

COVERS AND DIRECT LIMITS: A CONTRAMODULE-BASED APPROACH

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ABSTRACT. We present applications of contramodule techniques to the Enochs conjecture about covers and direct limits, both in the categorical tilting context and beyond. In the n -tilting-cotilting correspondence situation, if \mathbf{A} is a Grothendieck abelian category and the related abelian category \mathbf{B} is equivalent to the category of contramodules over a topological ring \mathfrak{R} belonging to one of certain four classes of topological rings (e. g., \mathfrak{R} is commutative), then the left tilting class is covering in \mathbf{A} if and only if it is closed under direct limits in \mathbf{A} , and if and only if all the discrete quotient rings of the topological ring \mathfrak{R} are perfect. More generally, if M is a module satisfying a certain telescope Hom exactness condition (e. g., M is Σ -pure- Ext^1 -self-orthogonal) and the topological ring \mathfrak{R} of endomorphisms of M belongs to one of certain seven classes of topological rings, then the class $\text{Add}(M)$ is closed under direct limits if and only if every countable direct limit of copies of M has an $\text{Add}(M)$ -cover, and if and only if M has perfect decomposition. In full generality, for an additive category \mathbf{A} with (co)kernels and a precovering class $\mathbf{L} \subset \mathbf{A}$ closed under summands, an object $N \in \mathbf{A}$ has an \mathbf{L} -cover if and only if a certain object $\Psi(N)$ in an abelian category \mathbf{B} with enough projectives has a projective cover. The 1-tilting modules and objects arising from injective ring epimorphisms of projective dimension 1 form a class of examples which we discuss.

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INTRODUCTION

0.0. The main result (or one of the main results) of Bass' 1960 paper [6] can be stated as follows: given an associative ring R , every left R -module has a projective cover if and only if the class of projective modules is closed under direct limits in the category of left R -modules. Subsequently, in 1981 Enochs proved that any precovering class of modules closed under direct limits is covering [14, Theorems 2.1 and 3.1], and in the late 1990s he asked the question whether every covering class of modules is closed under direct limits (see [19, Section 5.4]; cf. [4, Section 5]).

A hypothetical general positive answer to this question is sometimes called “the Enochs conjecture”. A positive answer in many particular cases was recently obtained by Angeleri Hügel, Šaroch, and Trlifaj [4, Theorem 5.2 and Corollary 5.5], based on set-theoretical tools developed by Šaroch in [31]. (An alternative elementary proof of a part of the results of [4] is suggested in the preprint [9].) The aim of this paper is to offer a new approach to proving particular cases of the Enochs conjecture, based on the recently developed techniques of contramodules and categorical tilting theory [25, 26, 27, 23, 24, 28, 9].

0.1. The general idea of our approach can be explained as follows. Firstly, we extend Bass' theorem about projective covers from the categories of modules over associative rings to some other abelian categories \mathbf{B} with enough projective objects. This is the subject of the paper [24].

Secondly, let A be an associative ring and M be a left A -module. More generally, M could be an object of a good enough additive/abelian category \mathbf{A} in lieu of $A\text{-mod}$. We consider the full subcategory $\mathbf{Add}(M) \subset \mathbf{A}$ consisting of all the direct summands of coproducts of copies of M in \mathbf{A} . The aim is to prove the Enochs conjecture for the class of objects $\mathbf{Add}(M)$ in \mathbf{A} .

For this purpose, we find an abelian category \mathbf{B} such that the full subcategory $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ of projective objects in \mathbf{B} is equivalent to the full subcategory $\mathbf{Add}(M) \subset \mathbf{A}$. Then we transfer our knowledge about the Enochs conjecture for the class of projective objects \mathbf{B}_{proj} in \mathbf{B} to the class of objects $\mathbf{Add}(M)$ in \mathbf{A} .

In fact, we do more. Extending the discussion in [4] to the category-theoretic context, we consider covers in cotorsion pairs, self-pure-projective and \varinjlim -pure-rigid objects, and objects with perfect decomposition. Under certain assumptions, we prove that the class $\mathbf{Add}(M)$ is covering in \mathbf{A} if and only if the object $M \in \mathbf{Add}(M)$ has a perfect decomposition. This is based on some results of the papers [28] and [9].

