distributions. The most classical result is Frobenius theorem, stating necessary and sufficient conditions for a vector distribution to be integrable, i.e., to admit a tangent submanifold.

We then discuss also distributions which satisfy an “opposite” assumption than the one of Frobenius, i.e., bracket-generating distributions and the corresponding Chow theorem.

Despite being named for Ferdinand Georg Frobenius work in 1877, the two implications of Frobenius theorem were proven by Feodor Deahna, 1840, and Alfred Clebsch, 1866.

Similarly Chow theorem is named after Wei-Liang Chow who proved it in 1939, but Petr Konstantovich Rashevskii proved it independently in 1938. (cf. The Arnold principle)

6.1 Diffeomorphisms built with flows of vector fields

Most of the results of this chapter are based on local diffeomorphisms built with compositions of flows of vector fields. We start with a proposition computing the differential of such a map.

Proposition 6.1. *Let M be a smooth n -dimensional manifold and $X_1, \dots, X_n$ be linearly independent vector fields at a point $q_0 \in M$. Then the map*

$$\psi : \mathbb{R}^n \rightarrow M, \quad \psi(t_1, \dots, t_n) = e^{t_1 X_1} \circ \dots \circ e^{t_n X_n}(q_0),$$

is a local diffeomorphism at 0. Moreover we have, denoting $t = (t_1, \dots, t_n)$,

$$\frac{\partial \psi}{\partial t_i}(t) = \left(e_*^{t_1 X_1} \dots e_*^{t_{i-1} X_{i-1}} X_i \right) (\psi(t)). \quad (6.1)$$

Proof. The map ψ is clearly smooth. It is easy to compute the differential of the map at $t = 0$.

$$\frac{\partial \psi}{\partial t_i}(0) = \frac{d}{ds} \Big|_{s=0} \psi(0, \dots, 0, s, 0, \dots, 0) = \frac{d}{ds} \Big|_{s=0} e^{s X_i}(q_0) = X_i(q_0) \quad (6.2)$$

Hence the partial derivatives of ψ are linearly independent, thus ψ is a local diffeo. For t in a neighborhood of 0 it is less trivial to compute the differential. First let us observe that by definition of the map ψ we have that

$$e^{-t_i X_i} \dots \circ e^{-t_1 X_1} (\psi(t)) = e^{t_{i+1} X_{i+1}} \dots \circ e^{t_n X_n} (q_0) \quad (6.3)$$

Hence we have

$$\begin{aligned} \frac{\partial \psi}{\partial t_i}(t) &= \frac{d}{ds} \Big|_{s=0} \psi(t_1, \dots, t_i + s, \dots, t_n) \\ &= \frac{d}{ds} \Big|_{s=0} e^{t_1 X_1} \circ \dots \circ e^{(t_i + s) X_i} \circ \dots \circ e^{t_n X_n} (q_0) \\ &= \frac{d}{ds} \Big|_{s=0} e^{t_1 X_1} \circ \dots \circ e^{t_i X_i} \circ e^{s X_i} \circ e^{t_{i+1} X_{i+1}} \dots \circ e^{t_n X_n} (q_0) \end{aligned}$$

and using (6.3)

$$\begin{aligned} \frac{\partial \psi}{\partial t_i}(t) &= \frac{d}{ds} \Big|_{s=0} e^{t_1 X_1} \circ \dots \circ e^{t_i X_i} \circ e^{s X_i} \circ e^{-t_i X_i} \dots \circ e^{-t_1 X_1} (\psi(t)) \\ &= (e_*^{t_1 X_1} \dots e_*^{t_i X_i} X_i) (\psi(t)) = (e_*^{t_1 X_1} \dots e_*^{t_{i-1} X_{i-1}} X_i) (\psi(t)). \end{aligned}$$

where in the last identity we used (5.6) and the fact that the differential of a composition is the composition of differentials. Notice that $e_*^{tX} X = X$ for every X . $\square$

6.2 Rectification of vector fields

As a direct consequence of the previous result, every vector field locally around a point which is not singular can be “rectified”, in the sense that its flow in coordinates is given by straight lines.

Corollary 6.2 (Rectification of a vector field). *Let X be a vector field on M and $q_0 \in M$ such that $X(q_0) \neq 0$. Then there exists coordinates $(x_1, \dots, x_n)$ defined by $\varphi : U \subset M \rightarrow \mathbb{R}^n$ on a neighborhood U of q_0 such that $\varphi_* X = \partial / \partial x_1$.*

Proof. Let $X_1 := X$ be the first element of a family of vector fields $X_1, \dots, X_n$ that are linearly independent at $q_0 \in M$. Notice that the existence of such a family is guaranteed by Exercise 4.15.

Let now $\psi : \mathbb{R}^n \rightarrow M$ be the map build in Proposition 6.1 associated with $X_1, \dots, X_n$. Let $V \subset \mathbb{R}^n$ be the neighborhood of q_0 where ψ is a local diffeomorphism on $U = \psi(V)$. Notice that the identity (6.1) for $i = 1$ gives

$$\frac{\partial \psi}{\partial t_1}(t) = X_1(\psi(t)). \quad (6.4)$$

It follows that

$$\psi_* \frac{\partial}{\partial t_1} = X_1.$$

Using $\varphi = \psi^{-1} : U \subset M \rightarrow V \subset \mathbb{R}^n$ as a coordinate map we have $\varphi_* X_1 = \partial / \partial t_1$. $\square$

Another consequence of Proposition 6.1 is a characterization of families $X_1, \dots, X_n$ of vector fields that can be simultaneously “rectified”, i.e., if they appear as a family of coordinate vector fields.

Theorem 6.3. *Let M be a smooth n -dimensional manifold and $X_1, \dots, X_n$ be linearly independent vector fields at a point $q_0 \in M$. Then there exists local coordinates $\varphi : U \subset M \rightarrow \mathbb{R}^n$ in a neighborhood U of q_0 such that*

$$\varphi_* X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n, \quad (6.5)$$

if and only if $[X_i, X_j] = 0$ for every $i, j = 1, \dots, n$.

Proof. (i) If a local coordinate map $\varphi : U \rightarrow \mathbb{R}^n$ satisfying (6.5) exists then

$$[X_i, X_j] = \left[\varphi_*^{-1} \frac{\partial}{\partial x_i}, \varphi_*^{-1} \frac{\partial}{\partial x_j} \right] = \varphi_*^{-1} \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$$

since the coordinate vector fields commute.

(ii). Let us consider the map ψ associated to $X_1, \dots, X_n$ and defined by

$$\psi : \mathbb{R}^n \rightarrow M, \quad \psi(t_1, \dots, t_n) = e^{t_1 X_1} \circ \dots \circ e^{t_n X_n}(q_0),$$

which is a local diffeomorphism at 0. For $t = (t_1, \dots, t_n)$ small we have

$$\frac{\partial \psi}{\partial t_i}(t) = \left(e_*^{t_1 X_1} \dots e_*^{t_{i-1} X_{i-1}} X_i \right) (\psi(t)). \quad (6.6)$$

Remember that $[X, Y] = 0$ implies $e_*^{tX} Y = Y$ (cf. Lemma 5.17). Hence if we assume $[X_i, X_j] = 0$ for every $i, j = 1, \dots, n$, then we have

$$\frac{\partial \psi}{\partial t_i}(t) = X_i(\psi(t)). \quad (6.7)$$

and the statement is proved using $\varphi = \psi^{-1}$. □

Example 6.4. Let us consider $X = x\partial_x + y\partial_y$ and find coordinates that rectify X around $(1, 0)$. Notice that $Y = -y\partial_x + x\partial_y$ is linearly independent from X at $(1, 0)$.

Following the construction a map that rectify X is the inverse of the map

$$\psi : \mathbb{R}^2 \rightarrow M, \quad \psi(t_1, t_2) = e^{t_1 X} \circ e^{t_2 Y}(1, 0)$$

It is easy to compute

$$e^{t_2 Y}(1, 0) = (\cos t_2, \sin t_2), \quad e^{t_1 X}(x_0, y_0) = (e^{t_1} x_0, e^{t_1} y_0)$$

Hence

$$\psi(t_1, t_2) = (e^{t_1} \cos t_2, e^{t_1} \sin t_2)$$

One might check that $\psi_* \partial / \partial t_1 = X$. It also holds $\psi_* \partial / \partial t_2 = Y$. This last fact is a consequence of $[X, Y] = 0$.

6.3 Frobenius theorem

In this section we prove Frobenius theorem about vector distributions.

Definition 6.5. Let M be a smooth manifold. A *vector distribution* D of rank m on M is a family of vector subspaces $D_q \subset T_q M$ where $\dim D_q = m$ for every q .

A vector distribution D is said to be *smooth* if, for every point $q_0 \in M$, there exists a neighborhood U of q_0 and a family of vector fields $X_1, \dots, X_m$ defined on U such that

$$D_q = \text{span}\{X_1(q), \dots, X_m(q)\}, \quad \forall q \in U. \quad (6.8)$$

Definition 6.6. A smooth vector distribution D on M is said to be *flat* if for every point $q_0 \in M$ there exists a neighborhood U of q_0 and a local diffeomorphism $\varphi : U \rightarrow \mathbb{R}^n$ such that $\varphi_{*,q}(D_q) = \mathbb{R}^m \times \{0\}$ for all $q \in U$.

Example 6.7. Let us consider in $\mathbb{R}^3$ the distribution $D = \text{span}\{X, Y\}$ spanned by the two vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$$

Let us show that this distribution is not flat. If D is flat, there exists a local diffeomorphism $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ around the origin (playing the role of the inverse of φ) which satisfies

$$\psi_* \frac{\partial}{\partial x} = X, \quad \psi_* \frac{\partial}{\partial y} = Y$$

Writing explicitly this condition one obtain the system of first order PDEs

$$\begin{cases} \frac{\partial \psi_1}{\partial x} = 1 \\ \frac{\partial \psi_2}{\partial x} = 0 \\ \frac{\partial \psi_3}{\partial x} = -\frac{\psi_2}{2} \end{cases} \quad \begin{cases} \frac{\partial \psi_1}{\partial y} = 0 \\ \frac{\partial \psi_2}{\partial y} = 1 \\ \frac{\partial \psi_3}{\partial y} = +\frac{\psi_1}{2} \end{cases}$$

which has no solution since from the last lines we obtain the contradiction $\frac{\partial^2 \psi_3}{\partial x \partial y} \neq \frac{\partial^2 \psi_3}{\partial y \partial x}$.

In general it can be quite complicated to establish existence or non existence of solution of PDEs. Frobenius theorem reduces the flatness of a distribution to the following computable property.

Definition 6.8. A smooth vector distribution D (of rank m) on M is said to be *involutive* if for every point $q_0 \in M$, there exists a neighborhood U of q_0 and a family of vector fields $\{X_1, \dots, X_m\}$ satisfying (6.8) such that

$$[X_i, X_k] = \sum_{j=1}^m a_{ij}^k X_j, \quad \forall i, k = 1, \dots, m. \quad (6.9)$$

for some smooth functions a_{ij}^k on M .

Example 6.9. The distribution $D = \text{span}\{X, Y\}$ in $\mathbb{R}^3$ spanned by the two vector fields

$$X = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$$

is not involutive since

$$[X, Y] = \frac{\partial}{\partial z}$$

which is linearly independent on X and Y at every point.

Exercise 6.10. Prove that a smooth vector distribution D is involutive if and only if for every point $q_0 \in M$ there exists a neighborhood U of q_0 and *for every* family of vector fields $\{X_1, \dots, X_m\}$ satisfying (6.8) we have

$$[X_i, X_k] = \sum_{j=1}^m a_{ij}^k X_j, \quad \forall i, k = 1, \dots, m. \quad (6.10)$$

for some smooth functions a_{ij}^k on M .

Now we can state the main result of this section.

Theorem 6.11 (Frobenius Theorem). *A smooth distribution is involutive if and only if it is flat.*

To prove Frobenius theorem we first need a lemma.

Lemma 6.12. *Assume that D is involutive and locally spanned by $X_1, \dots, X_m$. Then for every $k = 1, \dots, m$, we have $e_*^{tX_k} D = D$.*

Notice that this means that for every $k = 1, \dots, m$ and $q \in M$ we have

$$e_*^{tX_k}(D_q) = D_{e^{tX_k}(q)}$$

Proof of Lemma 6.12. Let us define the time-dependent vector fields

$$Y_i^k(t) := e_*^{tX_k} X_i.$$

Using (6.10) and (5.13) we compute

$$\dot{Y}_i^k(t) = e_*^{tX_k} [X_i, X_k] = \sum_{j=1}^m e_*^{tX_k} (a_{ij}^k X_j) = \sum_{j=1}^m a_{ij}^k(t) Y_j^k(t),$$

where we set $a_{ij}^k(t) = a_{ij}^k \circ e^{-tX_k}$. More explicitly, this means that for every $q \in M$

$$\frac{d}{dt}(e_*^{tX_k} X_i)(q) = \sum_{j=1}^m a_{ij}^k(e^{-tX_k}(q))(e_*^{tX_k} X_j)(q),$$

Notice that for every fixed $q \in M$ and $k \in \{1, \dots, m\}$ this is a system of non-autonomous linear differential equations in the vector space $T_q M$ of the form

$$\frac{d}{dt} x_i(t) = \sum_{j=1}^k a_{ij}(t) x_j(t)$$

Thus denote by $A^k(t) = (a_{ij}^k(t))_{i,j=1}^m$ and consider for every k the unique solution to the matrix Cauchy problem

$$\dot{M}(t) = A^k(t)M(t), \quad M(0) = I. \quad (6.11)$$

which we denote by $M^k(t) = (m_{ij}^k(t))_{i,j=1}^m$. Then we have

$$Y_i^k(t) = \sum_{j=1}^m m_{ij}^k(t) Y_j^k(0),$$

since $Y_j^k(0) = X_j$, this implies for every $i, k = 1, \dots, m$,

$$e_*^{tX_k} X_i = \sum_{j=1}^m m_{ij}^k(t) X_j,$$

which proves the claim. $\square$

Proof of the Frobenius Theorem. The statement is local, hence it is sufficient to prove the statement on a neighborhood U of an arbitrary point $q_0 \in M$. We can assume the vector fields are complete.

(i). Assume first that the distribution is flat. Then there exists a diffeomorphism $\varphi : U \rightarrow \mathbb{R}^n$ such that $D_q = \varphi_{*,q}^{-1}(\mathbb{R}^m \times \{0\})$. It follows that for all $q \in U$ we have

$$D_q = \text{span}\{X_1(q), \dots, X_m(q)\}, \quad \text{where } X_i(q) := \varphi_{*,q}^{-1} \left(\frac{\partial}{\partial x_i} \right).$$

and we have for $i, k = 1, \dots, m$,

$$[X_i, X_k] = \left[\varphi_{*,q}^{-1} \left(\frac{\partial}{\partial x_i} \right), \varphi_{*,q}^{-1} \left(\frac{\partial}{\partial x_k} \right) \right] = \varphi_{*,q}^{-1} \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right] = 0.$$

(ii). Let us now prove that if D is involutive then it is flat. As before, it is not restrictive to work on a neighborhood U where

$$D_q = \text{span}\{X_1(q), \dots, X_m(q)\}, \quad \forall q \in U. \quad (6.12)$$

and (6.10) are satisfied. Complete the family $X_1, \dots, X_m$ to a basis of the tangent space

$$T_q M = \text{span}\{X_1(q), \dots, X_m(q), Z_{m+1}(q), \dots, Z_n(q)\},$$

in a neighborhood of q_0 and set $\psi : \mathbb{R}^n \rightarrow M$ defined by

$$\psi(t_1, \dots, t_m, s_{m+1}, \dots, s_n) = e^{t_1 X_1} \circ \dots \circ e^{t_m X_m} \circ e^{s_{m+1} Z_{m+1}} \circ \dots \circ e^{s_n Z_n}(q_0).$$

By construction ψ is a local diffeomorphism at $(t, s) = (0, 0)$ and for (t, s) close to $(0, 0)$ we have that (cf. Proposition 6.1) for every $i = 1, \dots, m$

$$\frac{\partial \psi}{\partial t_i}(t, s) = (e_*^{t_1 X_1} \dots e_*^{t_i X_i} X_i)(\psi(t, s)),$$

These vectors are linearly independent and, thanks to Lemma 6.12, belong to D . Hence

$$D_q = \psi_* \left(\text{span} \left\{ \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m} \right\} \right), \quad q = \psi(t, s),$$

and the claim is proved by considering $\varphi := \psi^{-1}$. $\square$

Reformulating Frobenius theorem in terms of submanifolds, one immediately obtains the following corollary.

Corollary 6.13. *Let D be an involutive distribution of rank m on a smooth manifold M of dimension $n \geq m$. Then, for every $q \in M$, there exists a (locally defined) submanifold S of dimension m passing through q and that is tangent to D at every point, i.e., $T_q S = D_q$ for every $q \in S$.*

Proof. Since D is involutive we can consider the map ψ built in the previous proof. In coordinates we consider the m -dimensional submanifold

$$\mathbb{R}^m \times \{0\} = \{s_{m+1} = \dots = s_n = 0\},$$

and by setting $S = \psi(\mathbb{R}^m \times \{0\})$, since ψ is in particular an immersion, one has at every point $q \in S$

$$T_q S = \psi_* \left(\text{span} \left\{ \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_m} \right\} \right) = D_q.$$

□

This in particular says that if we move from a given point $q_0 \in M$ only with curves that are tangent to D , then we are confined in a m -dimensional submanifold S of M .

The converse (integrability implies involutivity) is also true thanks to the following lemma.

Lemma 6.14. *Let $N \subset M$ be an embedded submanifold and X, Y in $\text{Vec}(M)$ be vector fields tangent to N , i.e., $X(q), Y(q) \in T_q N$, for every $q \in N$. Then $[X, Y]$ is tangent to N .*

Proof. A vector field X in M is tangent to N if and only if it is i -related to a vector field on N , i.e., there exists V such that $X|_N = i_* V$ for $V \in \text{Vec}(N)$ where here $i : N \rightarrow M$ is the canonical inclusion. If $X|_N = i_* V$ and $Y|_N = i_* W$ then $[X, Y]|_N = [i_* V, i_* W]|_N = (i_* [V, W])|_N = i_* [V, W]$ hence $[X, Y]$ is also tangent to N . □

6.4 Rashevski-Chow theorem: local version

We now state a theorem about “totally non integrable” distributions, and we start by introducing bracket-generating families of vector fields.

Definition 6.15. Let M be a smooth manifold and D a smooth vector distribution. D is said *bracket generating* if for every local basis in a neighborhood U of q we have

$$D_q = \text{span}\{X_1(q), \dots, X_m(q)\}, \quad \forall q \in U. \quad (6.13)$$

and

$$\text{span}\{[X_{i_1}, \dots, [X_{i_{j-1}}, X_{i_j}]](q) : 1 \leq i_\ell \leq m, \ell \leq j, j \in \mathbb{N}\} = T_q M \quad (6.14)$$

We denote the left hand side by

$$\text{Lie}_q D := \text{span}\{[X_{i_1}, \dots, [X_{i_{j-1}}, X_{i_j}]](q) : 1 \leq i_\ell \leq m, \ell \leq j, j \in \mathbb{N}\}$$

these are all possible Lie brackets of the basis evaluated at q .

Exercise 6.16. Show that $\text{Lie}_q D$ is well defined, i.e., does not depend on the choice of the basis.

A direct consequence of Lemma 6.14 is the following

Corollary 6.17. *Let $N \subset M$ be an embedded submanifold and D be a smooth vector distribution such that $D_q \subset T_q N$ for every $q \in N$. Then for every $q \in N$ we have $\dim \text{Lie}_q D \leq \dim N$.*

We can now prove the main technical lemma of this section

Lemma 6.18 (Rashevski-Chow lemma). *Let M be an n -dimensional manifold and D locally spanned by $X_1, \dots, X_k$ be a bracket generating distribution of rank k .*

Then for every $q_0 \in M$ and every neighborhood V of the origin in $\mathbb{R}^n$ there exist $\tau = (\tau_1, \dots, \tau_n) \in V$, and a choice of n vector fields $X_{i_1}, \dots, X_{i_n}$, such that τ is a regular point of the map

$$\psi : \mathbb{R}^n \rightarrow M, \quad \psi(t_1, \dots, t_n) = e^{t_n X_{i_n}} \circ \dots \circ e^{t_1 X_{i_1}}(q_0).$$

Remark 6.19. Observe that, if $k < n$, then $\tau \neq 0$. Indeed, the image of the differential of ψ at $t = 0$ is

$$\text{im } \psi_{*,0} = \text{span}_{q_0} \{X_{i_j} \mid j = 1, \dots, n\} \subseteq \text{span}_{q_0} \{X_i \mid i = 1, \dots, k\} = D_{q_0}$$

and the differential of ψ cannot be surjective if $D_{q_0} \neq T_{q_0} M$. In particular, in the choice of $X_{i_1}, \dots, X_{i_n}$, the same vector field is allowed to appear more than once.

The following proof indeed works for any family of vector fields $\mathcal{F} = \{X_1, \dots, X_N\}$ such that $\text{Lie}_q \mathcal{F} = T_q M$, without necessarily asking that the vector fields are linearly independent.

Proof of Lemma 6.18. We prove the lemma by steps. Let $\mathcal{F} = \{X_1, \dots, X_m\}$.

1. There exists a vector field $X_{i_1} \in \mathcal{F}$ such that $X_{i_1}(q_0) \neq 0$, otherwise all vector fields in $\mathcal{F}$ vanish at q_0 and $\dim \text{Lie}_{q_0} \mathcal{F} = 0$, which contradicts the bracket-generating condition. Then, for $|s|$ small enough, the map

$$\phi_1 : s_1 \mapsto e^{s_1 X_{i_1}}(q_0),$$

is a local diffeomorphism onto its image Σ_1 . If $\dim M = 1$ the Lemma is proved.

2. Assume $\dim M \geq 2$. Then there exist t_1^1 arbitrarily close to 0, and $X_{i_2} \in \mathcal{F}$ such that, if we denote by $q_1 = e^{t_1^1 X_{i_1}}(q_0)$, the vector $X_{i_2}(q_1)$ is not tangent to Σ_1 . Otherwise, by Lemma 6.14, $\dim \text{Lie}_{q_0} \mathcal{F} = 1$, which contradicts the bracket-generating condition. Then the map

$$\phi_2 : (s_1, s_2) \mapsto e^{s_2 X_{i_2}} \circ e^{s_1 X_{i_1}}(q_0),$$

is a local diffeomorphism near $(t_1^1, 0)$ onto its image Σ_2 . Indeed the vectors

$$\left. \frac{\partial \phi_2}{\partial s_1} \right|_{(t_1^1, 0)} \in T_{q_1} \Sigma_1, \quad \left. \frac{\partial \phi_2}{\partial s_2} \right|_{(t_1^1, 0)} = X_{i_2}(q_1),$$

are linearly independent by construction. If $\dim M = 2$ the Lemma is proved.

3. Assume $\dim M \geq 3$. Then there exist (t_2^1, t_2^2) arbitrarily close to $(t_1^1, 0)$, and $X_{i_3} \in \mathcal{F}$ such that, if $q_2 = e^{t_2^2 X_{i_2}} \circ e^{t_2^1 X_{i_1}}(q_0)$ we have that $X_{i_3}(q_2)$ is not tangent to Σ_2 . Otherwise, by Lemma 6.14, $\dim \text{Lie}_{q_1} \mathcal{D} = 2$, which contradicts the bracket-generating condition. Then the map

$$\phi_3 : (s_1, s_2, s_3) \mapsto e^{s_3 X_{i_3}} \circ e^{s_2 X_{i_2}} \circ e^{s_1 X_{i_1}}(q_0),$$

is a local diffeomorphism near $(t_2^1, t_2^2, 0)$. Indeed the vectors

$$\left. \frac{\partial \phi_3}{\partial s_1} \right|_{(t_2^1, t_2^2, 0)}, \left. \frac{\partial \phi_3}{\partial s_2} \right|_{(t_2^1, t_2^2, 0)} \in T_{q_2} \Sigma_2, \quad \left. \frac{\partial \phi_3}{\partial s_3} \right|_{(t_2^1, t_2^2, 0)} = X_{i_3}(q_2),$$

are linearly independent since the last one is transversal to $T_{q_2} \Sigma_2$ by construction, while the first two are linearly independent since $\phi_3(s_1, s_2, 0) = \phi_2(s_1, s_2)$ and ϕ_2 is a local diffeomorphism at (t_2^1, t_2^2) which is close to $(t_1^1, 0)$.

Repeating the same argument n times (with $n = \dim M$), the lemma is proved. $\square$

Corollary 6.20. *The map $\hat{\psi} : \mathbb{R}^n \rightarrow M$ defined by*

$$\hat{\psi}(t_1, \dots, t_n) = e^{-\tau_1 X_{i_1}} \circ \dots \circ e^{-\tau_n X_{i_n}} \circ \psi(t_1, \dots, t_n),$$

is a diffeomorphism from a neighborhood $\widehat{W}$ of $\tau \in V$ to a neighborhood of $\hat{\psi}(\tau) = q_0$.

Here $\tau = (\tau_1, \dots, \tau_n)$ is a point one obtains from the previous Lemma.

Thanks to Lemma 6.18 there exists a neighborhood $\widehat{V} \subset V$ of $\hat{s}$ such that $\hat{\psi}$ is a diffeomorphism from $\widehat{V}$ to $\psi(\widehat{V})$. We stress that in general $q_0 = \psi(0)$ does not belong to $\psi(\widehat{V})$, cf. Remark 6.19.

Corollary 6.21. *Let M be a smooth manifold and D be a bracket generating distribution.*

Then D is totally non-integrable, i.e., there is no submanifold N of M such that $D_q = T_q N$ for every q in a neighborhood of a point.

This means that if we move from a given point q with curves that are tangent to D , then we can reach an open neighborhood in M .

We have that D bracket-generating implies D non-integrable. The converse in full generality is not true.

Example 6.22. Let us consider the smooth function

$$\phi(t) = \begin{cases} e^{-1/t^2}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

This is a C^∞ function that vanish at $t = 0$ with all derivatives equal to zero at that point.

Consider the rank 2 distribution D in $M = \mathbb{R}^3$ given by $D = \text{span}\{X, Y\}$ with

$$X = \partial_x, \quad Y = \partial_y + \phi(x) \partial_z$$

It is easy to see that

$$[X, Y] = \phi'(x) \partial_z, \quad [Y, [X, Y]] = 0$$

while, more in general,

$$\underbrace{[X, \dots, [X, Y]]}_j = \phi^{(j)}(x) \partial_z$$

Hence the structure is not bracket generating on the plane $P = \{x = 0\}$. However, D is totally non integrable since if $N \subset \mathbb{R}^3$ is an integral manifold then it should be contained in $P = \{x = 0\}$. But if $x = 0$ then X is not tangent to P .

So non-integrable does not imply bracket generating at every point. One can prove the following partial converse

- (a) if D is non integrable then D is bracket-generating on an open dense set of M
- (b) if M and D are analytic, D is non-integrable implies D bracket-generating at every point.

Exercise 6.23 (Sphere rolling on a plane).

Chapter 7

Tensors and Differential forms

Duality in mathematics is not a theorem, but a “principle”.

Duality in Mathematics and Physics, 2007.

Sir M.F. Atiyah (1929–2019)

7.1 Cotangent space

Covectors are dual object to vectors, i.e., linear functionals defined on the tangent space. The space of all covectors at a point $q \in M$, is called cotangent space.

Definition 7.1. Let M be a n -dimensional smooth manifold. The *cotangent space* at a point $q \in M$ is the set

$$T_q^*M := (T_qM)^* = \{\lambda : T_qM \rightarrow \mathbb{R}, \lambda \text{ linear}\}.$$

If $\lambda \in T_q^*M$ and $v \in T_qM$, we will denote by $\langle \lambda, v \rangle := \lambda(v)$ the evaluation of the covector λ on the vector v .

Covectors are not just abstract object but corresponds to differential of scalar functions.

Example 7.2. Let $f : M \rightarrow \mathbb{R}$ be a smooth function and $q \in M$. The differential $f_{*,q} : T_qM \rightarrow \mathbb{R}$ of the scalar function f at q will be denoted d_qf or $df|_q$, it satisfies

$$\langle d_qf, v \rangle := \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)), \quad v \in T_qM, \quad (7.1)$$

where γ is any smooth curve such that $\gamma(0) = q$ and $\dot{\gamma}(0) = v$, is an element of T_q^*M .

Recall that if (U, φ) is a chart and $(x_1, \dots, x_n)$ is the associated coordinate system if

$$v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q$$

then (7.1) satisfies

$$\langle d_qf, v \rangle := \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \Big|_q f = \sum_{i=1}^n v_i \frac{\partial \hat{f}}{\partial x_i}(x) \quad (7.2)$$

where $\widehat{f} = f \circ \varphi^{-1}$ and $x = \varphi(q)$. If we choose as function f the coordinate function x_i and as a vector v the vector $\frac{\partial}{\partial x_j} \Big|_q$ we have

$$\left\langle d_q x_i, \frac{\partial}{\partial x_j} \Big|_q \right\rangle = \frac{\partial}{\partial x_j} \Big|_q x_i = \delta_{ij}$$

which proves that $d_q x_1, \dots, d_q x_n$ is a basis of $T_q^* M$ that is dual to the coordinate basis of $T_q M$.

