Atomicity, coherence of information, and point-free structures

Basil A. Karádais
Mathematisches Institut, Ludwig-Maximilians-Universität
Theresienstraße 39, 80333 München Germany
karadais@math.lmu.de

Abstract
We prove basic facts about the properties of atomicity and coherence for Scott information systems, and we establish direct connections between coherent information systems and well-known point-free structures.

1 Introduction
Domain theory has been a well-established branch of mathematics for several years now, one that exhibits a wide array of applications, as one sees for example in [4]. In particular it bears great significance regarding the denotational semantics of functional programming, which was historically one of the reasons that the theory emerged in the first place.

In Dana Scott’s [20], Scott domains—that is, pointed directed-complete algebraic cpo’s—are represented by information systems. These are supposed to structure atomic tokens of information according to their consistency and entailment; the latter models deduction of information and preorders the carrier, so that the actual objects of the domain are then recovered as ideals. Scott’s information systems have served as a natural approach to the domain-theoretic treatment of semantics, at least from the computer scientists’ viewpoint, since they provide the means to discuss higher-type algorithms in a tangible way, namely in terms of their finite approximations, that is, their tokens and the finite consistent sets of tokens that they consist of. In the context of information systems, the principle of finite support for computation finds one of its uttermost formulations: an algorithm is but a consistent and deductively closed collection of concrete, finite pieces of information. But there are also theoretical merits, as information systems are more basic than the domains they induce: properties of ideals reduce to properties of tokens and consistent sets, thus providing the possibility of elementary methods of argumentation, a prime example being the solution of domain equations up to identity (see [20][25] and also [24]), rather than up to isomorphism, as previous category-theoretic arguments could already show (see [21]).

Favoring the tangible nature of information systems—and more particularly of non-flat and coherent information systems that free algebras given by constructors induce—rather than favoring the more abstract mode of the extensively studied domains, is tied to the development of a constructive formal theory of partial computable functionals, one that should lend itself as naturally and intuitively as possible to an implementation in a proof assistant (for more details see [6]). This objective demands a solid understanding of the model, which in turns needs an in-depth
study of information systems in their own sake: this would lead to a bottom-up, constructive, and implementable redevelopment of domain theory for higher-type computability. Helmut Schwichtenberg started this redevelopment in [15] by introducing a cartesian-closed class of information systems which feature coherence as well as atomicity. The author pursued the issue further in his Ph.D. thesis [8]. Here we nevertheless retain a top-down approach and present results from [8] which directly relate coherent information systems to point-free structures that have been put to successful use in the past by the community. The work has a rather cartographic flavor which we deem necessary in order to clearly understand the nature of information systems used in practice from a point-free viewpoint.

**Preview**

In section 2 we recall basic facts and observations concerning information systems. We introduce the notions of atomicity and coherence and we show that atomic and coherent versions of information systems feature more ideals than the generic version. In section 3 we consider well-known point-free structures, namely domains, precusl’s and formal topologies, and impose appropriate coherence properties on them to show that they correspond to coherent information systems. In section 4 we gather some relevant notes and an outlook on future work.

**2 Scott information systems**

A (Scott) information system is a triple $\rho = (\text{Tok}_\rho, \text{Con}_\rho, \vdash_\rho)$ where $\text{Tok}_\rho$ is a countable set of tokens, $\text{Con}_\rho \subseteq \mathcal{P}(\text{Tok}_\rho)$ is a collection of consistent sets, also called (formal) neighborhoods and $\vdash_\rho \subseteq \text{Con}_\rho \times \text{Tok}_\rho$ is an entailment relation, such that consistency is reflexive and closed under subsets; entailment is reflexive and transitive; consistency propagates through entailment (we drop the subscripts when we can afford it):

$$\begin{align*}
\{a\} &\in \text{Con}, \\
U \in \text{Con} \land V \subseteq U &\rightarrow V \in \text{Con}, \\
a \in U &\rightarrow U \vdash a, \\
U \vdash V \land V \vdash c &\rightarrow U \vdash c, \\
U \in \text{Con} \land U \vdash b &\rightarrow U \cup \{b\} \in \text{Con} ;
\end{align*}$$

here $U \vdash V$ is a shorthand for $\forall b \in V. \ U \vdash b$. Among the properties that follow directly from the definition we have the following.

$$U \vdash V \land U \vdash V' \rightarrow U \vdash V \cup V'. \quad (1)$$

An ideal in $\rho$ is a set $u \subseteq \text{Tok}$ which is consistent and closed under entailment in the sense that

$$U \in \text{Con} \land (U \vdash a \rightarrow a \in u),$$

for all $U \subseteq \text{Tok}$ and $a \in \text{Tok}$. Denote the empty ideal by $\bot$ and the collection of all ideals of $\rho$ by $\text{Id}_{\text{Tok}}$. Define the deductive closure of a neighborhood $U \in \text{Con}_\rho$ by

$$\text{CL}_{\rho}(U) := \{a \in \text{Tok} \mid U \vdash_\rho a\} .$$

When $\rho$ is clear from the context we just write $U$. Write $\overline{\text{Con}}_\rho$ for the collection of all such closures. It is clear that $U \in \text{Id}_{\text{Tok}}$ for any $U \in \text{Con}_\rho$. Write $U \sim_\rho V$ if both $U \vdash_\rho V$ and $V \vdash_\rho U$; the following holds.

**Lemma 1.** It is $U \sim V$ if and only if $U = V$. 

2
Proof. The right direction follows from transitivity and the left one from reflexivity and transitivity of entailment. □

2.1 Two examples

Consider the strings over the alphabet \{l, r, m\}, the empty one denoted by \(\epsilon\). Define \(\mathcal{L}\) by \(\text{Tok}_\mathcal{L} := \{\epsilon, l, r\}\), \(\text{Con}_\mathcal{L} := \mathcal{P}(\text{Tok}_\mathcal{L})\), and

\[
\begin{align*}
\{l\} \vdash_\mathcal{L} l, & \quad \{r\} \vdash_\mathcal{L} r, & \quad \{\epsilon\} \vdash_\mathcal{L} \epsilon, \\
\{l, r\} \vdash_\mathcal{L} l, r, \epsilon, & \quad \{r, \epsilon\} \vdash_\mathcal{L} r, \epsilon, & \quad \{l, \epsilon\} \vdash_\mathcal{L} l, \epsilon, \\
\{l, r, \epsilon\} & \vdash_\mathcal{L} l, r, \epsilon.
\end{align*}
\]

Understanding that \(\{l, r\} \vdash_\mathcal{L} \epsilon\) is the only nontrivial entailment at hand, it is direct to see that \(\text{Id}_\mathcal{L} = \mathcal{P}(\text{Tok}_\mathcal{L}) \setminus \{l, r\}\).

Further define \(\mathcal{L}'\) by \(\text{Tok}_{\mathcal{L}'} := \{\epsilon, l, m, r, lm, lr, mr\}, U \in \text{Con}_{\mathcal{L}'}\), \(U \vdash_{\mathcal{L}'} b\) when there is a token \(a \in \text{Tok}_{\mathcal{L}'}\) which is a superword of all \(b \in U\), and \(U \vdash_{\mathcal{L}'} b\) when there exists an \(a \in U\) which is a superword of \(b\). Its ideals, \(\text{Id}_{\mathcal{L}'}\), are the following:

\[
\begin{align*}
\perp, & \quad \{\epsilon\}, \\
\{l, \epsilon\}, & \quad \{m, \epsilon\}, & \quad \{r, \epsilon\}, \\
\{l, m, \epsilon\}, & \quad \{l, r, \epsilon\}, & \quad \{m, r, \epsilon\}, \\
\{lm, l, m, \epsilon\}, & \quad \{lr, l, r, \epsilon\}, & \quad \{mr, m, r, \epsilon\}.
\end{align*}
\]

2.2 Function spaces and approximable maps

Let \(\rho\) and \(\sigma\) be information systems. Define their function space \(\rho \to \sigma\) by the following.

