Exercises on generators, cogenerator, exact functors.

Exercise 1. Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be functors between abelian categories $\mathcal{C}, \mathcal{D}$ such that (L, R) is an adjoint pair.

Prove that if R is exact, then L preserves projectives; that is, if P is a projective object of $\mathcal{C}$, then $L(P)$ is a projective object of $\mathcal{D}$. Dually, prove that if L is exact, then R preserves injective objects.

Exercise 2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories $\mathcal{C}, \mathcal{D}$.

(1) Prove that F is faithful iff $F(C) \neq 0$ for every non-zero object C of $\mathcal{C}$
(2) Prove that a projective object P of $\mathcal{C}$ is a generator iff, for every non-zero object $C \in \mathcal{C}$ there is a non-zero morphism $P \to C$.

Recall that P is a projective object (a generator) if the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact (faithful).

Exercise 3. An object V of a category $\mathcal{C}$ is called *a cogenerator* if $\mathrm{Hom}_{\mathcal{C}}(-, V)$ is faithful.

(1) Prove that, if $\mathcal{C}$ admits products, an object V is a cogenerator iff for every object $C \in \mathcal{C}$ there is a monomorphism $\mu\colon C \to V^I$, for some set I.
(2) An object V of an abelian category $\mathcal{C}$ is injective if $\mathrm{Hom}_{\mathcal{C}}(-, V)$ is exact.

    Prove that an injective object V is a cogenerator iff for every non-zero object $C \in \mathcal{C}$ there is a non-zero morphism $C \to V$.

Exercise 4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories $\mathcal{C}, \mathcal{D}$.

We say that F reflects exactness if $A \to B \to C$ is exact in $\mathcal{C}$ provided that $F(A) \to F(B) \to F(C)$ is exact in $\mathcal{D}$.

Prove that, if F is exact and fully faithful, then F reflects exactness.

Exercise 5. Let $\mathcal{A}$ be a small additive category and let $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}b)$ denote the category of additive contravariant functors from $\mathcal{A}$ to the category $\mathcal{A}b$ of abelian groups.
For each object A in $\mathcal{A}$, let h_A be the functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$.

Prove that the functor $P = \coprod_{A \in \mathcal{A}} h_A$ is faithful and a project object of $\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathcal{A}b)$.