Normal Form of Finite Algebraic Approximations*

Basil Karádais

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1 Intuition

Think of the set of natural numbers as it is inductively defined by the zero and the successor; it looks like this:

$$\{0, S0, S^2 0, \ldots\}$$

View now the naturals as "atomic pieces of information" ('atomic' to distinguish them from subsets): if one is given 5, he is informed that "the successor operator has been applied to zero five times". It is fair enough to say that the set $\{0, S0, S^2 0, \ldots\}$ is exceptionally precise in this sense. Since it feels natural to want to deal with "not enough information" as well, we add a "least info" element $*$ to the set of naturals, to mean "atomic information which, if completed, could provide the number n". In this way we have just added the notion of *partiality* to the natural numbers: indeed, $*$ can be thought of as a "partial number", or, more appropriately, as an "approximation of a natural number". The initial set has become

$$\{*, 0, S0, S^2 0, \ldots\}$$

Furthermore though, we want to have different degrees of "incomplete information", to signify different size of information needed: to obtain 2, one has to apply the successor operator twice to 0, whereas 1 is obtained only by one such application. It feels now natural to add different elements $*_n$ to the set of natural numbers, of different degrees of "information incompleteness", one for every "full info element" n, and in such a way that the set will remain inductive. To do this we let $*_n := S^n *$, and our already overabused set of natural numbers looks now like that:

$$\{*, 0, S*, S0, S^2 *, S^2 0, \ldots\}$$

Notice that every $n = S^n$ is approximated by as many as $n + 1$ "incomplete atomic info" elements, namely by $*, S*, \ldots, S^n *$. One could try now to imagine how an "arithmetical function containing incomplete info" would look like. It's not particularly straightforward, but this is the very central general idea: to approximate (infinite) partial functionals over natural numbers using proper finite "approximations", ie, "pieces of information".

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It turns out that the formalization of the central idea through 'atomic coherent information systems' (to be defined below), makes it indeed possible to control functionals in terms of finite approximations. What we restrict our attention to here, is making these approximations as economic as possible, that is, with no redundant information: the subset $\{S^3*, S*\}$ informs us that the successor has been applied three times *and* that the successor has been applied one time — clearly, the second information is redundant.

2 Acises and Function Spaces

An *atomic coherent information system graph*, or simplier an *acis (graph)*, is a triad $\alpha = (T, \diamond, \triangleright)$, where T is the *carrier*, a non-empty countable set, the elements of which are called *atoms*[1], $\diamond$ is the *consistency*, a reflexive and symmetric binary relation on T and $\triangleright$ is the *entailment*, a reflexive and transitive binary relation on T, such that *consistency propagates through entailment*, ie

$$\forall_{a,b,c \in T} \, . \, a \diamond b \wedge b \triangleright c \to a \diamond c$$

For $U, V \subseteq T$ write $U \diamond a := \forall_{b \in U} \, b \diamond a$, $U \diamond V := \forall_{a \in V} \, U \diamond a$, $U \triangleright a := \exists_{b \in U} \, b \triangleright a$ and $U \triangleright V := \forall_{a \in V} \, U \triangleright a$. The class of *finite approximations*[2] in α is defined by

$$U \in \mathsf{Con} :\Leftrightarrow (U \subseteq_f T) \wedge (\forall_{a,b \in U} \, a \diamond b)$$

Lemma 1 (soundness of entailment). *From reflexivity and propagation of consistency, it follows that*

1. $\forall_{a,b \in T} \, . \, a \triangleright b \to a \diamond b$
2. $\forall_{a,b,c \in T} \, . \, (a \triangleright b \wedge a \triangleright c) \to b \diamond c$

Let $\alpha = (T_\alpha, \diamond_\alpha, \triangleright_\alpha)$ and $\beta = (T_\beta, \diamond_\beta, \triangleright_\beta)$ be two acises. Define the *function space* $\alpha \to \beta = (T, \diamond, \triangleright)$ by

$$\begin{aligned} T &:= \mathsf{Con}_\alpha \times T_\beta \\ (U,a) \diamond (V,b) &:\Leftrightarrow U \diamond_\alpha V \to a \diamond_\beta b \\ (U,a) \triangleright (V,b) &:\Leftrightarrow V \triangleright_\alpha U \wedge a \triangleright_\beta b \end{aligned}$$

We will show that the function space $\alpha \to \beta$ is an acis itself ([Sch05]).