- If \(U \in \text{Con}_\rho\) and \(b \in \text{Tok}_\sigma\) then \(\langle U, b \rangle \in \text{Tok}_{\rho \to \sigma}\).
- Let \(U_1, \ldots, U_l \in \text{Con}_\rho\), \(b_1, \ldots, b_l \in \text{Tok}_\sigma\), and \(J := \{1, \ldots, l\}\); if for all \(I \subseteq J, \bigcup_{i \in I} U_i \in \text{Con}_\rho\) implies \(\bigcup_{i \in J} \{b_i\} \in \text{Con}_\sigma\), then \(\{\langle U_j, b_j \rangle \mid j \in J\} \in \text{Con}_{\rho \to \sigma}\).
- Let \(U_1, \ldots, U_l, U \in \text{Con}_\rho\), \(b_1, \ldots, b_l, b \in \text{Tok}_\sigma\), and \(J := \{1, \ldots, l\}\); if for some \(I \subseteq J\), it is both \(U \vdash_{\rho} U_i\) for all \(i \in I\) and \(\langle b_i \mid i \in I \rangle \vdash_{\sigma} b\), then \(\langle \langle U_j, b_j \rangle \mid j \in J\rangle \vdash_{\rho \to \sigma} \langle U, b \rangle\).

The definition of entailment can be formulated in terms of application between neighborhoods: \(\langle \langle U_1, b_1 \rangle, \ldots, U_l, b_l \rangle \rangle : U := \{b_i \mid U \vdash_{\rho} U_i, i \in J\}; \) so

- if \(\langle \langle U_1, b_1 \rangle, \ldots, U_l, b_l \rangle \rangle : U \vdash_{\sigma} b\), then \(\langle \langle U_1, b_1 \rangle, \ldots, U_l, b_l \rangle \rangle \vdash_{\rho \to \sigma} \langle U, b \rangle\).
These make \( \rho \to \sigma \) an information system.

A relation \( r \subseteq \text{Con}_\rho \times \text{Tok}_\sigma \) is called an \textit{approximable map} from \( \rho \) to \( \sigma \) if it is consistently defined and deductively closed:

\[
\forall_{b \in V} (U, b) \in r \Rightarrow V \in \text{Con}_\sigma ,
\]
\[
U' \slant \rho U \land \forall_{b \in V} (U, b) \in r \land V \vdash_\sigma b' \Rightarrow (U', b') \in r .
\]

Write \( \text{Apx}_{\rho \to \sigma} \) for these relations. Approximable maps provide an alternative description of ideals in a function space; in particular, it is \( \text{Apx}_{\rho \to \sigma} = \text{Ide}_{\rho \to \sigma} \).

As already remarked by Scott in [20, §4.1] (where actually the converse route was taken), any approximable map \( r \) from \( \rho \) to \( \sigma \) induces a relation \( \hat{r} \subseteq \text{Con}_\rho \times \text{Con}_\sigma \) by letting \( (U, V) \in \hat{r} \) if and only if \( (U, b) \in r \) for all \( b \in V \).

\textbf{Fact 2.} Let \( \hat{r} \) be an approximable map from \( \rho \) to \( \sigma \). For the relation \( \hat{r} \) the following hold:

\[
(\emptyset, \emptyset) \in \hat{r} ,
\]
\[
(U, V) \in \hat{r} \land (U, V') \in \hat{r} \Rightarrow (U, V \cup V') \in \hat{r} ,
\]
\[
U' \slant \rho U \land (U, V) \in \hat{r} \Rightarrow V' \vdash_\sigma b' \Rightarrow (U', b') \in \hat{r} .
\]

Conversely, if \( R \subseteq \text{Con}_\rho \times \text{Con}_\sigma \) satisfies the above, then the relation \( \hat{R} \subseteq \text{Con}_\rho \times \text{Tok}_\sigma \) defined by

\[
(U, b) \in \hat{R} : = \exists_{V \in \text{Con}_\sigma} (b \in V \land (U, V) \in R)
\]

is an approximable map from \( \rho \) to \( \sigma \).

In what follows we will identify \( r \) with \( \hat{r} \) and \( R \) with \( \hat{R} \).

\section{2.3 Atomicity and coherence of information}

We turn to a short discussion of two basic properties that an information system may have. Let \( \rho = (\text{Tok}_\rho, \text{Con}_\rho, \vdash_\rho) \) be an arbitrary information system. Define its \textit{atomic entailment} by \( U \vdash^A_\rho b \) when \( \{a\} \vdash_\rho b \), for some \( a \in U \); define its \textit{coherent neighborhoods} by \( U \in H\text{Con}_\rho \), when \( \{a, a'\} \subseteq \text{Con}_\rho \), for all \( a, a' \in U \). Further, define the coherent entailment \( U \vdash^H_\rho b \) inductively, by

\[
U \vdash^H_\rho b \lor b \in U \lor \exists_{V \in H\text{Con}_\rho} \left( U \vdash^H_\rho V \land V \vdash^H_\rho b \right) .
\]

Write \( Ap \) and \( Hp \) for the triples \( (\text{Tok}_\rho, \text{Con}_\rho, \vdash^A_\rho) \) and \( (\text{Tok}_\rho, H\text{Con}_\rho, \vdash^H_\rho) \) respectively. If \( Ap = \rho \), call \( \rho \) \textit{atomic}, and if \( Hp = \rho \), call it \textit{coherent}; the system \( \mathcal{L} \) in [22] is atomic but incoherent, since \( \{l, m\}, \{l, r\}, \{m, r\} \in \text{Con}_\mathcal{L} \) but \( \{l, m, r\} \notin \text{Con}_\mathcal{L} \), whereas \( \mathcal{E} \) is coherent but nonatomic because of \( \{l, r\} \vdash^\mathcal{E} e \).

These two notions have proven important for program semantics in several relevant works. Atomic-coherent information systems were used in [18], where various fundamental results were established, like density, preservation of values, and adequacy, while in [6] definability was established as well. A version of atomic information systems is also used in [3] to model intuitionistic linear logic. For the general setting of [19, Chapter 6], atomic-coherent systems turn out to be particularly appropriate when modeling type systems over data types like the natural or the boolean numbers, whose tokens are built by at most unary constructors (see [8, §1.4]), while in general it suffices to work within coherent systems.

Back on an abstract level, there are some direct observations to make, like that it is in general \( \vdash^A_\rho \subseteq \vdash_\rho \) and \( \text{Con}_\rho \subseteq H\text{Con}_\rho \), or that for any \( U \in \text{Con}_\rho \) it
is \( \text{CL}_{\text{Ap}}(U) \subseteq \text{CL}_{\text{H}}(U) \) as well as \( \text{CL}_{\text{M}}(U) = \text{CL}_{\text{H}}(U) \). The important and most basic observation is the following.

**Proposition 3.** Let \( \rho \) be an information system. The triples \( \text{Ap} \) and \( \text{H} \rho \) are both information systems. Furthermore, let \( \sigma \) be an information system; if \( \sigma \) is atomic then \( \rho \to \sigma \) is also atomic; if \( \sigma \) is coherent, then \( \rho \to \sigma \) is also coherent.

**Proof.** In order to show that \( \text{Ap} \) is an information system we have to check the laws of the definition concerning entailment. For reflexivity, if \( U \in \text{Con} \) then \( \{ a \} \vdash a \) for all \( a \in U \), so \( U \vdash \text{Ap} \) for all \( a \in U \). For transitivity, let \( U \vdash A \land V \vdash A \); by atomicity we get an \( \{ a \} \vdash a \), as well as a \( b \in V \) such that \( \{ b \} \vdash c \); it follows that there is an \( a_b \in U \), such that \( \{ a_b \} \vdash c \). Finally, consistency propagates through atomic entailment, since it does so through entailment in general.