Remark 7.3. Let $T : V \rightarrow W$ be a linear map between (finite-dimensional) vector spaces. The dual map is $T^* : W^* \rightarrow V^*$ defined by duality $\langle T^* \eta, v \rangle = \langle \eta, T(v) \rangle$. Fix a basis $\{e_1, \dots, e_n\}$ for V and $\{f_1, \dots, f_m\}$ for W . Then T is represented by the matrix A with respect to these basis, with A invertible, then T^* is represented by A^T with respect to the dual basis. Namely

$$T(e_i) = \sum_{j=1}^m a_{ij} f_j, \quad T^*(f_j^*) = \sum_{i=1}^n a_{ij} e_i^*$$

indeed assuming the first identity one gets

$$\langle T^* f_j^*, e_i \rangle = \langle f_j^*, T(e_i) \rangle = \sum_{k=1}^m \langle f_j^*, a_{ik} f_k \rangle = a_{ij}$$

Exercise 7.4 (Change of coordinates for covectors). Given two coordinate sets $(U, \{x_i\})$ and $(U', \{x'_i\})$ with $q \in U \cap U'$ show that if $\lambda \in T_q^* M$ writes as

$$\lambda = \sum_{i=1}^n p_i dx_i \Big|_q = \sum_{j=1}^n p'_j dx'_j \Big|_q.$$

then

$$p_i = \sum_{j=1}^n \frac{\partial x'_j}{\partial x_i} p'_j.$$

(Here, if φ, φ' denotes the coordinate maps, the quantity $\frac{\partial x'_j}{\partial x_i}$ is the jacobian of $\varphi' \circ \varphi^{-1}$).

The differential of a smooth map yields a linear map between tangent spaces. The dual of the differential gives a linear map between cotangent spaces.

Definition 7.5. Let $F : M \rightarrow N$ be a smooth map and $q \in M$. The *pullback* of F at point $F(q)$, where $q \in M$, is the map

$$F^* : T_{F(q)}^* N \rightarrow T_q^* M, \quad \lambda \mapsto F^* \lambda,$$

defined by duality in the following way

$$\langle F^* \lambda, v \rangle := \langle \lambda, F_* v \rangle, \quad \forall v \in T_q M, \forall \lambda \in T_{F(q)}^* N.$$

Notice that we can also write $F^* \lambda = \lambda \circ F_*$.

The cotangent bundle

The cotangent bundle is defined as the disjoint union

$$T^*M = \bigcup_{q \in M} T_q^*M$$

endowed with the natural projection

$$\pi : T^*M \rightarrow M, \quad \pi(\lambda) = q, \text{ if } \lambda \in T_q^*M$$

Proposition 7.6. *T^*M has the natural structure of smooth vector bundle of rank $n = \dim M$.*

Proof. The proof is similar to the argument for TM . Given charts $\{(U_i, \varphi_i)\}$ on M , then $\pi^{-1}(U_i)$ is an open set of T^*M and the reunion of these sets is an open cover of T^*M . We define coordinates

$$\bar{\varphi}_i : \pi^{-1}(U_i) \rightarrow \varphi_i(U_i) \times \mathbb{R}^n$$

as follows

$$\bar{\varphi}_i(\lambda) = (x_1, \dots, x_n, p_1, \dots, p_n)$$

if $\pi(\lambda) = q$ with $\varphi_i(q) = (x_1, \dots, x_n)$, and moreover

$$\lambda = \sum_{i=1}^n p_i dx_i \Big|_q$$

Notice that if $U_i \cap U_j \neq \emptyset$ then $\pi^{-1}(U_i) \cap \pi^{-1}(U_j) \neq \emptyset$ and

$$\bar{\varphi}_i \circ \bar{\varphi}_j^{-1} : \varphi_j(U_i \cap U_j) \times \mathbb{R}^n \rightarrow \varphi_i(U_i \cap U_j) \times \mathbb{R}^n$$

where $\bar{\varphi}_i \circ \bar{\varphi}_j^{-1}(x, p) = (x', p')$ if (cf. with Exercise 7.29)

$$x' = \varphi_i \circ \varphi_j^{-1}(x), \quad p = \frac{\partial(\varphi_i \circ \varphi_j^{-1})}{\partial x} p'$$

which completes the proof. □

Differential 1-forms

Definition 7.7. A *differential 1-form* on a smooth manifold M is a smooth section of T^*M i.e., a map

$$\omega : q \mapsto \omega(q) \in T_q^*M,$$

that associates with every point q in M a cotangent vector at q . We denote by $\Lambda^1(M)$ the set of differential forms on M .

In coordinates a differential form is written as follows, where ω_i are smooth functions

$$\omega = \sum_{i=1}^n \omega_i dx_i.$$

Since differential 1-forms are dual objects to vector fields, it is well defined the action of $\omega \in \Lambda^1(M)$ on $X \in \text{Vec}(M)$ pointwise, defining a function on M .

$$\langle \omega, X \rangle : q \mapsto \langle \omega(q), X(q) \rangle. \tag{7.3}$$

Exercise 7.8. Prove that a differential form ω is *smooth* if and only if, for every smooth vector field $X \in \text{Vec}(M)$, the function $\langle \omega, X \rangle \in C^\infty(M)$.

Definition 7.9. Let $F : M \rightarrow N$ be a smooth map and $g : N \rightarrow \mathbb{R}$ be a smooth function. The *pullback* F^*g is the smooth function on M defined by

$$F^*g = g \circ F, \quad q \in M.$$

If ω is a 1-form on N . The *pullback* $F^*\omega$ is the 1-form on M defined by

$$F^*\omega = \omega \circ F_*$$

The differential of a function

The differential of a function automatically defines a 1-form. We want to find the expression in coordinates, i.e. the functions ω_i such that

$$df = \sum_{i=1}^n \omega_i dx_i$$

By definition of dual basis

$$\omega_j = \left\langle df, \frac{\partial}{\partial x_j} \right\rangle = \frac{\partial f}{\partial x_j}$$

where the last element of the identity is written in coordinates. Then

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

Remark 7.10. We might think to our background in calculus and to the gradient formula

$$\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \tag{7.4}$$

but this quantity indeed is not well-defined. Show that (7.4) is not invariant by change of coordinates.

Some properties of differential and pullback.

Proposition 7.11. Let $F : M \rightarrow N$ be smooth, $\eta \in \Omega^1(N)$ and $g \in C^\infty(N)$, then

$$F^*(g\eta) = (g \circ F)F^*\eta \tag{7.5}$$

$$F^*dg = d(g \circ F) = dF^*g \tag{7.6}$$

where we recall that we have set $F^*g = g \circ F$.

Proof. The first property is just linearity of F^* . Thanks to (i) and the fact that every η is locally written as $\eta = \sum_{j=1}^m \eta_j dy_j$ it is sufficient to check (ii) for $g = y_j$ a coordinate function. In this case

$$F^*dy_j = \sum_{i=1}^n \frac{\partial F_j}{\partial x_i} dx_i = dF_j = d(y_j \circ F)$$

by denoting $F_j = y_j \circ F$ the j -th component of F in coordinates. Another way is to observe that $d(g \circ F) = dg \circ F_*$ so that $\langle F^*dg, X \rangle = \langle dg, F_*X \rangle = dg \circ F_*(X)$. $\square$

Remark 7.12. Combining the above formulas we get that if $F : M \rightarrow N$ is smooth and $\eta \in \Omega^1(N)$ is written in coordinates

$$\eta = \sum_{j=1}^m \eta_j dy_j$$

then

$$F^*\eta = \sum_{j=1}^m (\eta_j \circ F) F^* dy_j = \sum_{j=1}^m (\eta_j \circ F) dF_j$$

where again $F_j := y_j \circ F$ denotes the j -th component of F in coordinates.

Exercise 7.13. Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the function $F(x, y, z) = (x^2 y, y \sin z)$ and $\omega = u dv + v du$ in $\Omega^1(\mathbb{R}^2)$. Compute $F^*\omega$.

7.2 Tensors and tensor fields

Definition 7.14. A (covariant) k -tensor on a vector space V is a multilinear map

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \rightarrow \mathbb{R}.$$

The space of all such tensors is denoted $T^k(V)$. Basic examples are: linear maps (1-tensors), bilinear maps (2-tensors), determinant on $\mathbb{R}^n$ (n -tensor).

We introduce the tensor product as follows: let $\eta_1, \eta_2 \in V^* = T^1(V)$ we define the two tensor $\eta_1 \otimes \eta_2$ as follows

$$\eta_1 \otimes \eta_2 : V \times V \rightarrow \mathbb{R}, \quad \eta_1 \otimes \eta_2(v, w) := \eta_1(v) \eta_2(w)$$

Note that the operation is not commutative in general, since $\eta_1 \otimes \eta_2 \neq \eta_2 \otimes \eta_1$, but it is associative since $(\eta_1 \otimes \eta_2) \otimes \eta_3 = \eta_1 \otimes (\eta_2 \otimes \eta_3)$. As a consequence one can define the tensor product $\eta_1 \otimes \cdots \otimes \eta_k$ of k elements in $V^* = T^1(V)$ and this is an element of $T^k(V)$.

Let V be a vector space and $e_1, \dots, e_n$ be a basis of V and $e_1^*, \dots, e_n^*$ its dual basis. We notice that every bilinear form $B : V \times V \rightarrow \mathbb{R}$ can be written in an unique way as follows

$$B = \sum_{i,j=1}^n B_{ij} e_i^* \otimes e_j^*$$

where $B_{k\ell} = B(e_k, e_\ell)$. Indeed applying the right hand side to (e_k, e_ℓ) one gets

$$\left(\sum_{i,j=1}^n B_{ij} e_i^* \otimes e_j^* \right) (e_k, e_\ell) = \sum_{i,j=1}^n B_{ij} (e_i^* \otimes e_j^*) (e_k, e_\ell) = \sum_{i,j=1}^n B_{ij} \delta_{ik} \delta_{j\ell} = B_{k\ell}$$

This unique decomposition shows that $T^2(V)$ has dimension n^2 and a basis for $T^2(V)$ is given by the family $\{e_i^* \otimes e_j^* \mid 1 \leq i, j \leq n\}$. More in general we have the following property.

Lemma 7.15. Let V be a vector space with basis $e_1, \dots, e_n$. Let $e_1^*, \dots, e_n^*$ be its dual basis. Then the family of elements $T^k(V)$

$$\{e_{i_1}^* \otimes \dots \otimes e_{i_k}^* \mid 1 \leq i_1, \dots, i_k \leq n\},$$

is a basis for the vector space of k -tensors $T^k(V)$. In particular $\dim T^k(V) = n^k$.

The proof is a consequence of the observation that every $T \in T^k(V)$ can be written in a unique way as

$$T = \sum_{1 \leq i_1, \dots, i_k \leq n} T_{i_1 \dots i_k} e_{i_1}^* \otimes \dots \otimes e_{i_k}^* \quad (7.7)$$

where $T_{i_1 \dots i_k} = T(e_{i_1}, \dots, e_{i_k})$. These are called the components of the k -tensor T . In coordinates a tensor is represented by a sort of *generalized matrix* with k indices.

Exercise 7.16. Prove that if T satisfies (7.7) and it is also written as

$$T = \sum_{1 \leq j_1, \dots, j_k \leq n} T'_{j_1 \dots j_k} f_{j_1}^* \otimes \dots \otimes f_{j_k}^* \quad (7.8)$$

with respect to another basis $f_1, \dots, f_n$ satisfying $e_i = \sum_{j=1}^n a_{ij} f_j$, then we have

$$T'_{j_1 \dots j_k} = \sum_{1 \leq i_1, \dots, i_k \leq n} T_{i_1 \dots i_k} a^{i_1 j_1} \dots a^{i_k j_k}$$

where we denoted by a^{ij} the elements of the inverse of the matrix, namely $\sum_j a_{ij} a^{jl} = \delta_{il}$. with respect to a basis $e_1, \dots, e_n$

One can define the tensor product $T_1 \otimes T_2$ of $T_1 \in T^k(V)$ and $T_2 \in T^l(V)$, as follows

$$T_1 \otimes T_2(v_1, \dots, v_k, w_1, \dots, w_l) = T_1(v_1, \dots, v_k) T_2(w_1, \dots, w_l)$$

Clearly $T_1 \otimes T_2 \in T^{k+l}(V)$. Again, the tensor product just introduced is associative but not commutative. One can then consider $\otimes$ as a product for the associative (but not commutative) *graded algebra*

$$T^\bullet(V) = \bigoplus_{k \geq 0} T^k(V).$$

A *graded algebra* is an algebra $(\mathcal{A}, \cdot)$ which can be decomposed as $\mathcal{A} = \bigoplus_{k \geq 0} \mathcal{A}^k$ and for which the internal product respect the grading, i.e., it satisfies $\mathcal{A}^k \cdot \mathcal{A}^l \subset \mathcal{A}^{k+l}$.

Tensor product of vector spaces

Following the previous construction we have seen that the space $T^k(V)$ is generated by elements of the form $e_{i_1}^* \otimes \dots \otimes e_{i_k}^*$ so that it is suggestive to write

$$T^k(V) = \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} \quad (7.9)$$

even if we still have not defined the tensor product of vector spaces $V^* \otimes \cdots \otimes V^*$. If we accept the identity (7.9) for a moment, using the canonical isomorphisms between the vector space V and its bidual V^{**} given by

$$\beta : V \rightarrow V^{**}, \quad \beta(v)[\eta] := \eta(v), \quad \eta \in V^*$$

one can similarly build the set of *contravariant l -tensors*

$$T_l(V) := T^l(V^*) = V^{**} \otimes \cdots \otimes V^{**} \simeq V \otimes \cdots \otimes V.$$

Formally, $T \in T_l(V)$ is just a multilinear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \rightarrow \mathbb{R}.$$

Pushing this idea, we can define also *mixed tensors of type (k, l)* as follows

$$T_l^k(V) := \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ times}}$$

as the set of multilinear maps

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \times \underbrace{V^* \times \cdots \times V^*}_{l \text{ times}} \rightarrow \mathbb{R}.$$

If $T \in T_l^k(V)$ then we can write along a basis $e_1, \dots, e_n$ of V and the corresponding dual basis

$$T = \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{1 \leq j_1, \dots, j_l \leq n} T_{i_1 \dots i_k}^{j_1 \dots j_l} e_{i_1}^* \otimes \cdots \otimes e_{i_k}^* \otimes e_{j_1} \otimes \cdots \otimes e_{j_l} \quad (7.10)$$

where $T_{i_1 \dots i_k}^{j_1 \dots j_l}$ are the components of the tensor.

Remark 7.17. This is a good point to use the “physicists convention” about the position of indexes by using upper indices on vectors for dual elements and writing the dual basis as $\{e^1, \dots, e^n\}$ instead of $\{e_1^*, \dots, e_n^*\}$ (notice the corresponding position for components). So, with this convention, a vector and a covector should be written in components as follows

$$v = \sum_{i=1}^n v^i e_i, \quad \eta = \sum_{i=1}^n \eta_i e^i$$

or, even more shortly using Einstein summation, $v = v^i e_i$ and $\eta = \eta_i e^i$. Formula (7.10) then simplifies to

$$T = T_{i_1 \dots i_k}^{j_1 \dots j_l} e^{i_1} \otimes \cdots \otimes e^{i_k} \otimes e_{j_1} \otimes \cdots \otimes e_{j_l}.$$

We will not always use this convention in the following, but it is important to get used to this notation to read the literature.

Finally, we can also define the space of all tensors on V .

$$T(V) := \bigoplus_{k, l \geq 0} T_l^k(V).$$

Notice that $(T(V), \otimes)$ becomes a non-commutative associative algebra (only now the product of elements of the space belong to the space). It is a graded algebra in the sense that the product is compatible with the grading as follows. If $T \in T_l^k(V)$ and $S \in T_{l'}^{k'}(V)$ then $T \otimes S \in T_{l+l'}^{k+k'}(V)$.

On tensor product of spaces

Similarly to the previous construction, one could also define tensor products of different vector spaces $V \otimes W$, which are bilinear maps $V^* \times W^* \rightarrow \mathbb{R}$, or also $V^* \otimes W$, bilinear maps $V \times W^* \rightarrow \mathbb{R}$, etc.

Remark 7.18 (An abstract way to define $V \otimes W$). Consider the free vector space $\mathbb{R}\langle V \times W \rangle$ over $V \times W$, i.e. the vector space where every element $(v, w) \in V \times W$ is an element of a basis. Then we introduce the equivalence relation (where k is an arbitrary element of a field of scalars)

$$\begin{aligned} k(v, w) &\sim (kv, w) \sim (v, kw), \\ (v_1 + v_2, w) &\sim (v_1, w) + (v_2, w), \\ (v, w_1 + w_2) &\sim (v, w_1) + (v, w_2). \end{aligned}$$

Then we set $V \otimes W := \mathbb{R}\langle V \times W \rangle / \sim$ and we denote by $v \otimes w$ the equivalence class $[(v, w)]_{\sim}$. We have automatically satisfied

$$\begin{aligned} k(v \otimes w) &= kv \otimes w = v \otimes kw, \\ (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2. \end{aligned}$$

Notice that by definition every element of $V \otimes W$ is a linear combination of elements of the form $v \otimes w$, but it is not true in general that every element of $V \otimes W$ is equal to $v \otimes w$ for some v, w .

Exercise 7.19. Let $e_1, \dots, e_4$ be a basis of $\mathbb{R}^4$. Prove that $e_1 \otimes e_2 + e_3 \otimes e_4$ cannot be written as $v \otimes w$ for some vectors v, w .

Exercise 7.20. Prove that there is a unique isomorphisms between $(V \otimes W) \otimes Z$ and $V \otimes (W \otimes Z)$ sending $(v \otimes w) \otimes z$ and $v \otimes (w \otimes z)$. In particular we can write $V \otimes W \otimes Z$.

As a final example let us prove.

Lemma 7.21. Let V, W be finite dim vector spaces. Then $V^* \otimes W$ is isomorphic to $\text{Hom}(V, W)$

Proof. Let us build a linear isomorphisms $L : V^* \otimes W \rightarrow \text{Hom}(V, W)$. Of course it is enough to define it on elements of the form $\eta \otimes w$ for $\eta \in V^*$ and $w \in W$.

We set $L(\eta \otimes w) \in \text{Hom}(V, W)$ as follows $L(\eta \otimes w)[v] = \eta(v)w$. Then it is very easy to show that this map is injective. Indeed if $L(\eta \otimes w)$ is the zero map then we want to show that $\eta \otimes w = 0$. This means that either $w = 0$ or $\eta = 0$. But if $L(\eta \otimes w)$ is the zero map then $\eta(v)w = 0$ for every v . If $w \neq 0$ then this implies $\eta(v) = 0$ for all v and the statement is proved. $\square$

Remark 7.22. More in general an element of

$$T_l^k(V) := \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \dots \otimes V}_{l \text{ times}}$$

can also be thought as a linear map $V^{\otimes k} \rightarrow V^{\otimes l}$ where $V^{\otimes j} := \underbrace{V \otimes \dots \otimes V}_{j \text{ times}}$.

Symmetric and Alternating tensors

We focus again on tensors in $T^k(V)$, for a given k . Given a permutation $\sigma \in S_k$ we set

$$T^\sigma(X_1, \dots, X_k) = T(X_{\sigma(1)}, \dots, X_{\sigma(k)})$$

Definition 7.23. Given a tensor $T \in T^k(V)$ we say that

- T is *symmetric* if for every $\sigma \in S_k$ we have $T = T^\sigma$,
- T is *alternating* (or *skew-symmetric*) if for every $\sigma \in S_k$ we have $T = \text{sgn}(\sigma)T^\sigma$.

Given $T \in T^k(V)$ we can define its symetrization and skew-symetrization as

$$\text{Sym}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} T^\sigma, \quad \text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T^\sigma.$$

Exercise 7.24. Prove that, given a tensor $T \in T^k(V)$, then

- $\text{Sym}(T)$ is symmetric, $\text{Alt}(T)$ is skew-symmetric,
- T is symmetric if and only if $T = \text{Sym}(T)$
- T is alternating if and only if $T = \text{Alt}(T)$.

It is meaningful to speak about symmetric tensors $S^k(V)$ and alternating tensor $\Lambda^k(V)$.

Lemma 7.25. A tensor $T \in T^k(V)$ is alternating if and only if one of the following holds true

- $T(X_1, \dots, X_k) = 0$ whenever $X_1, \dots, X_k$ are linearly dependent.
- $T(X_1, \dots, X_k) = 0$ whenever two elements are equal

Given T and S in $T^k(V)$ we define their alternating product, also called *wedge product*

$$T \wedge S = \frac{(k+l)!}{k!l!} \text{Alt}(T \otimes S) \tag{7.11}$$

For instance we have, given elements v, w of V

$$v \wedge w = \frac{2!}{1!1!} \left(\frac{1}{2} (v \otimes w - w \otimes v) \right) = v \otimes w - w \otimes v$$

so that the strange constant in (7.11) is chosen in order to simplify constants in the wedge product. We warn the reader that this convention is not standard in all textbooks.

Denoting $\Lambda^k(V)$ the set of alternating tensors in $T^k(V)$ and then

$$\Lambda(V) := \bigoplus_{k \geq 0} \Lambda^k(V)$$

we have that $\wedge$ defines a product in the graded algebra $\Lambda(V)$ thanks to the following

Lemma 7.26. *We have the following properties in $\Lambda(V)$*

- (i) $\wedge$ is $\mathbb{R}$ -bilinear
- (ii) $T \wedge S = (-1)^{rs} S \wedge T$ if T, S of order r, s
- (iii) $(T \wedge S) \wedge R = T \wedge (S \wedge R)$

Notice that (ii) follows from the identity

$$\lambda_1 \wedge \dots \wedge \lambda_r \wedge \eta = (-1)^r \eta \wedge \lambda_1 \wedge \dots \wedge \lambda_r$$

Notice also that to prove (iii) one has to prove that

$$\text{Alt}(\text{Alt}(T \otimes S) \otimes R) = \text{Alt}(T \otimes \text{Alt}(S \otimes R))$$

so there is something to check. If that is true then we have well-defined

$$T \wedge S \wedge R = \frac{(k+l+r)!}{k!l!r!} \text{Alt}(T \otimes S \otimes R)$$

Lemma 7.27. *Given covectors $\lambda_1, \dots, \lambda_k$ and vectors $v_1, \dots, v_k$ then*

$$\lambda_1 \wedge \dots \wedge \lambda_k(v_1, \dots, v_k) = \det(\langle \lambda_i, v_j \rangle)$$

Proof. Recall that if $M = (m_{i,j})_{i,j=1}^n$ is a matrix then we have the expansion for the determinant

$$\det(M) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^n m_{i,\sigma(i)}$$

We use (iii) and

$$\begin{aligned} \lambda_1 \wedge \dots \wedge \lambda_k(v_1, \dots, v_k) &= \frac{k!}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \lambda_{\sigma(1)} \otimes \dots \otimes \lambda_{\sigma(k)}(v_1, \dots, v_k) \\ &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \langle \lambda_{\sigma(1)}, v_1 \rangle \dots \langle \lambda_{\sigma(k)}, v_k \rangle \\ &= \det \left(\langle \lambda_i, v_j \rangle_{i,j=1}^n \right) \end{aligned}$$

□

Lemma 7.28. *The space $\Lambda^k(V)$ is a vector subspace of $T^k(V)$, with $\dim \Lambda^k(V) = \binom{n}{k}$. A basis of $\Lambda^k(V)$ is given by*

$$\{e_{i_1}^* \wedge \dots \wedge e_{i_k}^* \mid 1 \leq i_1 < \dots < i_k \leq n\}. \quad (7.12)$$

Proof. If two elements are repeated in $e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$ then this is zero since alternating. If we exchange position, we have a minus sign by (ii). Hence the only elements which might be lin.ind, up to reordering, are those listed in (7.12). The proof that they are linearly independent follows by noticing that

$$e_{i_1}^* \wedge \dots \wedge e_{i_k}^*(e_{j_1}, \dots, e_{j_k}) = \delta_I^J$$

where δ_I^J is the Kronecker delta for multindex $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ which is equal to the $\text{sgn}(\sigma)$ if $J = \sigma(I)$ for some permutation σ , and zero otherwise.

□

Tensor fields

To give a concrete content to this theory let us consider tensor fields. We define the *tensor bundle*

$$T_l^k(M) := \bigcup_{q \in M} T_l^k(T_q M).$$

We have clear identifications

$$\begin{aligned} T_0^0 M &= M \times \mathbb{R}, \\ T_0^1 M &= T^* M, \quad T_1^0 M = TM. \end{aligned}$$

These are vector bundles. We can consider *smooth tensor fields*, i.e., smooth sections of these vector bundles which are denoted by $\mathcal{T}_l^k(M)$ or also $\Gamma(T_l^k(M))$. If

$$\sigma : M \rightarrow T_l^k(M)$$

is a smooth tensor field we can write locally in coordinates $\{x_1, \dots, x_n\}$ on an open set U

$$\sigma = \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{1 \leq j_1, \dots, j_l \leq n} \sigma_{i_1 \dots i_k}^{j_1 \dots j_l} dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_l}}$$

We recover smooth functions for $(k, l) = (0, 0)$, differential 1-forms for $(k, l) = (1, 0)$ and vector fields for $(k, l) = (0, 1)$.

Exercise 7.29 (Change of coordinates for tensor fields). Given two coordinate sets $(U, \{x_i\})$ and $(U', \{x'_i\})$ with $q \in U \cap U'$ show that if

$$\sigma = \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{1 \leq j_1, \dots, j_l \leq n} \sigma_{i_1 \dots i_k}^{j_1 \dots j_l} dx_{i_1} \otimes \dots \otimes dx_{i_k} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x_{j_l}}$$

and

$$\sigma = \sum_{1 \leq \ell_1, \dots, \ell_k \leq n} \sum_{1 \leq h_1, \dots, h_l \leq n} (\sigma')_{\ell_1 \dots \ell_k}^{h_1 \dots h_l} dx'_{\ell_1} \otimes \dots \otimes dx'_{\ell_k} \otimes \frac{\partial}{\partial x'_{h_1}} \otimes \dots \otimes \frac{\partial}{\partial x'_{h_l}}$$

Prove that

$$(\sigma')_{\ell_1 \dots \ell_k}^{h_1 \dots h_l} = \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{1 \leq j_1, \dots, j_l \leq n} \sigma_{i_1 \dots i_k}^{j_1 \dots j_l} \frac{\partial x_{i_1}}{\partial x'_{\ell_1}} \dots \frac{\partial x_{i_k}}{\partial x'_{\ell_k}} \frac{\partial x'_{h_1}}{\partial x_{j_1}} \dots \frac{\partial x'_{h_l}}{\partial x_{j_l}}$$

From now on we focus on tensors of the form $(k, 0)$, with special emphasis on alternating ones. The following lemma should be now well understood, but we invite the reader to check details.

Lemma 7.30. *Let M be smooth manifold and $\sigma : M \rightarrow T^k(M)$ a smooth tensor field. Then σ is smooth if and only if one of the following equivalence conditions is satisfied*

- in every coordinate charts the coefficients $\sigma_{i_1 \dots i_k}$ are smooth functions,
- for every smooth vector fields $X_1, \dots, X_k$ we have that

$$\sigma(X_1, \dots, X_k)(q) = \sigma_q(X_1|_q, \dots, X_k|_q)$$

is a smooth function.

Given two tensor fields σ and τ and $f \in C^\infty(M)$ then $f\sigma$ and $\sigma \otimes \tau$ are also tensor fields.

Exercise 7.31. Write the coordinate representation of $f\sigma$ and $\sigma \otimes \tau$ in terms of the ones of σ, τ .

If $F : M \rightarrow N$ is a smooth map we can pull-back tensors and tensor fields of type $(k, 0)$

$$F^* : T^k(T_{F(q)}N) \rightarrow T^k(T_qM)$$

by applying duality to every element for tensors

$$(F^*T)(v_1, \dots, v_k) = T(F_*v_1, \dots, F_*v_k)$$

and tensor fields

$$(F^*\sigma)(X_1, \dots, X_k)|_q = \sigma(F_*X_1, \dots, F_*X_k)|_{F(q)}$$

Some properties which we leave as exercises.

Proposition 7.32. *We have the following properties for $F : M \rightarrow N$ a smooth map*

- (i) F^* is $\mathbb{R}$ -linear on sections
- (ii) $F^*(\sigma \otimes \tau) = F^*\sigma \otimes F^*\tau$
- (iii) $F^*(g\sigma) = (g \circ F)F^*\sigma$

Notice that in general there is neither push-forward nor pull-back of mixed tensor fields through smooth maps (but if F is a diffeomorphism one can define both).

7.3 Differential forms

Consider the set of alternating tensors of type $(k, 0)$ on T_qM and build the vector bundle

$$\Lambda^k(M) := \bigcup_{q \in M} \Lambda^k(T_qM)$$

We set $\Omega^k(M)$ the smooth sections of $\Lambda^k(M)$, called differential k -forms. In coordinates $\omega \in \Omega^k(M)$ writes as

$$\omega = \sum_{i_1, \dots, i_k=1}^n \omega_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where $\omega_{i_1 \dots i_k}$ are smooth functions on M .

Proposition 7.33. *Suppose $F : M \rightarrow N$ is smooth. We have the following properties*

- (i) $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$ is $\mathbb{R}$ -linear
- (ii) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$
- (iii) $F^*(g\omega) = (g \circ F)F^*\omega$

Remark 7.34. In particular we have a formula to compute the pull-back: if on N with coordinates $\{y_j\}$ we have

$$\omega = \sum_{j_1, \dots, j_k=1}^n \omega_{j_1 \dots j_k} dy_{j_1} \wedge \dots \wedge dy_{j_k}$$

where $\omega_{j_1 \dots j_k}$ are smooth functions on N . Then we get

$$F^* \omega = \sum_{j_1, \dots, j_k=1}^n (\omega_{j_1 \dots j_k} \circ F) d(y_{j_1} \circ F) \wedge \dots \wedge d(y_{j_k} \circ F),$$

which means, denoting F_j the j -th coordinate of F

$$F^* \omega = \sum_{j_1 \dots j_k} (\omega_{j_1 \dots j_k} \circ F) dF_{j_1} \wedge \dots \wedge dF_{j_k},$$

Exercise 7.35. Consider the map

$$F :]0, +\infty[\times]0, 2\pi[\rightarrow \mathbb{R}^2, \quad F(r, \theta) = (r \cos \theta, r \sin \theta)$$

Prove that $F^*(dx \wedge dy) = r dr \wedge d\theta$

Notice that $\Omega^n(M)$ on an n -dimensional manifold has dimension 1, hence every top dimensional form is a smooth multiple of $dx_1 \wedge \dots \wedge dx_n$ in coordinates.