One specific feature of our approach is that we consider topologies on (the opposite ring to) the ring of endomorphisms $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ of the object M . In particular, the endomorphism ring of a module M over an associative ring A always has the so-called *finite topology*. Under certain assumptions, we prove that the class

$\text{Add}(M)$ is covering in \mathbf{A} if and only if all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.

0.2. The time has come to explain what our assumptions are. There are three kinds of assumptions. Firstly, given an object M in a category \mathbf{A} , there should exist a topology on the ring \mathfrak{R} of endomorphisms of M for which the abelian category \mathbf{B} could be described as the category of left \mathfrak{R} -contramodules. This always holds when $\mathbf{A} = A\text{-mod}$ is the category of modules over an associative ring, and more generally, when \mathbf{A} is a locally finitely generated abelian category (and in some other cases, too).

Secondly, the topological ring \mathfrak{R} has to satisfy one of the technical assumptions (a), (b), (c), or (d) under which the main results of the paper [24] are proved. In particular, the condition (a) says that the ring \mathfrak{R} is commutative (and when it is not, there are three other alternatives (b), (c), or (d) which may happen to hold for \mathfrak{R}).

Alternatively, there are three conditions (e), (f), and (g), under any one of which some of our results in this paper can be proved using the main results of the papers [28, 29]. In particular, (e) says that \mathfrak{R} has a countable base of neighborhoods of zero.

Thirdly, there is a more conceptual assumption which we call “telescope Hom exactness condition”, abbreviated as THEC. This condition is not very restrictive. It says that right exactness of the telescope sequences computing countable direct limits of copies of the object M in \mathbf{A} is preserved by the functor $\text{Hom}_{\mathbf{A}}(M, -)$. All Σ -pure-rigid and all self-pure-projective objects (hence, in particular, all n -tilting objects) in abelian categories with exact countable direct limits satisfy THEC.

0.3. Having mentioned the assumptions, we can now formulate our main result.

Theorem 0.1. *Let \mathbf{A} be a locally presentable additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Denote by \mathbf{B} the abelian category with enough projective objects such that the full subcategory $\text{Add}(M) \subset \mathbf{A}$ is equivalent to the full subcategory of projective objects $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$. Assume that there exists a (complete, separated, right linear) topological ring structure on the ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ such that the abelian category \mathbf{B} is equivalent to the abelian category of left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}$ (this always holds for $\mathbf{A} = A\text{-mod}$). Finally, assume that the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d) of the paper [24] (e. g., this holds if \mathfrak{R} is commutative). Then the following conditions are equivalent:*

- (1) *the class of objects $\text{Add}(M) \subset \mathbf{A}$ is covering;*
- (2) *every countable direct limit of copies of M has an $\text{Add}(M)$ -cover in \mathbf{A} ;*
- (3) *the class of objects $\text{Add}(M)$ is closed under direct limits in \mathbf{A} ;*
- (4) *the class \mathbf{B}_{proj} is covering in \mathbf{B} ;*
- (5) *any countable direct limit of copies of the projective generator $\mathfrak{R} \in \mathbf{B}$ has a projective cover in \mathbf{B} ;*
- (6) *the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} ;*
- (7) *the object $M \in \mathbf{A}$ has a perfect decomposition;*
- (8) *all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;*
- (9) *all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.*

Replacing the assumption of one of the conditions (a–d) with that of one of the conditions (e), (f), or (g) (e. g., if \mathfrak{R} has a countable base of neighborhoods of zero), the eight conditions (1–8) are equivalent.

Notice that, even in the case of the category of modules $\mathbf{A} = A\text{-mod}$, one can sometimes choose between several topologies on the ring \mathfrak{R} for which the category \mathbf{B} in Theorem 0.1 is equivalent to $\mathfrak{R}\text{-contra}$. In particular, when the A -module M is *self-small*, i. e., the natural map of abelian groups $\bigoplus_{i=0}^{\infty} \text{Hom}_A(M, M) \rightarrow \text{Hom}_A(M, \bigoplus_{i=0}^{\infty} M)$ is an isomorphism, it suffices to endow the ring \mathfrak{R} with the *discrete* topology. Then the condition (b) is satisfied.

Furthermore, suppose that a left A -module $M = \sum_{i=1}^{\infty} E_i$ is the sum of a countable family of its submodules $E_i \subset M$ such that the A -modules E_i are *weakly finitely generated* (known also as “small” or “dually slender”). This means that for any family of left A -modules $(N_x)_{x \in X}$, the natural map $\bigoplus_x \text{Hom}_A(E_i, N_x) \rightarrow \text{Hom}_A(E_i, \bigoplus_x N_x)$ is an isomorphism for every $i = 1, 2, \dots$. Then one can endow the ring \mathfrak{R} with the *weakly finite* topology, and the condition (e) is satisfied (cf. [9, Section 7.2]).

