Padova, June 20, 2012

FROM THE THEORY

1. Let $\mathcal{C}$ be a preadditive category having a zero object.

(1) Give the definition of kernel and cokernel of a morphism in $\mathcal{C}$.
(2) Prove that every kernel is a monomorphism (and every cokernel is an epimorphism).
(3) Give the definition of a pushout and prove that a cokernel is a pushout.

2. Let $\mathcal{C}$ be a preadditive category and I a small category.
Let $F\colon \to \mathcal{C}$ be a functor.

(1) Give the definition of $\lim F$ and prove that, when it exists, $\lim F$ is unique up to isomorphism.
(2) Assume that I is a discrete category (i.e. the only morphisms in I are the identities). Prove that $\lim F$ is isomorphic to the product $\prod_{i\in I} F(i)$.

3. Let $\mathcal{C}, \mathcal{D}$ be abelian categories and $L\colon \mathcal{C} \to \mathcal{D}$, $R\colon \mathcal{D} \to \mathcal{C}$ be functors.

(1) Complete the definition:
    (L, R) is an adjoint pair if ...
    (Express also by diagrams the naturality of the bijections involved)
(2) Define the unit and the counit of the adjunction.
(3) Assume that R is exact and P is a projective object of $\mathcal{C}$. Show that $L(P)$ is a projective object of $\mathcal{D}$.

4. Let X and Y be cochain complexes with terms in an abelian category $\mathcal{A}$ and let $f, g\colon X \to Y$ be cochain maps.
Complete the definitions:

(1) f is null homotopic if...
(2) f and g are homotopic maps if ...
(3) f is a homotopy equivalence if ...
(4) Prove that two homotopic maps induce the same morphisms $\mathrm{H}^n(X) \to \mathrm{H}^n(Y)$, for every $n \in \mathbb{Z}$.

EXERCISES:

5. Let $\mathcal{C}$ be a category and let I be a category consisting of the three objects $1, 2, 3$ and morphisms the identities together with $\alpha\colon 1 \to 2$, $\beta\colon 2 \to 3$. Consider the category $\mathcal{C}^I$ of the functors $F\colon I \to \mathcal{C}$ with morphisms the natural transformations between functors.

(1) Describe the objects of $\mathcal{C}^I$ as diagrams in $\mathcal{C}$.
(2) Describe the morphisms in $\mathcal{C}^I$.

6. Let $\mathcal{A}$ be an abelian category and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \| {\scriptstyle 1_C} & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \longrightarrow & 0
\end{array}$$

be a commutative diagram in $\mathcal{A}$ with exact rows.
Show that the left square is a pull-back (and a push-out) diagram.

7. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a faithful functor and let f be a morphism in $\mathcal{A}$.

(1) Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).
(2) Assume that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism.

8. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules and let $p.d.(M)$, $i.d.(M)$ and $f.d.(M)$ denote the projective, injective, flat dimension of the module M.

Prove that $p.d.(B) \leq \max\{p.d.(A), p.d.(C)\}$ and that equality holds except when $p.d.(C) = p.d.(A) + 1$.

Note that the same result holds for the injective and the flat dimensions.