Proposition 7.36. *Let $F : M \rightarrow N$ is smooth map between n -dimensional manifolds.*

$$F^*(g dy_1 \wedge \dots \wedge dy_n) = (g \circ F)(\det DF) dx_1 \wedge \dots \wedge dx_n$$

where DF is the Jacobian matrix of F in the corresponding coordinates.

Proof. By the previous result

$$F^*(g dy_1 \wedge \dots \wedge dy_n) = (g \circ F) dF_1 \wedge \dots \wedge dF_n$$

so it is enough to prove

$$dF_1 \wedge \dots \wedge dF_n = (\det DF) dx_1 \wedge \dots \wedge dx_n$$

It is enough to check the last identity on a basis. But from Lemma 7.27

$$dF_1 \wedge \dots \wedge dF_n \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) = \det \left(dF_i \left(\frac{\partial}{\partial x_j} \right) \right) = \det DF.$$

so that $dF_1 \wedge \dots \wedge dF_n = (\det DF) dx_1 \wedge \dots \wedge dx_n$. □

Exercise 7.37. Let $\{x_i\}$ and $\{x'_j\}$ be two set of coordinates. Prove that

$$dx'_1 \wedge \dots \wedge dx'_n = \det \left(\frac{\partial x'_j}{\partial x_i} \right) dx_1 \wedge \dots \wedge dx_n.$$

Exterior derivative.

We introduce the *exterior derivative* d . This is a linear map $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ for every $k \geq 0$. Recall that $\Omega^0(M) = C^\infty(M)$.

Theorem 7.38. *There exists a unique linear map $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ for every $k \geq 0$ such that*

(i) *if $f \in \Omega^0(M) = C^\infty(M)$, then $df \in \Omega^1(M)$ is its differential,*

(ii) *if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta,$$

(iii) $d \circ d = 0$,

Moreover

(iv) d is local: if $\omega = \omega'$ on $U \subset M$ then $d\omega = d\omega'$ on U ,

(v) d commutes with restrictions $d\omega|_U = d(\omega|_U)$,

(vi) d in coordinates express as

$$d\omega = \sum_{i_1 \dots i_k} d\omega_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{i_1 \dots i_k, j} \frac{\partial \omega_{i_1 \dots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}. \quad (7.13)$$

We denote by $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ if $I = (i_1, \dots, i_k)$ increasing multiindex.

Proof. Assume M has only one chart. Then use the formula (7.13) to define d . Clearly d is linear, local and commutes with restrictions and satisfies (i). We have to check only (ii) and (iii).

First notice that $d(fdx_I) = df \wedge dx_I$ for every multiindex I , even if I is not increasing. Then to check (ii) by linearity it is enough to check for $\omega = fdx_I$ and $\eta = gdx_J$ for I, J increasing multiindex. We have

$$\begin{aligned} d(\omega \wedge \eta) &= d(fgdx_I \wedge dx_J) \\ &= (fdg + gdf) \wedge dx_I \wedge dx_J \\ &= (-1)^k fdx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge gdx_J \\ &= (-1)^k \omega \wedge d\eta + d\omega \wedge \eta. \end{aligned}$$

For (iii) since

$$d(d\omega) = \sum_{i_1 \dots i_k} d(d\omega_{i_1 \dots i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

it is enough to prove that $d(df) = 0$. We have

$$\begin{aligned} d(df) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i \\ &= \sum_{1 \leq i < j \leq n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = 0. \end{aligned}$$

Let us now prove that if there exists an operator D satisfying (i)-(iii) then in coordinates it satisfies (vi). From this everything follows. (We leave other details to the reader.)

Let D be such an operator and apply to a k -form in coordinates, from property (ii)

$$\begin{aligned} D\omega &= \sum_{i_1 \dots i_k} D\omega_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} + \\ &\quad + \sum_{i_1 \dots i_k} \omega_{i_1 \dots i_k} \wedge D(dx_{i_1} \wedge \dots \wedge dx_{i_k}) \end{aligned}$$

The first line is (vi) since $D = d$ on functions. The second line is zero since by (ii)

$$D(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = Ddx_{i_1} \wedge (dx_{i_2} \wedge \dots \wedge dx_{i_k}) - dx_{i_1} \wedge D(dx_{i_2} \wedge \dots \wedge dx_{i_k})$$

and $Ddx_{i_1} = DDx_{i_1} = 0$ by (i), and the other terms is also zero iterating same argument. $\square$

Exercise 7.39. The differential of an arbitrary 1-form in $\mathbb{R}^3$. If a, b, c are smooth functions and

$$\omega = a dx + b dy + c dz$$

then we have

$$d\omega = da \wedge dx + db \wedge dy + dc \wedge dz$$

that is

$$d\omega = \left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z} \right) dx \wedge dz + \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right) dx \wedge dy + \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} \right) dy \wedge dz$$

Remark 7.40 (An algebraic comment). If $A = \oplus_k A^k$ is a graded associative algebra (not necessarily commutative), a linear map $D : A \rightarrow A$ has degree m if $D(A^k) \subset A^{k+m}$ for each k . It is a derivation (resp. antiderivation) if

$$D(xy) = (Dx)y + x(Dy), \quad (\text{resp. } D(xy) = (Dx)y + (-1)^k x(Dy))$$

whenever $x \in A^k$ and $y \in A^l$.

We can summarize the above properties of the differential d on $C^\infty(M)$ saying that it extends to a unique antiderivation on $\Omega(M)$ of degree 1 and such that d squares to zero.

Proposition 7.41. Let $F : M \rightarrow N$ be smooth and $\omega \in \Omega^k(N)$. Then

$$F^*(d\omega) = d(F^*\omega)$$

Proof. For $k = 0$ setting $F^*f = f \circ F$ this is item (i) in Proposition 7.11. For $k \geq 1$ we reduce to it. Since d is local it is sufficient to prove the formula in local coordinates. Since in coordinates

$$\omega = \sum \omega_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

by linearity it is sufficient to prove for ω of the form $f dx_{i_1} \wedge \dots \wedge dx_{i_k}$. But $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ so it is enough to prove for $\omega = f dx_i$. We have $d\omega = df \wedge dx_i$ and

$$F^*(d\omega) = F^*df \wedge F^*dx_i = dF^*f \wedge F^*dx_i = d(f \circ F) \wedge F^*dx_i$$

On the other hand

$$d(F^*\omega) = d((f \circ F)F^*dx_i) = d(f \circ F) \wedge F^*dx_i + (f \circ F)dF^*dx_i = d(f \circ F) \wedge F^*dx_i$$

where we used Proposition 7.11 and $dF^*dx_i = ddF^*x_i = 0$ since $d^2 = 0$. $\square$

Exercise 7.42. Consider the map

$$F :]0, +\infty[\times]0, 2\pi[\rightarrow \mathbb{R}^2, \quad F(r, \theta) = (r \cos \theta, r \sin \theta)$$

We proved that $F^*(dx \wedge dy) = r dr \wedge d\theta$. On the other hand

$$dx \wedge dy = d\left(\frac{1}{2}(xdy - ydx)\right), \quad r dr \wedge d\theta = d\left(\frac{1}{2}r^2 d\theta\right)$$

and $F^*(xdy - ydx) = r^2 d\theta$.

7.4 Lie derivatives of tensors and differential forms

Definition 7.43. Let $X \in \text{Vec}(M)$ and $\omega \in \Lambda^k M$, where $k \geq 0$. We define the *Lie derivative* of τ covariant tensor with respect to X as the operator

$$\mathcal{L}_X : \mathcal{T}^k(M) \rightarrow \mathcal{T}^k(M), \quad \mathcal{L}_X \tau = \frac{d}{dt} \Big|_{t=0} (e^{tX})^* \tau. \quad (7.14)$$

Notice that

$$\mathcal{L}_X \tau = \frac{d}{dt} \Big|_{t=0} (e^{tX})^* \tau = \lim_{t \rightarrow 0} \frac{1}{t} \left((e^{tX})^* (\tau_{e^{tX}(q)}) - \tau_q \right)$$

We define the *Lie derivative* of $\omega \in \Omega^k(M)$ with respect to X as the operator

$$\mathcal{L}_X : \Lambda^k(M) \rightarrow \Lambda^k(M), \quad \mathcal{L}_X \omega = \frac{d}{dt} \Big|_{t=0} (e^{tX})^* \omega. \quad (7.15)$$

We stress that the Lie derivative of a k -form along a vector field defines a new k -form.

For $k = 0$ this definition coincides with the Lie derivative of smooth functions, $\mathcal{L}_X f = Xf$, for $f \in C^\infty(M)$.

Lemma 7.44. *We have the following properties of the Lie derivative:*

$$(i) \quad \mathcal{L}_X(\omega_1 \wedge \omega_2) = (\mathcal{L}_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathcal{L}_X \omega_2),$$

$$(ii) \quad \mathcal{L}_X(f\omega) = (Xf)\omega + f\mathcal{L}_X \omega$$

$$(iii) \quad \mathcal{L}_X \circ d = d \circ \mathcal{L}_X$$

Property (i) can be also expressed by saying that $\mathcal{L}_X$ is a *derivation* of the exterior algebra of k -forms.

Proof. (i) and (ii) follows differentiating at $t = 0$ the following identities

$$(e^{tX})^*(\omega_1 \wedge \omega_2) = (e^{tX})^* \omega_1 \wedge (e^{tX})^* \omega_2 \quad (7.16)$$

$$(e^{tX})^*(f\omega) = (e^{tX})^* f \cdot (e^{tX})^* \omega_2 = (f \circ e^{tX})(e^{tX})^* \omega \quad (7.17)$$

(iii) follows since d commutes with F^* and is linear. $\square$

Given a k -form and a vector field, one can also introduce their inner product, defining a $(k-1)$ -form as follows.

Definition 7.45. Let $X \in \text{Vec}(M)$ and $\omega \in \Omega^k(M)$, with $k \geq 1$. We define the *inner product of ω and X* as the operator $i_X : \Omega^k M \rightarrow \Omega^{k-1} M$, such that

$$(i_X \omega)(Y_1, \dots, Y_{k-1}) := \omega(X, Y_1, \dots, Y_{k-1}), \quad Y_i \in \text{Vec}(M). \quad (7.18)$$

We have the following property, whose proof is left to the reader.

Proposition 7.46. *The operator i_X is an anti-derivation, in the following sense:*

$$i_X(\omega_1 \wedge \omega_2) = (i_X \omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge (i_X \omega_2), \quad \omega_i \in \Omega^{k_i} M, \quad i = 1, 2. \quad (7.19)$$

We end this section proving two classical formulas, usually referred as Cartan's formulas.

Proposition 7.47 (Cartan's formula). *Let $X \in \text{Vec}(M)$. The following identity holds true*

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \quad (7.20)$$

Proof. Set $D_X := i_X \circ d + d \circ i_X$. It is easy to check that D_X is a derivation on the algebra of k -forms, since i_X and d are anti-derivations. Let us show that D_X commutes with d . Indeed, using the fact that $d^2 = 0$, one gets

$$d \circ D_X = d \circ i_X \circ d = D_X \circ d.$$

Since any k -form can be expressed in coordinates as $\omega = \sum \omega_{i_1 \dots i_k} dx_{i_1} \dots dx_{i_k}$, it is sufficient to prove that $\mathcal{L}_X$ coincides with D_X on functions. This last property is easily verified, since

$$D_X f = i_X(df) + \underbrace{d(i_X f)}_{=0} = \langle df, X \rangle = Xf = \mathcal{L}_X f. \quad \square$$

Corollary 7.48. *Let $X, Y \in \text{Vec}(M)$ and $\omega \in \Lambda^1 M$, then*

$$d\omega(X, Y) = X \langle \omega, Y \rangle - Y \langle \omega, X \rangle - \langle \omega, [X, Y] \rangle. \quad (7.21)$$

Proof. On one hand Definition 7.43 implies, by Leibniz rule¹

$$\begin{aligned} \langle \mathcal{L}_X \omega, Y \rangle_q &= \left. \frac{d}{dt} \right|_{t=0} \langle (e^{tX})^* \omega, Y \rangle_q \\ &= \left. \frac{d}{dt} \right|_{t=0} \langle \omega, e_*^{tX} Y \rangle_{e^{tX}(q)} \\ &= X \langle \omega, Y \rangle - \langle \omega, [X, Y] \rangle. \end{aligned}$$

On the other hand, Cartan's formula (7.20) gives

$$\begin{aligned} \langle \mathcal{L}_X \omega, Y \rangle &= \langle i_X(d\omega), Y \rangle + \langle d(i_X \omega), Y \rangle \\ &= d\omega(X, Y) + Y \langle \omega, X \rangle. \end{aligned}$$

Comparing the two identities one gets (7.21). $\square$

¹Let $f(x, y)$ be a function of two variables. Then

$$\left. \frac{d}{dt} \right|_{t=0} f(t, t) = \frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial y}(0, 0) = \left. \frac{d}{dt} \right|_{t=0} f(t, 0) + \left. \frac{d}{dt} \right|_{t=0} f(0, t)$$

Exercise 7.49. Prove that for $\omega \in \Omega^k(M)$ and $Y \in \text{Vec}(M)$ we have

$$\begin{aligned} \mathcal{L}_Y(\omega(X_1, \dots, X_k)) &= (\mathcal{L}_Y\omega)(X_1, \dots, X_k) + \omega(\mathcal{L}_Y X_1, \dots, X_k) \\ &\quad \dots + \omega(X_1, \dots, \mathcal{L}_Y X_k) \end{aligned}$$

for every $X_1, \dots, X_k \in \text{Vec}(M)$. Which also means

$$\begin{aligned} (\mathcal{L}_Y\omega)(X_1, \dots, X_k) &= Y(\omega(X_1, \dots, X_k)) - \omega([Y, X_1], \dots, X_k) \\ &\quad \dots - \omega(X_1, \dots, [Y, X_k]) \end{aligned}$$

Exercise 7.50. Prove the following Leibniz rule formula: for $X \in \text{Vec}(M)$, $\omega \in \Lambda^k M$, and $f \in C^\infty(M)$

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge i_X\omega \tag{7.22}$$

Chapter 8

Orientation, Integration on manifolds

Stokes' theorem, first appeared in print in 1854.

George Stokes had for several years been setting the Smith's Prize Exam at Cambridge, in February, 1854, examination, question #8 is the following: [a version of Stokes' theorem]

V. Katz, The History of Stokes' Theorem
Mathematics Magazine, Vol. 52, No. 3 (May, 1979)

8.1 Orientation

Let V be a vector space. An orientation of V is an equivalence class of ordered basis. Two basis $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ belong to the same equivalence class if the matrix of the change of basis has positive determinant.

$$e'_i = \sum_{j=1}^n a_{ij} e_j, \quad \det(a_{ij}) > 0$$

Of course this is an equivalence relation and we have two equivalence classes.

One can think to a orientation also as a choice of an “orthogonal vector” to the vector space “up” or “down”. It is better formalized as follows.

Lemma 8.1. *Let V be a vector space of dim $n \geq 1$. A non zero element Ω of $\Lambda^n(V)$ defines an orientation for V : all ordered basis $e_1, \dots, e_n$ such that $\Omega(e_1, \dots, e_n) > 0$.*

The same can be done on manifolds, with top dimensional differential forms.

Definition 8.2. Two charts (U, φ) and (U', φ') are *equioriented* if $\det D(\varphi' \circ \varphi^{-1}) > 0$ on $\varphi(U \cap U')$. An atlas is *oriented* if every pair of charts are equioriented. A manifold M is *orientable* if M admits an oriented atlas. An *orientation* of M is a choice of an oriented atlas of M .

Lemma 8.3. *If M is covered by two charts (U, φ) and (U', φ') with $U \cap U'$ connected, then M is orientable.*

Proof. Notice that by assumption $\det D(\varphi' \circ \varphi^{-1})$ is not zero since it is a local diffeo, hence its sign is constant on every connected set, hence on $U \cap U'$. If $\det D(\varphi' \circ \varphi^{-1}) > 0$ ok, otherwise choose φ' and $\bar{\varphi} = (x_1, \dots, x_n, x_{n-1})$ which then are equioriented. $\square$

Corollary 8.4. S^n is orientable for every $n \geq 1$.

For $n > 1$ it is given by the previous Lemma 8.3. For $n = 1$ one can check it explicitly on the standard atlas.

Exercise 8.5 (The Mobius band is not orientable). Let us consider the Mobius band M as the infinite strip $M := \mathbb{R} \times [0, 1] / \sim$ with the identification $(x, 0) \simeq (-x, 1)$, endowed with the quotient topology. We denote by $[x, y] = \pi(x, y)$ the image of a point under the canonical projection. Consider on M the two open sets

$$U_1 = \pi(\mathbb{R} \times]0, 1[), \quad U_2 = \pi(\mathbb{R} \times [0, 1/2[\cup]1/2, 1])$$

Define the charts

$$\varphi_1([x, y]) = (x, y), \quad \varphi_2([x, y]) = \begin{cases} (x, y), & 0 \leq y < 1/2 \\ (-x, y - 1), & 1/2 \leq y < 1 \end{cases}$$

It is easy to see that φ_2 is well defined on M since $\varphi_2([x, 0]) = (x, 0) = \varphi_2([-x, 1])$. These are homeomorphism onto their images and the change of charts satisfies

$$\varphi_2 \circ \varphi_1^{-1}(x, y) = \begin{cases} (x, y), & 0 < y < 1/2 \\ (-x, y - 1), & 1/2 < y < 1 \end{cases}$$

with $\det D(\varphi_2 \circ \varphi_1^{-1}) = 1 > 0$ on $0 < y < 1/2$ and $\det D(\varphi_2 \circ \varphi_1^{-1}) = -1 < 0$ on $1/2 < y < 1$.

This is a not oriented atlas. Indeed all atlas need to show this behaviour. To be more formal, let $(U_i, \varphi_i)_{i \in I}$ be an oriented atlas on M . Define a “sign” function $\sigma : [0, 1] \rightarrow \{-1, 1\}$ as follows. For $y \in [0, 1]$ we consider some $i = i(y)$ such that $[0, y] \in U_i$ and set $\sigma(y) = \text{sgn} \det D(\varphi_i^{-1} \circ \pi)(0, y)$. It is easy to check that σ is well-defined and locally constant, hence constant on $[0, 1]$ which is connected. On the other hand the identification $[x, y] = [-x, y + 1]$ imposes $\sigma(0) = -\sigma(1)$, which gives a contradiction.

Exercise 8.6. Let $F : M \rightarrow N$ be a local diffeomorphism between oriented manifolds. Then F preserves orientations if and only if for every pair of oriented charts (U, φ) and (V, ψ) such that $F(U) \subset V$ we have $\det(D\hat{F}) > 0$ on $\varphi(U)$, where as usual $\hat{F} = \psi \circ F \circ \varphi^{-1}$.

Definition 8.7. A *volume form* on a n -dimensional smooth manifold M is a non vanishing n -form $\nu \in \Omega^n(M)$.

Notice that given a volume form every other n -form ω is written as $\omega = f\nu$ with $f \in C^\infty(M)$. If $f \neq 0$ then ω is also a volume form.

Lemma 8.8. For a volume form ν and a change of charts

$$\nu \left(\frac{\partial}{\partial x'_1}, \dots, \frac{\partial}{\partial x'_n} \right) = \det \left(\frac{\partial x'_j}{\partial x_i} \right) \nu \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

Proof. In coordinates $\nu = f dx'_1 \wedge \dots \wedge dx'_n$ for some $f \neq 0$. Hence it is enough to prove

$$dx'_1 \wedge \dots \wedge dx'_n = \det \left(\frac{\partial x'_j}{\partial x_i} \right) dx_1 \wedge \dots \wedge dx_n \quad (8.1)$$

But this is a consequence of the formula

$$dx'_j = \sum_{i=1}^n \frac{\partial x'_j}{\partial x_i} dx_i$$

and Lemma 7.27. □

Proposition 8.9. *A smooth manifold M is orientable if and only if M admits a volume form*

Proof. Fix a volume form $\nu \in \Omega^n(M)$, never vanishing. We build the atlas $\mathcal{A}$ of all positive charts, i.e., charts (U, φ) such that U connected and (notice that the sign is constant on a connected U)

$$\nu \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) > 0.$$

The atlas $\mathcal{A}$ is oriented since for any two charts (U, φ) and (U', φ') we have that property

$$\nu \left(\frac{\partial}{\partial x'_1}, \dots, \frac{\partial}{\partial x'_n} \right) = \det \left(\frac{\partial x'_j}{\partial x_i} \right) \nu \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

hence every change of charts must have positive determinant since ν is positive on oriented frames..

Conversely given an atlas $\mathcal{A}$ of equioriented charts we take a partition of unity $\{\psi_\alpha\}$ subordinated to it and we set

$$\nu = \sum_{\alpha} \psi_{\alpha} dx_1^{\alpha} \wedge \dots \wedge dx_n^{\alpha}$$

This is well defined on all M (the sum is finite at every point). We have to show that ν is never vanishing. But this follows from the formula (8.1). Fix a point q and a chart (U, φ) containing that point. In a neighborhood we read

$$\nu = \sum_{\alpha} \psi_{\alpha} \det \left(\frac{\partial x_j^{\alpha}}{\partial x_i} \right) dx_1 \wedge \dots \wedge dx_n$$

Notice that all terms are ≥ 0 and at least one is > 0 . Hence ν does not vanish. □

Example 8.10. We know that S^n is orientable. We can find a volume form on S^n as follows. Consider in $\mathbb{R}^{n+1}$ the n -form

$$\omega = \sum_{i=0}^n (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

where the “hat” stands for removing the corresponding term. The restriction of this n form to S^n is still a n -form which is never vanishing since for every $v_1, \dots, v_n \in T_x S^n$ we have

$$\omega_x(v_1, \dots, v_n) = \det(x, v_1, \dots, v_n) \neq 0$$

interpreting x and the v_i as elements of $\mathbb{R}^{n+1}$ (check as exercise!). For instance in S^2 this gives the volume form in coordinates (x, y, z) in $\mathbb{R}^3$

$$\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$$

Exercise 8.11. Let $F : M \rightarrow N$ be a smooth map between manifolds of the same dimension, and let ω be a volume form on N . Prove that if $F^*\omega$ is a volume form on M then F is a local diffeomorphism.

Recall that a manifold M is said paralelizable if TM is trivial.

Proposition 8.12. *Every paralelizable manifold is orientable*

Proof. Let $X_1, \dots, X_n$ be n vector fields that are linearly independent everywhere. Consider the dual basis $\eta_1, \dots, \eta_n$ of differential 1-forms. Then $\Omega := \eta_1 \wedge \dots \wedge \eta_n$ is a never vanishing volume form. $\square$

Corollary 8.13. *Every Lie group G is orientable.*

Proof. Every Lie group is paralelizable since TG is diffeomorphic to $G \times \mathfrak{g}$ hence to $G \times \mathbb{R}^n$ where $n = \dim G$. In other words one can choose $v_1, \dots, v_n$ in $\mathfrak{g}$ and build n left-invariant vector fields $X_1, \dots, X_n$ where $X_i(g) = L_{g*}v_i$ that are automatically everywhere linearly independent. The dual basis $\eta_1, \dots, \eta_n$ satisfies $L_g^*\eta_i = \eta_i$ for every $i = 1, \dots, n$. Hence $\Omega = \eta_1 \wedge \dots \wedge \eta_n$ satisfies $L_g^*\Omega = \Omega$ for every g . $\square$

Proposition 8.14. *The projective spaces $\mathbb{P}^n$ are orientable if and only if n is odd.*

Proof. We prove that $\mathbb{P}^n$ cannot be orientable if n even. We consider $\mathbb{P}^n$ as the quotient of S^n with the group $\{1, i\}$ where $1(x) = x$ and $i(x) = -x$. If we denote by $p : S^n \rightarrow \mathbb{P}^n$ the covering. Assume we have a volume form ν for $\mathbb{P}^n$ then $\eta = p^*\nu$ is a volume form for S^n . Hence $\eta = p^*\nu = f\omega$ where ω is the volume form of the Example 8.10 and f is a never vanishing function.

Notice that $p \circ i = p$ hence $p^* = i^*p^*$: applied to ν this gives $i^*\eta = \eta$. But we have also $i^*\omega = (-1)^{n+1}\omega = -\omega$ if n even. So

$$\eta = i^*\eta = i^*(f\omega) = -(f \circ i)\omega$$

This shows that η changes sign, and being smooth it vanishes at some point. Contradiction.

To complete the proof one can either (a) prove by hand that the atlas is orientable if n odd (b) prove that the form ω we have defined on the sphere descends to a volume form on the projective space for even dimensional spaces (cf. the more general Exercise 8.15). $\square$

Exercise 8.15. Let M be manifold and G discrete subgroup acting properly and smoothly on M . let $p : M \rightarrow M/G$. If ω is a differential form on M such that $g \cdot \omega = \omega$ for every $g \in G$ then there exists on M/G a unique differential form η such that $p^*\eta = \omega$. If ω is a volume form then η is a volume form as well.

8.2 Integration on manifolds

Given a function $f : M \rightarrow \mathbb{R}$ smooth, assuming that f has compact support in a chart (U, φ) , one might think to define its integral as follows through the chart

$$\int_U f = \int_{\varphi(U)} f \circ \varphi^{-1} dx$$

where the second integral is computed in $\mathbb{R}^n$. This is actually not well-defined since if the support is also contained in a second chart (U', φ') then we should have also

$$\int_U f = \int_{\varphi'(U)} f \circ (\varphi')^{-1} dx$$

which is not possible: indeed by the change of variable formula in $\mathbb{R}^n$

$$\int_{\Phi(V)} g(x) dx = \int_V g(\Phi(x)) |\det D\Phi(x)| dx$$

applied to $g = f \circ (\varphi')^{-1}$, $\Phi = \varphi' \circ \varphi^{-1}$ and $V = \varphi(U)$, one get that

$$\int_{\varphi'(U)} f \circ (\varphi')^{-1} dx = \int_{\varphi(U)} f \circ \varphi^{-1} |\det D(\varphi' \circ \varphi^{-1})(x)| dx$$

and in general there is no reason for having $|\det D(\varphi' \circ \varphi^{-1})(x)| = 1$ everywhere. Thinking from a computational point of view, we have to integrate not functions, but the correct object in order to absorb the determinant into the change of charts.

Integrating differential forms. The correct operation is integrating n -forms on open sets of $\mathbb{R}^n$, as follows. Let $U \subset \mathbb{R}^n$ open and $\eta = f(x) dx_1 \wedge \dots \wedge dx_n$ be a n -form with compact support in U . Then we set

$$\int_U \eta = \int_U f(x) dx_1 \wedge \dots \wedge dx_n = \int_U f(x) dx$$

where dx denotes here the Lebesgue measure.

Let now M be a manifold of dimension n and assume (U, φ) is a chart with $\omega \in \Omega^n(M)$ with compact support in U . It would be natural to set (recall that $\varphi : U \subset M \rightarrow \varphi(U) \subset \mathbb{R}^n$)

$$\int_U \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

with the right hand side a well-defined integral of the n -form $\eta = (\varphi^{-1})^* \omega$ with compact support in $\varphi(U)$. is this definition well posed? Let ω be a n -form with compact support in the intersection $U \cap U'$ of two charts (U, φ) (U', φ') . On one side we are setting

$$\int_U \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

while on the other one

$$\int_U \omega = \int_{\varphi'(U')} (\varphi'^{-1})^* \omega$$

We claim that we have equality between the two definitions if and only if the charts are equioriented. Indeed let us write

$$(\varphi^{-1})^* \omega = f(x) dx, \quad (\varphi'^{-1})^* \omega = g(y) dy$$

then writing $F(x) = y$ (i.e., we are setting $F := \varphi' \circ \varphi^{-1}$) we have by Proposition 7.36

$$f(x) dx = F^*(g(y) dy) = (g \circ F) \det(DF) dx$$

by our previous considerations. On the other hand the change of variable formula says

$$\int_{U'} g(y)dy = \int_U (g \circ F(x)) |\det(DF)| dx$$

with the absolute value! It works if and only if the charts are equioriented. We have proved

Proposition 8.16. *Let M be smooth manifold oriented and ω an n -dim form with compact support. Let (U, φ) (U', φ') two equioriented charts such that the support of ω is contained in $U \cap U'$. Then*

$$\int_{\varphi'(U')} (\varphi'^{-1})^* \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

In particular the integral is independent on the chart and we can set on an oriented manifold

$$\int_U \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

Proposition 8.17. *Let M be smooth manifold oriented and ω an n -dim form with compact support. Let $\{(U_i, \varphi_i)\}$ be an oriented atlas and $\{\psi_i\}$ partition of unity subordinated to it. Then the quantity*

$$\sum_{i \in \mathbb{N}} \int_M \psi_i \omega$$

is well defined and independent on atlas and partition of unity.

Proof. Since $\text{supp}(\omega)$ is compact and the $\{\psi_i\}$ partition of unity then the sum is finite at every point and every form $\psi_i \omega$ has compact support in his U_i . If we take two partition of unity $\{\psi_i\}$ and $\{\psi'_j\}$ we can compare

$$\sum_{i \in \mathbb{N}} \int_M \psi_i \omega, \quad \sum_{j \in \mathbb{N}} \int_M \psi'_j \omega$$

since both sums can write as

$$\sum_{i, j \in \mathbb{N}} \int_M \psi_i \psi'_j \omega$$

and each terms has support in $U_i \cap U'_j$ hence independent by previous results. $\square$

Definition 8.18. The integral of $\omega \in \Omega^n(M)$ on M is then defined as follows

$$\int_M \omega := \sum_{i \in \mathbb{N}} \int_M \psi_i \omega$$

where $\{\psi_i\}$ is an arbitrary partition of unity.

