Padova, June 18, 2013

Rings and Modules

FROM THE THEORY

1. In an additive category $\mathcal{C}$ consider a diagram:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & C \end{array}$$

(1) Define a pullback (P, g_1, f_1) ($g_1 \colon P \to B$, $f_1 \colon P \to A$) of the diagram.
(2) Realize the kernel of a morphism and the product of two objects as suitable pullbacks (when they exist).
(3) Prove that if f is mono, then f_1 is mono.
(4) (Optional) If $\mathcal{C}$ is abelian, prove that if f is epi, then f_1 is epi.

2. Let C be an object of a category $\mathcal{C}$.

(1) Complete the definition: B is a suboject of C if there is a monomorphism $B \overset{f}{\to} C$ representing an equivalence class of monomorphisms, i.e. ...
(2) If $A \overset{f}{\to} C$ and $B \overset{g}{\to} C$ are two subobjects of C, write $A \subseteq B$ in C if there is a morphism $\alpha \colon A \to B$ such that $f = g \circ \alpha$.
(3) Let now $\mathcal{C}$ be an abelian category and $A \overset{f}{\to} B \overset{g}{\to} C$ be a sequence of morphisms.
Prove that $g \circ f = 0$ if and only if $\mathrm{Im} f \subseteq \mathrm{Ker} g$.

3. Let $\mathcal{C}h(\mathcal{A})$ be the category of cochain complexes with terms in the abelian category $\mathcal{A}$.

Let $f \colon X \to Y$ be a cochain map in $\mathcal{C}h(\mathcal{A})$.

(1) Complete the definition: f is a "quasi-isomorphism" if ...
(2) Define the complex $\mathrm{Cone}(f)$.
(3) Prove that f is a quasi-isomorphism if and only if $\mathrm{Cone}(f)$ is an acyclic complex.

4. Let R be a ring and let M be a right R-module. Denote by M^* the left R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Prove that M is a flat right R-module if and only if M^* is an injective left R-module.

EXERCISES:

5. Let $F\colon \mathcal{C} \to \mathit{Sets}$ be a contravariant functor from a category $\mathcal{C}$ to the category of sets. F is said to be representable if there is an object $Z \in \mathcal{C}$ such that F is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, Z)$.

Assume that F is a representable functor. Prove that the object Z is uniquely determined up to isomorphism.

6. Let

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & L & \xrightarrow{j} & M & \xrightarrow{h} & N & & \end{array}$$

be a commutative diagram in an abelian category $\mathcal{A}$ with exact rows.

Show that there is a morphism $\lambda\colon C \to M$ such that $h \circ \lambda = \gamma$ if and only if there is a morhism $\theta\colon B \to L$ such that $\theta \circ f = \alpha$.

7. Let $L\colon \mathcal{C} \to \mathcal{D}$ $R\colon \mathcal{D} \to \mathcal{C}$ be functors between additive categories such that (L, R) is an adjoint pair.

Prove that R is faithful if and only if the counit morphism

$$\xi_D\colon LR(D) \to D$$

is an epimorphism for every $D \in \mathcal{D}$.

8. Let $\mathcal{C}h(\mathcal{A})$ be the category of cochain complexes with terms in an abelian category $\mathcal{A}$. For every object $A \in \mathcal{A}$ and every $n \in \mathbb{Z}$ consider the complexes

$$D^n(A) = \ldots \to 0 \to A^n \overset{id_A}{\to} A^{n+1} \to 0 \to \ldots,$$
$$S^n(A) = \ldots \to 0 \to 0 \to A^n \to 0 \to 0 \to \ldots,$$

with $A^n = A^{n+1} = A$.

Let (X, d_X) be an arbitrary complex in $\mathcal{C}h(\mathcal{A})$. Show that:

(1) $\mathrm{Hom}_{\mathcal{C}h(\mathcal{A})}(X, D^n(A)) \cong \mathrm{Hom}_{\mathcal{A}}(X^{n+1}, A)$.

(2) $\mathrm{Hom}_{\mathcal{C}h(\mathcal{A})}(X, S^n(A)) \cong \mathrm{Hom}_{\mathcal{A}}(X^n/B^n(X), A)$, where for every $n \in \mathbb{Z}$, $B^n(X)$ is the n^{th}-boundary of X.