Lemma 2. *Let $\alpha = (T_\alpha, \diamond, \triangleright)$ be an acis. Then*

1. $U \triangleright V_1 \wedge U \triangleright V_2 \to V_1 \diamond V_2$
2. $U \diamond V \wedge V \triangleright W \to U \diamond W$

[1] Also called *tokens*.
[2] Also called *consistent sets* and *formal neighborhoods* (as in [Sch05]).

Proof. For the first clause: Suppose that $U \rhd V_1$ and $U \rhd V_2$; this unfolds to $\forall_{b_1 \in V_1} \exists_{a_1 \in U} . a_1 \rhd b_1$ and $\forall_{b_2 \in V_2} \exists_{a_2 \in U} . a_2 \rhd b_2$; since $a_2 \diamond a_1$, by propagation we have $\forall_{b_1 \in V_1} \forall_{b_2 \in V_2} \exists_{a_2 \in U} . a_2 \diamond b_1 \wedge a_2 \rhd b_2$; since $b_1 \diamond a_2$, by propagation of consistency again, we obtain $\forall_{b_1 \in V_1} \forall_{b_2 \in V_2} . b_1 \diamond b_2$, that is, $V_1 \diamond V_2$.

For the second clause: Let $U \diamond V$ and $V \rhd W$; we have $U \cup V \rhd U$ and $U \cup V \rhd W$, so by the previous clause we take $U \diamond W$. $\square$

Proposition 3. *The function space between two acises is again an acis.*

Proof. The axioms of $\diamond$ and $\rhd$ are easy to check. For the axiom of propagation: Suppose that $(U, a) \diamond (V, b)$ and $(V, b) \rhd (W, c)$; by definition of consistency and entailment in the function space we have $U \diamond_\alpha V \to a \diamond_\beta b$ and $W \rhd_\alpha V \wedge b \rhd_\beta c$; we want to show that $(U, a) \diamond (W, c)$, or equivalently that $U \diamond_\alpha W \to a \diamond_\beta c$; let $U \diamond_\alpha W$; by the second clause of lemma 2, since $U \diamond_\alpha W \wedge W \rhd_\alpha V$, we have $U \diamond_\alpha V \to a \diamond_\beta b \wedge b \rhd_\beta c$ and $U \diamond_\alpha V$; by modus ponens we get $a \diamond_\beta b \wedge b \rhd_\beta c$; propagation in β yields $a \diamond_\beta c$; so we have proven that $U \diamond_\alpha W \to a \diamond_\beta c$, that is $(U, a) \diamond (W, c)$. $\square$

Define the *(finite) application* on finite approximations $\cdot : \mathsf{Con}_{\alpha \to \beta} \times \mathsf{Con}_\alpha \to \mathsf{Con}_\beta$, by

$$\{(U_i, a_i)\}_{i \in I} \cdot U :=_\beta \{a_i \mid U \rhd_\alpha U_i\}$$

Lemma 4. *For the application operation the following hold:*

1. *It is well-defined, that is*

$$\{(U_i, a_i)\}_i U \in \mathsf{Con}_\beta$$

2. *For all $U \in \mathsf{Con}_\alpha$, it is*

$$\{(U_i, a_i)\}_i \rhd_{\alpha \to \beta} \{(V_j, b_j)\}_j \leftrightarrow \{(U_i, a_i)\}_i \cdot U \rhd_\beta \{(V_j, b_j)\}_j \cdot U$$

3. *For all $\{(U_i, a_i)\}_i \in \mathsf{Con}_{\alpha \to \beta}$ it is*

$$U \rhd_\alpha V \to \{(U_i, a_i)\}_i \cdot U \rhd_\beta \{(U_i, a_i)\}_i \cdot V$$