Remark 8.19. It is easy to see that if M is oriented and if we denote by $\overline{M}$ the manifold M oriented with the opposite orientation then for every volume form ω we have

$$\int_{\overline{M}} \omega = - \int_M \omega.$$

Moreover if M is oriented with the orientation induced by the volume form ν , then

$$\int_M \nu > 0.$$

Remark 8.20. Given a volume form ν it makes sense to define

$$\int_M f \nu$$

If ν is the orientation of M sometimes we write by abuse of notation $\int_M f$. In particular if M is compact $\text{vol}_\nu(M) = \int_M \nu$.

Another immediate consequence of our construction we have the following version of the change of variable formula.

Proposition 8.21. *Let $F : M \rightarrow F(M) \subset N$ be an orientation preserving diffeomorphism of M onto its image $F(M)$. Then*

$$\int_{F(M)} \omega = \int_M F^* \omega$$

It is enough to consider differential forms with compact support in one chart and then one is reduced to the case of differential forms in $\mathbb{R}^n$ where this is the change of variable formula for the Lebesgue integral. Concretely this is what one does.

Proposition 8.22. *Let M be a smooth oriented manifold. Suppose the support of ω is contained in $\cup_i U_i$ with U_i compact and let V_i be compact domain in $\mathbb{R}^n$ with $F_i : V_i \rightarrow U_i$ parametrizations such that*

- $F_i(V_i) = U_i$ and F_i preserves the orientation (in the interior of the compact sets)
- U_i and U_j intersect only on the boundary

Then we have

$$\int_M \omega = \sum_i \int_{V_i} F_i^* \omega$$

Exercise 8.23. Let us compute the integral over S^2 of

$$\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$$

This is the volume of the sphere and gives 4π . Let us consider the parametrization

$$F(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$$

defined on the (open) rectangle $D = (0, \pi) \times (0, 2\pi)$. We have that the chart F is positively oriented with respect to ω (or that F preserves the orientation if we orient $D \subset \mathbb{R}^2$ with standard orientation). It is enough to check at one point: for instance $F(\pi/2, 0) = (1, 0, 0)$

$$\omega_{(1,0,0)} = dy \wedge dz$$

and

$$F_*(\partial_\varphi|_{(\pi/2,0)}) = \frac{d}{dt} \Big|_{t=0} F(\pi/2 + t, 0) = \frac{d}{dt} \Big|_{t=0} (0, \cos(t), -\sin(t)) = (0, 0, -1) := v_1$$

$$F_*(\partial_\theta|_{(\pi/2,0)}) = \frac{d}{dt}\bigg|_{t=0} F(\pi/2, t) = \frac{d}{dt}\bigg|_{t=0} (\cos(t), \sin(t), 0) = (0, 1, 0) := v_2$$

and

$$dy \wedge dz(v_1, v_2) = 1.$$

We have also

$$F^*\omega = \sin \varphi d\varphi \wedge d\theta$$

and (removing a zero measure set¹)

$$\int_{S^2} \omega = \int_D F^*\omega = \int_0^\pi \int_0^{2\pi} \sin \varphi d\theta d\varphi = 4\pi$$

Notice that we can integrate n -forms on n -dim manifolds, for every n . Hence in an n -dimensional manifold we can integrate k -forms on k -dimensional submanifolds. But we need to speak about orientation of submanifolds.

Remark 8.24. If M is not orientable one can still integrate *densities*. These are not tensors but they behave well (with absolute value) under change of charts. See [Lee13].

The Hairy Ball theorem

We end with an important application of the integration theory, known as Hairy Ball theorem. The proof we present here is due to J. Milnor.

Theorem 8.25 (Hairy Ball theorem). *There exists no never vanishing smooth vector field on S^n when n is even.*

Proof. Let us think to S^n as subset of $\mathbb{R}^{n+1}$. Then a vector field on S^n is a map $X : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that $x \cdot X(x) = 0$ where the dot denotes the scalar product in $\mathbb{R}^{n+1}$. Assume there exists a never vanishing vector field. Then we can consider $X(x)/\|X(x)\|$ and assume without loss of generality it has norm equal to 1 everywhere.

Let us consider the following family of maps for $\varepsilon > 0$

$$f_\varepsilon : S^n(1) \rightarrow S^n(\sqrt{1+\varepsilon^2}), \quad f_\varepsilon(x) = x + \varepsilon X(x)$$

We first need an auxiliary lemma.

Lemma 8.26. *For $\varepsilon > 0$ small enough the map f_ε is a global diffeomorphism.*

Proof of the Lemma. Let us consider the n -form in $\mathbb{R}^{n+1}$ (which we recall it restricts to a volume form on every sphere)

$$\omega = \sum_{i=0}^n (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

¹one can prove using a change of charts that zero-measure set are well defined on a manifold and that $\int_A \omega$ for every zero measure set A in M and ω volume form.

and let us consider $f_\varepsilon^* \omega$ the pull-back on $S^n(1)$ of the restriction of ω to $S^n(r)$ with $r = \sqrt{1 + \varepsilon^2}$. Using the formula

$$f_\varepsilon^* \omega = \sum_{i=0}^n (-1)^n (x_i \circ f_\varepsilon) d(x_0 \circ f_\varepsilon) \wedge \dots \wedge d(\widehat{x_i \circ f_\varepsilon}) \wedge \dots \wedge d(x_n \circ f_\varepsilon)$$

it is not difficult to see that $f_\varepsilon^* \omega$ is a *polynomial* with respect to ε of degree $\leq n + 1$ and being f_0 equal to the identity map we can indeed write

$$f_\varepsilon^* \omega = \omega + \varepsilon \eta_\varepsilon$$

where η_ε is a family of smooth n forms which is polynomial with respect to ε of degree $\leq n$. In particular, since we speak about volume forms, there exists a family of functions g_ε such that $\eta_\varepsilon = g_\varepsilon \omega$ and

$$f_\varepsilon^* \omega = (1 + \varepsilon g_\varepsilon) \omega$$

Since spheres are compact, this implies that for $\varepsilon > 0$ small enough $f_\varepsilon^* \omega$ is a volume form on S^n . In particular f_ε is a local diffeomorphism for $\varepsilon > 0$ small enough thanks to Exercise 8.11.

Let us show that for $\varepsilon > 0$ small enough the map f_ε is also injective. If this is not the case we would have a sequence $\varepsilon_k \rightarrow 0$ and sequences of distinct points $x_k, y_k \in S^n(1)$ such that $f_{\varepsilon_k}(x_k) = f_{\varepsilon_k}(y_k)$ which can be rewritten as

$$\frac{x_k - y_k}{\|x_k - y_k\|} = -\varepsilon_k \frac{X(x_k) - X(y_k)}{\|x_k - y_k\|}$$

This is a contradiction since the left hand side has norm 1, while the norm of the right hand side tends to zero thanks to the inequality

$$\|X(x_k) - X(y_k)\| \leq C \|x_k - y_k\|$$

which holds² since X is a smooth vector field and $S^n(1)$ is compact.

We have proved that for $\varepsilon > 0$ small enough the map f_ε is an injective local diffeomorphism. For such $\varepsilon > 0$ it is a local diffeomorphism, it is an open map and $f_\varepsilon(S^n(1))$ is open. Since the source space is compact, then $f_\varepsilon(S^n(1))$ is closed. Hence the image $f_\varepsilon(S^n(1))$ is open and closed and, by connectedness, f_ε is surjective. A bijective local diffeomorphism is by construction a global diffeomorphism.³ $\square$

To end the proof, let us now compute in two different ways the integral

$$\int_{S^n(r)} \omega$$

On the one hand using the change of variables $y = F(x) = rx$ we have that $F^* \omega = r^{n+1} \omega$ and

$$\int_{S^n(r)} \omega = \int_{F(S^n(1))} \omega = \int_{S^n(1)} F^* \omega = r^{n+1} \int_{S^n(1)} \omega = c_n r^{n+1}$$

²one can take $C = \sup\{\|DX(x)\|, x \in S^n(1)\}$

³Recall that any proper continuous map $f : X \rightarrow Y$ between smooth manifold is closed. If X is compact, every continuous map $f : X \rightarrow Y$ is proper. Indeed in Y , compact sets are closed (assuming Y is Hausdorff). f is continuous, so the inverse image of a closed set is closed. But a closed subset of a compact (Hausdorff) space is compact. So the inverse image of a compact set is compact.

On the other hand using f_ε as a change of variables (recall that $r = \sqrt{1 + \varepsilon^2}$)

$$\int_{S^n(r)} \omega = \int_{f_\varepsilon(S^n(1))} \omega = \int_{S^n(1)} f_\varepsilon^* \omega = P(\varepsilon)$$

where P is some polynomial in ε . This implies for $\varepsilon > 0$ small enough

$$c_n(1 + \varepsilon)^{\frac{n+1}{2}} = P(\varepsilon)$$

which is a contradiction for n even. □

Remark 8.27. Notice that on odd dimensional spheres there always exists a non vanishing vector field. Indeed consider for $n \geq 1$ the vector field in $\mathbb{R}^{2n}$

$$X = x_2 \partial_{x_1} - x_1 \partial_{x_2} + \dots + x_{2n} \partial_{x_{2n-1}} - x_{2n-1} \partial_{x_{2n}}$$

It is easy to see that this vector field is never vanishing and is tangent to (i.e., restricts to a well defined vector field on) the sphere $S^{2n-1} \subset \mathbb{R}^{2n}$.

The Haar measure on compact Lie groups

Corollary 8.28. *If G is a compact Lie group there exists a unique left-invariant volume form Ω such that $\int_G \Omega = 1$, called the Haar measure on G .*

A Lie group G is called *unimodular* if all left-invariant volume forms Ω are also right-invariant.

Remark 8.29. Some consequences: S^n is not paralelizable (and TS^n is not trivial) for n even, . In particular there cannot exists a Lie group structure on S^n for n even. Indeed one can prove that the only spheres which carry a Lie group structure are S^1 and S^3 (together with $S^0 \simeq \mathbb{Z}_2$).

Proof. It is enough to take any volume form ν and then set $\Omega = c^{-1}\nu$ with $c = \int_G \nu$. Notice that c is finite since G is compact. □

We stress that on non compact Lie groups it is possible to speak about *Haar measures* (plural!) for left invariant volume forms, but there is not the possibility of fixing a normalization in general since the volume might be infinite.

8.3 Manifolds with boundary

A manifold with boundary is a space which is locally modeled by open sets of $\mathbb{R}^n$ and open sets of the half-space (open being understood in the subspace topology). We denote the half space in $\mathbb{R}^n$

$$\mathbb{H}^n = \mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}.$$

Notice that $\partial\mathbb{H}^n = \{x_n = 0\}$, understood as topological boundary.

A *boundary chart* for a topological space M is a pair (U, φ) where $U \subset M$ open and $\varphi : U \rightarrow V$ homeomorphism on an open set V of $\mathbb{H}^n$ with the subspace topology $\mathbb{H}^n \subset \mathbb{R}^n$. In other terms $V = A \cap \mathbb{H}^n$ for some open set A in $\mathbb{R}^n$.

Repeating all the theory and replacing “charts” with “boundary charts” we have a definition of *smooth manifold with boundary*. This is a topological space M which is second countable and Hausdorff, locally homomorphic at every point to an open set of $\mathbb{H}^n$.

The boundary ∂M of a manifold with boundary M is the set of points q of M for which there exists a chart (U, φ) with $q \in \varphi^{-1}(\partial\mathbb{H}^n \cap \varphi(U))$. This is well defined in the sense that if, for a given $q \in M$, there exists a chart (U, φ) such that $q \in \varphi^{-1}(\partial\mathbb{H}^n \cap \varphi(U))$, then $q \in (\varphi')^{-1}(\partial\mathbb{H}^n \cap \varphi'(U'))$ for all other charts (U', φ') containing q . We invite the reader to check it.

Lemma 8.30. *If M is a n -dimensional manifold with boundary then ∂M is a $(n-1)$ -dimensional manifold without boundary.*

The property just stated of the boundary says $\partial(\partial M) = \emptyset$, hence we can say $\partial \circ \partial = 0$. As the exterior derivative d , the operator ∂ squares to zero. The Stokes theorem will add one more reason to feel that d and ∂ are related one to each other (in a suitable sense, they are dual operators).

Exercise 8.31. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. Assume that 0 is a regular value for f , hence $f^{-1}(0)$ is a smooth $(n-1)$ -dim manifold. Prove that $M = \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ is a manifold with boundary $\partial M = f^{-1}(0)$.

For instance $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a manifold with boundary of dimension n and $\partial B^n = S^{n-1}$, which is correctly a $(n-1)$ -dimensional manifold without boundary.

Induced orientation on the boundary

A first result is a result about orientation of hypersurfaces. Recall that given a n -form ω on M and a vector field X we can define a $(n-1)$ -form

$$i_X \omega(Y_1, \dots, Y_{n-1}) = \omega(X, Y_1, \dots, Y_{n-1})$$

Given a hypersurface $S \subset M$, we say that a vector field is *transverse* to S if $X(q) + T_q S = T_q M$ for every $q \in M$

Proposition 8.32. *Let M be an oriented smooth n -dim manifold and S be an hypersurface of M . Let X be a transverse vector field to S .*

- (a) *there exists a unique orientation on S induced by X and compatible with M , in the sense that $Y_1, \dots, Y_{n-1}$ is positive on S if and only if $X, Y_1, \dots, Y_{n-1}$ is positive on M .*
- (b) *this orientation is induced by the volume form $i_X \omega|_S$, where ω is an orientation for M .*

Example 8.33. S^n is an hypersurfaces of $\mathbb{R}^{n+1}$ which can be oriented compatibly with the standard orientation of $\mathbb{R}^{n+1}$ induced by the transverse vector field

$$X = \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$$

This orientation is the one induced by $i_X\omega$ where $\omega = dx_1 \wedge \dots \wedge dx_n$ hence

$$\begin{aligned} i_X\omega &= i_X(dx_0 \wedge \dots \wedge dx_n) \\ &= \sum_{i=0}^n (-1)^i dx_0 \wedge \dots \wedge i_X dx_i \wedge \dots \wedge dx_n \\ &= \sum_{i=0}^n (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \end{aligned}$$

which is exactly the volume form we used before on the sphere.

Not every smooth hypersurface admits an everywhere transverse vector field (this is indeed equivalent to the existence of a nonzero section of a suitably defined normal bundle!). But we have the following important result.

Proposition 8.34. *Let M be a smooth oriented manifold with boundary. Then there exists a smooth transverse vector field X to ∂M .*

Proof. Let $q \in \partial M$. We say that $v \in T_q M$ is inward pointing if v is the tangent vector to a curve $\gamma : [0, \varepsilon] \rightarrow M$ that is contained in M , outward pointing otherwise (that is if $-v$ is inward pointing). In charts these vectors are exactly those with $x_n > 0$ (resp. $x_n < 0$).

Cover now a neighborhood of ∂M by smooth boundary charts $\{(U_i, \varphi_i)\}$. In each chart fix $N_i = -\partial/\partial x_n$. Then set with a partition of unity

$$X = \sum_{i \in \mathbb{N}} \psi_i N_i$$

This is a smooth vector field on ∂M . Let us check it is outward pointing. Fix a point q and a chart $(y_1, \dots, y_n)$ at this point. Each N_i defined at q is outward pointing hence $dy_n(N_i) < 0$. Hence we have

$$dy_n(X)|_q = \sum_{i \in \mathbb{N}} \psi_i(q) dy_n(N_i)|_q < 0$$

because all terms are ≤ 0 and at least one is < 0 . □

Corollary 8.35. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. Assume that 0 is a regular value for f , hence $f^{-1}(0)$ is a smooth $(n-1)$ -dim manifold. Then $f^{-1}(0) = \partial M$ is orientable, where $M = \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$.*

This could also more easily be done via some Riemannian metric defining a transverse vector field.

Definition 8.36 (Stokes orientation of the boundary). Let M be oriented smooth manifold. Then ∂M is orientable and the orientation induced by any transverse vector field outward pointing is well-defined. This is called the *Stokes orientation* of ∂M .

We have to consider these three main examples

- The case of a domain Ω in $\mathbb{R}^2$ and its boundary.

- The sphere S^n is the boundary of the ball B^n . Its Stokes orientation is exactly the one already considered.
- The boundary $\partial\mathbb{H}^n$ of $\mathbb{H}^n$. Notice that $N = -\partial_{x_n}$ is an outward pointing vector. Hence the boundary is positively oriented by the chart $(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 0)$ only if the family of vectors

$$(-\partial_{x_n}, \partial_{x_1}, \dots, \partial_{x_{n-1}})$$

is positively oriented in $\mathbb{R}^n$, which gives $(-1)^n$, in the sense that, the induced orientation by the standard chart coincides with the Stokes orientation only if n is even, and its opposite if n is odd.

8.4 Stokes theorem

We can now prove the main theorem of the chapter.

Theorem 8.37. *Let M be a smooth oriented n -dim manifold with boundary and let ω be a compactly supported smooth $(n-1)$ -dim form on M . Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here ∂M is understood with the induced orientation given by M . If M has no boundary then the right hand side is zero.

Proof. We split the proof into three part (i) $M = \mathbb{H}^n$ (ii) M is covered by a single chart (iii) the general case.

(i) The fact that ω has compact support means that

$$\omega = \sum_{i=1}^n \omega_i dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

with $\text{supp}(\omega)$ contained in a rectangle $A := [-R, R]^{n-1} \times [0, R]$. We have

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

hence

$$d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_n$$

We start the integration over $\mathbb{H}^n$ and we do it in such a way that (a) we split the part of the sum which goes normal to the boundary with the others

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \int_{\mathbb{H}^n} \sum_{i=1}^{n-1} (-1)^{i-1} \frac{\partial \omega_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_n \\ &\quad + \int_{\mathbb{H}^n} (-1)^{n-1} \frac{\partial \omega_n}{\partial x_n} dx_1 \wedge \dots \wedge dx_i \wedge \dots \wedge dx_n \end{aligned}$$

and (b) we integrate first along the variable x_i in the first addend of the sum, and wrt x_n in the last one

$$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \dots \left(\int_{-R}^R \frac{\partial \omega_i}{\partial x_i} dx_i \right) dx_1 \dots \widehat{dx_i} \dots dx_n \\ &\quad + (-1)^{n-1} \int_{-R}^R \dots \int_{-R}^R \left(\int_0^R \frac{\partial \omega_n}{\partial x_n} dx_n \right) dx_1 \dots dx_{n-1} \end{aligned} \quad (8.2)$$

The first addend of the sum is zero since for every $i = 1, \dots, n-1$

$$\int_{-R}^R \frac{\partial \omega_i}{\partial x_i} dx_i = \omega_i(R) - \omega_i(-R) = 0.$$

Similarly, the last one gives $\int_0^R \frac{\partial \omega_n}{\partial x_n} dx_n = -\omega_n(x_1, \dots, x_{n-1}, 0)$ since at $x_n = R$ it vanishes. Thus (8.2) reduces to

$$\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{-R}^R \dots \int_{-R}^R \omega_n(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{n-1}$$

(Notice that this term is zero in the case the support of ω does not intersect the boundary). The other side of the equality is

$$\int_{\partial \mathbb{H}^n} \omega = \sum_{i=1}^n \int_{\partial \mathbb{H}^n} \omega_i(x_1, \dots, x_{n-1}, 0) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

Since x_n is constantly equal to zero on $\partial \mathbb{H}^n$ all the terms containing dx_n vanish, and only the $i = n$ term survives

$$\int_{\partial \mathbb{H}^n} \omega = \int_{\partial \mathbb{H}^n} \omega_n(x_1, \dots, x_{n-1}, 0) dx_1 \wedge \dots \wedge dx_{n-1}$$

Considering the fact that the coordinates $(x_1, \dots, x_{n-1})$ are positively oriented on $\partial \mathbb{H}^n$ if n is even and negatively if n odd we have

$$\int_{\partial \mathbb{H}^n} \omega = (-1)^n \int_{-R}^R \dots \int_{-R}^R \omega_n(x_1, \dots, x_{n-1}, 0) dx_1 \dots dx_{n-1}$$

which proves the statement if M is the half space.

(ii) If M is covered by a single chart (U, φ) (which we assume to be an oriented chart). We have $M = \varphi(U)$ and $\partial M = \varphi^{-1}(\partial \mathbb{H}^n \cap \varphi(U))$

$$\int_M d\omega = \int_{\varphi(U)} (\varphi^{-1})^* d\omega = \int_{\varphi(U)} d(\varphi^{-1})^* \omega$$

now we use the result in $\mathbb{H}^n$ and we have (notice $\partial \varphi(U) = \varphi(U) \cap \partial \mathbb{H}^n$)

$$\int_M d\omega = \int_{\partial \varphi(U)} (\varphi^{-1})^* \omega = \int_{\varphi^{-1}(\partial \mathbb{H}^n \cap \varphi(U))} \omega = \int_{\partial M} \omega$$

(iii) Assume now that $\text{supp}(\omega)$ is compact in M and cover it with finitely many oriented charts $\{(U_i, \varphi_i)\}_{i=1}^N$ and let ψ_i a subordinate smooth partition of unity, with $\sum_{i=1}^N \psi_i = 1$. Then we have

$$\begin{aligned}\int_{\partial M} \omega &= \sum_{i=1}^N \int_{\partial M} \psi_i \omega = \sum_{i=1}^N \int_M d(\psi_i \omega) \\ &= \sum_{i=1}^N \int_M (d\psi_i \wedge \omega + \psi_i d\omega) \\ &= \int_M d\left(\sum_{i=1}^N \psi_i\right) \wedge \omega + \int_M \left(\sum_{i=1}^N \psi_i\right) d\omega = \int_M d\omega\end{aligned}$$

□

Remark 8.38. The assumption of compact support for ω can clearly be removed if M is compact. If M is non compact the theorem does not hold for arbitrary non compactly supported forms, as there can be evident integrability issues for both terms of the equality to be defined. Even if both terms are defined, they can be different. For instance if $M = [0, +\infty[$ and $f = 1$ is a 0-form (a function) then $\int_M df = 0$ while $\int_{\partial M} f = -1$.

There are some special cases. For instance in $\mathbb{R}^2$ on a region D we have the following

Corollary 8.39. (*Green formula in $\mathbb{R}^2$*) Let D be a regular domain in $\mathbb{R}^2$ and $\omega = Pdx + Qdy$. Then

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\partial D} Pdx + Qdy$$

As a particular case if D is a region enclosed by a curve γ oriented in the counter clockwise sense

$$\text{Area}(D) = \frac{1}{2} \int_{\gamma} xdy - ydx$$

Remark 8.40. Notice that if D is a region enclosed by a curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ oriented in the counter clockwise sense and we define in $\mathbb{R}^2$ the vector field $X(x, y) = (Q(x, y), -P(x, y))$ we can rewrite as a divergence Theorem (check the next chapter)

$$\int_D \text{div}(X) dx dy = \int_{\gamma} X \cdot \nu$$

There are similar result in $\mathbb{R}^3$. The reader is invited to write down explicit formulas and find back calculus formulas.

Another example: the Curl Theorem. The line integral of a vector field over a loop γ is equal to the flux of its curl through the surface S whose boundary is $\partial S = \gamma$.

$$\int_S (\nabla \times F) dA = \int_{\partial S} F \cdot ds$$

where dA is the area element on S and ds is the lin element on $\gamma = \partial S$. Recall that $\nabla \times F = \text{curl}(F)$ is equal to

$$\nabla \times F = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Remark 8.41. Exactly as one can define smooth manifolds with boundary by locally modeling topological spaces (with suitable assumptions) on open sets of $\mathbb{H}^n$, one can define “manifold with corners” by locally modeling on open sets of

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_i \geq 0\}.$$

There holds a version of Stokes theorem for manifold with corners. In particular this allows for curves that are piecewise smooth.

Line integrals

We can integrate 1-forms $\omega \in \Omega^1(M)$ along smooth curves $\gamma : [a, b] \rightarrow M$. By definition of integral we can use γ as a chart and we have

$$\int_{\gamma} \omega := \int_{[a,b]} \gamma^* \omega.$$

Recall that if in coordinates $\omega = \sum_{i=1}^n \omega_i dx_i$ then $\gamma^* \omega = \sum_{i=1}^n \omega_i(\gamma(t)) \langle dx_i, \dot{\gamma}(t) \rangle$ hence we get the following explicit formula for the integral

$$\int_{\gamma} \omega = \int_a^b \langle \omega_{\gamma(t)}, \dot{\gamma}(t) \rangle dt. \quad (8.3)$$

Clearly the integral is linear and independent on the parametrization. If $\gamma : [a, b] \rightarrow M$ is piecewise smooth we can extend the definition (8.3) by adding the pieces

Proposition 8.42 (Fundamental theorem of calculus). *Let $f : M \rightarrow \mathbb{R}$ and $\gamma : [a, b] \rightarrow M$ piecewise smooth. Then*

$$\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a))$$

Proof. Indeed we have

$$\int_{\gamma} df = \int_a^b \langle df_{\gamma(t)}, \dot{\gamma}(t) \rangle dt = \int_a^b (f \circ \gamma)'(t) dt = f(\gamma(b)) - f(\gamma(a))$$

which can be also seen as a consequence of the Stokes theorem. □

Notice in particular that the value of the integral $\int_{\gamma} df$ is independent on the path joining $x = \gamma(a)$ and $y = \gamma(b)$. In particular if the curve γ is closed but non trivial, the integral is always zero. We can be more precise.

Proposition 8.43. *Let $\omega \in \Omega^1(M)$. The two following properties are equivalent*

- (a) $\int_{\gamma} \omega = 0$ for every closed curve γ in M ,
- (b) $\omega = df$ for some smooth $f : M \rightarrow \mathbb{R}$. (i.e., ω is exact)

Proof. Only (a) implies (b) is needed. Fix an arbitrary point $q_0 \in M$, we want to prove that the function f satisfying is

$$f(q) = \int_{q_0}^q \omega$$

where the integral is computed along any curve joining q_0 with q (well-defined thanks to (a)). Notice that if we replace q_0 by any other point q_1 in M we get a different function $\tilde{f}$, which differs by a constant, hence $d\tilde{f} = df$. Hence it is sufficient to prove that in coordinates we have

$$\frac{\partial f}{\partial x_i}(x) = \omega_i(x)$$

□

Notice that if $\omega = df$ then in coordinates

$$\omega = \sum_{i=1}^n \omega_i dx_i, \quad \omega_i = \frac{\partial f}{\partial x_i}$$

Hence

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial \omega_j}{\partial x_i}$$

This means that ω is closed. Hence exact implies closed. Closed differential 1-forms are not necessarily exact.

Example 8.44. Let us consider the 1-form in $\mathbb{R}^2$

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

This is closed but not exact on $\mathbb{R}^2 \setminus \{0\}$ since the integral over the circle of radius 1 parametrized as $t \mapsto (\cos t, \sin t)$ is 2π (the length of the circle!). It is exact on every ball not containing zero since $\omega = d\theta$ where $\theta = \arctan(y/x)$.

On closed and exact forms

Definition 8.45. We say that a differential k -form $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$, we say that ω is *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Every exact form is closed since if $\omega = d\eta$ then $d\omega = d^2\eta = 0$. The converse is not true.

Corollary 8.46. *Let M be a compact smooth manifold without boundary. Then the integral over M of an exact form is zero.*

Proof. If $\omega = d\eta$ we have since $\partial M = \emptyset$

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$$

□

Hence we have a way to check whether a closed form is exact or not by restating the previous results.

Corollary 8.47. *Let M be a smooth manifold and ω be a closed k -form. If there exists S compact submanifold without boundary such that $\int_S \omega \neq 0$ then ω is not exact.*

Example 8.48. Let us consider the 1-form in $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

This 1-form is not exact on $\mathbb{R}^2 \setminus \{0\}$ since the integral over the circle of radius 1 parametrized as $t \mapsto (\cos t, \sin t)$ is 2π (it is the length of the circle!). The form ω is closed since

$$d\omega = \left[\frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) + \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) \right] dx \wedge dy = 0$$

Corollary 8.49. *Let M be a compact smooth manifold with boundary. Then the integral over ∂M of a closed form is zero.*

Proof. If $d\omega = 0$ we have

$$\int_{\partial M} \omega = \int_M d\omega = 0 \quad \square$$

Example 8.50. The circle of radius 1 is not a boundary in $\mathbb{R}^2 \setminus \{0\}$! Indeed if you want to see it as a boundary in $\mathbb{R}^2 \setminus \{0\}$ you must consider it as a connected component of the boundary ∂D of an annulus type domain D .

Homotopy invariance

Definition 8.51. Two smooth paths $\gamma_0, \gamma_1 : [0, 1] \rightarrow M$ are smoothly homotopic if there exists a smooth map $H : [0, 1] \times [0, 1] \rightarrow M$ such that $H(0, t) = \gamma_0(t)$ and $H(1, t) = \gamma_1(t)$.

Thanks to Stokes theorem (version with corners!) we have

Theorem 8.52. *Let ω be a closed 1-form on a smooth manifold M , i.e., $d\omega = 0$. Let γ_1, γ_2 be two smooth homotopic path with the same endpoints. Then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

Proof. Since ω is closed we have

$$\int_{[0,1]^2} d(H^*\omega) = \int_{[0,1]^2} H^*d\omega = 0$$

On the other hand by Stokes theorem (version with corners!)

$$\int_{[0,1]^2} d(H^*\omega) = \int_{\partial[0,1]^2} H^*\omega$$

and

$$\int_{\partial[0,1]^2} H^* \omega = \sum_{i=1}^4 \int_{H \circ \Gamma_i} \omega$$

where Γ_i for $i = 1, \dots, 4$ are counter clockwise parametrization of the 4 edges of the square $[0, 1]^2$. Analyzing the terms two of them are zero since the image under H are constant curves, then we have

$$\sum_{i=1}^4 \int_{H \circ \Gamma_i} \omega = \int_{H \circ \Gamma_1} \omega - \int_{H \circ \Gamma_3} \omega = \int_{\gamma_0} \omega - \int_{\gamma_1} \omega$$

□

Remark 8.53. One can adapt the proof to just a continuous homotopy between two curves that are piecewise smooth.

The difference between closed and exact forms is topological, hence global. In the same spirit we have this important result, which we will only sketch for the moment.