0.4. Specializing to the tilting context, we prove the following theorem with our methods (cf. [4, Theorem 5.2 and Corollary 5.5]).

Theorem 0.2. *Let \mathbf{A} be a Grothendieck abelian category and $T \in \mathbf{A}$ be an n -tilting object. Let (\mathbf{L}, \mathbf{E}) denote the induced n -tilting cotorsion pair in \mathbf{A} , and let \mathbf{B} denote the heart of the related n -tilting t -structure on $\mathbf{D}(\mathbf{A})$. Assume that there exists a (complete, separated, right linear) topology on the ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(T, T)^{\text{op}}$ such that the abelian category \mathbf{B} is equivalent to the abelian category of left \mathfrak{R} -contramodules $\mathfrak{R}\text{-contra}$ (this always holds when \mathbf{A} is a locally weakly finitely generated abelian category). Finally, assume that the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d). Then the following conditions are equivalent:*

- (1) *the class \mathbf{L} is covering in \mathbf{A} ;*
- (2) *any countable direct limit of copies of T has an \mathbf{L} -cover in \mathbf{A} ;*
- (3) *the class \mathbf{L} is closed under direct limits in \mathbf{A} ;*
- (4) *the class $\text{Add}(T)$ is covering in \mathbf{A} ;*
- (5) *any countable direct limit of copies of T has an $\text{Add}(T)$ -cover in \mathbf{A} ;*
- (6) *the class $\text{Add}(T)$ is closed under direct limits in \mathbf{A} ;*
- (7) *any or all of the equivalent conditions (4–6) of Theorem 0.1 hold for the category $\mathbf{B} = \mathfrak{R}\text{-contra}$;*
- (8) *the object $T \in \mathbf{A}$ has a perfect decomposition;*
- (9) *all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;*
- (10) *all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.*

Replacing the assumption of one of the conditions (a–d) with that of one of the conditions (e), (f), or (g), the nine conditions (1–9) are equivalent.

0.5. In the full generality (without any of the assumptions mentioned in Section 0.2), we make the following simple observations.

Let \mathbf{A} be an additive category with cokernels and (weak) kernels, and $\mathbf{L} \subset \mathbf{A}$ be a precovering class of objects closed under direct summands. Viewing \mathbf{L} as a full

subcategory in \mathbf{A} , we notice that \mathbf{L} has weak kernels, too. So there exists a unique abelian category \mathbf{B} with enough projectives such that the full subcategory of projectives in \mathbf{B} is equivalent to \mathbf{L} [16, Corollary 1.5], [21, Proposition 2.3]. Furthermore, the equivalence of categories $\mathbf{B}_{\text{proj}} \cong \mathbf{L}$ can be naturally extended to a pair of adjoint functors $\Phi: \mathbf{B} \rightarrow \mathbf{A}$ and $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ (where Ψ is the right adjoint).

Let $N \in \mathbf{A}$ be an object. Then N has an \mathbf{L} -cover in \mathbf{A} if and only if the object $\Psi(N) \in \mathbf{B}$ has a projective cover. More specifically, given an object $L \in \mathbf{L}$ and the related object $\Psi(L) = P \in \mathbf{B}_{\text{proj}}$, a morphism $l: L \rightarrow N$ is an \mathbf{L} -cover if and only if the morphism $\Psi(l): P \rightarrow \Psi(N)$ is a projective cover. Hence the class \mathbf{L} is covering in \mathbf{A} if and only if all the objects in the essential image of the functor Ψ have projective covers in \mathbf{B} .

0.6. In the final sections of the paper, we discuss the class of examples for Theorem 0.2 provided by the tilting modules and objects arising from injective homological ring epimorphisms of projective dimension 1. Here our discussion is based on the paper [8].

In fact, there are two classes of examples. Let $u: R \rightarrow U$ be an injective homological epimorphism of associative rings such that the projective dimension of the left R -module U does not exceed 1. Then the left R -module $U \oplus U/R$ is 1-tilting. If the ring R is commutative, then the condition (d) is satisfied for the topological ring \mathfrak{S} of endomorphisms of the R -module $U \oplus U/R$, and Theorem 0.2 is applicable for $\mathbf{A} = R\text{-mod}$ and $T = U \oplus U/R$.

