Padova, July 6 2012

FROM THE THEORY

1. Let $\mathcal{B}$ be a small additive category and let $\mathrm{Hom}(\mathcal{B}^{\mathrm{op}}, \mathcal{A}b)$ be the abelian category of the additive contravariant functors $F\colon \mathcal{B} \to \mathcal{A}b$. For every object B of $\mathcal{B}$ let $h_B = \mathrm{Hom}_{\mathcal{B}}(-, B)$.

(1) State Yoneda's Lemma and prove that h_B is a projective object of $\mathrm{Hom}(\mathcal{B}, \mathcal{A}b)$.

(2) Prove that the functor

$$h\colon \mathcal{B} \to \mathrm{Hom}(\mathcal{B}^{\mathrm{op}}, \mathcal{A}b), \quad B \mapsto h_B$$

is fully faithful.

(3) Assume that $\mathcal{B}$ is abelian. Prove that h is a left exact functor.

2. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories and $F\colon \mathcal{A} \to \mathcal{B}$ a right exact functor. Assume that $\mathcal{A}$ has enough projectives.

(1) Define projective resolutions of objects of $\mathcal{A}$.

(2) Give the definition of the left derived functors

$$L_i F\colon \mathcal{A} \to \mathcal{B}.$$

(3) Complete the definition:
An object $Q \in \mathcal{A}$ is said F-acyclic if ...

3. State and prove the Horseshoe Lemma.

4. Consider the category Mod-R of right R-modules.

(1) Give the definition of the flat dimension, w.d. (M) of a right R-module M.

(2) Prove that a right R-module M is flat if and only if

$$\mathrm{Tor}_1^R(M, R/I) = 0$$

for every left ideal I of R.

EXERCISES:

5. Let $\mathcal{C}$ be an abelian category.

(1) Let α be a kernel of some morphism. Prove that $\alpha = \ker(\mathrm{coker}\alpha)$.
(2) Let α be a morphism in $\mathcal{C}$. Prove that if α is an epimorphism, then $\alpha = \mathrm{coker}(\ker\alpha)$.

6. Let $\mathcal{A}$ be an abelian category and

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

be a commutative diagram in $\mathcal{A}$.

Show that the outer rectangle is a pull-back (a push-out) if each of the two squares is a pull-back (a push-out).

7. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories.

Let X be a complex with terms in $\mathcal{B}$ and let $G\colon \mathcal{B} \to \mathcal{C}$ be an exact functor.

(1) Prove that $H^i(G(X)) \cong G(H^i(X))$.
(2) Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has enough injectives and denote by R^iF the right derived functors of F. Prove that for every $A \in \mathcal{A}$, $R^i(G \circ F(A)) \cong G(R^iF(A))$.

8. Let $\mathcal{C}h(\mathcal{A})$ be the category of cochain complexes with terms in the abelian category $\mathcal{A}$. For every object $A \in \mathcal{A}$ let $\tilde{A}$ be the complex

$$\ldots \to 0 \to 0 \to A \overset{id}{\to} A \to 0 \to 0 \to \ldots'$$

with the first A in degree 0. Let (X, d_X) be an arbitrary complex in $\mathcal{C}h(\mathcal{A})$.

(1) Show that $\mathrm{Hom}_{\mathcal{C}h(\mathcal{A})}(\tilde{A}, X) \cong \mathrm{Hom}_{\mathcal{A}}(A, X^0)$.
(2) Conclude that every cochain map $\tilde{A} \to X$ is null homotopic.