Theorem 8.54 (Poincaré Lemma for 1-forms). *Let $U \subset \mathbb{R}^n$ be open starshaped, then any closed 1-form ω is exact on U .*

Proof. Fix $x_0 \in U$ and defines $f(x)$ as the line integral of ω along the line segment from $x_0 \in U$ to x . Using the fact that ω is closed (and differentiation under the integral sign) one shows that $df = \omega$. □

Indeed the same is true for k -forms.

Theorem 8.55 (Poincaré Lemma for k -forms). *Let $U \subset \mathbb{R}^n$ be open starshaped, then any closed k -form ω is exact on U .*

This means that closed is equivalent to locally exact. We end this chapter by defining the *de Rham cohomology groups*

Definition 8.56. Let M be a smooth manifold, let $Z^k(M)$ be the set of closed k -forms and $B^k(M)$ the set of exact k -forms. We set

$$H_{dR}^k(M) = \frac{Z^k(M)}{B^k(M)}$$

where $\omega_1 \sim \omega_2$ in $Z^k(M)$ if $\omega_1 = \omega_2 + d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Thanks to the Poincaré Lemma we can conclude that

$$H_{dR}^0(\mathbb{R}^n) = \mathbb{R}, \quad H_{dR}^k(\mathbb{R}^n) = 0, \quad k \geq 1$$

Exercise 8.57. Prove that if $F : M \rightarrow N$ is a smooth map then F^* maps closed forms into closed forms and exact forms into exact forms. Hence induces a well-defined map $F^* : H_{dR}^k(N) \rightarrow H_{dR}^k(M)$. Prove that two diffeomorphic manifold have isomorphic de Rham cohomology groups

More in general we have

Proposition 8.58. *If M is connected, then $H_{dR}^0(M) \simeq \mathbb{R}$. If M is a connected, orientable and compact manifold. Then $H_{dR}^n(M) \simeq \mathbb{R}$.*

Indeed the de Rham cohomology groups are not only preserved by diffeomorphisms but also homotopies. The de Rham theorem states that these groups coincide with the singular cohomology groups that one can define with algebraic topology approach. These results goes beyond the scope of these lecture notes and we invite the interested reader to have a look at further results in this directions.

Chapter 9

Riemannian manifolds

*The investigation of this more general kind would require no really different principles [...]
I restrict myself, therefore, to those manifolds in which the line element
is expressed as the square root of a quadratic differential.*

Ueber die Hypothesen, welche der Geometrie zu Grunde liegen, 1867

Bernhard Riemann, 1826 – 1866

A (covariant) 2-tensor $T : V \times V \rightarrow \mathbb{R}$ on a finite dimensional vector space V is said to be *non-degenerate* if $T(v, w) = 0$ for all $w \in V$ implies $v = 0$. This is equivalent to ask one of the following two equivalent conditions:

- the map $L : V \rightarrow V^*$ given by $L(v) = T(v, \cdot)$ is an isomorphism,
- for every basis $e_1, \dots, e_n$ of V the matrix $(T(e_i, e_j))_{i,j=1,\dots,n}$ is invertible.

Endowing a vector space with a non-degenerate 2-tensor hence defines a natural isomorphism $V \rightarrow V^*$ induced by T . This is what happens for instance given an inner product on V , which can be thought as a non degenerate symmetric (and positive) 2-tensor $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$. The above isomorphism is just a finite dimensional version of the so-called Riesz representation theorem $x \in V \mapsto \phi_x : V \rightarrow \mathbb{R}$ such that $\phi_x(v) = \langle v, x \rangle$.

Extending this idea to manifolds, every non-degenerate 2-tensor field T on M will induce a natural isomorphism from TM to T^*M producing a one to one correspondence between differential one forms and vector fields.

9.1 Riemannian structure

A *Riemannian metric* on a smooth manifold M is a covariant 2-tensor field g (i.e., a section of the tangent bundle of covariant 2 tensor on M) that is symmetric and positive definite. This means that for every $q \in M$ we have

$$g : T_q M \times T_q M \rightarrow \mathbb{R},$$

which is a symmetric and positive definite bilinear form, i.e., an inner product. Sometimes the inner product is also denoted by $\langle v | w \rangle_q$ (or simply $\langle v | w \rangle$ when the notation is clear), the vertical bar distinguishing the inner product of vectors from the duality product of a covector on a vector.

This permits to define norm of vectors and angle between vectors as in Euclidean spaces

$$\|v\|^2 = g(v, v), \quad \cos(\widehat{vw}) = \frac{g(v, w)}{\|v\|\|w\|}.$$

A Riemannian metric in coordinates on a chart (U, φ) is written as

$$g = \sum_{i,j=1}^n g_{ij}(x) dx_i \otimes dx_j, \quad (9.1)$$

where $(g_{ij}(x))$ is a symmetric and positive definite matrix whose entries are smooth functions $g_{ij} \in C^\infty(U)$. It is very common to use also the notation without the tensor product symbol

$$g = \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j,$$

which can be formally deduced from (9.1) by setting $dx_i dx_j = \text{Sym}(dx_i \otimes dx_j) = \frac{1}{2}(dx_i \otimes dx_j + dx_j \otimes dx_i)$. Notice that in terms of the standard basis induced by coordinates we have the identity of functions

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Notice that $\mathbb{R}^n$ admits the standard Riemannian (Euclidean) metric $\bar{g}$ defined by

$$g = dx_1^2 + \dots + dx_n^2 = \sum_{i=1}^n dx_i^2.$$

Proposition 9.1. *Every smooth manifold admits a Riemannian metric.*

Proof. Take a covering of the manifold $\{U_\alpha, \varphi_\alpha\}$ and a partition of unity $\{\psi_\alpha\}$ and set

$$g = \sum_{\alpha} \psi_{\alpha} g_{\alpha},$$

where $g_{\alpha} = \varphi_{\alpha}^* \bar{g}$ is the pullback in U_{α} of the Euclidean metric on $\mathbb{R}^n$. It is an easy check that all conditions are satisfied. $\square$

Given a Riemannian metric g on a smooth manifold M we say that (M, g) is a *Riemannian manifold*.

Example 9.2. Let $F : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ be an immersion describing an immersed (locally) submanifold $S = F(U) \subset \mathbb{R}^n$. Then we can consider the metric induced by $\mathbb{R}^n$ on S that is the pullback $g = F^* \bar{g}$. Indeed this means for v, w in $T_x \mathbb{R}^k$ their inner product is the Euclidean inner product on the image

$$g(v, w) = \bar{g}(F_* v, F_* w)$$

To compute let us consider

$$F^* \left(\sum_{i=1}^n dx_i^2 \right) = \sum_{i=1}^n dF_i^2 = \sum_{i=1}^n dF_i \otimes dF_i$$

which is computed by writing

$$F^*\bar{g} = \sum_{i,j,k=1}^n \left(\frac{\partial F_i}{\partial u_j} du_j \right) \otimes \left(\frac{\partial F_i}{\partial u_k} du_k \right) = \sum_{j,k=1}^n \left(\sum_{i=1}^n \frac{\partial F_i}{\partial u_j} \frac{\partial F_i}{\partial u_k} \right) du_j \otimes du_k$$

which gives $g = F^*\bar{g}$ with metric

$$g_{jk} = \sum_{i=1}^n \frac{\partial F_i}{\partial u_j} \frac{\partial F_i}{\partial u_k} = \left\langle \frac{\partial F}{\partial u_j}, \frac{\partial F}{\partial u_k} \right\rangle_{\mathbb{R}^n}$$

Example 9.3 (Surfaces in $\mathbb{R}^3$). Notice that specifying the previous example to the case of a parametrized regular surface $F : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ in dimension 3, and denoting (u, v) coordinates in the $\mathbb{R}^2$ we have that

$$\begin{aligned} g_{11} &= \left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial u} \right\rangle_{\mathbb{R}^3} = E, & g_{22} &= \left\langle \frac{\partial F}{\partial v}, \frac{\partial F}{\partial v} \right\rangle_{\mathbb{R}^3} = G \\ g_{12} &= g_{21} = \left\langle \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \right\rangle_{\mathbb{R}^3} = F \end{aligned}$$

so that the metric $g = F^*\bar{g}$ agrees with the so called *first fundamental form* of Gauss

$$g = E du^2 + 2F dudv + G dv^2.$$

The metric g , thought as a non degenerate bilinear form on $T_q M$, induces an isomorphism between tangent and cotangent spaces

$$I : T_q M \rightarrow T_q^* M, \quad v \mapsto g(v, \cdot).$$

In the standard basis induced by coordinates the matrix associated to I is computed as follows

$$g \left(\frac{\partial}{\partial x_i}, \cdot \right) = \sum_{j=1}^n a_{ij} dx_j$$

and it is easy to see that $a_{ij} = g_{ij}$ by applying both sides to $\frac{\partial}{\partial x_j}$.

Remark 9.4. The isomorphisms

$$I : T_q M \rightarrow T_q^* M, \quad I^{-1} : T_q^* M \rightarrow T_q M,$$

which are represented respectively by $G = (g_{ij})$ and $G^{-1} = (g^{ij})$ in coordinates, are also called *musical isomorphism* because using proper indices notation $I(v) = v^\flat$ lower indices and $I^{-1}(\eta) = \eta^\sharp$ raises indices. Indeed, with Einstein summation notation

$$\begin{aligned} v &= v^i \frac{\partial}{\partial x_i}, & v^\flat &= v_j dx^j, & v_j &= g_{ij} v^i. \\ \eta &= \eta_j dx^j, & \eta^\sharp &= \eta^i \frac{\partial}{\partial x_i}, & \eta_i &= g^{ij} \eta_j. \end{aligned}$$

Remark 9.5. On a Riemannian manifold (M, g) we can always define locally an orthonormal frame $X_1, \dots, X_n$ of vector fields such that

$$g(X_i, X_j) = \delta_{ij}.$$

Indeed it is enough to apply the Gram Schmidt algorithm to the smooth family of vectors given by the frame. In particular given a non vanishing vector field X we can find a local orthonormal frame $X_1, \dots, X_n$ such that $X = X_1$.

Definition 9.6. Let $f \in C^\infty(M)$, the *Riemannian gradient* of f is the vector field $\nabla f = I^{-1}(df)$, that is the vector field satisfying

$$g(\nabla f, v) = df(v), \quad \forall v \in TM.$$

In coordinates it is simple to check that

$$\nabla f = \sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Exercise 9.7. Let $X_1, \dots, X_n$ be local o.n frame for the metric and let $\eta_1, \dots, \eta_n$ be the dual basis. Prove that

$$g = \sum_{i=1}^n \eta_i \otimes \eta_i.$$

Prove that the differential of a function f and its Riemannian gradient are written as

$$df = \sum_{i=1}^n (X_i f) \eta_i, \quad \nabla f = \sum_{i=1}^n (X_i f) X_i.$$

Definition 9.8. An *isometry* between two Riemannian manifolds (M, g) and (M', g') is a smooth diffeomorphism $F : M \rightarrow M'$ such that $F^* g' = g$. This implies that for every $q \in M$ and $v, w \in T_q M$

$$g_q(v, w) = g'_{F(q)}(F_* v, F_* w).$$

Using local diffeomorphisms we can define local isometries. We stress that in this definition isometries are smooth. Isometries preserves length of vectors and angles between them.

Definition 9.9. A *conformal transformation* between two Riemannian manifolds (M, g) and (M', g') is a smooth diffeomorphism $F : M \rightarrow M'$ such that $F^* g' = e^f g$ for some $f \in C^\infty(M)$.

Two metrics that are conformal defines the same angles but do not define the same length.

Exercise 9.10. Let $\pi : S^n \rightarrow \mathbb{R}^n$ be the stereographic projection from the north pole $N \in S^n$. Prove that if g_{S^n} is the standard metric on S^n and $\bar{g}$ the Euclidean metric on $\mathbb{R}^n$ then on $S^n \setminus \{N\}$ we have $g_{S^n} = f \pi^* \bar{g}$ for some positive $f \in C^\infty(S^n)$.

Riemannian volume.

If M is orientable Riemannian manifold we can define a natural volume, called *Riemannian volume* on M that is the volume dV_g defined by the normalization $dV_g(X_1, \dots, X_n) = 1$ on some (equivalently, every) positive orthonormal basis.

Proposition 9.11. *The Riemannian volume is written in coordinates*

$$dV_g = \sqrt{\det g_{ij}} dx_1 \wedge \dots \wedge dx_n.$$

Proof. Let us write the volume in a coordinate chart

$$dV = f dx_1 \wedge \dots \wedge dx_n,$$

and we look for the function f . Consider a local orthonormal frame $X_1, \dots, X_n$ and write it in coordinates as follows

$$X_i = \sum_{j=1}^n b_{ij} \frac{\partial}{\partial x_j}.$$

Then we have by Lemma 7.27

$$1 = dV(X_1, \dots, X_n) = \det(B) dV_g \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) = \det(B) f.$$

Since $g(X_i, X_k) = \delta_{ik}$ we have that

$$\delta_{ik} = g \left(\sum_{j=1}^n b_{ij} \frac{\partial}{\partial x_j}, \sum_{l=1}^n b_{kl} \frac{\partial}{\partial x_l} \right) = \sum_{j,l=1}^n b_{ij} g_{jl} b_{kl},$$

that means the matrix identity $BGB^T = I$. In particular $\det(B)^2 = \det(G)^{-1}$, which completes the proof. $\square$

Riemannian divergence.

We start by defining the divergence of a vector field with respect to a volume form.

Definition 9.12. Let $\omega \in \Omega^n(M)$ be a volume form. Then for a vector field X on M we denote by $\operatorname{div}_\omega X$ the function such that $\mathcal{L}_X \omega = (\operatorname{div}_\omega X) \omega$.

Notice that the divergence can be characterized as follows: given $\Omega \subset M$ consider $\Omega_t = e^{tX}(\Omega)$ the image of the set under the flow. Then

$$\left. \frac{d}{dt} \right|_{t=0} \operatorname{vol}_\omega(\Omega_t) = \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega_t} \omega = \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega} e^{tX*} \omega = \int_{\Omega} (\operatorname{div}_\omega X) \omega,$$

so taking an infinitesimal set around a point q , the divergence is saying if the flow of X is increasing or decreasing the volume of the set.

Exercise 9.13. Prove that given $f > 0$ smooth positive function we have the identity

$$\operatorname{div}_{f\omega} X = \operatorname{div}_\omega X + \frac{Xf}{f}.$$

Notice that, if $d_q f = 0$ at some point, then the divergence of X at q is *independent* with respect to the volume form.

On a Riemannian manifold we simply write $\operatorname{div} X$ for the divergence with respect to the Riemannian volume dV_g .

Exercise 9.14. Prove that the Riemannian divergence of a vector field $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$ is written in coordinates as

$$\operatorname{div}(X) = \frac{1}{\sqrt{\det g_{ij}}} \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(\sqrt{\det g_{ij}} X_k \right),$$

where we assume orientability such that $\det g_{ij} > 0$.

Exercise 9.15. Let $X_1, \dots, X_n$ be local o.n frame for the metric and let $\eta_1, \dots, \eta_n$ be the dual basis. Prove that

$$dV_g = \eta_1 \wedge \dots \wedge \eta_n$$

For a vector field Y compute first $\mathcal{L}_Y(dV_g)$ and then $\operatorname{div}(Y)$ in terms of a local orthonormal frame assuming that

$$[X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k.$$

We can reinterpret the Stokes Theorem for top dimensional forms as the following statement.

Theorem 9.16. Let (M, g) be an orientable Riemannian manifold with boundary ∂M , let dV_g the Riemannian volume and dS_g the surface measure on ∂M with the Stokes orientation. Then for every compactly supported vector field X we have

$$\int_M (\operatorname{div} X) dV_g = \int_{\partial M} g(X, N) dS_g$$

where N is an outward unit normal vector field to ∂M .

Proof. Since X is compactly supported also $\operatorname{div} X$ is compactly supported and

$$\int_M (\operatorname{div} X) dV_g = \int_M \mathcal{L}_X dV_g = \int_M d(i_X dV_g) = \int_{\partial M} i_X dV_g$$

where we used $\mathcal{L}_X dV_g = d(i_X dV_g) + i_X d(dV_g)$ but $d(dV_g) = 0$ since dV_g is a top dimensional form.

The proof is completed by observing that $i_X dV_g = g(X, N) dS_g$. To prove this last statement we can write dV_g as follows

$$dV_g = \eta \wedge \nu_1 \wedge \dots \wedge \nu_{n-1}$$

where $\eta, \nu_1, \dots, \nu_{n-1}$ is the dual basis of a basis of the tangent space $N, X_1, \dots, X_{n-1}$, where N is orthogonal to S and $X_1, \dots, X_{n-1}$ is a basis for TS . Due to $\eta(Y) = g(N, Y)$ for every Y we have the conclusion. $\square$

Remark 9.17. Under the same assumptions if f is a compactly supported function we can apply the statement to fX and get

$$\int_M f(\operatorname{div} X) dV_g + \int_M Xf dV_g = \int_{\partial M} f g(X, N) dS_g.$$

Laplace-Beltrami operator.

One can define the Laplace-Beltrami operator on a Riemannian manifold (M, g) as follows

$$\Delta f = \operatorname{div}(\nabla f),$$

where as usual the divergence is computed with respect to the Riemannian volume. From previous formulae we have that in coordinates

$$\Delta f = \frac{1}{\sqrt{\det g}} \sum_{k,l=1}^n \frac{\partial}{\partial x_k} \left(\sqrt{\det g} g^{kl} \frac{\partial f}{\partial x_l} \right).$$

Applying the divergence theorem to the vector field ∇f one gets the Green formula

$$\int_M f \Delta f dV_g + \int_M |\nabla f|^2 dV_g = \int_{\partial M} f \frac{\partial f}{\partial \nu} dS_g,$$

denoting $\frac{\partial f}{\partial \nu} = g(\nabla f, N)$. This shows for instance that:

Proposition 9.18. *Let (M, g) be a compact oriented Riemannian manifold without boundary. Let Δ denote its Laplace-Beltrami operator. Then*

- (a) *every harmonic function (i.e., $\Delta f = 0$) on M is constant,*
- (b) *if $\lambda \neq 0$ is an eigenvalue for Δ (i.e., $\Delta f = \lambda f$), then $\lambda < 0$.*

In particular the spectrum of Δ is contained in the interval $(-\infty, 0]$.

9.2 The metric space structure

We can introduce the length of a *piecewise smooth curve* $\gamma : [0, T] \rightarrow M$ as follows

$$L(\gamma) = \int_0^T \|\dot{\gamma}(t)\| dt.$$

The length is invariant by reparametrization. Indeed let $\alpha : [0, T] \rightarrow [0, T']$ be a smooth reparametrization. Then setting $\tilde{\gamma}(t) = \gamma(\alpha(t))$ we have that $\|\tilde{\gamma}'(t)\| = \|\gamma'(\alpha(t))\| |\alpha'(t)|$ and

$$L(\tilde{\gamma}) = \int_0^{T'} \|\tilde{\gamma}'(t)\| dt = \int_0^{T'} \|\gamma'(\alpha(t))\| |\alpha'(t)| dt = \int_0^T \|\gamma'(\tau)\| d\tau = L(\gamma),$$

where we have set the change of variable $\tau = \alpha(t)$. A curve that has finite length can always be reparametrized by arc length parameter $s : [0, L(\gamma)] \rightarrow [0, T]$ defined by

$$s(t) = \int_0^t \|\dot{\gamma}(t)\| dt.$$

Remark 9.19. Differentiating with respect to t the previous formula we get the coordinate expression

$$\frac{ds}{dt} = \sqrt{\sum_{i,j=1}^n g_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt}},$$

which justifies the historical/classical notation for the metric

$$ds^2 = \sum_{i,j=1}^n g_{ij} dx_i dx_j.$$

The Riemannian distance. We can introduce the *Riemannian distance* for $x, y \in M$

$$d(x, y) = \inf\{L(\gamma) \mid \gamma : [0, T] \rightarrow M, \gamma(0) = x, \gamma(T) = y\}.$$

Theorem 9.20. *Let (M, g) be a Riemannian manifold and d be the induced distance. Then (M, d) is a metric space whose metric topology coincides with the manifold topology.*

The proof needs a lemma comparing length with respect to g and length in coordinates.

Lemma 9.21. *Let g be any Riemannian metric on $\mathbb{R}^n$. Then for every compact $K \subset \mathbb{R}^n$ there exists a constant such that*

$$c_1 \|v\|_{\mathbb{R}^n} \leq \|v\|_g \leq c_2 \|v\|_{\mathbb{R}^n},$$

for every $x \in K$ and $v \in T_x \mathbb{R}^n$.

Proof. The set of tangent vectors w to $\mathbb{R}^n$ of Euclidean norm = 1 and based in K is compact, and the norm associated to g is smooth. Hence there exists constant such that $c_1 \leq \|w\|_g \leq c_2$ for every such (Euclidean) unitary w . Consider now an arbitrary v attached at a point of K . Applying the inequality to $w = v/\|v\|_{\mathbb{R}^n}$ one gets that w is (Euclidean) unitary so that

$$c_1 \leq \left\| \frac{v}{\|v\|_{\mathbb{R}^n}} \right\|_g \leq c_2$$

which implies the conclusion by multiplying every factor by $\|v\|_{\mathbb{R}^n}$. $\square$

Now we can prove Theorem 9.20.

Proof. The fact that d is non negative follows from the definition. Also d is symmetric since for every curve $\gamma : [0, T] \rightarrow M$ joining x to y we can consider $\tilde{\gamma} : [0, T] \rightarrow M$ defined by $\tilde{\gamma}(t) = \gamma(T - t)$ joining y to x . The fact that we have chosen as class of curves those that are piecewise smooth also proves the triangular inequality. We have only to prove that $x \neq y$ implies $d(x, y) > 0$, and that the two topologies coincides. We split the two implications.

(a). Lemma 9.21 says that for every smooth curve γ sufficiently short to stay in a single chart then we can compare its length with the Euclidean length in the chart and

$$c_1 L_{\mathbb{R}^n}(\gamma) \leq L_g(\gamma) \leq c_2 L_{\mathbb{R}^n}(\gamma).$$

Now let U be open in the manifold topology, fix $x_0 \in U$ and a open set $V \subset U$ with compact closure $\bar{V} = K$ such that the Euclidean ball $B(x_0, \delta) \subset V$. If $x \notin K$ then for every curve joining x_0 with x we have

$$L_g(\gamma) > c_1 L_{\mathbb{R}^n}(\gamma) \geq c_1 \delta.$$

Taking the infimum over all curves joining x_0 with x we have $d(x_0, x) \geq \varepsilon$ for $\varepsilon = c_1 \delta$. This implies $B^g(x_0, \varepsilon) \subset \bar{V} \subset U$. Notice that this part of the argument in particular says that $x \neq y$ implies $d(x, y) > 0$.

(b). The other implication is similar. Consider a point in a metric ball $x \in B^g(x_0, \varepsilon)$ with $x \notin \bar{B}^g(x_0, \varepsilon/2)$. Then $d(x_0, x) > \varepsilon/2$ and since this is an infimum, for every curve joining x_0 to x we have $L_g(\gamma) > \varepsilon/2$. By the inequality $L_{\mathbb{R}^n}(\gamma) > \varepsilon/2c_2$ so that $B(x_0, \delta) \subset B^g(x_0, \varepsilon)$ for $\delta = \varepsilon/2c_2$. $\square$

Example 9.22. Let us consider the group of positive affine transformation of the real line

$$f(t) = at + b, \quad a > 0, b \in \mathbb{R}$$

This is a group with the composition as a product, which rewrites as

$$(a, b) \cdot (a', b') = (aa', ab' + b)$$

Putting coordinates $(x, y) = (b, a)$ we have coordinates on the upper half-plane $\mathbb{H}^2$ endowed with the Lie group structure

$$(x, y) \cdot (x', y') = (x + yx', yy')$$

Notice that the neutral element of the group is $e = (0, 1)$ and $(x, y)^{-1} = (-xy^{-1}, y^{-1})$. The left-invariant vector fields X, Y with coincide with ∂_x, ∂_y at the identity are

$$X = y\partial_x, \quad Y = y\partial_y$$

and the left-invariant Riemannian metric

$$g = \frac{1}{y^2}(dx^2 + dy^2)$$

which is a model for the hyperbolic plane. Write in complex coordinates $z = x + iy$

$$g = \frac{dx^2 + dy^2}{y^2} = -4 \frac{dzd\bar{z}}{(z - \bar{z})^2}$$

Exercise 9.23. Prove that all maps of the form $T : z \mapsto \frac{az+b}{cz+d}$ with real coefficients and $ad - bc = 1$ are isometries. In particular all such transformations are obtained by composing the following: translations, dilations of positive factor and inversion (with reflection) with respect to the unit circle

$$T_b : z \mapsto z + b, \quad T_a : z \mapsto a^2 z, \quad T_{\pm} : z \mapsto \pm \frac{1}{z}$$

we have

$$T_- T_b T_a : z \mapsto -\frac{1}{a^2 z + b}.$$

From now on we remove the g in the notation of the length $L = L_g$ and balls $B(x, r) = B^g(x, r)$.

Definition 9.24. Let $\gamma : [0, T] \rightarrow M$ such that $\gamma(0) = x$ and $\gamma(T) = y$, we say that γ is a *length-minimizer* if $d(x, y) = \ell(\gamma)$.

Exercise 9.25. Prove that a piece of vertical segment is length-minimizer in the example. What about a piece of horizontal segment?

9.3 Length-minimizers and geodesics

The existence of length-minimizers is not guaranteed in general. Counterexamples are very easy to build, such as $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric.

Theorem 9.26. Let (M, g) Riemannian manifold and assume (M, d) is a complete metric space. Then for every $x, y \in M$ there exists a length-minimizer joining x and y .

Remark 9.27. A characterization of metric completeness for Riemannian manifolds holds:

- (a) (M, d) is complete as a metric space
- (b) all closed balls $\overline{B}(x, r)$ are compact

We need the following version of Arzela-Ascoli theorem in metric spaces.

Proposition 9.28 (Arzela-Ascoli). *In a compact metric space, any sequence of curves with uniformly bounded lengths contains a uniformly converging subsequence.*

Another key property is the semicontinuity of the length-functional: if γ_n is a sequence of curves with fixed endpoints which converges uniformly to γ then

$$L(\gamma) \leq \liminf_{n \rightarrow \infty} L(\gamma_n).$$

With these two ingredients we can prove Theorem 9.26.

Proof of Theorem 9.26. Consider $x, y \in M$ and consider a minimizing sequence γ_n of curves parametrized by constant speed on $[0, 1]$ such that $L(\gamma_n) \rightarrow d(x, y)$ for $n \rightarrow \infty$. It is not restrictive to assume that $L(\gamma_n) < d(x, y) + 1 =: R$ for every n . In particular all curves are contained in the compact set $\overline{B}(x, R)$. Since every sequence of curves with uniformly bounded lengths contains a uniformly converging subsequence, we can assume that $\gamma_n \rightarrow \gamma$ uniformly. By semicontinuity of the length

$$L(\gamma) \leq \liminf_{n \rightarrow \infty} L(\gamma_n) = d(x, y),$$

which implies that γ is a length-minimizer. □

Necessary conditions

Let us discuss necessary conditions to be minimizer. Recall that every curve can be reparametrized by constant speed.

Lemma 9.29. *Let $T > 0$ be fixed. A piecewise smooth curve $\gamma : [0, T] \rightarrow M$ is a length-minimizer and has constant speed if and only if γ is a minimizer of the functional*

$$E(\gamma) = \frac{1}{2} \int_0^T \|\dot{\gamma}(t)\|^2 dt.$$

Proof. We can assume without loss of generality that γ is smooth since the integral is additive. By the Cauchy-Schwartz inequality we have the general inequality

$$\ell(\gamma)^2 \leq 2E(\gamma)T. \tag{9.2}$$

Notice that Cauchy-Schwartz inequality also says that we have equality in (9.2) if and only if γ has constant speed. The conclusion then easily follows. □

We can now look to minimizers of the energy functional. Let us consider a short curve whose support is contained in a single chart (U, φ) and let $x(t) = \varphi(\gamma(t))$. In coordinates

$$E(\gamma) = \frac{1}{2} \int_0^T \sum_{i,j=1}^n g_{ij}(x(t)) \dot{x}_i(t) \dot{x}_j(t) dt.$$

Lemma 9.30. A solution $x : [0, T] \rightarrow \mathbb{R}^n$ to the problem

$$\min \frac{1}{2} \int_0^T \sum_{i,j=1}^n g_{ij}(x(t)) \dot{x}_i(t) \dot{x}_j(t) dt, \quad x(0) = x, \quad x(T) = y, \quad (9.3)$$

satisfies the system of equations

$$\ddot{x}_k + \sum_{i,j=1}^n \Gamma_{ij}^k \dot{x}_i \dot{x}_j = 0, \quad k = 1, \dots, n. \quad (9.4)$$

where Γ_{ij}^k are smooth functions defined by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n g^{mk} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right)$$

Remark 9.31. These equations makes sense only in local coordinates. For the moment no geometric interpretation of the equation (9.4). We will solve this problem later.