Assume additionally that the flat dimension of the right R -module U does not exceed 1. Then we consider the full subcategory $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ of what we call *left u -comodules* in the category of left R -modules $R\text{-mod}$. The category \mathbf{A} is a Grothendieck abelian category, and the left R -module U/R is a 1-tilting object in \mathbf{A} . If the ring R is commutative, then so is the topological ring $\mathfrak{R} = \text{Hom}_R(U/R, U/R)^{\text{op}}$, and Theorem 0.2 is applicable for $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ and $T = U/R$.

0.7. In conclusion, let us say a few words about how our results compare to those of the paper [4]. Our results are both more and less general than the results of [4]. On the one hand, the paper [4] only deals with cotorsion pairs in module categories, while we work in more general additive and abelian categories. On the other hand, the main results of the present paper require one of the rather restrictive conditions (a), (b), (c), (d), (e), (f), or (g), while there are no comparable assumptions in [4].

Even for module categories $\mathbf{A} = A\text{-mod}$, our Theorem 0.1 is both stronger and weaker than the results of [4]. On the one hand, we do not assume that the object M belongs to the kernel of a cotorsion pair. The running assumption in [4] is that of a cotorsion pair (\mathbf{L}, \mathbf{E}) in $A\text{-mod}$ such that the right-hand class \mathbf{E} is closed under direct limits. Under this assumption, any module $M \in \mathbf{L} \cap \mathbf{E}$ satisfies our telescope Hom exactness condition (in fact, it is enough that \mathbf{E} be closed under countable coproducts). So, in this respect, our setting is more general.

On the other hand, the assertions of [4, Theorem 5.2 and Corollary 5.5] tell more than those of our theorems. In particular, [4, Corollary 5.5 (5)] allows to conclude

that the module in the kernel of the cotorsion pair is Σ -pure-split, while we only prove that our object M has a perfect decomposition.

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1. CONTRAMODULES OVER TOPOLOGICAL RINGS

Cocomplete abelian categories with enough projective objects, and more specifically contramodule categories, play a key role in this paper. In this section, we briefly recall the basic material related to contramodules over complete, separated topological rings with right linear topologies. More details can be found in [24, Section 1], [23, Section 2], [25, Introduction and Section 5], [26, Section 6], and [22, Section 1].

1.1. Linear topological abelian groups. A topological abelian group A is said to have a *linear topology* if open subgroups form a base of neighborhoods of zero in A . A topological abelian group A with a linear topology (a “linear topological abelian group”, for brevity) is *separated* if the natural map $\lambda_A: A \rightarrow \varprojlim_{U \subset A} A/U$, where U ranges over the open subgroups of A , is injective, and A is *complete* if the map λ_A is surjective. Obviously, A is separated if and only if the intersection of all its open subgroups is zero.

For any abelian group A and a set X , we use the notation $A[X] = A^{(X)}$ for the coproduct of X copies of A . The elements of $A[X]$ are interpreted as finite formal linear combinations of elements of X with the coefficients in A .

Let \mathfrak{A} be a complete, separated linear topological abelian group. For any set X , we denote by $\mathfrak{A}[[X]]$ the projective limit

$$\mathfrak{A}[[X]] = \varprojlim_{\mathfrak{U} \subset \mathfrak{A}} (\mathfrak{A}/\mathfrak{U})[X],$$

where \mathfrak{U} ranges over all the open subgroups of \mathfrak{A} . Equivalently, $\mathfrak{A}[[X]]$ is the group of all infinite formal linear combinations $\sum_{x \in X} a_x x$ of elements of the set X with the coefficients $a_x \in \mathfrak{A}$ such that the family of coefficients $(a_x)_{x \in X}$ converges to zero in \mathfrak{A} in the following sense: for any open subgroup $\mathfrak{U} \subset \mathfrak{A}$, the set of all indices $x \in X$ for which $a_x \notin \mathfrak{U}$ must be finite.

