Exercises on morphisms and functors

Exercise 1. Let A, B be two objects of a category $\mathcal{C}$. Consider the functors

$$h_A = \mathrm{Hom}_{\mathcal{C}}(-, A)\colon \mathcal{C} \to \mathbf{Sets}, \quad h_B = \mathrm{Hom}_{\mathcal{C}}(-, B)\colon \mathcal{C} \to \mathbf{Sets}.$$

Prove that h_A and h_B are naturally isomorphic if and only if A and B are isomorphic in $\mathcal{C}$.

Conclude that, if a functor F is corepresentable, then it is represented by a unique object of $\mathcal{C}$ up to isomorphisms.

Exercise 2. Let R be a ring and let $\mathcal{C}_R$ be the category with a single object $*$ and with $\mathrm{Hom}_{\mathcal{C}}(*, *) = R$.

Prove that the category $\mathcal{H}(\mathcal{C}_R, \mathcal{A}b)$ of the additive contravariant functors $F\colon \mathcal{C}_R \to \mathcal{A}b$ is isomorphic to the category of right R-modules.

Exercise 3. Let $\mathcal{C}$ be a category with a zero object $0_{\mathcal{C}}$. Prove that, for every pair of objects A, B of $\mathcal{C}$, there is a unique zero morphism $A \to B$.

Exercise 4. Prove that an equalizer (a coequalizer) is a monomorphism (epimorphism). In particular a kernel is a monomorphism and a cokernel is an epimorphism.

Exercise 5. Let $(P, \leq)$ be a partial ordered set and let $\mathcal{P}$ be the category with objects the elements of P and with Hom-sets for every pair p, q of objects of P, defined by $\mathrm{Hom}(p, q) = \emptyset$ if $p \not\leq q$ and a unique morphism if $p \leq q$.

Prove the product of two objects $p, q \in P$ is their greatest lower bound $p \wedge q$ (if it exists), and their coproduct is the least upper bound $p \vee q$ (if it exists).

Exercise 6. Let $A \xrightarrow{f} B$ be a morphism in a category $\mathcal{C}$.

Prove that $B \xrightarrow{h} C$ is a cokernel of f if and only if C corepresents the functor

$$\mathrm{Ker} f^*\colon \mathrm{Hom}_{\mathcal{C}}(B, -) \xrightarrow{\mathrm{Hom}(f,-)} \mathrm{Hom}_{\mathcal{C}}(A, -).$$

Exercise 7. Let A, B be two objects of a category $\mathcal{C}$.

Prove that a coproduct $A \coprod B$ exists in $\mathcal{C}$ if and only if it corepresents the functor

$$\mathrm{Hom}_{\mathcal{C}}(A, -) \times \mathrm{Hom}_{\mathcal{C}}(B, -).$$