Proof. We use the following fact from Calculus of Variation (see Appendix 9.4): solutions of (9.3) satisfy the Euler-Lagrange equations for the functional

$$\frac{1}{2} \int_0^T L(x(t), \dot{x}(t)) dt, \quad L(x, \dot{x}) = \sum_{i,j=1}^n g_{ij}(x) \dot{x}_i \dot{x}_j.$$

These equations are written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_m} - \frac{\partial L}{\partial x_m} = 0, \quad m = 1, \dots, n.$$

We have $\frac{\partial L}{\partial \dot{x}_m} = \sum_{i=1}^n g_{im} \dot{x}_i$ hence

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_m} = \sum_{i,j=1}^n \frac{\partial g_{im}}{\partial x_j} \dot{x}_i \dot{x}_j + g_{im} \ddot{x}_i = \sum_{i,j=1}^n \frac{1}{2} \frac{\partial g_{im}}{\partial x_j} \dot{x}_i \dot{x}_j + \frac{1}{2} \frac{\partial g_{jm}}{\partial x_i} \dot{x}_i \dot{x}_j + g_{im} \ddot{x}_i,$$

where we used symmetry of the first term. Moreover

$$\frac{\partial L}{\partial x_m} = \sum_{i,j=1}^n \frac{1}{2} \frac{\partial g_{ij}}{\partial x_m} \dot{x}_i \dot{x}_j.$$

We have

$$\sum_{i,j=1}^n g_{im} \ddot{x}_i + \frac{1}{2} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right) \dot{x}_i \dot{x}_j = 0,$$

thus multiplying for g^{mk} (which we recall is the inverse of the metric)

$$\ddot{x}_k + \sum_{m,i,j=1}^n \frac{1}{2} g^{mk} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right) \dot{x}_i \dot{x}_j = 0.$$

□

Notice that (9.4) can be rewritten as a first-order system in TM as follows

$$\begin{cases} \dot{x}_k = v_k, \\ \dot{v}_k = -\Gamma_{ij}^k(x) v_i v_j \end{cases}, \quad k = 1, \dots, n \quad (9.5)$$

Geodesics.

We call *geodesics* curves that satisfy the necessary condition, i.e., critical points of the length functional. It follows from the previous considerations that:

Corollary 9.32. *For every $q \in M$ and $v \in T_q M$ there exists a unique geodesic $\gamma_{q,v} : [0, T[\rightarrow M$ defined on some open interval such that $\gamma(0) = q$ and $\dot{\gamma}(0) = v$.*

Exercise 9.33. Assuming that both sides of the equality are defined, prove the following homogeneity property $\gamma_{q,v}(ts) = \gamma_{q,tv}(s)$.

By standard continuity theorems for ODEs with respect to initial data, there exists an open subset $\mathcal{U}$ in TM such that for $v \in TM$ and $q = \pi(v)$ the corresponding solution of the ODE is defined for $T \geq 1$.

Definition 9.34. We define the *exponential map* $\exp : \mathcal{U} \rightarrow M$ defined by

$$\exp(q, v) := \exp_q(v) = \gamma_{q,v}(1)$$

where γ_v is the unique geodesic such that $\gamma_{q,v}(0) = q$ and $\dot{\gamma}_{q,v}(0) = v$.

The exponential map defines good coordinates in a neighborhood of the base point.

Proposition 9.35. *Let $q \in M$. The map $\exp_q : T_q M \rightarrow M$ is a local diffeomorphism at $v = 0$ and $d_0 \exp_q : T_q M \rightarrow T_q M$ is the identity map.*

Proof. It follows from the computation

$$d_0 \exp_q(v) = \left. \frac{d}{dt} \right|_{t=0} \exp_q(tv) = \left. \frac{d}{dt} \right|_{t=0} \gamma_{q,tv}(1) = \left. \frac{d}{dt} \right|_{t=0} \gamma_{q,v}(t) = v$$

□

Coordinates induced by $\exp_q$ are called *normal coordinates*. In these coordinates geodesics starting from q becomes straight lines by construction. We then state two important theorems.

Theorem 9.36. *Assume that (M, d) is a complete metric space. Then $\exp$ is defined on whole TM , i.e., all geodesics can be defined on $[0, +\infty[$.*

Proof. Let $q_0 \in M$ be arbitrary. It enough to show that any geodesic $\gamma_{v_0}(t)$ starting from $q_0 \in M$ with initial velocity $v_0 \in T_{q_0} M$ with $\|v_0\| = 1$ is defined for all $t \in \mathbb{R}$. Recall that the pair $(x(t), v(t))$, coordinates of $(\gamma_{v_0}(t), \dot{\gamma}_{v_0}(t))$, satisfy the geodesic equation in coordinates.

Let $\gamma_{v_0}(t)$ be defined on $[0, T[$, and assume that it is not extendable to some interval $[0, T + \varepsilon[$. For any sequence $t_j \nearrow T$ the sequence $(\gamma_{v_0}(t_j))_j$ is a Cauchy sequence on M since

$$d(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|.$$

The sequence $(\gamma_{v_0}(t_j))_{j \in \mathbb{N}}$ is then convergent to a point $q_1 \in M$ by completeness. Since $(\dot{\gamma}(t_j))_{j \in \mathbb{N}}$ stays in a compact set, there exists a subsequence (which we denote by the same symbol) such that $(\gamma_{v_0}(t_j), \dot{\gamma}_{v_0}(t_j)) \rightarrow (q_1, v_1)$. This contradicts the fact that $(x(t), v(t))$ is not extendable by standard result of ODEs. □

A crucial property of geodesics is that short arcs are global length-minimizers, even among piecewise smooth curves.

Theorem 9.37. *Let $\gamma : [0, T] \rightarrow M$ be a geodesic. Then there exists $\varepsilon > 0$ such that $\gamma|_{[0, \varepsilon]}$ is the unique length-minimizer among all piecewise-smooth curves joining $\gamma(0)$ and $\gamma(\varepsilon)$.*

This has the following crucial implication.

Corollary 9.38. *Every piecewise smooth curve $\gamma : [0, T] \rightarrow M$ that is a length-minimizer is of class C^∞ .*

Proof. Write a proof one day □

This is based on the Gauss Lemma

Lemma 9.39. *Let $q \in M$ and let $v \in T_q M$ such that $q' = \exp_q(v)$ is defined. Let $w \in T_q M \simeq T_v(T_q M)$ then*

$$g(d_v \exp_q(v), d_v \exp_q(w)) = g(v, w)$$

Remark 9.40. Notice that $\exp_q : T_q M \rightarrow M$ is *not* a local isometry, i.e., not all geodesics in a neighborhood of q becomes straight lines, but only those passing through q !

9.4 Appendix: on the Euler-Lagrange equations

Assume we are interested in minimizing the quantity

$$I(x(t)) = \int_0^T L(x(t), \dot{x}(t)) dt$$

Then if $x(t)$ is a minimizer we have $I(x(t)) \leq I(x(t) + \varepsilon h(t))$ for every $\varepsilon \geq 0$ and arbitrary $h(t)$. Assuming some smoothness we need to have

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} I(x(t) + \varepsilon h(t)) = 0$$

We have

$$\begin{aligned} I(x(t) + \varepsilon h(t)) &= \int_0^T L(x(t) + \varepsilon h(t), \dot{x}(t) + \varepsilon \dot{h}(t)) dt \\ &= \int_0^T L(x(t), \dot{x}(t)) dt + \varepsilon \int_0^T \frac{\partial L}{\partial x} h(t) + \frac{\partial L}{\partial \dot{x}} \dot{h}(t) dt + o(\varepsilon) \end{aligned}$$

With an integration by parts

$$\int_0^T \frac{\partial L}{\partial x} h(t) + \frac{\partial L}{\partial \dot{x}} \dot{h}(t) dt = \int_0^T \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) h(t) dt$$

hence the first order term in ε is zero if for every function $h(t)$ we have

$$\int_0^T \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) h(t) dt = 0$$

which implies the integrand is zero

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

Chapter 10

Connections, parallel transport and curvature

*Nous pensons que, après avoir surmonté les difficultés de initiation, on se convaincra aisément que la généralité [...] contribue non seulement à l'élégance mais aussi à l'agilité et à la perspicuité des démonstrations et des conclusions.*¹

Méthodes de calcul différentiel absolu et leurs applications, 1900

Gregorio Ricci-Curbastro 1853 – 1925

Tullio Levi Civita, 1873 – 1941

On a manifold, in general there is no canonical way to identify tangent spaces (or, more generally, fibers of a vector bundle) at different points. Thus, one has to expect that a notion of derivative for vector fields (or sections of a vector bundle), depends on a certain choice. The additional structure required to correctly define these notions is the one of connection.

10.1 Affine connections and parallel transport

Recall that we had a way to differentiate vector fields along another one

$$\mathcal{L}_X Y = [X, Y]$$

This corresponds to compute for $\gamma(t) = e^{tX}(q)$ and $Y(t) = Y(\gamma(t))$

$$\mathcal{L}_X Y|_q = \lim_{t \rightarrow 0} \frac{e_*^{-tX} Y(t) - Y(0)}{t}$$

This seems a good object but it has a drawback:

- the map $X \mapsto \mathcal{L}_X Y$ it is not a tensor, i.e., it is not $C^\infty(M)$ linear since $\mathcal{L}_{fX} Y \neq f \mathcal{L}_X Y$.

In this section M is just a smooth manifold, no metric g at the moment.

¹Transl. We think that after having overcome the difficulties at the beginning, one will easily convince himself that the generality [...] contributes not only to the elegance but also to the agility and the perspicaciousness of the proofs and the conclusions

Definition 10.1. A *linear connection* on TM is $\nabla : \text{Vec}(M) \times \text{Vec}(M) \rightarrow \text{Vec}(M)$ bilinear and satisfying the following properties

- (a) $\nabla_{fX}(Y) = f\nabla_X Y$
- (b) $\nabla_X(gY) = (Xg)Y + g\nabla_X Y$

Definition 10.2. Given a local frame $E_1, \dots, E_n$ (not necessary o.n.) we define the *Christoffel symbols* of ∇ associated to the frame as the set of functions Γ_{ij}^k satisfying

$$\nabla_{E_i} E_j = \sum_{k=1}^n \Gamma_{ij}^k E_k$$

As we will see the name is not occasional. The first observation is that $\nabla_X Y$ is local.

Lemma 10.3. *The value $\nabla_X Y|_q$ depends only on the value of X at q and on the values of Y on a neighborhood of q .*

Proof. Let $E_1, \dots, E_n$ local frame (not necessary o.n.) and write

$$X = \sum_{i=1}^n f_i E_i, \quad Y = \sum_{j=1}^n g_j E_j$$

Here we treat f_i and g_j as functions defined on a neighborhood of a point q . Then by using the rules

$$\nabla_X Y = \sum_{i=1}^n f_i \nabla_{E_i} Y \tag{10.1}$$

$$\begin{aligned} &= \sum_{i,j=1}^n f_i (E_i g_j) E_j + f_i g_j \nabla_{E_i} E_j \\ &= \sum_{i,j,k=1}^n (Xg_k + f_i g_j \Gamma_{ij}^k) E_k \end{aligned} \tag{10.2}$$

□

Thanks to last formula (10.2) we can reinforce the previous locality statement

Corollary 10.4. *The value of $\nabla_X Y|_q$ actually depends only on the value of X at q and on the values of Y on a curve that is tangent to X at q .*

We can also deduce existence

Proposition 10.5. *Every smooth manifold admits linear connection on TM*

Proof. The proof combines partition of unity with the following observation: given any local frame $E_1, \dots, E_n$ and a set of n^3 functions Γ_{ij}^k the formula (10.2) defines a connection in the open set. □

Example 10.6. In $\mathbb{R}^n$ we have a canonical connection $\bar{\nabla}$ defined as

$$\bar{\nabla}_X Y = \sum_{i=1}^n (XY^i) \frac{\partial}{\partial x_i}$$

the ordinary differentiation along X of the coefficients of Y (corresponds to the choice $\Gamma_{ij}^k = 0$).

Differentiation along curves and parallel transport

Hence given a linear connection ∇ , a regular curve $\gamma : [0, T] \rightarrow M$ (with $\dot{\gamma} \neq 0$) and a smooth vector field Y defined only along γ , it is well defined the vector field $\nabla_{\dot{\gamma}}Y$ along γ .

Remark 10.7 (coordinate formula). Consider a local frame $E_1, \dots, E_n$ and write

$$\dot{\gamma}(t) = \sum_{i=1}^n v_i(t) E_i|_{\gamma(t)}, \quad Y(\gamma(t)) = \sum_{j=1}^n y_j(t) E_j|_{\gamma(t)}$$

then from the previous formulae we get

$$\nabla_{\dot{\gamma}}Y|_{\gamma(t)} = \sum_{k=1}^n \left(\dot{y}_k(t) + \sum_{i,j=1}^n v_i(t) y_j(t) \Gamma_{ij}^k(\gamma(t)) \right) E_k|_{\gamma(t)}.$$

If there exists a smooth vector field X such that $X|_{\gamma(t)} = \dot{\gamma}(t)$ then $\nabla_{\dot{\gamma}}Y = \nabla_XY$ on γ .

Definition 10.8. We say that a vector field Y along a smooth curve $\gamma : [0, T] \rightarrow M$ is *parallel* with respect to ∇ if $\nabla_{\dot{\gamma}}Y = 0$ along γ .

Proposition 10.9. *Given any linear connection ∇ . Let $\gamma : [0, T] \rightarrow M$ be a smooth curve and $v_0 \in T_{\gamma(0)}M$. There exists a unique smooth vector field V along γ such that $V|_{\gamma(0)} = v_0$ and V parallel along γ with respect to ∇ .*

Proof. We have to solve the non autonomous linear system of differential equations

$$\dot{y}_k(t) + \sum_{i,j=1}^n \Gamma_{ij}^k(\gamma(t)) \dot{\gamma}_i(t) y_j(t) = 0, \quad k = 1, \dots, n$$

which can be written setting $A_{k,j}(t) = \sum_{i=1}^n \Gamma_{ij}^k(\gamma(t)) \dot{\gamma}_i(t)$ as

$$\dot{y}_k(t) + \sum_{j=1}^n A_{k,j}(t) y_j(t) = 0, \quad k = 1, \dots, n$$

Since the differential equation is linear then the solution is global, i.e., defined on $[0, T]$. $\square$

The map which associates v_0 with $V|_{\gamma(t)}$ is called the *parallel transport* of v_0 along γ .

Proposition 10.10. *Let $\gamma : [0, T] \rightarrow M$ be a smooth curve and let $v_0 \in T_{\gamma(0)}M$. Then the map $P_{0,t}^\gamma : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M$ defined by $v_0 \mapsto v_t := V|_{\gamma(t)}$ is a linear isomorphism.*

Proof. The fact that $P_{0,t}^\gamma$ is linear comes from the fact that the flow of a linear (nonautonomous) equation is linear. $\square$

Remark 10.11. Similarly we can define $P_{s,t}^\gamma$ for all s, t and, we have the relations

$$P_{s,s}^\gamma = id, \quad P_{t,r}^\gamma \circ P_{s,t}^\gamma = P_{s,r}^\gamma, \quad (P_{s,t}^\gamma)^{-1} = P_{t,s}^\gamma.$$

The connection ∇ and the parallel transport $P_{0,t}^\gamma$ are intimately related.

Proposition 10.12. Let $\gamma : [0, T] \rightarrow M$ be a smooth curve $\gamma(0) = q$ and $\dot{\gamma}(0) = X_0$. For Y vector field along γ we have

$$\nabla_{X_0} Y(q) = \lim_{t \rightarrow 0} \frac{(P_{0,t}^\gamma)^{-1} Y(\gamma(t)) - Y(q)}{t}$$

Proof. Fix a basis $v_1, \dots, v_n$ of tangent vectors at $q = \gamma(0)$, and set $V_i(t) := P_{0,t}^\gamma(v_i)$. By construction the vector fields V_i are parallel along γ hence $\nabla_{\dot{\gamma}} V_i = 0$. Write $Y(\gamma(t)) = \sum_{i=1}^n y_i(t) V_i(t)$. We have

$$P_{0,t}^{-1} Y(\gamma(t)) = \sum_{i=1}^n y_i(t) v_i$$

We have

$$\lim_{t \rightarrow 0} \frac{(P_{0,t}^\gamma)^{-1} Y(\gamma(t)) - Y(q)}{t} = \lim_{t \rightarrow 0} \sum_{i=1}^n \frac{y_i(t) - y_i(0)}{t} v_i = \dot{y}_i(0) v_i$$

On the other hand

$$\nabla_{X_0} Y(q) = (\nabla_{X_0} y_i(t)) V_i|_{t=0} + y_i(0) (\nabla_{X_0} V_i)|_{t=0} = \dot{y}_i(0) v_i$$

where we used that $\nabla_{X_0} y_i(t) = \dot{y}_i(t)$ □

Covariant derivative

A connection ∇ permits to differentiate tensors T of type (k, l) giving a tensor ∇T of type $(k+1, l)$. The formula for tensor of type $(k, 0)$ is as follows

$$\nabla T(X_1, \dots, X_n, Y) = Y(T(X_1, \dots, X_n)) - \sum_{i=1}^n T(X_1, \dots, \nabla_Y X_i, \dots, X_n)$$

and the general case is similar. Given T and a vector field X we define the covariant derivative

$$(\nabla_Y T)(X_1, \dots, X_n) := \nabla T(X_1, \dots, X_n, Y)$$

Notice that on 0-tensors (functions) we have $\nabla_X f = Xf$. For a covariant 1 tensor (differential form) ω we have

$$(\nabla_X \omega)Y = \nabla \omega(Y, X) = X\omega(Y) - \omega(\nabla_X Y)$$

while for covariant 2 tensors we have

$$(\nabla_X \tau)(Y, Z) = \nabla \tau(Y, Z, X) = X\tau(Y, Z) - \tau(\nabla_X Y, Z) - \tau(Y, \nabla_X Z)$$

Notice that the compatibility with the metric is defined by $\nabla g = 0$.

10.2 The Levi-Civita connection

Definition 10.13. A linear connection ∇ on a Riemannian manifold is said *compatible with the metric* if

$$X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \quad (10.3)$$

This is equivalent to ask that g is parallel, i.e., $\nabla g = 0$.

Definition 10.14. The *torsion* of a connection is the $(2, 1)$ tensor $T : \text{Vec}(M) \times \text{Vec}(M) \rightarrow \text{Vec}(M)$ defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Theorem 10.15. *On a Riemannian manifold (M, g) there exists a unique linear connection that is compatible with the metric and with zero torsion.*

Proof. The proof is based on the following key fact, which is proved by the following algorithm which we invite the reader to check:

- (i) writing three times identity (10.3) for the ordered triples $\{X, Y, Z\}$, $\{Y, Z, X\}$ and $\{Z, X, Y\}$,
- (ii) compute $(1) + (2) - (3)$,
- (iii) use the $T = 0$ identity $\nabla_X Y - \nabla_Y X = [X, Y]$,

One obtains the following key formula.

Lemma 10.16 (Koszul formula). *For a connection ∇ which is compatible with the metric g we have the identity*

$$2 \langle \nabla_X Y | Z \rangle = X \langle Y | Z \rangle + Y \langle Z | X \rangle - Z \langle X | Y \rangle + \langle [X, Y] | Z \rangle - \langle [Y, Z] | X \rangle + \langle [Z, X] | Y \rangle \quad (10.4)$$

The Koszul formula says in particular that if ∇ exists it is unique. The existence is guaranteed by formula (10.4) once one proves that the right hand side correctly defines a connection, which is left as an exercise. $\square$

Definition 10.17. The connection uniquely defined by Theorem 10.15 is called Levi-Civita connection of the Riemannian manifold (M, g) .

Remark 10.18. Two particular cases: Koszul formula to a frame that is commuting $[E_i, E_j] = 0$ only the first line is non zero.

$$2 \langle \nabla_X Y | Z \rangle = X \langle Y | Z \rangle + Y \langle Z | X \rangle - Z \langle X | Y \rangle \quad (10.5)$$

On the other side, applying Koszul formula to a o.n. frame $g(X_i, X_j) = \delta_{i,j}$ then only the commutator shows up.

$$2 \langle \nabla_X Y | Z \rangle = \langle [X, Y] | Z \rangle - \langle [Y, Z] | X \rangle + \langle [Z, X] | Y \rangle \quad (10.6)$$

Exercise 10.19. The Christoffel symbols of the Levi Civita connection associated with the frame $E_i = \frac{\partial}{\partial x_i}$ satisfy

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n g^{mk} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right)$$

These are the symbols found in the previous chapter.

This permits to interpret as geodesics as those curves which the acceleration in the sense of the Levi Civita connection is zero.

Proposition 10.20. *A smooth curve $\gamma : [0, T] \rightarrow M$ parametrized by constant speed is a geodesic if and only if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$*

Of course we can also take a local o.n. frame and use it to build a connection.

Lemma 10.21. *Let $X_1, \dots, X_n$ be a local o.n. frame such that $[X_i, X_j] = c_{ij}^k X_k$. Prove that the Christoffel symbols of the Levi Civita connection associated to this frame $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k$ are written as*

$$\Gamma_{ij}^k = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j)$$

In particular we have the property $\Gamma_{ij}^k = -\Gamma_{ik}^j$.

Proof. Use the Koszul formula for o.n. frames

$$2\langle \nabla_X Y | Z \rangle = \langle [X, Y] | Z \rangle - \langle [Y, Z] | X \rangle + \langle [Z, X] | Y \rangle \quad (10.7)$$

onto the basis $X = X_i$, $Y = X_j$ and $Z = X_k$. □

Proposition 10.22. *Let $\gamma : [0, T] \rightarrow M$ be a smooth curve and $v_0 \in T_{\gamma(0)}M$. The map $P_{0,t}^\gamma : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M$ defined by $v_0 \mapsto V|_{\gamma(t)}$ with the Levi-Civita connection is a linear isometry.*

Proof. Let us rewrite the differential equation in terms of an orthonormal frame

$$\dot{y}_k(t) + \sum_{j=1}^n A_{k,j}(t)y_j(t) = 0, \quad k = 1, \dots, n$$

setting $A_{k,j}(t) = \sum_{i=1}^n \Gamma_{ij}^k(\gamma(t))\dot{\gamma}_i(t)$. Since the frame is orthonormal the corresponding symbols satisfy $\Gamma_{ij}^k = -\Gamma_{ik}^j$ hence $A_{k,j}(t) = -A_{j,k}(t)$ is skew-symmetric. The flow is defined by an orthogonal matrix, hence $P_{0,t}^\gamma$ is an isometry. □

Remark 10.23. It follows the geometric interpretation of the parallel transport on a 2D surface: a field $v(t)$ along a geodesic is parallel if and only if $\|v(t)\|$ is constant and the angle between $v(t)$ and $\dot{\gamma}(t)$ is constant. If γ is not a geodesic we can use approximations.

Exercise 10.24. Compute the Christoffel symbols of the Levi Civita connection associated to the left invariant frame on the hyperbolic plane. Prove that its geodesics are either vertical lines or semicircles (centered on $y = 0$).

Remark 10.25. The Levi-Civita connection of the ordinary $\mathbb{R}^n$ is the canonical connection $\bar{\nabla}$ and acts as

$$\bar{\nabla}_X Y = \sum_{i=1}^n X Y^i \frac{\partial}{\partial x_i}$$

the ordinary differentiation along X of the coefficients of Y (corresponds to $\Gamma_{ij}^k = 0$)

Proposition 10.26. *Let (M, g) be an embedded submanifold of Euclidean $\mathbb{R}^N$ with the induced metric. Then the Levi-Civita connection ∇ for M is written as*

$$\nabla_X Y = \pi^\perp(\bar{\nabla}_X Y)$$

where for every $x \in M$ we have considered the orthogonal projection

$$\pi^\perp : \mathbb{R}^N \simeq T_x \mathbb{R}^N \rightarrow T_x M$$

and $\bar{\nabla}$ is the canonical connection in $\mathbb{R}^n$.

Proof. First observe that the formula correctly defines a connection on M (check is left to the reader)

$$\nabla_X Y = \pi^\perp(\bar{\nabla}_X Y)$$

To prove that ∇ is torsion free, for every X, Y tangent to M we have

$$\nabla_X Y - \nabla_Y X = \pi^\perp(\bar{\nabla}_X Y - \bar{\nabla}_Y X) = \pi^\perp[X, Y] = [X, Y]$$

since $[X, Y]$ is also tangent to M . The fact that ∇ is metric: for every X, Y, Z tangent to M , extend them to $\mathbb{R}^N$ and compute

$$Xg(Y, Z) = X\bar{g}(Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z) \quad (10.8)$$

but we have for X, Y, Z tangent to M , extend them to $\mathbb{R}^N$ and compute

$$g(\nabla_X Y, Z) = g(\pi^\perp \bar{\nabla}_X Y, Z) \quad (10.9)$$

$$= \bar{g}(\pi^\perp \bar{\nabla}_X Y, Z) \quad (10.10)$$

$$= \bar{g}(\pi^\perp \bar{\nabla}_X Y, Z) + \bar{g}((1 - \pi^\perp) \bar{\nabla}_X Y, Z) \quad (10.11)$$

$$= \bar{g}(\bar{\nabla}_X Y, Z) \quad (10.12)$$

□

Corollary 10.27. *Let (M, g) a Riemannian manifold which is an hypersurface in $\mathbb{R}^N$ with the induced metric. Then a smooth curve $\gamma : [0, T] \rightarrow M$ parametrized by constant speed is a geodesic on M if and only if $\ddot{\gamma}(t) \perp T_{\gamma(t)}M$.*

Indeed on hypersurfaces of $\mathbb{R}^N$ the equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ means $\pi^\perp(\bar{\nabla}_{\dot{\gamma}} \dot{\gamma}) = 0$, but since $\nabla_{\dot{\gamma}} \dot{\gamma} = \ddot{\gamma}$ this means $\ddot{\gamma} \perp T_{\gamma(t)}M$ when we see $\ddot{\gamma}$ as a vector of $\mathbb{R}^N$.

Exercise 10.28. Let $f : M \rightarrow \mathbb{R}$ be a smooth function. Observe that the covariant derivative ∇f of f with respect to the Levi-Civita connection is the Riemannian gradient. Indeed

$$(\nabla f)(X) = \nabla_X f = Xf.$$

The *Riemannian Hessian* of a function f is defined as $\nabla^2 f = \nabla(\nabla f)$. We have

$$\nabla^2 f(Y, X) = \nabla_X((\nabla f)(Y)) = \nabla_X(Yf) = X(Yf) - (\nabla_X Y)f$$

Notice that the Hessian is symmetric if and only if the torsion is zero

$$\nabla^2 f(X, Y) - \nabla^2 f(Y, X) = Y(Xf) - (\nabla_Y X)f - X(Yf) + (\nabla_X Y)f = T(X, Y)f$$

10.3 The Riemann curvature tensor

We first start by observing that the parallel transport (associated with the Levi-Civita connection) along a closed curve is in general non zero. One can consider for instance a geodesic triangle on the sphere S^2 .

We introduce the curvature (3,1) tensor $R(X, Y)$ also called *Riemann tensor*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (10.13)$$

Let us first check that R is indeed a tensor, i.e., the value of $R(X, Y)Z$ at a point depends only on the value of X, Y, Z at the point itself.

Proposition 10.29. *R is skew-symmetric wrt X, Y and $C^\infty(M)$ -linear in every variable.*

Proof. The skew-symmetry is immediate from the formula. Next we prove that R is $C^\infty(M)$ -linear. By skew-symmetry, it is sufficient to prove that R is linear in the first argument, namely that

$$R(fX, Y)Z = fR(X, Y)Z, \quad \text{where } f \in C^\infty(M). \quad (10.14)$$

Applying the definition of ∇ and the Leibniz rule for the Lie bracket one gets

$$\begin{aligned} R(fX, Y) &= \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX, Y]} Z \\ &= f \nabla_X \nabla_Y Z - \nabla_Y (f \nabla_X Z) - \nabla_{f[X, Y] - (Yf)X} Z \\ &= f \nabla_X \nabla_Y Z - (Yf) \nabla_X Z - f \nabla_Y \nabla_X Z - f \nabla_{[X, Y]} Z + (Yf) \nabla_X Z \end{aligned}$$

which proves $R(fX, Y) = fR(X, Y)$. □

Remark 10.30. Another observation proving that R is the (3,1) tensor is that

$$R(X, Y)Z = (\nabla^2 Z)(Y, X) - (\nabla^2 Z)(X, Y). \quad (10.15)$$

where ∇Z is the (1,1) tensor $(\nabla Z)(X) = \nabla_X Z$ and $\nabla^2 Z = \nabla(\nabla Z)$.

Proposition 10.31. *Let $X_1, \dots, X_n$ local o.n. frame. We define the coefficients*

$$R_{ijk}^l = \langle R(X_i, X_j)X_k, X_l \rangle$$

we have that

$$R_{ijk}^l = X_i(\Gamma_{jk}^l) - X_j(\Gamma_{ik}^l) + \Gamma_{ia}^l \Gamma_{jk}^a - \Gamma_{ja}^l \Gamma_{ik}^a - c_{ij}^a \Gamma_{ak}^l$$

In particular if $\Gamma_{ij}^k = 0$ for all i, j, k , then $R = 0$.

Exercise 10.32. Prove a similar formula to the previous one but for the curvature coefficients $\langle R(\partial_i, \partial_j)\partial_k, \partial_l \rangle$ in terms of the Christoffel symbols of the coordinate frame.

Flat manifolds

Theorem 10.33. *The following conditions are equivalent:*

- (a) $R(X, Y)Z = 0$ for every X, Y, Z
- (b) there exists a local orthonormal frame such that $\nabla_{X_i} X_j = 0$
- (c) there exists a local orthonormal frame such that $[X_i, X_j] = 0$.

Proof. The only non trivial fact is (a) implies (b). Indeed (b) implies (c) since the torsion free condition gives

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$$

and (c) implies (a) since $[X_i, X_j] = 0$ implies $c_{ij}^k = 0$ hence $\Gamma_{ij}^k = 0$ for all i, j, k .

To prove (a) implies (b) we do in the case $n = 2$ (the general case is similar). Do as follows: it is enough to prove that we can build a frame such that $\nabla_{\partial_i} X_j = 0$ in a neighborhood. Fix X_1, X_2 as ∂_1, ∂_2 at q and build the parallel transport on the segment $(x_1, 0)$ by parallel transport along the x_1 axis and then at every (x_1, x_2) by parallel transport along the second.

We have clearly $\nabla_{\partial_1} X_j|_{(x_1, 0)} = 0$ and $\nabla_{\partial_2} X_j|_{(x_1, x_2)} = 0$ for $j = 1, 2$. The second means $\nabla_{\partial_2} X_j = 0$ locally. We do not know if $\nabla_{\partial_1} X_j|_{(x_1, x_2)}$ is zero if $x_2 \neq 0$.

