Padova, July 8, 2013

Rings and Modules

FROM THE THEORY

1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories.

(1) Complete the definition: F is an equivalence if ....

(2) Prove that if F is an equivalence, then F is fully faithful and essentially surjective.

2. Let f be a morphism in a category $\mathcal{C}$. Assume that $f = \text{coker} g$ for some morphism g and that $\ker f$ exists.

Prove that $f = \text{coker}(\ker f)$.

3. Let $\mathcal{C}h(\mathcal{A})$ be the category of cochain complexes with terms in the abelian category $\mathcal{A}$.

Let $f\colon X \to Y$ be a cochain map in $\mathcal{C}h(\mathcal{A})$.

(1) Complete the definition: f is a "quasi-isomorphism" if ...

(2) Complete the definition: f is a homotopy equivalence if ...

(3) Prove that if f is a homotopy equivalence then f is a quasi-isomorphism but that the converse doesn't hold.

4. Denote by $r.gl.dim\ R$ the right global dimension of a ring R and by $p.dim\ M$ the projective dimension of the right R-module M.

Prove that $r.gl.dim\ R = \sup\{p.dim\ (R/I) \mid I \text{ is a right ideal of } R\}$.

EXERCISES:

5. Let R be a ring and M_R be a right R-module.

Let $\{N_i\}_{i\in I}$ be a directed system of submodules of M.

Prove that $\varinjlim_{i\in I} M/N_i \cong M/N$ where $N = \sum_{i\in I} N_i$.

6. $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors between categories $\mathcal{C}$, $\mathcal{D}$ such that $F \circ G$ is naturally isomorphic to $1_{\mathcal{D}}$ and $G \circ F$ is naturally isomorphic to $1_{\mathcal{C}}$.

Prove that the pairs (F, G) and (G, F) are adjoint pairs.

7. (Kaplansky's Lemma) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules and let $p.dim\ M$ denote the projective dimension of the module M.

Prove that the following hold:

(1) If two of $p.dim\ A$, $p.dim\ B$, $p.dim\ C$ are finite so is the third.
(2) One and only one of the following cases may occur
  (a) $p.dim\ A < p.dim\ B = p.dim\ C$.
  (b) $p.dim\ B < p.dim\ A = p.dim\ C - 1$.
  (c) $p.dim\ A = p.dim\ B \geq p.dim\ C - 1$.

8. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of cochain complexes with terms in an abelian category $\mathcal{A}$.

Assume that the sequence is degreewise splitting, i.e.

$$0 \to L^n \xrightarrow{f^n} M^n \xrightarrow{g^n} N^n \to 0$$

is splitting for every $n \in \mathbb{Z}$.

Prove that there is a cochain map $h\colon N[-1] \to L$ such that:

$$d_M^n = \begin{pmatrix} d_L^n & h^{n+1} \\ 0 & d_N^n \end{pmatrix}$$

where d_L, d_M, d_N are the differentials of L, M, N, respectively.