Proof. For the first clause: Let $\{(U_i, a_i)\}_i U = \{a_i \mid U \rhd_\alpha U_i\}$; we want to show that $\forall_{i_1, i_2 \in I} a_{i_1} \diamond_\beta a_{i_2}$; since $\{(U_i, a_i)\}_{i \in I} \in \mathsf{Con}_{\alpha \to \beta}$, it is $\forall_{i_1, i_2 \in I} (U_{i_1}, a_{i_1}) \diamond_{\alpha \to \beta} (U_{i_2}, a_{i_2})$, or, equivalently, $\forall_{i_1, i_2 \in I} (U_{i_1} \diamond_\alpha U_{i_2} \to a_{i_1} \diamond_\beta a_{i_2})$; by lemma 2 (modus ponens) we have what we wanted.[3]

For the second clause: For the right direction, let $\{(U_i, a_i)\}_{i \in I} \rhd_{\alpha \to \beta} \{(V_j, b_j)\}_{j \in J}$, which by definition is $\forall_{j \in J} \exists_{i \in I} (V_j \rhd_\alpha U_i \wedge a_i \rhd_\beta b_j)$; we want to show that $\{(U_i, a_i)\}_i \cdot U \rhd_\beta \{(V_j, b_j)\}_j \cdot U$, which by definition is $\{a_i \mid U \rhd_\alpha U_i\} \rhd_\beta \{b_j \mid U \rhd_\alpha V_j\}$, which is provided by the assumption. For the other way round, let $\{(U_i, a_i)\}_i \cdot U \rhd_\beta \{(V_j, b_j)\}_j \cdot U$, or $\{a_i \mid U \rhd_\alpha U_i\} \rhd_\beta \{b_j \mid U \rhd_\alpha V_j\}$; we have to show that $\{(U_i, a_i)\}_{i \in I} \rhd_{\alpha \to \beta} \{(V_j, b_j)\}_{j \in J}$, which by definition is $\forall_{j \in J} \exists_{i \in I} (V_j \rhd_\alpha U_i \wedge a_i \rhd_\beta b_j)$; for every $l \in J$ we may put $U := V_l$ and the assumption then yields $\{a_i \mid V_l \rhd_\alpha U_i\} \rhd_\beta \{b_j \mid V_l \rhd_\alpha V_j\}$; since $V_l \rhd_\alpha V_i$, there is a $k \in I$ so that $V_l \rhd_\alpha U_k$ and $a_k \rhd_\beta b_l$.

For the third clause: Let $U \rhd_\alpha V$; due to transitivity of entailment we have $\forall_i . V \rhd_\alpha U_i \to U \rhd_\alpha U_i$, which proves what we need. $\square$

[3] That finite application is indeed a function, ie, it maps a pair of approximations to a unique approximation, is easy to see.

3 Algebraic acises

Let $A = \{C_1, \ldots, C_k\}$ be an *algebra (given by constructors)* and $*$ be a ground-type atom meaning "least info"; here, each constructor comes with a type of finite arity. We define the *algebraic preacis on A* to be the triad $\tilde{A}_* = (T_{A_*}, \diamond_{A_*}, \triangleright_{A_*})$ where

$$
\begin{aligned}
a \in T_{A_*} &\quad :\Leftrightarrow \quad a = * \vee (\underset{i}{\exists}\, a = C_i\vec{a} \wedge \underset{j}{\forall}\, a_j \in T_{A_*}) \\
a \diamond_{A_*} b &\quad :\Leftrightarrow \quad a = * \vee b = * \vee ((\underset{i}{\exists}.\ a = C_i\vec{a} \wedge b = C_i\vec{b}) \wedge \underset{j}{\forall}\, a_j \diamond_{A_*} b_j) \\
a \triangleright_{A_*} b &\quad :\Leftrightarrow \quad b = * \vee ((\underset{i}{\exists}.\ a = C_i\vec{a} \wedge b = C_i\vec{b}) \wedge \underset{j}{\forall}\, a_j \triangleright_{A_*} b_j)
\end{aligned}
$$

