# Exercises on limits, colimits and adjoint pairs

**Exercise 1.** Let $\mathcal{C}$ be an abelian category. Let $F\colon I \to \mathcal{C}$ be a functor from a small category $I$ to $\mathcal{C}$ and assume that both $\varinjlim F$ and $\varprojlim F$ exist in $\mathcal{C}$.
Prove that for every object $C$ in $\mathcal{C}$ there are isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}(C, \varprojlim F) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}}(C, F(i));$$

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim F, C) \cong \varprojlim \mathrm{Hom}_{\mathcal{C}}(F(i), C)$$

where the limits and colimits in the right hand side are taken in category of abelian groups.

**Exercise 2.** Let $\mathcal{T}$ denote the full subcategory of **Ab** consisting of the torsion abelian groups and let $\iota\colon \mathcal{T} \to \mathbf{Ab}$ denote the embedding functor. Let $t\colon \mathbf{Ab} \to \mathcal{T}$ be the functor sending an abelian group $G$ to its torsion submodule.

Show that that the pair $(\iota, t)$ is an adjoint pair with $t$ right adjoint to $\iota$.

**Exercise 3.** Let $m, n \in \mathbb{N}$ and let:

$$0 \to \{A_n, f_{mn}\}_{m \leq n} \xrightarrow{\phi} \{B_n, g_{mn}\}_{m \leq n} \xrightarrow{\psi} \{C_n, h_{mn}\}_{m \leq n} \to 0$$

be a short exact sequence of countable inverse systems of $R$-modules. Show that, if $f_{mn}$ are epimorphisms for all $m \leq n$, then $\varprojlim \psi$ is an epimorphism.
Hint: without loss of generality, assume that $A_n \leq B_n$ and that $f_{mn}$ are the restrictions of $g_{mn}$ to $A_n$.

**Exercise 4.** Let $\mathcal{C}$ be an additive category admitting all (filtrant) direct limits.

(1) An object $C$ of $\mathcal{C}$ is said to be of *finite type* if the natural map:

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(C, F(i)) \to \mathrm{Hom}_{\mathcal{C}}(C, \varinjlim F(i))$$

is injective, for all functors $F\colon I \to \mathcal{C}$ from a filtered partially ordered set $I$.
Show that, if $\mathcal{C}$=Mod-$R$, for some ring $R$, then the above definition coincides with the notion of finitely generated module.
(Hint: use the fact that, if $\mathcal{F}$ denotes the set of finitely generated submodules of a module $C$, then $C/\sum_{A\in\mathcal{F}} A \cong \varinjlim_{A\in\mathcal{F}} C/A$.)

(2*) An object $C$ of $\mathcal{C}$ is said to be of *finite presentation* if for every filtered partially ordered set $I$ the natural map:

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(C, F(i)) \to \mathrm{Hom}_{\mathcal{C}}(C, \varinjlim F(i))$$

is bijective, for all functors $F\colon I \to \mathcal{C}$.
Show that, if $\mathcal{C}$=Mod-$R$, for some ring $R$, then the above definition coincides with the notion of finitely presented module.

(Recall that a module $M_R$ is finitely presented if there is a short exact sequence $0 \to K \to R^n \to M \to 0$, with $K$ finitely generated.
$*$ means difficult. See e.g. "Foundations of Module and Ring Theory" by R. Wisbauer, Ch. 5, 25.4.)

**Exercise 5.** Let $M$ be a finitely presented module and $\phi\colon F \to M$ an epimorphism with $F$ a finitely generated module.

Prove that Ker$\phi$ is finitely generated.

**Exercise 6.** Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors between abelian categories $\mathcal{C}, \mathcal{D}$ such that $(L, R)$ is an adjoint pair.

Let $F\colon I \to \mathcal{D}$ be a functor from a small category $I$ to $\mathcal{D}$ and assume that $(\varprojlim F, p_i)$ exists in $\mathcal{D}$.
Show that $(R(\varprojlim F), R(p_i))$ is a limit of $R \circ F\colon I \to \mathcal{C}$.