We now show that \( \text{H} \rho \) is an information system. Reflexivity for coherent consistency is immediate from the definition, since all singletons are already in \( \text{Con} \). For closure of coherent consistency under subsets, let \( U \in \text{HCon} \); by closure under subsets for neighborhoods, it is in \( \text{Con} \); so it is a finite set; for \( a, a' \in V \), since \( V \subseteq U \), it is \( \{ a, a' \} \in \text{Con} \), so \( V \in \text{HCon} \). Reflexivity of coherent entailment follows directly from the definition. For the transitivity of coherent entailment let’s consider the most complicated case, where \( U, V \in \text{HCon} \) with

\[
\forall b \in V \forall W_v \in \text{HCon} \quad \left( U \vdash H \, W_u \land W_u \vdash H \, b \right) \land \exists W_v \in \text{HCon} \quad \left( V \vdash H \, W_v \land W_v \vdash H \, c \right).
\]

By applying the induction hypothesis first for \( W_U \) via \( V \) to \( W_V \) and then for \( W_U \) via \( W_V \) to \( c \), we get \( W_U \vdash H \, c \), so by definition it is \( U \vdash H \, c \). For propagation of coherent consistency through coherent entailment, let \( U \vdash H \, b \) for some \( U \in \text{HCon} \); if \( U \vdash b \) then \( U \cup \{ b \} \in \text{Con} \subseteq \text{HCon} \); if \( b \in U \), then \( U \cup \{ b \} = U \in \text{HCon} \); if \( U \vdash H \, V \) and \( V \vdash H \, b \) for some \( V \in \text{HCon} \), then, by the induction hypothesis, it is \( U \cup V \in \text{HCon} \) and \( U \cup V \vdash H \, b \); by the induction hypothesis again, \( U \cup V \cup \{ b \} \in \text{HCon} \); by closure under subsets, \( U \cup \{ b \} \in \text{HCon} \).

Now assume that \( \sigma \) is atomic. Let \( \{ (U_b, b) \} \in \text{Con}_{\rho \to \sigma} \) and \( (U, b) \in \text{Tok}_{\rho \to \sigma} \). By definition, it is \( \{ (U_b, b) \} \mid j \in J \} \vdash \rho \to \sigma \) \( (U, b) \) if \( \{ (U, b) \} \mid j \in J \} \) \( (U, b) \). Let \( J_0 \) be the subset of these indices \( j \in J \) for which \( U \vdash j \); by atomicity in \( \sigma \), \( \{ b_j \} \mid j \in J_0 \} \vdash \sigma \) \( b \) implies that \( \{ b_j \} \vdash j \), for a specific \( j \in J_0 \); then \( \{ (U, b_j) \} \vdash \rho \to \sigma \) \( (U, b) \), for the same \( j \). The case of coherence is treated in an equally straightforward way.  

Both atomicity and coherence may provide extra ideals. For example, consider again the information systems \( \mathcal{E} \) and \( \mathcal{F} \) from [2.1] it is direct to check that

\[
\text{Id}_{\mathcal{E}} \not\ni \{ l, r \} \in \text{Id}_{\mathcal{AE}}
\]

and also that

\[
\text{Id}_{\mathcal{F}} \not\ni \{ l, m, r, e \} \in \text{Id}_{\mathcal{AHF}}.
\]

So atomic and coherent information systems may be ideal-wise richer. We now show that this is the only direction that we get richer in ideals and also that this is the richest we can get in this manner. Write \( \rho \preceq \sigma \) if there is an embedding of \( \text{Id}_\rho \) into \( \text{Id}_\sigma \), and \( \rho \simeq \sigma \) if \( \text{Id}_\rho \) and \( \text{Id}_\sigma \) are isomorphic.

**Theorem 4.** Let \( \rho \) be an information system.

1. It is \( \rho \preceq \text{Ap} \) and \( \rho \preceq \text{H} \rho \).

\(^1\)A proof can be found in [19] Chapter 6. In [23], the preservation of both atomicity and coherence by the formation of the function space is shown simultaneously for a version of information system called acis.
2. Atomicity and coherence are idempotent, in the sense that $A(Ap) \simeq Ap$ and $H(Hp) \simeq Hp$.

Proof. We show that $\text{lde}_p \subseteq \text{lde}_{Ap}$; let $u \in \text{lde}_p$; if $U \subseteq^f u$, then $U \in \text{Con}$ by the consistency in $p$; if further $U \vdash^A b$, then $U \vdash b$ since $\vdash^A \subseteq \vdash$, so $b \in u$ by deductive closure in $p$.

We now show that $\text{lde}_p \subseteq \text{lde}_{Hp}$; let $u \in \text{lde}_p$; if $U \subseteq^f u$, then $U \in \text{Con} \subseteq H\text{Con}$ by the consistency in $p$; if further $U \vdash^A b$, then $U \vdash b$ since $U \in \text{Con}$ again, so $b \in u$ by deductive closure in $p$.

Now we show idempotence. For atomicity, the only thing we have to show is that $\vdash^A \equiv \vdash^A$. By definition, it is $U \vdash^A b$ if and only if there is an $a \in U$ such that $\{a\} \vdash^A b$; this in turn means that there is an $a' \in \{a\}$ such that $\{a'\} \vdash b$; clearly it must be $a = a'$, so we’ve found an $a \in U$ such that $\{a\} \vdash b$. So it is $A(Ap) \simeq Ap$—actually with the trivial isomorphism.

For the idempotence of coherence, we have $\text{Con}_{Hp} = H\text{Con}_p$. Let $U \in H(H\text{Con}_p)$; then

$$\forall_{a,d' \in U} \{a,d'\} \in H\text{Con}_p \Leftrightarrow \forall_{a,d' \in U} \{a,d'\} \in \text{Con}_p,$$

which means that $U \in H\text{Con}_p$, hence $U \in \text{Con}_{Hp}$. The step ($\ast$) holds since two-element sets that are consistent are also coherently consistent and vice versa. Finally, let $U \vdash^H b$; if it is $U \vdash^H b$ or $b \in U$, then we’re done; if there is a $V \in H(H\text{Con}_p)$ such that $U \vdash^H V$ and $V \vdash^H b$, then, as we just saw, it is $V \in H\text{Con}_p$, which means that $U \vdash^H V$ and $V \vdash^H b$, so by transitivity, $U \vdash^H b$. $\square$

3 Coherent point-free structures

Links between domain theory and point-free topology have been noticed and studied by several people already (see [17], [14], [10], [11], [22]). Our main objective here is to find direct correspondences between the information systems that we find useful in practice and respective point-free structures, domains included.

The basic problem that we face in such an endeavor lies, not surprisingly, in the very nature of atomicity and coherence, which are both defined in terms of tokens; but tokens are not observable in a point-free setting, where everything there starts with neighborhoods. With atomicity in particular, the problem seems impenetrable, since it is a property which fully exploits the presence of atomic pieces of data. Indeed, in order to express atomicity in a point-free setting, it seems that we can not avoid having to talk about “atomic neighborhoods”; regarding the relatively limited status of atomic systems in applications, this has not proved a convincing task up to now.

Coherence on the other hand, though based on comparisons of tokens by definition, lends itself to an easy characterization on the level of consistent sets.

**Lemma 5.** A finite set of tokens is a coherent neighborhood if and only if every two of its subsets have a coherent union: it is $U \in H\text{Con}$ if and only if $U \in \mathcal{P} (\text{Tok})$ and $U_1 \cup U_2 \in H\text{Con}$, for any $U_1, U_2 \subseteq U$.

Notice that we could equivalently say “$U_0 \in H\text{Con}$ for any $U_0 \subseteq U$” instead of “$U_1 \cup U_2 \in H\text{Con}$ for any $U_1, U_2 \subseteq U$”; we choose the latter to stress that the issue of validity of a neighborhood in a coherent system is raised from comparisons of its tokens, to comparisons of its subsets. The coherence conditions [2], [3], and [5], that we will pose in the following are all modeled after Lemma 5.

In this section we restrict ourselves to the case where we have countable carrier sets.
3.1 Domains

We start with the known correspondence of arbitrary Scott information systems and domains (of countable base), which we quickly recount here without proofs, to set the mood for what comes next (for details we refer to [1],[2]).

Let \( D = (D,\sqsubseteq,\perp,\text{lub}) \) be a domain and define \( I(D):=(\text{Tok},\text{Con},\vdash) \) by

\[
\text{Tok}: = D_c ,
\{u_i\}_{i\in I} \in \text{Con} : = \{u_i\}_{i\in I} \sqsubseteq D_c \land \text{lub} \{u_i\}_{i\in I} \in D ,
\{u_i\}_{i\in I} \vdash u \sqsubseteq \text{lub} \{u_i\}_{i\in I} .
\]

Note that, by the well-known fact that a bounded finite set of compact elements has a compact least upper bound, if \( \{u_i\}_{i\in I} \in \text{Con} \) then \( \text{lub} \{u_i\}_{i\in I} \in D_c \). Conversely, define \( D(\rho) : = (\text{ide}_\rho,\sqsubseteq,\perp,\cup) \).