For any complete, separated linear topological abelian group \mathfrak{A} and any map of sets $f: X \rightarrow Y$ there is a naturally induced “push-forward” map $\mathfrak{A}[[f]]: \mathfrak{A}[[X]] \rightarrow \mathfrak{A}[[Y]]$ taking a formal linear combination $\sum_{x \in X} a_x x$ to the formal linear combination $\sum_{y \in Y} b_y y$ with the coefficients $b_y = \sum_{x: f(x)=y} a_x$. Here the latter sum is understood as the limit of finite partial sums in the topology of \mathfrak{A} ; the convergence condition

on the family of elements $(a_x)_{x \in X}$ together with the conditions of separatedness and completeness of \mathfrak{A} guarantee that the coefficients b_y are well-defined (and form a family of elements $(b_y)_{y \in Y}$ which again converges to zero in \mathfrak{A}). This construction shows that the assignment $X \mapsto \mathfrak{A}[[X]]$ is a functor from the category of sets to the category of sets or even abelian groups.

1.2. Monads on Sets. A *monad* \mathbb{T} on the category of sets is a functor $\mathbb{T}: \mathbf{Sets} \rightarrow \mathbf{Sets}$ endowed with natural transformations of *monad unit* $\epsilon: \text{Id}_{\mathbf{Sets}} \rightarrow \mathbb{T}$ and *monad multiplication* $\phi: \mathbb{T} \circ \mathbb{T} \rightarrow \mathbb{T}$ satisfying the following associativity and unitality equations. The two natural maps $\mathbb{T}(\phi_X): \mathbb{T}(\mathbb{T}(\mathbb{T}(X))) \rightarrow \mathbb{T}(\mathbb{T}(X))$ and $\phi_{\mathbb{T}(X)}: \mathbb{T}(\mathbb{T}(\mathbb{T}(X))) \rightarrow \mathbb{T}(\mathbb{T}(X))$ should have equal compositions with the map $\phi_X: \mathbb{T}(\mathbb{T}(X)) \rightarrow \mathbb{T}(X)$ for any set X ,

$$\mathbb{T} \circ \mathbb{T} \circ \mathbb{T} \rightrightarrows \mathbb{T} \circ \mathbb{T} \rightarrow \mathbb{T},$$

and both the natural maps $\mathbb{T}(\epsilon_X): \mathbb{T}(X) \rightarrow \mathbb{T}(\mathbb{T}(X))$ and $\epsilon_{\mathbb{T}(X)}: \mathbb{T}(X) \rightarrow \mathbb{T}(\mathbb{T}(X))$ composed with the natural map ϕ_X should be equal to the identity endomorphism of the set $\mathbb{T}(X)$,

$$\mathbb{T} \rightrightarrows \mathbb{T} \circ \mathbb{T} \rightarrow \mathbb{T}.$$

Here ϵ_X denotes the map $X \rightarrow \mathbb{T}(X)$ assigned to an object $X \in \mathbf{Sets}$ by the natural transformation ϵ , and similarly, $\phi_X: \mathbb{T}(\mathbb{T}(X)) \rightarrow \mathbb{T}(X)$ is the map assigned to X by the natural transformation ϕ .

A *module* (or, in a more standard terminology, an algebra) over a monad $\mathbb{T}: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is a set C endowed with a map of sets $\pi_C: \mathbb{T}(C) \rightarrow C$, called the *monad action* map, satisfying the following associativity and unitality equations. The compositions of the two maps ϕ_C and $\mathbb{T}(\pi_C): \mathbb{T}(\mathbb{T}(C)) \rightarrow \mathbb{T}(C)$ with the map π_C should be equal to each other,

$$\mathbb{T}(\mathbb{T}(C)) \rightrightarrows \mathbb{T}(C) \rightarrow C,$$

and the composition of the map $\epsilon_C: C \rightarrow \mathbb{T}(C)$ with the map $\pi_C: \mathbb{T}(C) \rightarrow C$ should be equal to the identity map id_C ,

$$C \rightarrow \mathbb{T}(C) \rightarrow C.$$

A *morphism of \mathbb{T} -modules* $f: B \rightarrow C$ is a map of sets for which the following square diagram is commutative:

$$\begin{array}{ccc} \mathbb{T}(B) & \xrightarrow{\pi_B} & B \\ \mathbb{T}(f) \downarrow & & \downarrow f \\ \mathbb{T}(C) & \xrightarrow{\pi_C} & C \end{array}$$

The composition of morphisms of \mathbb{T} -modules is defined in the obvious way.