Define the *algebraic acis on A*[4] to be the triad $\tilde{A} = (T_A, \diamond_A, \triangleright_A)$ by restricting $\tilde{A}_*$ as follows:

$$
\begin{aligned}
T_A &:= T_{A_*} - \{*\} \\
\diamond_A &:= \diamond_{A_*} - \{(a,*), (*,a) \mid a \in T_{A_*}\} \\
\triangleright_A &:= \triangleright_{A_*} - \{(a,*) \mid a \in T_{A_*}\}
\end{aligned}
$$

Call the elements of T_{A_*} *preatoms* to distinguish them from the atoms in T_A.[5]

The algebra $\mathbb{B} = \{\mathtt{tt}, \mathtt{ff}\}$ of boolean numbers defines the preacis

$$
\begin{aligned}
b \in T_{\tilde{\mathbb{B}}_*} &\quad :\Leftrightarrow \quad b = * \vee b = \mathtt{tt} \vee b = \mathtt{ff} \\
b_1 \diamond_{\tilde{\mathbb{B}}_*} b_2 &\quad :\Leftrightarrow \quad b_1 = * \vee b_2 = * \vee b_1 = b_2 = \mathtt{tt} \vee b_1 = b_2 = \mathtt{ff} \\
b_1 \triangleright_{\tilde{\mathbb{B}}_*} b_2 &\quad :\Leftrightarrow \quad b_2 = * \vee b_1 = b_2 = \mathtt{tt} \vee b_1 = b_2 = \mathtt{ff}
\end{aligned}
$$

and the parametric algebra $\mathbb{L}(\pi) = \{\mathrm{Nil}_\pi, \mathrm{Cons}_\pi\}$ of lists of objects belonging to an arbitrary acis π, defines the preacis

$$
\begin{aligned}
l \in T_{\tilde{\mathbb{L}}_*(\pi)} &\quad :\Leftrightarrow \quad l = * \vee l = \mathrm{Nil}_\pi \\
&\qquad\quad \vee (l = \mathrm{Cons}_\pi(a, l') \wedge a \in T_\pi \wedge l' \in T_{\tilde{\mathbb{L}}_*(\pi)}) \\
l_1 \diamond_{\tilde{\mathbb{L}}_*(\pi)} l_2 &\quad :\Leftrightarrow \quad l_1 = * \vee l_2 = * \vee l_1 = l_2 = \mathrm{Nil}_\pi \\
&\qquad\quad \vee (l_1 = \mathrm{Cons}_\pi(a_1, l_1') \wedge l_2 = \mathrm{Cons}_\pi(a_2, l_2') \wedge a_1 \diamond_\pi a_2 \wedge l_1' \diamond_{\tilde{\mathbb{L}}_*(\pi)} l_2') \\
l_1 \triangleright_{\tilde{\mathbb{L}}_*(\pi)} l_2 &\quad :\Leftrightarrow \quad l_2 = * \\
&\qquad\quad \vee (l_1 = \mathrm{Cons}_\pi(a_1, l_1') \wedge l_2 = \mathrm{Cons}_\pi(a_2, l_2') \wedge a_1 \triangleright_\pi a_2 \wedge l_1' \triangleright_{\tilde{\mathbb{L}}_*(\pi)} l_2')
\end{aligned}
$$

The most important example of an algebraic acis though is the *acis of natural numbers*. Consider the algebra of natural numbers $\mathbb{N} = \{0, S\}$, where 0 is the zero constructor and S is the successor constructor. The preacis $\tilde{\mathbb{N}}_*$ and acis $\tilde{\mathbb{N}}$

[4] Also referred to as *algerba A with approximations* in [Sch05].