**Fact 6** (Representation theorem). If \( D \) is a domain and \( \rho \) an information system, then \( I(D) \) is an information system and \( D(\rho) \) a domain, where compact elements are given by deductive closures, that is, where \( D \).

Furthermore, if \( D \) is a domain then \( \text{ide}_D \cong D \), through the isomorphism pair \( u \mapsto \text{lub} u \) and \( u \mapsto \text{apx}(u) \), where \( \text{apx}(u) \) is the set of the compact approximations of \( u \).

Let now \( r \) be an approximable map from \( \rho \) to \( \sigma \). Define a mapping \( D(r) : D(\rho) \rightarrow D(\sigma) \) by

\[
D(r)(u) := \bigcup \{ V \in \text{Con}_\sigma : U \subseteq \rho (U,V) \in r \} .
\]

Conversely, if \( f \) is a continuous mapping from a domain \( D \) to a domain \( E \). Define a relation \( I(f) \subseteq \text{Con}_I(D) \times \text{Con}_{I(E)} \) by

\[
(U,V) \in I(f) : = \text{lub} V \leq f(\text{lub} U) .
\]

These establish a bijective correspondence, as the following statement expresses.

**Fact 7.** If \( r \) is an approximable map from \( \rho \) to \( \sigma \) then \( D(r) : D(\rho) \rightarrow D(\sigma) \) is a continuous mapping. Conversely, if \( f : D \rightarrow E \) is a continuous mapping then \( I(f) \) is an approximable map from \( I(D) \) to \( I(E) \). Furthermore, the collection of continuous mappings from \( D \) to \( E \) is in a bijective correspondence with the collection of approximable maps between \( I(D) \) and \( I(E) \).

**Coherent domains**

Let \( D = (D,\sqsubseteq,\perp) \) be a domain and \( \{u_i\}_{i\in I} \subseteq D_c \) an arbitrary finite set of compact elements. Call \( D \) a coherent domain if

\[
\text{lub} \{u_i\}_{i\in I} \in D_c \iff \bigvee_{i,j\in I} \text{lub} \{u_i,u_j\} \in D_c . \tag{2}
\]

Note that the choice of \( D_c \) instead of the more modest \( D \) is justified again by the basic fact that bounded finite sets of compacts have a compact least upper bound.

**Theorem 8.** Let \( D \) be a coherent domain and \( \rho \) a coherent information system. Then \( I(D) \) is a coherent information system and \( D(\rho) \) is a coherent domain.

**Proof.** Let \( D \) be a coherent domain and \( \{u_i\}_{i\in I} \in \text{Con}_{I(D)} \). By the definition of \( I(D) \), \( \{u_i\}_{i\in I} \subseteq D_c \) and \( \text{lub} \{u_i\}_{i\in I} \in D \). Since bounded finite sets of compacts have a compact least upper bound, it is \( \text{lub} \{u_i\}_{i\in I} \in D_c \). By the coherence,

\[
\bigvee_{i,j\in I} \text{lub} \{u_i,u_j\} \in D_c \quad \text{and we’re done.}
\]
Let now ρ be a coherent information system and \{U_i\}_{i \in I} \subseteq \text{Con}_ρ. By the coherence and Lemma 5 it is

\[ \bigcup_{i \in I} U_i \in \text{Con}_ρ \iff \exists_{i,j \in I} U_i \cup U_j \in \text{Con}_ρ, \]

so also

\[ \bigcup_{i \in I} \overline{U_i} \in \text{Con}_ρ \iff \forall_{i,j \in I} \overline{U_i} \cup \overline{U_j} \in \text{Con}_ρ, \]

and we’re done. □

3.2 Precusl’s

Precusl’s are structures that provide yet another representation of domains, this time with a more order-theoretic emphasis. Stoltenberg-Hansen et al in [22, Chapter 6] have used precusl’s as an alternative to information systems to solve domain equations up to identity. In their textbook one can also find a correspondence between precusl’s and information systems, which we recall here before moving to the issue of coherence.

A preordered conditional upper semilattice with a distinguished least element, or just precusl, is a ‘consistently complete preordered set with a distinguished least element’, that is, a quadruple \(P = (N, \sqsubseteq, \sqcup, \bot)\) where \(\sqsubseteq\) is a preorder on \(N\), \(\bot\) is a (distinguished) least element and \(\sqcup\) is a partial binary operation on \(N\) which is defined only on consistent pairs, that is, on pairs having an upper bound, and then yields a (distinguished) least upper bound:

\[
U \sqcup V \in N : = \exists_{W \in N} (U \sqsubseteq W \land V \sqsubseteq W),
\]

\[
U \sqcup V \in N \Rightarrow U \sqsubseteq U \sqcup V \sqsubseteq U \sqcup V
\]

\[
\land \forall_{W \in N} (U \sqsubseteq W \land V \sqsubseteq W \rightarrow U \sqcup V \sqsubseteq W).
\]

We think of \(N\) as “a set of formal basic opens”, \(\sqsubseteq\) as “formal inclusion”, \(\bot\) as “a formal empty set” and \(\sqcup\) as “a formal union”. Call a subset \(u \subseteq N\) a precusl ideal when it satisfies

\[
\bot \in u \land \forall_{V \in u} \exists_{U \subseteq u} (V \sqsubseteq U \rightarrow V \in u).
\]

Write \(\text{Ide}_P\) for the class of all precusl ideas of \(P\). Observe that the second of the three requirements for a precusl ideal expresses the property of being upward directed, so it follows that any finite subset in a precusl ideal will have a least upper bound in the ideal.

Let now \(P = (N, \sqsubseteq, \sqcup, \bot)\) be a precusl and define \(I(P) = (\text{Tok}, \text{Con}, \vdash)\) by

\[
\text{Tok} := N,
\]

\[
\forall \in \text{Con} := \forall \sqsubseteq N \land \bigcup \forall \in N,
\]

\[
\forall \vdash U := U \sqsubseteq \bigcup \forall.
\]

Conversely, let \(\rho = (\text{Tok}, \text{Con}, \vdash)\) be an information system and define \(P(\rho) = (N, \sqsubseteq, \sqcup, \bot)\) by

\[
N := \text{Con},
\]

\[
U \sqsubseteq V := V \vdash U,
\]

\[
\bot := \emptyset,
\]

\[
U \sqcup V := U \sqcup V \text{ if } U \sqcup V \in \text{Con}.
\]
The following is Theorem 6.3.4 of [22].

**Fact 9.** If $P$ is a precusl and $\rho$ an information system, then $I(P)$ is an information system and $P(\rho)$ is a precusl. Furthermore, it is $\text{lde}_P = \text{lde}_{I(P)}$ and $\text{lde}_{P} \cong \text{lde}_{P(\rho)}$.

A precusl approximable map from $P$ to $P'$ is a relation $\mathcal{R} \subseteq N \times N'$ which satisfies the following:

- $(\bot, \bot') \in \mathcal{R}$,
- $(U, V) \in \mathcal{R} \cap (U', V') \in \mathcal{R} \rightarrow (U, V \cup V') \in \mathcal{R}$,
- $U \subseteq U' \cap (U, V) \in \mathcal{R} \cap V' \subseteq V \rightarrow (U', V') \in \mathcal{R}$,

where $(U, V \cup V') \in \mathcal{R}$ naturally implies that $V \cup V'$ is defined. Write $\text{Apx}_{P \rightarrow P'}$ for all precusl approximable maps from $P$ to $P'$. For every $\mathcal{R} \in \text{Apx}_{P \rightarrow P'}$ define a relation $I(\mathcal{R}) \subseteq \text{Con}_{I(P)} \times \text{Con}_{I(P')}$ by

$$(U, V) \in I(\mathcal{R}) := \left( \bigcup \mathcal{U}, \bigcup \mathcal{V} \right) \in \mathcal{R}.$$  

Conversely, let $r$ be an approximable map from $\rho$ to $\sigma$. Define a relation $P(r) \subseteq N_{P(\rho)} \times N_{P(\sigma)}$ by

$$(U, V) \in P(r) := (U, V) \in r.$$  

One can show (see [22], pp. 151–2) that these establish a bijective correspondence.

