Exercises on additive and abelian categories

Exercise 1. For each abelian group G the torsion subgroup $t(G)$ of G consists of the elements $x \in G$ for which there is $0 \neq n \in \mathbb{N}$ with $nx = 0$. G is said to be a torsion group if $G = t(G)$.

(1) Show that every abelian group homomorphism $G \to H$ induces a homomorphism $t(G) \to t(H)$.
(2) Let $\mathbf{Ab}$ denote the category of abelian groups. Prove that $t \colon \mathbf{Ab} \to \mathbf{Ab}$ defines a functor.
(3) Prove that the monomorphism $\eta_G \colon t(G) \to G$, for every G defines a natural transformation $\eta \colon t \to 1_{\mathbf{Ab}}$.
(4) Show that t is a left exact functor

Exercise 2. Recall that an abelian group G is divisible if, for every $x \in G$ and every $0 \neq a \in \mathbb{Z}$ there is $y \in G$ such that $x = ay$.

Let $\mathcal{D}$ be the full subcategory of $\mathcal{A}b$ consisting of the divisible abelian groups.

Prove that $\mathcal{D}$ is additive, every morphism in $\mathcal{D}$ has kernel and cokernel, but $\mathcal{D}$ is not abelian.

(Hint: if f is a morphism in $\mathcal{D}$, then $\mathrm{Ker} f$ is the maximal divisible subgroup of the kernel of f in $\mathcal{A}b$.

The morphism $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism and an epimorphism in $\mathcal{D}$, but it is not an isomorphism.)

Exercise 3. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let $F \colon \mathcal{C} \to \mathcal{D}$ be a left and right exact functor (i.e. F preserves short exact sequences.)

Prove that, if $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence, then $\mathrm{Im} F(f) = F(\mathrm{Im} f)$.

Exercise 4. In an abelian category $\mathcal{C}$ consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0
\end{array}$$

Prove that if α and γ are monomorphisms (epimorphisms), then β is a monomorphism (epimorphism).

Exercise 5. Let

$$\begin{CD} P @>{p_2}>> C_2 \\ @V{p_1}VV @VV{f_2}V \\ C_1 @>>{f_1}> C \end{CD}$$

be a pull-back diagram. We have shown in the lectures that if f_1 is an epimorphism, then so is p_2. Prove that:

(1) if f_1 is a monomorphism, then so is p_2.
(2) if f_1 is a kernel of g, then p_2 is a kernel of $g \circ f_2$.

Exercise 6. Let

$$\begin{CD} C @>{f_1}>> C_1 \\ @V{f_2}VV @VV{\nu_1}V \\ C_2 @>>{\nu_2}> P \end{CD}$$

be a push-out diagram. Prove that:

(1) if f_1 is a monomorphism, then so is ν_2.
(2) if f_1 is an epimorphism, then so is ν_2.
(3) if f_1 is a cokernel of g, then ν_2 is a cokernel of $g \circ f_2$.

Exercise 7. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\Big\downarrow & & {\scriptstyle\beta}\Big\downarrow & & \Big\|{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows in an abelian category $\mathcal{C}$.

(1) Show that the left square is a pull-back.
(2) (difficult: we will prove it in some lecture) Show that the left square is a push out diagram.