[5] Notice that all preatoms are atoms except for $*$. Furthermore, equality $=_{A_*}$ is defined by

$$
a =_{A_*} b :\Leftrightarrow a = b = * \vee ((\underset{i}{\exists}.\ a = C_i\vec{a} \wedge b = C_i\vec{b}) \wedge \underset{j}{\forall}\, a_j =_{A_*} b_j)
$$

and naturally $=_A := =_{A_*} - \{(*,*)\}$. Equality for finite approximations then $U = V$, should be understood as *set equality*. For simplicity's sake though, we keep this implicit in what follows.

are defined as explained before. In particular, the preacis is defined by

$$\begin{aligned}
a \in T_{\tilde{\mathbb{N}}_*} \quad &:\Leftrightarrow \quad a = * \vee a = 0 \vee (a = Sa' \wedge a' \in T_{\tilde{\mathbb{N}}_*}) \\
a \diamond_{\tilde{\mathbb{N}}_*} b \quad &:\Leftrightarrow \quad a = * \vee b = * \vee a = b = 0 \vee (a = Sa' \wedge b = Sb' \wedge a' \diamond_{\tilde{\mathbb{N}}_*} b') \\
a \rhd_{\tilde{\mathbb{N}}_*} b \quad &:\Leftrightarrow \quad b = * \vee a = b = 0 \vee (a = Sa' \wedge b = Sb' \wedge a' \rhd_{\tilde{\mathbb{N}}_*} b')
\end{aligned}$$

For brevity, write ι for $\tilde{\mathbb{N}}$, S^n for $\underbrace{S \cdots S}_{n}$ and n for $S^n 0$.

The function space $\iota \to \iota$ is defined by

$$\begin{aligned}
(\{a_i\}_i, a) \in T_{\iota\to\iota} \quad &:\Leftrightarrow \quad \forall_i \forall_{i'} a_i \diamond_\iota a_{i'} \wedge a \in T_\iota \\
(\{a_i\}_i, a) \diamond_{\iota\to\iota} (\{b_j\}_j, b) \quad &:\Leftrightarrow \quad \forall_i \forall_j a_i \diamond_\iota b_j \to a \diamond_\iota b \\
(\{a_i\}_i, a) \rhd_{\iota\to\iota} (\{b_j\}_j, b) \quad &:\Leftrightarrow \quad \forall_i \exists_j b_j \rhd_\iota a_i \wedge a \rhd_\iota b
\end{aligned}$$

The function space $(\iota \to \iota) \to \iota$ is defined by

$$\begin{aligned}
&(\{(\{a_{k_i}\}_{k_i}, a_i)\}_i, a) \in T_{(\iota\to\iota)\to\iota} \\
&\qquad :\Leftrightarrow \left(\forall_i \forall_{i'} .\ \forall_{k_i} \forall_{k_{i'}} a_{k_i} \diamond_\iota a_{k_{i'}} \to a_i \diamond_\iota a_{i'} \right) \wedge a \diamond_\iota b \\
&(\{(\{a_{k_i}\}_{k_i}, a_i)\}_i, a) \diamond_{(\iota\to\iota)\to\iota} (\{(\{b_{k_j}\}_{k_j}, b_j)\}_j, b) \\
&\qquad :\Leftrightarrow \left(\forall_i \forall_j .\ \forall_{k_i} \forall_{k_j} a_{k_i} \diamond_\iota b_{k_j} \to a_i \diamond_\iota b_j \right) \to a \diamond_\iota b \\
&(\{(\{a_{k_i}\}_{k_i}, a_i)\}_i, a) \rhd_{(\iota\to\iota)\to\iota} (\{(\{b_{k_j}\}_{k_j}, b_j)\}_j, b) \\
&\qquad :\Leftrightarrow \left(\forall_i \exists_j .\ \forall_{k_j} \exists_{k_i} a_{k_i} \rhd_\iota b_{k_j} \wedge b_j \rhd_\iota a_i \right) \wedge a \rhd_\iota b
\end{aligned}$$