**Fact 10.** If $r$ is an approximable map from $\rho$ to $\sigma$ then $P(r)$ is a precusl approximable map from $P(\rho)$ to $P(\sigma)$. Conversely, if $\mathcal{R}$ is a precusl approximable map from $P$ to $P'$ then $I(\mathcal{R})$ is an approximable map from $I(P)$ to $I(P')$. Furthermore, it is $\text{Apx}_{P \rightarrow \sigma} \cong \text{Apx}_{P(\rho) \rightarrow P(\sigma)}$ and $\text{Apx}_{P \rightarrow P'} \cong \text{Apx}_{I(P) \rightarrow I(P')}$.  

**Coherent precusl’s**

Call a precusl coherent if it satisfies the following property for a finite collection $\mathcal{U} \subseteq N$:

$$\bigcup \mathcal{U} \subseteq N \iff \forall \{U, V \in \mathcal{U} \} \cup \bigcup \mathcal{U} \subseteq N.$$  

(3)

**Theorem 11.** If $P$ is a coherent precusl then $I(P)$ is a coherent information system. Conversely, if $\rho$ is a coherent information system then $P(\rho)$ is a coherent precusl.

**Proof.** Suppose first that $P$ is coherent, that is, such that (3) holds for all $\mathcal{U} \subseteq N$. Let $\mathcal{U} \subseteq \text{Con}_{I(P)}$; by the definition,

$$\mathcal{U} \subseteq N \cap \bigcup \mathcal{U} \subseteq N \cap \bigcup \mathcal{U} \subseteq N$$  

$$\bigcup \mathcal{U} \subseteq N \cap \bigcup \mathcal{U} \subseteq N$$

so $I(P)$ is a coherent information system.

Now suppose that $\rho$ is a coherent information system, that is, such that

$$U \in \text{Con} \iff \forall \{a, b \in U\} \in \text{Con}.$$  

(4)

---

2 The proofs of isomorphism between information system ideals and precusl ideals, which is omitted in [22] can be found under Proposition 3.12 of [8].
for all $U \not\subseteq f \text{Tok}$. Let $U \subseteq f N\rho$, that is, $U \subseteq f \text{Con}$; we have
\[
\bigsqcup U \in N\rho \iff \bigsqcup \forall U \subseteq f \text{Tok} \land \forall U \cup V \subseteq \text{Con}
\]
\[
\iff \bigsqcup \forall U \subseteq f \text{Tok} \land \forall U \cup V \in N\rho
\]
\[
\Rightarrow \forall U \cup V \in N\rho,
\]
where at $(\star)$ we used (4) and Lemma 5. Conversely, we have
\[
\bigsqcup \forall U \subseteq f \text{N\rho} \land \forall U \cup V \in N\rho \iff \bigsqcup \forall U \subseteq f \text{Con} \land \forall \{a, b\} \in \text{Con}
\]
\[
\Rightarrow \forall \{a, b\} \in \text{Con}
\]
\[
\bigsqcup \text{N\rho} \subseteq \text{Con}
\]
\[
\Rightarrow \bigsqcup U \subseteq f \text{N\rho}
\]
so $P(\rho)$ is indeed a coherent precusl.

3.3 Scott–Ershov formal topologies

The structure of a “formal topology” was defined by Giovanni Sambin as early as 1987 in [13], and as the area has developed a number of alternative definitions has appeared. Suitable for our purposes is a version of the definition in [10], whose main difference from Sambin’s original is the disposal of the “positivity predicate”. In fact, we depart a bit from this definition as well, in that we require the presence of a top element among the formal basic opens.

We will use order-theoretic notions which are dual to notions appearing before, namely a greatest or top element and greatest lower bounds of sets of elements; all these are to be understood in the usual order-theoretic way.

Define a formal topology as a triple $T = (N, \sqsubseteq, \preceq)$ where $N$ is the collection of formal basic opens, $\sqsubseteq \subseteq N \times N$ is a preorder with a top element $\top$, which formalizes inclusion between basic opens, and $\preceq \subseteq N \times \mathcal{P}(N)$, called covering, formalizes inclusion between arbitrary opens, and is reflexive, transitive, localizing, and extends formal inclusion between formal basic opens:
\[
U \preceq V \iff \forall U \in U \forall V \in V \Rightarrow \forall W \in \mathcal{P}(U \cap V)
\]
\[
W \sqsubseteq U \land W \sqsubseteq V.
\]

3Concerning the issue of the positivity predicate see [10, §2.4] or [15, Footnote 13] for relevant discussion. For the issue of a top element see Exercise 6.5.21 of [22]. A discussion on the abandonment of positivity as well as nomenclature discrepancies between our exposition and the literature can be found in [4].
\[\top \in u, \]
\[\forall U, V \in u \ (W \subseteq U \land W \subseteq V), \]
\[\forall U \in u \ (U \prec V \rightarrow \exists W \in u \ W \subseteq U \land W \subseteq V), \]

Dually to the case of preclus ideals, the second of the three requirements for a formal point expresses the property of being downward directed, so it follows that any finite subset in a formal point will have a greatest lower bound in the ideal. Denote the collection of formal points of \( \mathcal{F} \) by \( \text{Pt}_\mathcal{F} \).

Call a formal topology \( \mathcal{T} \) unary if
\[U \prec U \rightarrow \exists V \in U \ U \prec V,\]
where we write \( U \prec V \) for \( U \prec \{V\} \), and consistently complete if
\[\forall U, V \in N \ (\exists W \in N \ W \subseteq U \land W \subseteq V \rightarrow \exists W \in N \ W = U \cap V),\]
where \( \bigcap W \) denotes the greatest lower bound of \( W \). Finally, call \( \mathcal{T} \) a Scott–Ershov formal topology if it is both unary and consistently complete.

One can prove that every domain can be represented by the collection of formal points of a certain Scott–Ershov formal topology (see Theorem 4.35 of [10] and Theorem 6.2.15 of [22]). Here we proceed to link formal topologies directly to information systems, before we discuss coherence.

Let \( \mathcal{T} = (N, \subseteq, \prec) \) be a Scott–Ershov formal topology. Define \( I(\mathcal{T}) = (\text{Tok}, \text{Con}, \vdash) \) by
\[
\text{Tok} := N, \\
\forall W \in \text{Con} := W \subseteq f \bigcap N \in N, \\
\forall U \in N \ (\forall V \in V \in N \ W \subseteq U \land W \subseteq V \rightarrow \exists W \in N \ W = U \cap V).
\]

Conversely, let \( \rho = (\text{Tok}, \text{Con}, \vdash) \) be an information system. Define \( F(\rho) = (N, \subseteq, \prec) \) by
\[
N := \text{Con}, \\
U \subseteq V := U \vdash V, \\
U \prec V := \exists V \in \rho \ U \vdash V.
\]

Note that the definition is independent from the choice of representatives by Lemma[1]

**Proposition 12.** If \( \mathcal{T} \) is a Scott–Ershov formal topology and \( \rho \) an information system, then \( I(\mathcal{T}) \) is an information system and \( F(\rho) \) is a Scott–Ershov formal topology. Furthermore, it is \( \text{Pt}_\mathcal{T} = \text{Ide}_I(\mathcal{T}) \) and \( \text{Ide}_\rho \simeq \text{Pt}_F(\rho) \).

**Proof.** First let \( \mathcal{T} \) be a Scott–Ershov formal topology. We check the defining properties of an information system for \( I(\mathcal{T}) \). For reflexivity of consistency, let \( U \in N \); it is \( U \subseteq U \), so \( \{U\} \in N \) and \( \{U\} \in \text{Con} \) by definition. For closure under subsets, let \( W \in \text{Con} \) and \( W \subseteq V \), then \( \bigcap W \in N \) and \( \bigcap V \subseteq V \), so \( \bigcap V \subseteq U \) and \( V \subseteq \text{Con} \) by definition. For reflexivity of entailment, let \( W \in \text{Con} \) and \( U \subseteq V \); then \( \bigcap W \subseteq U \), so \( W \vdash U \) by definition. For transitivity of entailment, let \( W \vdash V \) and \( V \vdash W \); then \( \bigcap W \subseteq V \) and \( \bigcap V \subseteq W \) by transitivity we get \( \bigcap W \subseteq W \), so \( W \vdash W \) by definition. Finally, for propagation of consistency through
entailment, let \( \forall \in \text{Con} \) and \( \forall \vdash V \); by definition, \( \bigsqcup \forall \in N \) and \( \bigsqcup \forall \subseteq V \), so \( \forall(\forall \cup \{V\}) \in N \) and \( \forall \cup \{V\} \in \text{Con} \) by definition.