But since ∂_1, ∂_2 is coordinate frame and $\nabla_{\partial_1} \nabla_{\partial_2} = \nabla_{\partial_2} \nabla_{\partial_1}$. Then

$$\nabla_{\partial_2} \nabla_{\partial_1} X_j|_{(x_1, x_2)} = \nabla_{\partial_1} \nabla_{\partial_2} X_j|_{(x_1, x_2)} = \nabla_{\partial_1} 0 = 0$$

Hence $\nabla_{\partial_1} X_j|_{(x_1, x_2)}$ is the parallel transport along x_2 axis of $\nabla_{\partial_1} X_j|_{(x_1, 0)}$, which is zero. □

With this characterization we have immediately

Corollary 10.34. *(M, g) admits a local orthonormal frame such that $[X_i, X_j] = 0$, if and only if it is locally isometric to Euclidean $\mathbb{R}^n$.*

It is enough to prove that if there exists a local orthonormal frame such that $[X_i, X_j] = 0$ then we can build the local isometry.

If $[X_i, X_j] = 0$ then consider the map

$$\psi : \mathbb{R}^n \rightarrow M, \quad \psi(t_1, \dots, t_n) = e^{t_1 X_1} \circ \dots \circ e^{t_n X_n}(q_0),$$

which satisfies since $[X_i, X_j] = 0$ (cf. Chapter ?)

$$\psi_* \frac{\partial}{\partial t_i} = X_i$$

hence ψ is a local isometry.

Some curvature identities

The following identity, first discovered by G. Ricci-Curbastro, is known as the first Bianchi identity.

Proposition 10.35 (first Bianchi identity). *For every $X, Y, Z \in \text{Vec}(M)$ the following identity holds*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0. \quad (10.16)$$

Proof. We will show that (10.16) is a consequence of the Jacobi identity for vector fields (5.12). Using the fact that ∇ is a torsion-free connection we can write

$$\begin{aligned} [X, [Y, Z]] &= \nabla_X[Y, Z] - \nabla_{[Y, Z]}X \\ &= \nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[Y, Z]}X, \\ [Z, [X, Y]] &= \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X, Y]}Z, \\ [Y, [Z, X]] &= \nabla_Y\nabla_Z X - \nabla_Y\nabla_X Z - \nabla_{[Z, X]}Y, \end{aligned}$$

Then, adding these identities and using (5.12), one gets

$$\begin{aligned} 0 &= [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \\ &= \nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[Y, Z]}X \\ &\quad + \nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X, Y]}Z \\ &\quad + \nabla_Y\nabla_Z X - \nabla_Y\nabla_X Z - \nabla_{[Z, X]}Y \\ &= R(X, Y)Z + R(Y, Z)X + R(Z, X)Y. \end{aligned}$$

□

Exercise 10.36 (second Bianchi identity). Prove that for every $X, Y, Z, W \in \text{Vec}(M)$ one has

$$(\nabla_X R)(Y, Z, W) + (\nabla_Y R)(Z, X, W) + (\nabla_Z R)(X, Y, W) = 0.$$

(Hint: Expand the identity $\nabla_{[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]} W = 0$.)

Remark 10.37. The relations for the Christoffel symbols implies the following skew-symmetry property: for $X, Y, Z, W \in \text{Vec}(M)$

$$\langle R(X, Y)Z \mid W \rangle = -\langle R(X, Y)W \mid Z \rangle,$$

where $\langle \cdot \mid \cdot \rangle$ denotes the Riemannian inner product.

Let us introduce the notation

$$R(X, Y, Z, W) := \langle R(X, Y)Z \mid W \rangle.$$

Then, the first Bianchi identity (10.16) can be rewritten as follows: for $X, Y, Z, W \in \text{Vec}(M)$ one has

$$R(X, Y, Z, W) + R(Z, X, Y, W) + R(Y, Z, X, W) = 0. \quad (10.17)$$

Moreover, the skew-symmetry properties of the curvature tensor discussed in Proposition 10.29 and Remark 10.37 can be rewritten as follows

$$R(X, Y, Z, W) = -R(Y, X, Z, W), \quad R(X, Y, Z, W) = -R(X, Y, W, Z). \quad (10.18)$$

Proposition 10.38. *For every $X, Y, Z, W \in \text{Vec}(M)$ we have $R(X, Y, Z, W) = R(Z, W, X, Y)$.*

Proof. Using (10.17) four times we can write the identities

$$\begin{aligned} R(X, Y, Z, W) + R(Z, X, Y, W) + R(Y, Z, X, W) &= 0, \\ R(Y, Z, W, X) + R(W, Y, Z, X) + R(Z, W, Y, X) &= 0, \\ R(Z, W, X, Y) + R(X, Z, W, Y) + R(W, X, Z, Y) &= 0, \\ R(W, X, Y, Z) + R(Y, W, X, Z) + R(X, Y, W, Z) &= 0. \end{aligned}$$

Summing these identities and using (10.18), one gets $R(X, Z, W, Y) = R(W, Y, X, Z)$. $\square$

Proposition 10.39. *Assume that $R(X, Y, X, W) = 0$ for every $X, Y, W \in \text{Vec}(M)$. Then*

$$R(X, Y, Z, W) = 0 \quad \forall X, Y, Z, W \in \text{Vec}(M).$$

Proof. By assumptions and the skew-symmetry properties (10.18) of the Riemann tensor we have that $R(X, Y, Z, W) = 0$ whenever any two of the vector fields coincide. In particular

$$0 = R(X, Y + W, Z, Y + W) = R(X, Y, Z, W) + R(X, W, Z, Y). \quad (10.19)$$

Notice that the two extra terms that should appear developing the left hand side vanish, by assumptions. Then (10.19) can be rewritten as

$$R(X, Y, Z, W) = R(Z, X, Y, W).$$

This means that the quantity $R(X, Y, Z, W)$ is invariant by cyclic permutations of X, Y, Z . But the cyclic sum of these terms is zero thanks to (10.17), hence $R(X, Y, Z, W) = 0$. $\square$

From the properties of the Riemann curvature one obtains the following.

Corollary 10.40. *There is a well defined map*

$$\bar{R} : \wedge^2 T_q M \rightarrow \wedge^2 T_q M, \quad \bar{R}(X \wedge Y) := R(X, Y).$$

Moreover $\bar{R}$ is self-adjoint with respect to the scalar product on $\wedge^2 T_q M$ induced by the Riemannian scalar product, namely

$$\langle \bar{R}(X \wedge Y) \mid Z \wedge W \rangle = \langle X \wedge Y \mid \bar{R}(Z \wedge W) \rangle.$$

More curvatures and comparison

We define the *sectional curvature*

$$\text{Sec}(X, Y) = \frac{R(X, Y, Y, X)}{\|X\|^2 \|Y\|^2 - \langle X \mid Y \rangle^2}$$

which indeed depends only on the plane $\Pi = \text{span}\{X, Y\}$.

Remark 10.41. Indeed one can prove that Sec completely determines the Riemann tensor R .

We define the *Ricci curvature* which is a quadratic form computing an average of sectional curvatures containing a fixed vector

$$\text{Ric}(V) = \sum_{i=1}^n \text{Sec}(V, X_i)$$

where $X_1, \dots, X_n$ is an orthonormal basis.

This opens the big theory of comparison geometry, for instance let us state one single result.

Theorem 10.42 (Bonnet-Myers theorem, 1941). *Let (M, g) be a complete Riemannian manifold of dimension n whose Ricci curvature satisfies $\text{Ric}(V) \geq (n-1)K\|V\|^2$ for some $K > 0$. Then M is compact and*

$$\text{diam}(M) \leq \pi/\sqrt{k}.$$

Bonnet proved the version with inequality on all sectional curvatures. Myers weaken the result at Ricci curvatures.

In the previous theorem the equality case is attained for instance in the case of the sphere $S^n(r)$ of radius $r = 1/K$. A striking result is the following rigidity result.

Theorem 10.43 (Cheng, 1975). *Let (M, g) be a complete Riemannian manifold of dimension n whose Ricci curvature satisfies $\text{Ric}(V) \geq (n-1)K\|V\|^2$ for some $K > 0$. If $\text{diam}(M) = \pi/\sqrt{k}$, then M is isometric to the sphere $S^n(r)$ of radius $r = 1/K$.*

Appendix A

Problems and Exercises

The following exercises are taken from exams of Academic Years from 2020/21 to 2024/25.

Exercise A.1. Consider the two subsets of $\mathbb{R}^3$ defined by

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}, \quad E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 3z^2 = 3\}.$$

0. Prove that E and H are C^∞ submanifolds of $\mathbb{R}^3$

1. Is $E \cap H$ a smooth submanifold of $\mathbb{R}^3$? Is $E \cap H$ compact? Is $E \cap H$ connected?

Let now $g : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $g(x, y, z) = x^2 + y^2 + 3z^2 - 3$ and let $(a, b, c) \in \mathbb{R}^3$ such that $g(a, b, c) > 0$. (Notice that $E = g^{-1}(0)$).

2. Let C the subset of points $p = (x, y, z)$ of E such that the affine tangent hyperplane $T_p E$ passes through (a, b, c) . Prove that C is a smooth submanifold of $\mathbb{R}^3$.

Exercise A.2. Let ω be the differential 1-form in $\mathbb{R}^3$

$$\omega = dz + a(x, y)dx - b(x, y)dy$$

where $a, b : \mathbb{R}^2 \rightarrow \mathbb{R}$ be smooth functions depending only on x, y . Consider the vector distribution $D = \ker \omega$ (i.e., $D_q = \ker \omega_q$ for every $q = (x, y, z)$)

1. Find two everywhere linearly independent smooth vector fields such that $D = \text{span}\{X, Y\}$.
2. Compute $[X, Y]$ and give a necessary and sufficient condition (C1) on the functions a, b such that D is integrable.
3. Compute the 2-form $d\omega$ and give a necessary and sufficient condition (C2) on the functions a, b such that $\omega \wedge d\omega$ is a volume form in $\mathbb{R}^3$. What is the relation between (C1) and (C2)?

Exercise A.3. Let $\bar{g}$ denote the Euclidean metric in $\mathbb{R}^3$. Let $F : \mathbb{R}^+ \times]0, \pi[\times]0, 2\pi[\rightarrow \mathbb{R}^3$

$$F(\rho, \theta, \varphi) = (\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \theta)$$

be the map defining spherical coordinates in $\mathbb{R}^3$.

1. Compute the tensor $F^*\bar{g}$, then deduce that the standard Riemannian metric g_{S^2} on the sphere S^2 (which is the restriction of $\bar{g}$ to S^2) is written in the coordinates (θ, φ)

$$g_{S^2} = d\theta^2 + \sin^2 \theta d\varphi^2$$

2. compute the length of meridians (curves of the form $\varphi = \varphi_0$) and parallels (curves of the form $\theta = \theta_0$) with respect to the Riemannian metric g_{S^2} .
3. compute the Riemannian volume form vol_g on S^2 in the coordinates (θ, φ) and deduce the volume A of the sphere with respect to vol_g

$$A = \int_{S^2} d\text{vol}_g$$

Exercise A.4. The sphere $S^2 \subset \mathbb{R}^3$ described as

$$S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

can be endowed by the atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ where $U_P = S^2 \setminus \{P\}$ for $P \in \{N, S\}$ the north and south pole $N = (0, 0, 1), S = (0, 0, -1)$, and $\varphi_P : U_P \rightarrow \mathbb{R}^2$ the corresponding stereographic projection from P on the plane $\{x_3 = 0\}$.

1. Find the explicit expression for φ_N and φ_S and check the smooth compatibility between charts. Is it a maximal atlas?
2. Consider the map $F : \mathbb{C} \rightarrow \mathbb{C}$, $F(z) = z^2 + 1$ thought as a smooth map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. Consider the function $\bar{F} : S^2 \rightarrow S^2$ defined by

$$\bar{F}(x) = \begin{cases} \varphi_N^{-1} \circ F \circ \varphi_N(x), & x \neq N \\ N, & x = N \end{cases}$$

Is the function $\bar{F}$ smooth with respect to the C^∞ structure?

Hint: it may be useful to write $\varphi_N \circ \varphi_S^{-1}$ in complex coordinates.

Exercise A.5. Let ω be a differential k -form in $U = \mathbb{R}^n \setminus \{0\}$. For $t > 0$, let $H_t : U \rightarrow U$ be the map $H_t(x) = tx$. For $p \in \mathbb{N}$, we say that ω is p -homogeneous if $H_t^* \omega = t^p \omega$.

1. Express the p -homogeneity of ω in terms of its coefficients in coordinates

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

2. Let X be the vector field $X = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Prove that if ω is p -homogeneous then $L_X \omega = p\omega$.

Exercise A.6. Consider the $(2, 0)$ tensor in $\mathbb{R}^3$ defined by $\tau = dx \otimes dx + dy \otimes dy - dz \otimes dz$.

0. Define what is a Riemannian metric. Is τ a Riemannian metric on $\mathbb{R}^3$?

1. Let $i : H \hookrightarrow \mathbb{R}^3$ be the canonical inclusion of the surface

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1, z > 0\} \subset \mathbb{R}^3.$$

Prove that $i^*\tau$ defines a Riemannian metric g on H

2. Compute the Riemannian length (with respect to g) of the piece of curve defined by $H \cap \{x = 0\}$ and joining the two points $P = (0, -1, \sqrt{2})$ and $Q = (0, 1, \sqrt{2})$.

Exercise A.7. Denote S^2 the unit sphere of $\mathbb{R}^3$ and $\mathbb{P}^2(\mathbb{R})$ the real projective plane. Let F be the map

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad F(x, y, z) = (2xz, 2yz, 1 - 2z^2)$$

1. Show that F restricts to a map f from S^2 to S^2 . Is f of class C^∞ ?

2. Compute the linear map $f_* : T_p S^2 \rightarrow T_{f(p)} S^2$ for $p = (1, 0, 0) \in S^2$ (you might fix a basis of your choice in the tangent spaces).

3. Is f a local diffeomorphism? If not, find all critical points and critical values of f .

4. Denote the canonical projection $\pi : S^2 \rightarrow \mathbb{P}^2(\mathbb{R})$. Show that there exists a unique map $h : \mathbb{P}^2(\mathbb{R}) \rightarrow S^2$ of class C^∞ such that $f = h \circ \pi$

Exercise A.8. Let X, Y be two vector fields on a smooth manifold M . Consider the following two operators acting on differential forms on M

- L_X the Lie derivative of a differential form with respect to a vector field X ,
- i_Y the inner product of a differential form with respect to a vector field Y .

Prove the following identity on differential forms

$$L_X \circ i_Y - i_Y \circ L_X = i_{[X, Y]} \tag{A.1}$$

Hint: start by proving the required identity for a 1-differential form ω

Exercise A.9. Let (M, g) be a Riemannian manifold.

1. Recall the definition of Levi-Civita connection ∇ defined on M , and then the explicit formula for $\bar{\nabla}_X Y$, where $\bar{\nabla}$ is the Levi-Civita connection in the Euclidean space $(\mathbb{R}^n, \bar{g})$.
2. In the Euclidean space $\mathbb{R}^2$, compute $\bar{\nabla}_X X$ where $X = x\partial_y - y\partial_x$.
3. Prove that $\nabla_Y Y = 0$, where $Y = X|_{S^1}$ and ∇ is the Levi Civita connection on S^1 considered with the induced Riemannian metric of $\mathbb{R}^2$.

Exercise A.10. Let I_n the identity matrix of size n and

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The set $\text{Sp}(2n)$ of symplectic matrices is the subset of square matrices of size $2n$

$$\text{Sp}(2n) := \{M \in M_{2n}(\mathbb{R}) : M^T J M = J\} \subset M_{2n}(\mathbb{R}).$$

1. For $n = 1$, describe explicitly $\text{Sp}(2)$ and show it is a submanifold of $M_2(\mathbb{R}) \simeq \mathbb{R}^4$. Of which dimension?
2. Prove that $\text{Sp}(2n)$ is a submanifold of $M_{2n}(\mathbb{R})$ and compute its dimension.
3. Compute $T_I \text{Sp}(2n)$, the tangent space to $\text{Sp}(2n)$ at identity.

Hint: consider the map $F : M_{2n}(\mathbb{R}) \rightarrow X$, $F(M) = M^T J M$, for a suitable space X such that...

Exercise A.11. For which values of $\alpha \in \mathbb{R}$ the following 2-form Ω in $\mathbb{R}^3 \setminus \{0\}$ is closed?

$$\Omega = (x^2 + y^2 + z^2)^\alpha (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)$$

Exercise A.12. Consider on $\mathbb{R}^2$ the operation

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, y_2 + x_2 e^{y_1})$$

1. Prove that $(\mathbb{R}^2, \cdot)$ is a Lie group and find the identity e of the group (i.e., the neutral element)
2. Find the two left-invariant vector fields X_1, X_2 satisfying $X_i(e) = \partial_{x_i}$ for $i = 1, 2$
3. Compute $X_3 := [X_1, X_2]$. Is X_3 left-invariant?
4. Compute the Riemannian metric g for which X_1, X_2 is a global orthonormal frame
5. Compute the Riemannian volume associated with g of the unit square $Q = [0, 1] \times [0, 1]$

Exercise A.13. Consider the map $F : \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{R}$ given by $F(u, v) = (2u\bar{v}, |u|^2 - |v|^2)$. Consider S^3 as a subset of $\mathbb{C}^2 \simeq \mathbb{R}^4$ and similarly S^2 as a subset of $\mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$.

1. Prove that $\pi := F|_{S^3}$ is a well defined map from S^3 to S^2 .
2. Prove that π is C^∞ with respect to the smooth structures of S^3 and S^2 .
3. Prove that $\pi^{-1}(x)$ is a submanifold of S^3 for every $x \in S^2$. Of which dimension?

Exercise A.14. Consider the sphere S^2 embedded in $\mathbb{R}^3$. Let X be the vector field

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$$

1. Show that the restriction of the 2-form $i_X(dx \wedge dy \wedge dz)$ to the sphere is a volume form on S^2 . We denote this form α .
2. Show that on the complement of the equator $\{z = 0\} \cap S^2$, we have

$$\alpha = \frac{dx \wedge dy}{z}$$

3. Let φ_N be the stereographic projection from the north pole $N = (0, 0, 1)$ from $S^2 \setminus \{N\}$ to the plane $\{z = 0\}$. Write out φ_N^{-1} and calculate $\omega := (\varphi_N^{-1})^*\alpha$.
4. Compute $\int_{\mathbb{R}^2} \omega$

Exercise A.15. Let (M, g) be an orientable two dimensional Riemannian manifold. Let X_1, X_2 be an orthonormal basis for a metric g on M . Assume that for $\alpha_1, \alpha_2 \in C^\infty(M)$ we have on M

$$[X_1, X_2] = \alpha_1 X_1 + \alpha_2 X_2, \quad (\text{A.2})$$

1. Recall the definition of Riemannian volume. Prove that the Riemannian volume form can be written as $\text{vol}_g = \nu_1 \wedge \nu_2$ where ν_1, ν_2 are dual basis of X_1, X_2 .
2. Recall the definition of divergence of a vector field with respect to the Riemannian volume. Compute the divergence $\text{div}(X)$ and $\text{div}(Y)$ with respect to vol_g in terms of α_1, α_2 .

Exercise A.16. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ function and let M be its graph

$$M = \{(x, f(x)) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

1. Show that M is a smooth orientable manifold specifying an atlas and the dimension.
2. Is M an immersed/embedded submanifold of $\mathbb{R}^{n+1}$?
3. Let $i : M \rightarrow \mathbb{R}^{n+1}$ be the canonical inclusion. Compute $g := i^*\bar{g}$ the induced Riemannian metric on M , where $\bar{g}$ is the Euclidean metric in $\mathbb{R}^{n+1}$.
4. Compute Ω_M the Riemannian volume form of M
5. Compute $\int_U \Omega_M$ for the special case $n = 2$, $f(x) = \|x\|^2$ and $U = \{(x, f(x)) \in M \mid \|x\|^2 \leq 1\}$.

Hint: one might use the formula $\det(I + vv^T) = 1 + \|v\|^2$, for $v \in \mathbb{R}^n$ seen as a column vector, where we denote I the identity matrix and v^T the transpose of v .

Exercise A.17. Consider the 2-sphere $S^2 \subset \mathbb{R}^3$ endowed by the standard atlas¹ $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$.

1. for $P \in S^2$ describe $T_P S^2$ as a subset of $T_P \mathbb{R}^3$ and compute the linear map $(\varphi_N)_* : T_P S^2 \rightarrow \mathbb{R}^2$

¹this is $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$, $U_P = S^2 \setminus \{P\}$, $N = (0, 0, 1), S = (0, 0, -1)$ where $\varphi_P : U_P \rightarrow \mathbb{R}^2$ is the corresponding stereographic projection from P onto $\{x_3 = 0\} \simeq \mathbb{R}^2$.

2. Compute explicitly $X_i = (\varphi_N^{-1})_* Y_i$ defined on $U_N = S^2 \setminus \{N\}$, where $Y_i = \partial_{u_i}$ for $i = 1, 2$ is the constant vector field on $\mathbb{R}^2$, with coordinates (u_1, u_2) .
Hint: Solve the linear equation $(\varphi_N)_ X_i = Y_i$ with unknown X_i tangent to S^2 .*
3. Prove that X_i can be continuously extended to a C^∞ vector field $\bar{X}_i$ on S^2
4. Compute the Lie bracket $[\bar{X}_1, \bar{X}_2]$.

Exercise A.18. Consider the set $AL(\mathbb{R}^2)$ of affine lines in $\mathbb{R}^2$. Given a line ℓ of equation $ax + by = c$ we define the sets $U_1 = \{\ell \in AL(\mathbb{R}^2) \mid a \neq 0\}$, $U_2 = \{\ell \in AL(\mathbb{R}^2) \mid b \neq 0\}$ and the charts $\varphi_i : U_i \rightarrow \mathbb{R}^2$ defined for $i = 1, 2$ by

$$\varphi_1(\ell) = \left(\frac{b}{a}, \frac{c}{a} \right), \quad \varphi_2(\ell) = \left(\frac{a}{b}, \frac{c}{b} \right)$$

The atlas $\{(U_i, \varphi_i)\}_{i=1,2}$ gives the structure of smooth manifold to $AL(\mathbb{R}^2)$.

1. Let o be the origin of $\mathbb{R}^2$, for every affine line $\ell \in AL(\mathbb{R}^2)$ define $f(\ell) := \text{dist}^2(o, \ell)$, where dist denotes the Euclidean distance in $\mathbb{R}^2$ from a point to a line. Prove that the function f is C^∞ with respect to the smooth structure of $AL(\mathbb{R}^2)$.
2. Find all critical points of f given at point 2. Discuss their nature.

Exercise A.19. Consider $F : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n$ given by $F(x) = x/\|x\|$.

1. For $x_0 \neq 0$ and $v \in \mathbb{R}^n$ compute $DF(x_0)v$. Determine the rank of F and $\ker DF(x_0)$.

Let now $g : \mathbb{R}^n \rightarrow \mathbb{R}$ of class C^∞ and $M = \{x \in \mathbb{R}^n \mid g(x) = 0\}$. Assume that

- (a) $\langle \nabla g(x), x \rangle \neq 0$ for every $x \in M$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.
 - (b) for all $y \in \mathbb{R}^n \setminus \{0\}$ there exists unique $r > 0$ such that $ry \in M$.
2. Prove that M is a C^∞ embedded submanifold of $\mathbb{R}^n \setminus \{0\}$. Of which dimension?
 3. Prove that the restriction of F to M , regarded as a map $G := F|_M : M \rightarrow S^{n-1}$, is an injective immersion. Is G a local diffeomorphism?

Exercise A.20. Consider in $\mathbb{R}^3$ the differential 2-form $\omega = x dy \wedge dz - y dx \wedge dz + z dx \wedge dy$ and the vector field $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$.

1. Does there exist a differential 1-form ν in $\mathbb{R}^3$ such that $\omega = d\nu$?
2. Compute $L_X \omega$ and $i_X(d\omega)$, where L_X denotes the Lie derivative and i_X the interior product.
3. Compute the integral $\int_E L_X \omega$, where $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 4z^2 = 4\}$.

Exercise A.21. Consider the subsets of $\mathbb{R}^3$ given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + 1 = z^2, z > 0\}, \quad N = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + 4 = 4z\},$$

1. Is M a smooth manifold? Is $M \cap N$ an embedded smooth manifold of $\mathbb{R}^3$? Of which dimension?
2. Let $C = M \cap N$. Compute the tangent spaces $T_q M$ and $T_q C$ for $q = (0, 0, 1)$.
3. Find two vector fields X, Y tangent to M such that $\pi_* X = \partial_x$ and $\pi_* Y = \partial_y$ where $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is the projection onto the first two coordinates. Compute their Lie bracket $[X, Y]$.
4. Let g be the Riemannian metric induced on M by the restriction of the Euclidean metric $\bar{g}$ in $\mathbb{R}^3$. Compute $\nabla_X Y$ and $\nabla_Y X$, where ∇ is the Levi Civita connection of (M, g) .

Exercise A.22. Let us denote by $\mathbb{P}^1(\mathbb{R})$ the real projective line (points on $\mathbb{P}^1(\mathbb{R})$ are denoted $[x : y]$)

1. Recall the standard differential structure (i.e., the atlas) on $\mathbb{P}^1(\mathbb{R})$. Is $\mathbb{P}^1(\mathbb{R})$ orientable?
2. Establish an explicit smooth diffeomorphism between $\mathbb{P}^1(\mathbb{R})$ and S^1 .
3. Show that the projection $\pi : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{P}^1(\mathbb{R})$ defined by $\pi(x, y) = [x : y]$ is of class C^∞ .
4. Let $P(x), Q(x)$ be two real polynomials with no real root in common. Prove that the following map is smooth

$$F : \mathbb{R} \rightarrow \mathbb{P}^1(\mathbb{R}), \quad F(x) = \begin{cases} [\frac{P(x)}{Q(x)} : 1] & \text{if } Q(x) \neq 0 \\ [1 : 0] & \text{if } Q(x) = 0 \end{cases}$$

Give an example, for some choice of non constant $P(x)$ and $Q(x)$, such that F is an immersion and an example when F is not an immersion.

5. Show that $G : \mathbb{P}^1(\mathbb{R}) \setminus \{[1 : 0]\} \rightarrow \mathbb{P}^1(\mathbb{R})$ defined by

$$G([x : 1]) = F(x)$$

admits a continuous extension to $\bar{G} : \mathbb{P}^1(\mathbb{R}) \rightarrow \mathbb{P}^1(\mathbb{R})$. Prove that $\bar{G}$ is smooth. Give an example, for some choice of non constant $P(x)$ and $Q(x)$, such that $\bar{G}$ is a diffeomorphism.

Exercise A.23. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be smooth and consider the 1-differential form in $\mathbb{R}^3$ where we denote points (x, y, z)

$$\omega = (1 - f(x))dy - (1 + f(x))dz.$$

- 1.a. Find necessary and sufficient conditions (V) on f under which $\omega \wedge d\omega$ is a volume form on $\mathbb{R}^3$.
- 1.b. Assume conditions (V) holds. Find² a smooth vector field Z such that $i_Z \omega = 1$ and $i_Z d\omega = 0$.

²here $i_X \eta$ (resp. $L_X \eta$) denotes the interior product (resp. the Lie derivative) of a differential k -form η with respect a vector field X .

- 1.c. For the vector field Z computed in 1.b., then compute $L_Z\omega$.
2. Find necessary and sufficient conditions (C) on f in such a way that the distribution $D = \ker \omega$ is involutive. Assuming (C), find the integral manifold through the origin.

Exercise A.24. Consider $f : \mathbb{R} \rightarrow]0, +\infty[$ smooth positive and the subset of $\mathbb{R}^3$ given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = f^2(z)\}$$

1. Prove that M is an embedded smooth manifold. Of which dimension?
2. Let $i : M \rightarrow \mathbb{R}^3$ be the canonical inclusion. Compute $g = i^*\bar{g}$ the restriction of the Euclidean metric $\bar{g}$ in $\mathbb{R}^3$ to M .
3. Compute the length (with respect to the Riemannian metric g) of the curve γ_c given by the intersection of M and the plane $\{z = c\}$.
4. For which $c \in \mathbb{R}$ the curve γ_c is a geodesic?

Exercise A.25. Consider the map $F : \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$$F(x, y, z, t) = x^2 + y^2 + z^2 - t^2$$

1. Prove that $M := F^{-1}(1)$ is a regular submanifold of $\mathbb{R}^4$.
2. Let $\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be the map $\pi(x, y, z, t) = (x, y, z)$. Is $\pi|_M : M \rightarrow \mathbb{R}^3$ a submersion?
3. Prove that M is diffeomorphic to the product $S^2 \times \mathbb{R}$.
4. Does there exists a smooth vector field on M which is never vanishing?
(If yes, show an example. If not, justify your statement.)

Exercise A.26. Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0\} \subset \mathbb{R}^3$ and ω the 1-form on Ω defined by

$$\omega = \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx + z dz$$

Moreover let $i : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \Omega$ defined by

$$i(u, v) = \left(\frac{u}{\sqrt{u^2 + v^2}}, \frac{v}{\sqrt{u^2 + v^2}}, u^2 + v^2 \right)$$

1. Prove that ω is closed
2. For $I = [0, 2\pi]$ and $\gamma : I \rightarrow \mathbb{R}^2$ given by $\gamma(t) = (\cos t, \sin t)$ compute $\int_I (i \circ \gamma)^* \omega$
3. Show that $i^* \omega$ is not exact

Exercise A.27. Let $M = \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ be endowed with the Riemannian metric g such that the two vector fields $X_1 = \partial_x$ and $X_2 = x\partial_y$ define an orthonormal basis for g .