The function space $\iota \to (\iota \to \iota)$ is defined by

$$\begin{aligned}
&(\{a_{i_1}\}_{i_1}, \{a_{i_2}\}_{i_2}, a) \in T_{\iota\to(\iota\to\iota)} \\
&\qquad :\Leftrightarrow \forall_{i_1} \forall_{i_1'} a_{i_1} \diamond_\iota a_{i_1'} \wedge \forall_{i_2} \forall_{i_2'} a_{i_2} \diamond_\iota a_{i_2'} \wedge a \in T_\iota \\
&(\{a_{i_1}\}_{i_1}, \{a_{i_2}\}_{i_2}, a) \diamond_{\iota\to(\iota\to\iota)} (\{b_{j_1}\}_{j_1}, \{b_{j_2}\}_{j_2}, b) \\
&\qquad :\Leftrightarrow \forall_{i_1} \forall_{j_1} a_{i_1} \diamond_\iota b_{j_1} \to \forall_{i_2} \forall_{j_2} a_{i_2} \diamond_\iota b_{j_2} \to a \diamond_\iota b \\
&(\{a_{i_1}\}_{i_1}, \{a_{i_2}\}_{i_2}, a) \rhd_{\iota\to(\iota\to\iota)} (\{b_{j_1}\}_{j_1}, \{b_{j_2}\}_{j_2}, b) \\
&\qquad :\Leftrightarrow \forall_{i_1} \exists_{j_1} b_{j_1} \rhd_\iota a_{i_1} \wedge \forall_{i_2} \exists_{j_2} b_{j_2} \rhd_\iota a_{i_2} \wedge a \rhd_\iota b
\end{aligned}$$

where we write (U_1, U_2, a) for $(U_1, (U_2, a))$.

One should notice how the notions of consistency and entailment between atoms of higher types breaks down to consistency and entailment between atoms of ground type. Dub ι and its function spaces *arithmetical acises*.

4 Normal Form of Algebraic Approximations

By the very definition of entailment in an abstract acis, we have non-antisymmetricity, ie, we can have two different atoms entailing one another. An acis where antisymmetricity for entailment holds will be called *antisymmetrical acis*. By induction on the formation of atoms one can prove that

Lemma 5. *All algebraic acises are antisymmetrical.*[6]

Even in the case of an antisymmetrical acis though, non-antisymmetricity may appear in its finite approximations as well as in atoms and approximations of its function spaces. For an acis α, define the following equivalence on Con_α:

$$U \sim_\alpha V :\Leftrightarrow U \rhd_\alpha V \wedge V \rhd_\alpha U$$

Non-trivial examples of equivalent finite approximations in arithmetical acises are the following:

$$\begin{array}{rcl} \{S^2*\} & \sim_\iota & \{S^2*, S*\} \\ \{(\{S^2*\}, S^2*)\} & \sim_{\iota\to\iota} & \{(\{S^2*\}, S^2*), (\{2\}, S*)\} \\ \{(\{(\{0\}, S*)\}, 0)\} & \sim_{(\iota\to\iota)\to\iota} & \{(\{(\{0\}, S*)\}, 0)\, , (\{(\{0\}, S*), (\{0\}, 1)\}, 0)\} \\ \{(\{S^2*\}, \{S^2*\}, S*)\} & \sim_{\iota\to(\iota\to\iota)} & \{(\{S^2*, S*\}, \{S^2*, S*\}, S*)\} \end{array}$$

We would like to have a notion of "normal form" for approximations, such that every approximation would have a normal form and two approximations in normal form would be equivalent if and only if they were equal. This turns out to be easily feasible for algebraic acises and their function spaces, as we now show.[7]

The definition of the set NF_α of finite approximations in *normal form*, for acises α built on algebraic acises, is inductive on the formation of the acis:

- For an algebraic acis α, a finite approximation $U \in \mathsf{Con}_\alpha$ is in normal form if it *contains no entailments*, ie, if none of its elements entails some other:
$$\{a_i\}_i \in \mathsf{NF}_\alpha :\Leftrightarrow \underset{i}{\forall} \underset{j\neq i}{\forall} a_i \not\rhd_\alpha a_j$$

- For function spaces α, β built on algebraic acises, a finite approximation $\{(U_i, b_i)\}_i \in \mathsf{Con}_{\alpha\to\beta}$ is in normal form if all its lower-type objects are either already in normal form or else atoms and if it contains no entailments:

$$\{(U_i, a_i)\}_i \in \mathsf{NF}_{\alpha\to\beta} :\Leftrightarrow$$
$$\underset{i}{\forall} .\ U_i \in \mathsf{NF}_\alpha \wedge a_i \in \mathsf{NF}_\beta \cup T_\beta \wedge \underset{j\neq i}{\forall} (U_i, a_i) \not\rhd_{\alpha\to\beta} (U_j, a_j)$$

Proposition 6. *For all acises α built on algebraic acises the following hold:*

[6] A parametric algebraic acis, like $\tilde{\mathbb{L}}(\pi)$, is antisymmetric if the parameter acis π is antisymmetric (it doesn't even need to be algebraic). For simplicity's sake, we focus here on non-parametrical algebraic acises.