Now let \( \rho \) be an information system. We check the defining properties of a Scott–Ershov formal topology for \( F(\rho) \). That \( \sqsubseteq \) is a preorder with \( \top:=\emptyset \) is direct to see. For reflexivity of covering, let \( \overline{U} \in \forall \); since \( U \uparrow U \), it is \( \overline{U} \prec \forall \) by definition. For transitivity of covering, we have

\[
\overline{W} \prec \forall \wedge \overline{U} \prec \forall \Leftrightarrow \exists \overline{V} \in \forall \left( W \vdash U \wedge \exists \overline{V} \vdash V \right)
\]

\[
\Leftrightarrow \exists \overline{V} \vdash V \wedge \overline{W} \prec \forall.
\]

For localization, we have

\[
\overline{W} \prec \forall \wedge \overline{W} \prec \forall \Leftrightarrow \exists \overline{V} \in \forall \left( W \vdash U \wedge \exists \overline{V} \vdash V \right)
\]

\[
\Leftrightarrow \exists \overline{V} \in \forall \left( \overline{W} \subseteq U \wedge \exists \overline{V} \subseteq V \right)
\]

\[
\Leftrightarrow \overline{W} \in \forall \downarrow \forall
\]

\[
\Leftrightarrow \overline{W} \prec \forall \downarrow \forall.
\]

To show that the covering extends formal inclusion between formal basic opens, we have:

\[
\overline{W} \subseteq \overline{U} \wedge \overline{W} \prec \forall \Leftrightarrow W \vdash U \wedge \exists \overline{V} \vdash V
\]

\[
\Leftrightarrow \exists \overline{V} \vdash V \wedge \overline{W} \prec \forall
\]

So \( F(\rho) \) is indeed a formal topology. To show that it is unary is easy: let \( \overline{U} \prec \forall \); by definition there is a \( \forall \in \forall \), for which \( U \vdash V \), that is, \( \overline{U} \subseteq \forall \); by reflexivity and extension, we get \( \overline{U} \subseteq \{\forall\} \). To show, finally, that it is consistently complete, let \( \overline{U}, \overline{V}, \overline{W} \in N \), with \( \overline{W} \subseteq \overline{U} \) and \( \overline{W} \subseteq \overline{U} \); that is, \( W \vdash U \) and \( W \vdash V \); by (1), we get \( W \vdash U \cup V \), and so, \( \overline{W} \subseteq \overline{U} \cup \forall \); let \( \overline{U} \cap \forall:=\overline{U} \cap \forall \); that this does the job is direct to see.

We now show the bijective correspondence between information system ideals and formal points. For \( \text{Pt}(\exists) \subseteq \text{Id}_{\text{H}(\exists)} \), let \( u \in \text{Pt}(\exists) \) and \( \forall \subseteq \exists \). Since \( u \) is downward directed in \( \exists \), it is \( \bigsqcup \forall \subseteq \exists \), and so \( \forall \in \text{Con}_{\text{H}(\exists)} \) by definition. If further \( \forall \vdash \text{H}(\exists) U \), it is \( \bigsqcup \forall \subseteq U \) by definition, and \( \bigsqcup \forall \prec \{U\} \); hence \( U \in \forall \) by the third formal point property.

For \( \text{Id}_{\text{H}(\exists)} \subseteq \text{Pt}(\exists) \), let \( u \in \text{Id}_{\text{H}(\exists)} \). That \( \top \in \exists \subseteq u \), follows from downward closure in \( I(\exists) \). Let \( U, V \in u \); by the consistency in \( I(\exists) \), \( \{U, V\} \in \text{Con}_{\text{H}(\exists)} \), and then \( U \cap V \in N \) by definition; since \( \{U, V\} \vdash \text{H}(\exists) U \cap V \), it is \( U \cap V \in u \) by the deductive closure in \( I(\exists) \). If now \( U \in u \) and \( \forall \prec \exists \), then, since \( \exists \) is unary, we have \( \forall \in \forall \subseteq \exists \), \( \forall \subseteq \exists \), \( \forall \subseteq \exists \vdash \{U\} \vdash \text{H}(\exists) V \) by the definition, so that \( \forall \in \forall \subseteq \exists \), \( \forall \subseteq \exists \), \( \forall \subseteq \exists \) follows from the deductive closure in \( I(\exists) \).
For $\text{ldo}_\rho \simeq \text{Pt}_F(\rho)$ take the following isomorphism pair:

$$\text{ldo}_\rho \ni u \mapsto \mathcal{P}(u) \in \text{Pt}_F(\rho)$$

$$\text{Pt}_F(\rho) \ni u \mapsto \bigcup u \in \text{ldo}_\rho$$

where $\mathcal{P}(u) := \{ U \subseteq u \}$ contains the closures of subsets of $u$.

Indeed, for the right embedding, since $\emptyset \subseteq^f u$, it is $\top \in \mathcal{P}(u)$; for every $U, V \subseteq^f u$, since $U \cup V \subseteq^f u$, it is also $U \cap V \in \mathcal{P}(u)$; if $U \subseteq^f u$ and $U \prec V$, then $\exists_{V \in \mathcal{P}(u)} U \subseteq V$, that is, $\exists_{V \in \mathcal{P}(u)} U \vdash V$ by definition; then $\exists_{V \in \mathcal{P}(u)} V \subseteq^f u$ by deductive closure in $\rho$.

For the left embedding, if $\{ a_i \}_{i \in \mathbb{N}} \subseteq^f \bigcup u$, then $\forall_{i \in \mathbb{N}} \exists_{V \in \mathcal{P}(u)} U_i \vdash a_i$; since $u$ is downward directed in $F(\rho)$ we have that $\exists_{V \in \mathcal{P}(u)} \forall_{i \in \mathbb{N}} W \vdash U_i \vdash a$, so $\{ a_i \}_{i \in \mathbb{N}} \in \text{Con}$ by transitivity of entailment and $[1]$. If now $U \subseteq^f \bigcup u$ and $U \vdash a$, then, by definition, $U \subseteq \pi$, that is $U \prec \{ \pi \}$; by the third formal point property in $F(\rho)$, we have $\pi \in u$, so $a \in \bigcup u$.

That the two embeddings are mutually inverse is also quite direct. Indeed, let $u \in \text{ldo}_\rho$. We have

$$a \in \bigcup \mathcal{P}(u) \iff \exists_{U \subseteq^f u} U \vdash a \iff a \in u$$

where $(\ast)$ holds leftwards for $U := \{ a \}$, and

$$\{ a_i \}_{i \in \mathbb{N}} \in \mathcal{P}(\bigcup u) \iff \{ a_i \}_{i \in \mathbb{N}} \subseteq^f \bigcup u \iff \forall_{i \in \mathbb{N}} \exists_{U_i \subseteq^f u} U_i \vdash a_i$$

$$\iff \forall_{i \in \mathbb{N}} \exists_{U_i \subseteq^f u} \bigcap U_i \subseteq \pi_i \iff \forall_{i \in \mathbb{N}} \exists_{U_i \subseteq^f u} \bigcap U_i \subseteq \pi_i$$

$$\iff \{ \pi_i \}_{i \in \mathbb{N}} \in u \iff \{ a_i \}_{i \in \mathbb{N}} \subseteq u,$$

where $(\ast)$ hold leftwards for $U := \{ a_i \}$, $i < n$. \hfill \Box

An \textit{approximable map of Scott–Ershov formal topologies} from $\mathcal{F}$ to $\mathcal{F}'$ is a relation $\mathcal{R} \subseteq N \times N'$ which satisfies the following:

- $\{ \top, \top' \} \in \mathcal{R}$,
- $(U, V) \in \mathcal{R}$ and $(U, V') \in \mathcal{R}$ imply $(U, V \cap V') \in \mathcal{R}$,
- $U' \subseteq U$ and $(U, V) \in \mathcal{R}$ imply $U' \cap V \subseteq V' \Rightarrow (U', V') \in \mathcal{R}$.