1. Compute the Riemannian metric g and the corresponding Riemannian volume form vol_g .
2. Compute the length of the following curves for $x_0 > 0$, $y_0 \in \mathbb{R}$ and $0 < a < b$

$$\gamma_1 : [a, b] \rightarrow \mathbb{R}, \quad \gamma_1(t) = (x_0, t),$$

$$\gamma_2 : [a, b] \rightarrow \mathbb{R}, \quad \gamma_2(t) = (t, y_0),$$

3. Recall the definition of divergence of a vector field X with respect to the Riemannian volume vol_g . Compute the divergence $\text{div}(X_1)$ and the divergence $\text{div}(X_2)$.
4. Is γ_2 a length-minimizer for the Riemannian metric g ?

Exercise A.28. Consider the map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$F(x, y, z) = (x + y^2, x + 2y^2 + z^2)$$

1. Prove that $M := F^{-1}(0, 1)$ is an embedded submanifold of $\mathbb{R}^3$.
2. Prove that M is diffeomorphic to S^1 .
3. Let $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the map $\pi(x, y, z) = (x, y)$. Is $\pi|_M : M \rightarrow \mathbb{R}^2$ an immersion?
4. Let $g : M \rightarrow \mathbb{R}$ be defined as $g(x, y, z) = x$ for every $(x, y, z) \in M$. Find critical points of g .

Exercise A.29. Consider the following vector fields in $\mathbb{R}^3$

$$X = \partial_x - y\partial_z, \quad Y = \partial_y + x\partial_z, \quad Z = \partial_z$$

1. For $i = 1, 2$ discuss whether D_i is a flat distribution.

$$D_1 = \text{span}\{X, Y\}, \quad D_2 = \text{span}\{X, Z\}.$$

2. For flat distributions, find an integral manifold through a generic point $(x_0, y_0, z_0) \in \mathbb{R}^3$.
3. Find a non zero differential 1-form ω such that $\omega(X) = \omega(Y) = 0$. Compute $L_Z\omega$ and $i_Z d\omega$.

Exercise A.30. Consider the 2-sphere $S^2 \subset \mathbb{R}^3$ endowed by the standard³ atlas $\mathcal{A} = \{(U_N, \varphi_N), (U_S, \varphi_S)\}$ and the Riemannian metric g_{S^2} induced by the Euclidean metric $\bar{g}$ of $\mathbb{R}^3$. Consider the inverse of the chart $\varphi_N^{-1} : \mathbb{R}^2 \rightarrow S^2 \setminus \{N\}$. After writing explicitly φ_N and φ_N^{-1}

1. Compute the Riemannian metric $g := (\varphi_N^{-1})^* g_{S^2}$ defined on $\mathbb{R}^2$

³here $U_P = S^2 \setminus \{P\}$ for $P \in \{N, S\}$ the north and south pole $N = (0, 0, 1)$, $S = (0, 0, -1)$, and $\varphi_P : U_P \rightarrow \mathbb{R}^2$ the corresponding stereographic projection from P on the plane $\{x_3 = 0\} \simeq \mathbb{R}^2$.

2. Compute the corresponding Riemannian volume vol_g associated with $(\mathbb{R}^2, g)$. Verify that

$$\text{vol}_g = 4 \frac{du \wedge dv}{(1 + u^2 + v^2)^2}$$

3. Compute the integral $\int_{\mathbb{R}^2} \text{vol}_g$. Comment your result.

Exercise A.31. Consider $S^1 \subset \mathbb{R}^2$ and the map $F : S^1 \times S^1 \rightarrow \mathbb{R}^2$ defined for every $(x, y) \in S^1 \times S^1$

$$F(x, y) = x + y$$

1. Prove that $S^1 \times S^1$ is a smooth oriented manifold of dimension 2.
2. Show that F is a smooth map.
3. Is F a submersion? If not, find all critical points of F .
4. Is $M = F^{-1}(S^1)$ a smooth manifold of $S^1 \times S^1$? Of which dimension? Describe M .

Exercise A.32. Consider the differential one form in $M = S^1 \times \mathbb{R}^2$ with coordinates (θ, x, y)

$$\omega = \cos \theta \, dx + \sin \theta \, dy$$

1. Find a basis X_1, X_2 of vector fields of the distribution $D := \ker \omega$. Is the distribution D flat?
2. Prove that $\omega \wedge d\omega$ is a volume form on M .
3. Show that there exists a unique vector field X_0 such that⁴ $i_{X_0}\omega = 1$ and $i_{X_0}(d\omega) = 0$.
4. For every $t \in \mathbb{R}$, compute the flow e^{tX_0} of X_0 , and show that $(e^{tX_0})^*\omega = \omega$.

Exercise A.33. Let us consider the unit open disk $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ in the plane $\mathbb{R}^2$ with the Riemannian metric

$$g = \frac{4}{(1 - x^2 - y^2)^2} (dx^2 + dy^2)$$

1. Find an orthonormal frame X_1, X_2 for the Riemannian metric g on D .
2. Compute the Riemannian volume form $\Omega := \text{vol}_g$ and then the Riemannian area $\int_D \Omega$ of D .
3. Prove that $M = \{(u, v, w) \in \mathbb{R}^3 \mid w = \sqrt{u^2 + v^2 + 1}\} \subset \mathbb{R}^3$ is a smooth manifold and given

$$F : \mathbb{R}^3 \setminus \{w = -1\} \rightarrow \mathbb{R}^2, \quad F(u, v, w) = \left(\frac{u}{1+w}, \frac{v}{1+w} \right)$$

prove that $\Phi := F|_M$ is a diffeomorphism between M and D . (*Hint: write Φ^{-1}*)

⁴here i_X denotes the interior product with respect to a vector field X

4. Compute the two form $F^*\Omega$ on $\mathbb{R}^3 \setminus \{w = -1\}$. Is $\Phi^*\Omega$ a volume form on M ?

Exercise A.34. In the real projective plane $\mathbb{P}^2(\mathbb{R})$ consider the subset M given by

$$M = \{[x_0, x_1, x_2] \in \mathbb{P}^2(\mathbb{R}) \mid x_0^2 + x_1^2 - x_2^2 = 0\}.$$

1. Recall the atlas defining the differentiable structure for the real projective plane $\mathbb{P}^2(\mathbb{R})$.
2. Is M an embedded submanifold of $\mathbb{P}^2(\mathbb{R})$? Of which dimension? Is M connected?
3. Prove that the following function $F : M \rightarrow \mathbb{R}$ is (well-defined) and smooth

$$F([x_0, x_1, x_2]) = \frac{x_0^2 - x_1^2 - x_2^2}{x_0^2 + x_1^2 + x_2^2}$$

4. Is F a submersion? If not find all critical points of F .

Exercise A.35. Let D be the distribution in $\mathbb{R}^3$ spanned by the vector fields X and Y

$$X = \frac{\partial}{\partial x} + \frac{2xz}{1+x^2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

1. Prove that D is flat and find a 1-form ω such that $D = \ker \omega$.
2. Compute explicitly the flow e^{tX} of the vector field X . Is it true that $e_*^{tX} Y = Y$?
3. Find an integral manifold S_1 for D passing through $(0, 0, 0)$ and give equation(s) for S_1 .
4. Find an integral manifold S_2 for D passing through $(1, 1, 1)$ and give equation(s) for S_2

Exercise A.36. Let us consider the unit sphere S^2 and its standard atlas $(U_N, \varphi_N), (U_S, \varphi_S)$ given by the stereographic projections⁵

1. Write explicitly the chart $\varphi_N : U_N \rightarrow \mathbb{R}^2$
2. Compute the Riemannian metric on U_N given by $g := (\varphi_N)^*\bar{g}$ where $\bar{g} = dx^2 + dy^2$ is the Euclidean metric on $\mathbb{R}^2$.
3. Prove that there exists $\psi \in C^\infty(U_N)$ such that $g(v, w) = \psi \langle v, w \rangle_{\mathbb{R}^3}$, where $\langle v, w \rangle_{\mathbb{R}^3}$ is the inner product of $\mathbb{R}^3$. Compute ψ . Can ψ be extended to a smooth function on S^2 ?
4. Compute the length with respect to g of parallels of $S^2 \setminus \{N\}$ (i.e., intersections of S^2 with the plane $z = z_0$, for $-1 \leq z_0 < 1$).

⁵recall that $U_P = S^2 \setminus \{P\}$ for $P = N, S$ the north and south poles $N = (0, 0, 1), S = (0, 0, -1)$

Exercise A.37. Let $R > r > 0$. Let us consider the subset of $\mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}.$$

1. Prove that S is an embedded smooth manifold, that is orientable.
2. Consider $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $G(x, y, z) = x$. Is $G|_S$ a submersion? If not, compute its critical points.
3. Consider $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by $F(x, y, z) = (x + y, x - y)$. Is $F|_S$ a submersion? If not, compute its critical points.
4. Compute the Riemannian metric $g := i^*\bar{g}$ where $\bar{g}$ is the Euclidean metric on $\mathbb{R}^3$ and i is the inclusion $i : S \rightarrow \mathbb{R}^3$. (*Hint: introduce suitable angular coordinates on S*)
5. Compute the Riemannian volume form $\Omega = \text{vol}_g$ and compute $\int_S \Omega$. Is Ω exact?

Exercise A.38. Consider the sphere S^2 embedded in $\mathbb{R}^3$ with coordinates (x, y, z) in the usual way. Let $\varphi_N : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ denote stereographic projection from the north pole, and let X_i be the vector fields in $\mathbb{R}^2$ with coordinates (u, v)

$$X_1 = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, \quad X_2 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}$$

1. Compute explicitly the flow e^{tX_i} of the vector fields X_i in $\mathbb{R}^2$, for $i = 1, 2$. Compute $[X_1, X_2]$.
2. Consider $Y_i = (\varphi_N^{-1})_* X_i$ vector field on $S^2 \setminus \{N\}$, for $i = 1, 2$. Compute Y_1 and check that

$$Y_1 = -xz \frac{\partial}{\partial x} - yz \frac{\partial}{\partial y} + (1 - z^2) \frac{\partial}{\partial z}$$

3. Show that Y_1 coincides with the orthogonal projection of the vector field $Z = \frac{\partial}{\partial z}$ in $\mathbb{R}^3$ to the tangent space to $S^2 \setminus \{N\}$. Describe the flow e^{tY_1} of the vector field Y_1 in S^2 .
4. Compute Y_2 . Describe the flow e^{tY_2} of the vector field Y_2 in S^2 .
5. Compute the Lie bracket $[Y_1, Y_2]$. Discuss the result.

Exercise A.39. Consider $\mathbb{R}^3$ with coordinates x, y, z and $\mathbb{R}^2$ with coordinates u, v . Consider the map

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad F(x, y, z) = (x^2y + e^z, yz - x)$$

and fix $p = (1, 1, 1)$ in $\mathbb{R}^3$.

1. Compute $F_* \left(\frac{\partial}{\partial x} \Big|_p \right)$. Compute $F^*(du|_{F(p)})$.
2. Compute the matrix representing $F_*|_p : T_p\mathbb{R}^3 \rightarrow T_{F(p)}\mathbb{R}^2$ with respect to a suitably chosen basis. Is F a submersion at p ?

3. Discuss if $F^{-1}(0, 0)$ is a smooth submanifold of $\mathbb{R}^3$. If yes: of which dimension? is it bounded?
4. Consider the 1-dim submanifold of $\mathbb{R}^3$ given by $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 4y^2 = 1, z = 0\}$. Is the map $F|_C : C \rightarrow \mathbb{R}^2$ an immersion?
5. Given the differential 2-form $\Omega = du \wedge dv$ in $\mathbb{R}^2$, compute $\omega := F^*\Omega$. Then compute $\int_Q \omega$, where $Q =]0, 1[\times]0, 1[\times \{0\} \subset \mathbb{R}^3$.

Exercise A.40. Consider the following set of matrices

$$\mathcal{G} := \left\{ A = \begin{pmatrix} a & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a > 0, b, c \in \mathbb{R} \right\}$$

and denote by $GL_n(\mathbb{R})$ the set of $n \times n$ invertible matrices with real entries.

1. Prove that $GL_3(\mathbb{R})$ is a smooth manifold. Of which dimension?
2. Prove that $\mathcal{G}$ is a subgroup of $GL_3(\mathbb{R})$. Prove that $\mathcal{G}$ is a smooth manifold of dimension 3. (*Hint: write the group law in coordinates.*)
3. Let I be the identity matrix. Compute the tangent space $T_I \mathcal{G}$.
4. Write a set X_1, X_2, X_3 of linearly independent left-invariant vector fields on $\mathcal{G}$.
5. Compute $[X_i, X_j]$ for $i, j = 1, 2, 3$.

N.B. Questions 3. to 5. can be answered either in coordinates or in matrix notations.

Exercise A.41. Consider $\mathbb{R}^3$ with coordinates x, y, z , and the sets

$$S = \{(x, y, z) \mid x^2 + y^2 = 2z^2, z > 0\}, \quad S' = \{(x, y, z) \mid z = xy\}$$

Consider $F : S \rightarrow S'$ defined by $F(x, y, z) = (x, yz, xyz)$, for every $(x, y, z) \in S$.

1. Are S and S' embedded submanifold of $\mathbb{R}^3$? Define an atlas for both S and S' .
2. Prove that F is smooth using the atlas introduced above.
3. Let $p = (1, 1, 1) \in S$. Compute $F_*|_p : T_p S \rightarrow T_{F(p)} S'$. Is F a submersion at p ? Is F a global diffeomorphism?
4. Let $i : S' \rightarrow \mathbb{R}^3$ be the canonical inclusion. Compute $g := i^* \bar{g}$ the induced Riemannian metric on S' , where $\bar{g}$ is the Euclidean metric in $\mathbb{R}^3$.
5. Find a basis of orthonormal vector fields X, Y for S' with respect to g .

Exercise A.42. Prove or disprove the following formula for the Lie derivative:

$$L_f X \omega = f L_X \omega + df \wedge i_X \omega$$

where X is a vector field, f a smooth function and ω a differential k -form.

Exercise A.43. Consider the vector fields

$$X = \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}$$

1. Given a smooth submersion $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, prove that X is tangent to $S = f^{-1}(0)$ if and only if $Xf = 0$ at every point of S .
2. Is the distribution $D = \text{span}\{X, Y\}$ a flat distribution ?
3. Does there exists a smooth submersion $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial z} = 0 \\ \frac{\partial f}{\partial y} + x \frac{\partial f}{\partial z} = 0 \end{cases}$$

Exercise A.44. Consider $S^1 \subset \mathbb{R}^2$ and the map $F : S^1 \times \mathbb{R} \rightarrow S^2$ defined for every $(x_1, x_2, y) \in S^1 \times \mathbb{R}$

$$F(x_1, x_2, y) = \left(\frac{x_1}{\sqrt{1+y^2}}, \frac{x_2}{\sqrt{1+y^2}}, \frac{y}{\sqrt{1+y^2}} \right)$$

where S^2 is also regarded as a subspace of $\mathbb{R}^3$. Fix $p = (1, 0, \sqrt{3}) \in S^1 \times \mathbb{R}$.

1. Compute F_* at p with respect to a basis of the tangent spaces, which you will specify.
2. Is F a submersion? Is F a local diffeomorphism? Is F a global diffeomorphism?
3. Let C be a great circle of S^2 . Is $F^{-1}(C)$ a submanifold? Describe it (dimension, topology).
4. Let $g = F^* \bar{g}_{S^2}$ where $\bar{g}_{S^2}$ is the standard Riemannian metric on S^2 induced by $\mathbb{R}^3$. Compute the length of the curve $\{y = y_0\}$ on $S^1 \times \mathbb{R}$ with respect to g .

Exercise A.45. Consider three linearly independent differential 1-forms on a smooth manifold M , $\dim M = 3$, satisfying

$$d\omega_1 = \omega_2 \wedge \omega_3, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2 \quad (\text{A.3})$$

First assume $M = \mathbb{R}^3$ and $\omega_1 = dx$, $\omega_2 = dy - xdz$.

1. Find all ω_3 , if any, satisfying the above conditions (A.5).

Now the general case. Let X_1, X_2, X_3 be the dual basis of vector fields to $\omega_1, \omega_2, \omega_3$.

2. Compute $L_{X_j}\omega_3$, for $j = 1, 2, 3$, where L_X denotes the Lie derivative.
3. Compute $[X_i, X_j]$ for every $i, j = 1, 2, 3$.

Exercise A.46. Let $\gamma : [0, T] \rightarrow M$ be a smooth regular curve (i.e., $\dot{\gamma}(t) \neq 0$ for all $t \in [0, T]$) and let $X, Y \in \text{Vec}(M)$ be linearly independent vector fields. Consider $\eta(t) = e^{tX}(\gamma(t))$.

1. Write a formula to compute $\dot{\eta}(t)$, for every $t \in [0, T]$, in terms of X and γ .
2. Compute explicitly in $M = \mathbb{R}^3$ the curve η for $\gamma(t) = (t, 0, t)$ and $X = x\partial_y - y\partial_x$. Check the validity of the formula given in 1.
3. Consider the distribution $D = \text{span}\{X, Y\}$, and assume $\dot{\gamma}(t) \in D_{\gamma(t)}$ for every $t \in [0, T]$. Is it true that $\dot{\eta}(t) \in D_{\eta(t)}$ for every $t \in [0, T]$? Discuss your answer.

Exercise A.47. Let us consider the following subsets of $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$

$$S_1 = \{(x, [x]) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) \mid x \neq 0\}$$

$$S_2 = S_1 \cup (\{0\} \times \mathbb{P}^1(\mathbb{R}))$$

where we denote points as follows $x = (x_1, x_2) \in \mathbb{R}^2$ and $[y] = [y_1, y_2] \in \mathbb{P}^1(\mathbb{R})$.

1. Show that S_1 is a smooth embedded submanifold of $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$. Of which dimension?
2. Is S_2 a smooth embedded submanifold of $\mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R})$. If yes, of which dimension?
(Hint: find equations describing S_1 and ...)
3. Consider the two projections onto the two factors

$$p_1 : S_2 \rightarrow \mathbb{R}^2, \quad p_2 : S_2 \rightarrow \mathbb{P}^1(\mathbb{R}).$$

defined by $p_1(x, [y]) = x$ and $p_2(x, [y]) = [y]$. Are p_1, p_2 submersions?

Exercise A.48. Consider $M = \{(x, y, z) \in \mathbb{R}^3 \mid 1 + x^2 = y^2 + z^2\}$ and the 2-form in $\mathbb{R}^3$

$$\omega = dz \wedge dy + (z + y)(dx \wedge dy + dx \wedge dz)$$

1. Find a basis of $T_p M$ for every $p \in M$.
2. Find a never vanishing vector field X such that $i_X \omega = 0$
3. Is ω a volume form on M ?
4. Compute $\int_M \omega$

Exercise A.49. Consider in the Euclidean $(\mathbb{R}^3, \bar{g})$ the Levi Civita connection denoted $\bar{\nabla}$. Define

$$\nabla_X Y := \bar{\nabla}_X Y + X \times Y$$

where $\times$ denotes the standard cross product of vectors in $\mathbb{R}^3$.

1. Prove that $\nabla_X Y$ defines a (linear) connection.
2. Compute its Christoffel symbols Γ_{ij}^k with respect to the standard basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$
3. Prove that if $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, then γ is a straight line.

Exercise A.50. Let us consider a nonzero *column* vector $x \in \mathbb{R}^3 \setminus \{0\}$. Define the 3×3 matrix

$$A_x = \frac{1}{\|x\|^2} x x^T.$$

0. Prove that A_x is the matrix representing the orthogonal projection onto the direction of x .

Consider the map onto the 3×3 symmetric matrices $\text{Sym}_3(\mathbb{R})$

$$F : \mathbb{R}^3 \setminus \{0\} \rightarrow \text{Sym}_3(\mathbb{R}), \quad F(x) = A_x.$$

1. Justify the fact that F is smooth.
2. Compute the differential of F . Can we interpret $DF(x)v$ as an orthogonal projection?
3. Is F an immersion? Is F a submersion?
4. Justify that F descends to a map defined on the projective plane

$$\bar{F} : P^2(\mathbb{R}) \rightarrow \text{Sym}_3(\mathbb{R}), \quad \bar{F}([x]) = A_x.$$

5. Write $\bar{F}$ in coordinates (*one chart is ok*). Is $\bar{F}$ an immersion?

Exercise A.51. Let M be a smooth compact n -dimensional manifold with boundary. Consider two differential forms $\omega \in \Omega^{p-1}(M)$ and $\tau \in \Omega^{n-p}$ for some $1 \leq p \leq n$. Consider the following formula:

$$\int_M d\omega \wedge \tau = \int_{\partial M} \omega \wedge \tau + (-1)^p \int_M \omega \wedge d\tau \quad (\text{A.4})$$

1. Compute the left and right hand side of (A.4) for $n = 2$, $M = \{x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ and $p = 1$ with the choice $\omega = x^2 + y^2 \in \Omega^0(M)$ and $\tau = xdy - ydx \in \Omega^1(M)$. Is the formula true?
2. Prove formula (A.4) for general n and $p = 1$.
3. Prove formula (A.4) for general n and p .
4. If ω, τ are exact forms, is it true that $\omega \wedge \tau$ is exact?

5. What is the meaning of the formula for $M = [0, 1] \subset \mathbb{R}$ and $n = p = 1$?

Exercise A.52. Consider the sphere S^2 as a subset of $\mathbb{R}^3$ endowed with the standard metric g_{S^2} induced on S^2 by $\mathbb{R}^3$. Consider the map

$$F : D \subset \mathbb{R}^2 \rightarrow S^2, \quad F(u, v) = (u, v, \sqrt{1 - u^2 - v^2})$$

where $D = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ is the unit disk in $\mathbb{R}^2$.

1. Is F an immersion, submersion, local/global diffeomorphism? Discuss.
2. Compute F^*g_S the Riemannian metric defined on the disk D .
3. Compute the Riemannian volume form $\omega := \text{vol}_D$ with respect to F^*g_S
4. Compute $\int_D \omega$ and comment your result.

Exercise A.53. Let X be a vector field in $\mathbb{R}^2$ with coordinates (x, y) .

1. Find all possible such vector fields X satisfying the following conditions (here $[\cdot, \cdot]$ denotes the Lie bracket)
 - a. $[\frac{\partial}{\partial x}, X] = X$ and $[\frac{\partial}{\partial y}, X] = X$
 - b. $[\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, X] = X$
2. For vector fields satisfying condition 1.a. then compute $e_*^{t\frac{\partial}{\partial x}} X$

Hint. It could be useful for point 2. to consider a suitable change of coordinates.

Exercise A.54. Consider the set S of oriented segments of length 1 in the Euclidean plane $\mathbb{R}^2$.

1. Can S be endowed with a smooth manifold structure? Of which dimension?
2. Give an example of a submersion $F : S \rightarrow \mathbb{R}^2$, if it exists.

Exercise A.55. Let $n \geq 2$. Consider the map $F : GL(n) \rightarrow \mathbb{R}^2$ given by

$$F(A) = (\text{tr}(A), \det(A))$$

1. Prove that $GL(n)$ is a smooth manifold. Is the map F smooth?
2. Let first $n = 2$. Prove that $F^{-1}(0, 1)$ is a smooth manifold. Of which dimension?
3. Let now $n \geq 2$. Compute the differential of F_* at every point $A \in GL(n)$.

4. Is $(0, 1) \in \mathbb{R}^2$ is a regular value for F ?

Consider now $O(n)$ as a smooth submanifold of $GL(n)$

5. Describe the smooth structure of $O(n)$. Is $\Phi = F|_{O(n)} : O(n) \rightarrow \mathbb{R}^2$ smooth?
6. Is $(0, 1) \in \mathbb{R}^2$ is a regular value for Φ ? Highlight the differences with respect to questions 2.-4.

Exercise A.56. Consider the three vector fields on $\mathbb{R}^3$

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z} \quad (\text{A.5})$$

Recall the definition of Lie derivative $\mathcal{L}_X Y = \frac{d}{dt} \Big|_{t=0} e^{-tX} Y$; call this formula (*).

1. Compute the Lie bracket $[X_1, X_2]$, explain your computation.
2. Compute explicitly $\mathcal{L}_{X_1} X_2$ through formula (*) and check that it is equal to the previous result
3. Find explicitly $\omega_1, \omega_2, \omega_3$ the dual basis of 1-forms with respect to the basis X_1, X_2, X_3
4. Compute $d\omega_i$ for $i = 1, 2, 3$, and express them as linear combinations of $\omega_j \wedge \omega_k$.
5. Compute the Riemannian metric g on $\mathbb{R}^3$ such that X_1, X_2, X_3 are orthonormal.
6. Compute the Lie derivative $\mathcal{L}_{X_3} g$ where g is the $(2, 0)$ tensor obtained in question 5.

Exercise A.57. Consider the projection map from the sphere S^2 to the projective plane $\mathbb{P}^2(\mathbb{R})$

$$\pi : S^2 \rightarrow \mathbb{P}^2(\mathbb{R}), \quad \pi(x, y, z) = [x, y, z]$$

Also, denote by g_0 the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$ on the S^2 .

1. Describe the standard atlas of $\mathbb{P}^2(\mathbb{R})$.
2. Fixing also an atlas of S^2 , write the map π in coordinates. Compute its differential.
3. Is the map π is a local diffeomorphism?
4. Let $i : S^2 \rightarrow S^2$ the antipodal map $i(x) = -x$. Prove that $i^* g_0 = g_0$.

Exercise A.58. Consider in $\mathbb{R}^3$ with coordinates (x, y, z) a smooth function $f = f(x, y) \in C^\infty(\mathbb{R}^2)$ a smooth function and the two vector fields

$$X = \frac{\partial}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial z}$$

1. Prove that $D = \text{span}\{X, Y\}$ is an involutive distribution

2. Compute the flow e^{tX} and e^{tY} of the vector fields. Check that $e^{tX} \circ e^{sY} = e^{sY} \circ e^{tX}$.
3. Find an integral submanifold $S \subset \mathbb{R}^3$ of D passing through the point $(0, 0, 1)$.
4. Find an integral submanifold $S \subset \mathbb{R}^3$ of D passing through an arbitrary point (x_0, y_0, z_0) . Discuss whether the equation defining S is independent on the point.

Exercise A.59. Let M be a smooth n -dimensional manifold, ω be a volume form and X, Y be vector fields. Prove or disprove the following formula:

$$\operatorname{div}_\omega[X, Y] = X(\operatorname{div}_\omega Y) - Y(\operatorname{div}_\omega X)$$

(Recall the definition of divergence of a vector field with respect to a volume form: $\operatorname{div}_\omega X$ is the function f such that $L_X \omega = f\omega$, where L_X denotes the Lie derivative.)

Exercise A.60. Let $n \in \mathbb{N}, n \geq 2$. Consider the map $F : SO(n) \times SO(n) \rightarrow SO(n)$ given by

$$F(A, B) = AB^{-1}.$$

1. What is the dimension of $SO(n) \times SO(n)$? Is the map F smooth?
2. Compute the tangent space $T_A SO(n)$ for $A \in SO(n)$
3. Show that (or justify) $T_{(A, B)}(SO(n) \times SO(n)) \simeq T_A SO(n) \oplus T_B SO(n)$
4. Compute the differential of $F_{*, (A, B)}$ (*Hint: one might start from $(A, B) = (I, I)$*)
5. Is F an immersion? Is F a submersion?
6. Compute $\ker F_*$ at every point, and in particular compute its dimension.

Exercise A.61. Let $c \in \mathbb{R}$ and consider the subset of $\mathbb{R}^3$ defined by

$$M_c = \{(x, y, z) \in \mathbb{R}^3 \mid x^3 + cy^2 - 2cxz = 3c\}$$

1. For which values of $c \in \mathbb{R}$ the set M_c is an embedded smooth submanifold of $\mathbb{R}^3$?
2. For $c = 1$ find a basis of the tangent space to M_1 at every point.
3. Is it true that on the space $\mathbb{R}^3 \setminus \{x \neq 0\}$ the sets M_c defines a foliation of surfaces that are tangent to a distribution D ?
4. If yes, find a basis of vector fields X_1, X_2 generating the distribution.
5. Compute the Lie bracket $[X_1, X_2]$ and discuss the result.

Exercise A.62. We identify $\mathbb{R}^4 \simeq \mathbb{C}^2$ and write points (x_1, x_2, x_3, x_4) of $\mathbb{R}^4$ as (z_1, z_2) with the identifications $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. Consider the following subsets of $\mathbb{R}^4$

$$S = \{|z_1|^2 + |z_2|^2 = 1\}, \quad M = \{z_1 + z_2^2 = 0\}$$

1. Are S and M smooth submanifolds of $\mathbb{R}^4$?
2. Prove that for every point $q \in S \cap M$ one has

$$\dim(T_q S + T_q M) = 4$$

3. Prove that $S \cap M$ is a smooth submanifold. Of which dimension?
4. Consider now two smooth submersions $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $G : \mathbb{R}^n \rightarrow \mathbb{R}^k$ and denote by $S = F^{-1}(0)$ and $M = G^{-1}(0)$. Assume that for every point $q \in S \cap M$ one has

$$\dim(T_q S + T_q M) = n$$

Is it true that $S \cap M$ is a smooth submanifold? If yes, of which dimension?

Exercise A.63. (a). Given in $\mathbb{R}^n$ two vector fields X, Y and a differential 1-form ω in coordinates

$$X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=1}^n Y_i(x) \frac{\partial}{\partial x_i}, \quad \omega = \sum_{j=1}^n \omega_j(x) dx_j$$

compute the coordinate expression of the following quantities : $L_X \omega$ and $i_{[X,Y]} \omega$.

(b). Prove or disprove the validity of the following identity on differential k -forms ω for any $0 \leq k \leq n$

$$L_X i_Y \omega - i_Y L_X \omega = i_{[X,Y]} \omega$$

(which can be understood as operator identity $[L_X, i_Y] = i_{[X,Y]}$).

Exercise A.64. Let $Z \subset \mathbb{R}^n$ be a 1-dimensional submanifold.

1. Explain why looking at the tangent space to points in Z defines a function:

$$F : Z \rightarrow \mathbb{RP}^{n-1}$$

2. Prove that this function F is smooth.

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