[7] Normal forms for finite approximations in *flat* information systems were treated twenty years ago in [Sch86].

1. *For all* $U \in \mathsf{Con}_\alpha$ *there is a* $U' \in \mathsf{NF}_\alpha$ *so that* $U \sim_\alpha U'$.

2. *For all* $U, V \in \mathsf{NF}_\alpha$ *it is* $U \sim_\alpha V \leftrightarrow U = V$.

Proof. We prove the more general step cases. For the first clause: Let $\{(U_i, a_i)\}_{i\in I} \in \mathsf{Con}_{\alpha\to\beta}$ with $U_i \in \mathsf{NF}_\alpha$, $a_i \in \mathsf{NF}_\beta \cup T_\beta$, for every i; suppose that there are $k, l \in I$ such that $(U_k, a_k) \rhd_{\alpha\to\beta} (U_l, a_l)$; set $I' := I - \{l\}$; it is easy to see that $\{(U_i, a_i)\}_{i\in I} \sim_{\alpha\to\beta} \{(U'_i, a'_i)\}_{i'\in I'}$.

The left direction of the second clause is obvious. For the right direction let $\{(U_i, a_i)\}_{i\in I}, \{(V_j, b_j)\}_{j\in J} \in \mathsf{NF}_{\alpha\to\beta}$ be such that $\{(U_i, a_i)\}_{i\in I} \sim_{\alpha\to\beta} \{(V_j, b_j)\}_{j\in J}$; this unfolds to

$$\forall_j \exists_{i(j)} (U_{i(j)}, a_{i(j)}) \rhd_{\alpha\to\beta} (V_j, b_j) \wedge \forall_i \exists_{j(i)} (V_{j(i)}, b_{j(i)}) \rhd_{\alpha\to\beta} (U_i, a_i)$$

which is equivalent to $\forall_j \exists_i (V_j, b_j) \sim_{\alpha\to\beta} (U_i, a_i)$; by definition we get $\forall_j \exists_i .\ V_j \sim_\alpha U_i \wedge b_j \sim_\beta a_i$, which, by assumption and induction hypothesis, yields $\forall_j \exists_i (V_j, b_j) = (U_i, a_i)$; similarly we have $\forall_i \exists_j (U_i, a_i) = (V_j, b_j)$, which concludes the proof. □

Corollary 7. *For all acises* α *built on algebraic acises it is* $\mathsf{Con}_\alpha / \sim_\alpha \cong \mathsf{NF}_\alpha$.

We close with two remarks. First, notice that arithmetical approximations have fairly simple normal forms, since they are built on singletons of T_ι. Namely, normal forms in the arithmetical acises we introduced in the previous section, follow the patterns shown below:

$$\begin{array}{rcl} \mathsf{NF}_\iota & : & a \\ \mathsf{NF}_{\iota\to\iota} & : & \{(a_i^1, a_i^2)\}_i \\ \mathsf{NF}_{(\iota\to\iota)\to\iota} & : & \{(\{(a_{j_i}^1, a_{j_i}^2)\}_{j_i}, a_i^3)\}_i \\ \mathsf{NF}_{\iota\to(\iota\to\iota)} & : & \{(a_i^1, (a_i^2, a_i^3))\}_i \end{array}$$

where curly brackets of singletons have been omitted.

Secondly, application between finite algebraic approximations in normal form does not necessarily yield an approximation in normal form. A counter-example for arithmetical acises, is the application of $\{(\{S*\}, S*), (\{S^2*\}, S^2*)\} \in \mathsf{NF}_{\iota\to\iota}$ to $\{2\} \in \mathsf{NF}_\iota$, which yields $\{S^2*, S*\} \notin \mathsf{NF}_\iota$.

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