Write $\text{Apx}_{\mathcal{F} \rightarrow \mathcal{F}'}$ for all approximable maps of Scott–Ershov formal topologies from $\mathcal{F}$ to $\mathcal{F}'$. For every $\mathcal{R} \in \text{Apx}_{\mathcal{F} \rightarrow \mathcal{F}'}$ define a relation $I(\mathcal{R}) \subseteq \text{Con}_I(\mathcal{F}) \times \text{Con}_I(\mathcal{F}')$ by

$$(\mathcal{U}, \mathcal{V}) \in I(\mathcal{R}) := \left( \bigcap \mathcal{U}, \bigcap \mathcal{V} \right) \in \mathcal{R}.$$  

Conversely, let $\mathcal{R}$ be an approximable map from $\rho$ to $\sigma$. Define a relation $F(\mathcal{R}) \subseteq N_{F(\rho)} \times N_{F(\sigma)}$ by

$$(\mathcal{U}, \mathcal{V}) \in F(\mathcal{R}) := (U, V) \in \mathcal{R}.$$  

Again, it is easy to see that the definition does not rely on the choice of the representatives, due to deductive closure of $\mathcal{R}$.

We show that these establish a bijective correspondence.
Proposition 13. If \( r \) is an approximable map from \( \rho \) to \( \sigma \) then \( F(r) \) is an approximable map of Scott–Ershov formal topologies from \( F(\rho) \) to \( F(\sigma) \). Conversely, if \( \mathcal{R} \) is an approximable map of Scott–Ershov formal topologies from \( \mathcal{T} \) to \( \mathcal{T}' \) then \( I(\mathcal{R}) \) is an approximable map from \( I(\mathcal{T}) \) to \( I(\mathcal{T}') \). Furthermore, it is \( \text{Apx}_{\rho \to \sigma} \cong \text{Apx}_{F(\rho) \to F(\sigma)} \) and \( \text{Apx}_{\mathcal{T} \to \mathcal{T}'} \cong \text{Apx}_{I(\mathcal{T}) \to I(\mathcal{T}')} \).

Proof. Let \( r \) be an approximable map from \( \rho \) to \( \sigma \). Since, by Proposition 2, \( (\varnothing, \varnothing) \in r \), it is \( (\top, \top) \in F(r) \). If \((U, V), (U', V') \in F(r)\), then, by definitions, \((U, V) \cup V' \in r\), so \((U, V \cap V') \in F(r)\), and \((U, V \cap V') \in F(r)\). If \( U' \subseteq U\), \((U, V) \in F(r)\) and \( V \subseteq V'\), then, by definitions, \( U' \cup U\), \((U, V) \in r\) and \( V \subseteq V'\) respectively, so, \((U', V') \in r\) and \((U', V') \in F(r)\).

Conversely, let \( \mathcal{R} \) be an approximable map of Scott–Ershov formal topologies from \( \mathcal{T} \) to \( \mathcal{T}' \). Since \( (\top, \top) \in \mathcal{R}\), it is \( (\varnothing, \varnothing) \in I(\mathcal{R})\). If \( (\mathcal{W}, \mathcal{Y}) \in F(\mathcal{R})\), then, by definition, \((\wang \mathcal{W}, \wang \mathcal{Y})\), \((\wang \mathcal{W} \cap \wang \mathcal{Y}) \in \mathcal{R}\); since \( \mathcal{R} \) is an approximable map of Scott–Ershov formal topologies, \((\wang \mathcal{W}, \wang \mathcal{Y}) \cap (\wang \mathcal{Y}, \wang \mathcal{Y}) \in \mathcal{R}\), or, \((\glob \mathcal{W}, \glob \mathcal{Y} \cap \glob \mathcal{Y}) \in \mathcal{R}\), since \( \mathcal{R} \) is an approximable map of Scott–Ershov formal topologies, \((\glob \mathcal{W}, \glob \mathcal{Y} \cap \glob \mathcal{Y}) \in \mathcal{R}\), or, \((\mathcal{W}, \mathcal{Y} \cap \mathcal{Y}) \in \mathcal{R}\), and \( \mathcal{Y} \subseteq \mathcal{Y}'\), by definition we obtain \( \mathcal{Y} \subseteq \mathcal{Y}'\), \((\mathcal{W}, \mathcal{Y} \cap \mathcal{Y}) \in \mathcal{R}\), \( \mathcal{Y} \subseteq \mathcal{Y}'\), \((\mathcal{W}, \mathcal{Y} \cap \mathcal{Y}) \in \mathcal{R}\), \( \mathcal{Y} \subseteq \mathcal{Y}'\); then \((\mathcal{W}, \mathcal{Y} \cap \mathcal{Y}) \in \mathcal{R}\), so \((\mathcal{Y}', \mathcal{Y}') \in \mathcal{R}\).

We show that \( F: \text{Apx}_{\rho \to \sigma} \to \text{Apx}_{\mathcal{R}(\rho) \to \mathcal{R}(\sigma)} \) is bijective. To show injectivity, let \( F(r) = F(r') \); then \((U, V) \in r \iff (U, V) \in F(r') \); \((U, V) \in r \), \( (U, V) \in F(r') \), \((U, V) \in r' \), so \( r = r' \). To show surjectivity, let \( \mathcal{R} \in \text{Apx}_{\mathcal{R}(\rho) \to \mathcal{R}(\sigma)} \); set \((U, V) \in r \iff (U, V) \in \mathcal{R} \); it is straightforward to check that \( r \in \text{Apx}_{\rho \to \sigma} \) and \( F(r) = \mathcal{R} \).

We show finally that \( I: \text{Apx}_{\mathcal{T} \to \mathcal{T}'} \to \text{Apx}_{I(\mathcal{T}) \to I(\mathcal{T}')} \) is bijective. To show injectivity, let \( I(\mathcal{R}) = I(\mathcal{R}') \) and \( (U, V) \in \mathcal{R} \); then, by the definition of \( I \), there are \( \mathcal{W} \in \text{Con}_{\mathcal{T}(\mathcal{R})} \) and \( \mathcal{Y} \in \text{Con}_{\mathcal{T}(\mathcal{R}')}, \) such that \( U = \wang \mathcal{W}, V = \wang \mathcal{Y} \) and \((\mathcal{W}, \mathcal{Y}) \in I(\mathcal{R})\); by the assumption we get equivalently that \((\mathcal{W}, \mathcal{Y}) \in I(\mathcal{R})\), so \((U, V) \in \mathcal{R}\), and \( \mathcal{R} = \mathcal{R}' \). To show surjectivity, let \( r \in \text{Apx}_{I(\mathcal{T}) \to I(\mathcal{T}')} \); set \((U, V) \in \mathcal{R} \iff \mathcal{R} \in \text{Apx}_{\mathcal{T} \to \mathcal{T}'} \); then \((U, V) \in r \iff (U, V) \in \mathcal{R} \); it is \( \mathcal{R} \in \text{Apx}_{\mathcal{T} \to \mathcal{T}'} \), since (i) by \( r \in \text{Apx}_{I(\mathcal{T}) \to I(\mathcal{T}')} \) we get \((\varnothing, \varnothing) \in r\), which yields \((\top, \top) \in \mathcal{R}\), (ii) it is \((U, V) \in \mathcal{R} \).
where (⋆) holds for \( \mathcal{U} := \{ U \} \), and (⋆⋆) for \( \mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2 \), and (iii) it is
\[
\mathcal{U}^\prime \subseteq \mathcal{U} \land (U, V) \in R \land V \subseteq \mathcal{V}^\prime
\]
\[
\iff U^\prime \subseteq \mathcal{U} \land \exists \mathcal{W} \in \mathcal{W} \left( U = \bigcap \mathcal{W} \land V = \bigcap \mathcal{V} \land (\mathcal{U}, \mathcal{V}) \in r \right) \land V \subseteq \mathcal{V}^\prime
\]
\[
\iff \exists \mathcal{W}, \mathcal{V} \left( U^\prime = \bigcap \mathcal{W} \land \forall U \in \mathcal{W} \mathcal{V}^\prime \upharpoonright I(T) U \right)
\]
\[
\land \exists \mathcal{V} \left( U = \bigcap \mathcal{W} \land V = \bigcap \mathcal{V} \land (\mathcal{U}, \mathcal{V}) \in r \right)
\]
\[
\land \mathcal{V}^\prime \upharpoonright I(T) V
\]
\[
\iff \exists \mathcal{W}, \mathcal{V} \left( U^\prime = \bigcap \mathcal{W} \land V^\prime = \bigcap \mathcal{V} \land (\mathcal{U}, \mathcal{V}) \in r \right)
\]
\[
\iff (U^\prime, V^\prime) \in R,
\]
where (⋆) holds for \( \mathcal{U}^\prime := \{ U \} \) and because
\[
\mathcal{V} \subseteq \mathcal{V}^\prime \Rightarrow \bigcap \mathcal{V} \subseteq \mathcal{V}^\prime \Rightarrow \mathcal{V}^\prime \upharpoonright I(T) \mathcal{V}^\prime.
\]
and (⋆⋆) holds for \( \mathcal{V}^\prime := \{ V \} \) and because \( r \) is an approximable map; finally, direct application of the definitions gives
\[
(\mathcal{U}, \mathcal{V}) \in I(R) \iff \left( \bigcap \mathcal{W}, \bigcap \mathcal{V} \right) \in R \iff (\mathcal{U}, \mathcal{V}) \in r,
\]
which means that \( I(R) = r \). \( \Box \)

### Coherent Scott–Ershov formal topologies

Call a Scott–Ershov formal topology coherent if it satisfies the following property for a finite collection \( \mathcal{W} \subseteq \mathcal{f} N \):
\[
\bigcap \mathcal{W} \in N \iff \forall U, V \in \mathcal{W} U \cap V \in N . \tag{5}
\]

**Theorem 14.** If \( T \) is a coherent Scott–Ershov formal topology then \( I(T) \) is a coherent information system. Conversely, if \( \rho \) is a coherent information system then \( F(\rho) \) is a coherent Scott–Ershov formal topology.

**Proof.** Suppose first that \( T \) is coherent, that is, such that (5) holds for all \( \mathcal{W} \subseteq \mathcal{f} N \). Let \( \mathcal{W} \in \text{Con}_{I(T)} \); by definition,
\[
\mathcal{W} \subseteq \mathcal{f} N \land \bigcap \mathcal{W} \in N \iff \forall U, V \in \mathcal{W} U \cap V \in N
\]
\[
\iff \mathcal{W} \subseteq \mathcal{f} N \land \bigcup \mathcal{W} \subseteq \mathcal{f} N \land U, V \in N \land U \cap V \in N
\]
\[
\iff \mathcal{W} \subseteq \mathcal{f} N \land \bigcup \mathcal{W} \subseteq \mathcal{f} N \land \{ U, V \} \in \text{Con}_{I(T)} ,
\]
so \( I(T) \) is a coherent information system.

Now suppose that \( \rho \) is a coherent information system, that is, such that
\[
U \in \text{Con} \iff \forall_{a, b \in U} \{ a, b \} \in \text{Con} , \tag{4}
\]
for all $U \subseteq \text{Tok}$. Let $U \subseteq \mathcal{N}_F(\rho)$, that is, $U \subseteq \text{Con}$; we have

$$
\bigcap \mathcal{U} \in \mathcal{N}_F(\rho) \iff \bigcup \mathcal{U} \in \text{Con}
$$

$$
\iff \forall U, V \subseteq \bigcup \mathcal{U} \in \text{Con}
$$

$$
\iff \forall U, V \subseteq \mathcal{U} \cap \mathcal{V} \in \mathcal{N}_F(\rho)
$$

$$
\Rightarrow \forall U \subseteq \bigcup \mathcal{U} \in \mathcal{N}_F(\rho)
$$

where at (*) we used (4) and Lemma 5. Conversely, we have

$$
\mathcal{U} \subseteq \mathcal{N}_F(\rho) \land \forall U, V \subseteq \mathcal{U} \cap \mathcal{V} \in \mathcal{N}_F(\rho) \iff \mathcal{U} \subseteq \text{Con} \land \forall U, V \subseteq \mathcal{U} \cap \mathcal{V} \in \text{Con}
$$

$$
\iff \mathcal{U} \subseteq \text{Con} \land \forall a, b \in \bigcup \mathcal{U} \{a, b\} \in \text{Con}
$$

$$
\Rightarrow \mathcal{U} \subseteq \text{Con} \land \forall a, b \in \bigcup \mathcal{U} \{a, b\} \in \text{Con}
$$

$$
\iff \bigcup \mathcal{U} \in \text{Con}
$$

$$
\iff \bigcap \mathcal{U} \in \mathcal{N}_F(\rho)
$$

so $F(\rho)$ is indeed a coherent Scott–Ershov formal topology. $\square$

4 Notes

On the notion of atomicity

The defining property of a unary formal topology (page 11) looks similar to the atomicity property for an information system (page 4)—in fact, unary formal topologies are called “atomic” by Erik Palmgren in a preliminary version of [11]—but the two are not essentially related from our viewpoint.

The property of being unary for a formal topology expresses atomicity of compact covering, whereas in information systems we have atomicity of information flow: in the first case, an “atom” would be a formal basic open while in the second case, an atom (that is, a token) represents a simple piece of data.

In order to avoid confusions, one should notice how the transition from an information system to a point-free structure—domains included—implies jumping from the level of atomic pieces of data to (finitely determined) sets of atomic pieces of data: atomicity of information appears in the presence of atomic pieces of data, which become indiscernible when one moves to a point-free setting (see however the last note).

On the notion of coherence

Coherence in domain theory is in no way considered here for the first time. Coherent cpo’s appear already in Gordon Plotkin’s [12], where he attributes the notion to George Markowsky and Barry Rosen [9]. In the handbook chapter of Samson Abramsky and Achim Jung [1], coherence is studied in the more general setting of continuous domains. Viggo Stoltenberg-Hansen et al [22] introduce the notion too in an exercise. We should also mention Jean-Yves Girard’s coherence.
spaces [5], which he uses alternatively to Scott–Ershov domains. On the other hand, coherence has been considered in point-free topology as well, at least since Peter Johnstone’s [7], where coherent locales are discussed.

### Featuring Coquand and Plotkin

Both of the finite Scott information systems $C$ and $L$ of section 2 are elaborations of existing counterexamples. The first one is based on a remark by Thierry Coquand at the Mathematics, Algorithms, and Proofs summer school in Genoa, August 2006, against the choice of atomicity for information systems which are induced by algebras given by superunary constructors. The second system stems from Plotkin’s [12], where he uses the entailment graph of $L$ as an example of a “consistently complete” but not “coherent” complete partial order.

### On the absence of positivity

In 3.3 we associated information systems to a version of formal topologies where the positivity predicate is absent. The intuitive meaning of positivity in this context should be taken as inhabitation: a neighborhood is positive when it contains a point, that is, when it can be extended to an ideal. This is clearly an important and necessary concept to raise in a general constructive setting. Our restricted setting though fulfills inhabitation by design, since $U$, defined for every consistent set $U$, is an ideal; to suppress positivity is merely an Occam’s razor choice, reflecting our predilection for the barest possible setting within which to achieve the wanted connection starting from information systems. However, as Giovanni Sambin suggested in a private communication [16], starting from formal topologies with positivity, we may seek to establish similar connections, in particular with information systems as well as with appropriate positivity semilattices (that is, semilattices equipped with a monotone positivity predicate) replacing precusl’s.

### Nomenclature discrepancies


### Outlook

The issue of linking the theory of information systems and formal topology has many facets, at least as many as the various point-free structures that are currently studied by the community. Apart from the ones that we have covered in this chapter, further correspondences may be asked of various other settings, from the event structures of [23] to the apartness spaces of [2], and also with other versions of formal topologies, ones which accommodate the positivity predicate for instance (see note above).

In a different and a bit more obscure direction, there still remains the issue of a point-free formulation of atomicity. In [3] a more general notion of implicit atomicity emerged, which might shed some light on the matter, since it pushes the concept of atomic flow of information from the level of tokens to the level of eigen-neighborhoods, a special kind of neighborhoods which behave deductively as generalized tokens.
Acknowledgments

The questions answered in this work occurred to the author at the Third Workshop in Formal Topology in Padua, May 2007, organized by Elena Paccagnella, Silvia Pittarello and Giovanni Sambin, and some of the results were presented for the first time in a cozy Forschungstutorium held by Peter Schuster at LMU during the winter semester of 2007–8; at the time the author was supported by a Marie Curie Early Stage Training fellowship (MEST-CT-2004-504029). Helmut Schwichtenberg and Giovanni Sambin gave crucial feedback on earlier versions. Thanks to all of the above.

References