For any monad $\mathbb{T}: \mathbf{Sets} \rightarrow \mathbf{Sets}$, the category of \mathbb{T} -modules $\mathbb{T}\text{-mod}$ is complete and cocomplete. For any set X , the set $\mathbb{T}(X)$ with the action map $\pi_{\mathbb{T}(X)} = \phi_X$ is a \mathbb{T} -module; such \mathbb{T} -modules are called the *free \mathbb{T} -modules*. For any \mathbb{T} -module C ,

morphisms of \mathbb{T} -modules $\mathbb{T}(X) \longrightarrow C$ are in bijective correspondence with maps of sets $X \longrightarrow C$.

A monad $\mathbb{T}: \mathbf{Sets} \longrightarrow \mathbf{Sets}$ is said to be *additive* if the category of \mathbb{T} -modules $\mathbb{T}\text{-mod}$ is additive. In this case, the underlying set of every \mathbb{T} -module has a natural abelian group structure; so the forgetful functor $\mathbb{T}\text{-mod} \longrightarrow \mathbf{Sets}$ lifts naturally to a forgetful functor $\mathbb{T}\text{-mod} \longrightarrow \mathbf{Ab}$. For any additive monad \mathbb{T} , the category $\mathbb{T}\text{-mod}$ is abelian; the forgetful functor $\mathbb{T}\text{-mod} \longrightarrow \mathbf{Ab}$ is faithful, exact, and preserves all limits [22, Lemma 1.1]. For any additive monad \mathbb{T} , the abelian category of \mathbb{T} -modules $\mathbb{T}\text{-mod}$ has enough projective objects. A \mathbb{T} -module is projective if and only if it is a direct summand of a free \mathbb{T} -module.

1.3. Right linear topological rings. All *rings* in this paper are presumed to be associative and unital. A topological ring R is said to have a *right linear topology* if open right ideals form a base of neighborhoods of zero in R . A *two-sided linear topology* on R is a topology in which open two-sided ideals form a base of neighborhoods of zero. When the ring R is commutative, one simply says that “ R has a linear topology” if open ideals form a base of neighborhoods of zero. A topological ring with a right (resp., two-sided) linear topology is called *right* (resp., *two-sided*) *linear topological* (or just “linear topological”, if the ring is commutative).

Let \mathfrak{R} be a complete, separated right linear topological ring. Then the functor $\mathbb{T}_{\mathfrak{R}}: X \longmapsto \mathfrak{R}[[X]]$ has a natural structure of a monad on the category of sets. By the definition (see Section 1.2), this means that there are natural transformations of monad unit $\epsilon: \text{Id}_{\mathbf{Sets}} \longrightarrow \mathbb{T}_{\mathfrak{R}}$ and monad multiplication $\phi: \mathbb{T}_{\mathfrak{R}} \circ \mathbb{T}_{\mathfrak{R}} \longrightarrow \mathbb{T}_{\mathfrak{R}}$ satisfying the associativity and unitality equations.

For any set X , the natural “point measure” map $\epsilon_X: X \longrightarrow \mathfrak{R}[[X]]$ assigns to an element $x \in X$ the formal linear combination $\sum_{z \in X} r_z z$ with the coefficients $r_x = 1$ and $r_z = 0$ for $z \neq x$. The natural “opening of parentheses” map $\phi_X: \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ assigns to a formal linear combination $\sum_{y \in \mathfrak{R}[[X]]} r_y y$, where $y = \sum_{x \in X} s_{y,x} x \in \mathfrak{R}[[X]]$ and $r_y, s_{y,x} \in \mathfrak{R}$, the formal linear combination $\sum_{x \in X} t_x x \in \mathfrak{R}[[X]]$ with the coefficients $t_x = \sum_{y \in \mathfrak{R}[[X]]} r_y s_{y,x} \in \mathfrak{R}$. Here the infinite sum in the construction of the coefficient t_x is understood as the limit of finite partial sums in the topology of \mathfrak{R} , and the conditions of right linear topology, completeness, and separatedness imposed on the ring \mathfrak{R} guarantee the convergence.

1.4. Contramodules. A *left contramodule* over a complete, separated right linear topological ring \mathfrak{R} is a module (or, in the more standard terminology, an algebra) over the monad $\mathbb{T}_{\mathfrak{R}}$. In other words, a left \mathfrak{R} -contramodule \mathfrak{C} is a set endowed with a *left contraaction* map $\pi_{\mathfrak{C}}: \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ satisfying the associativity and unitality equations written down in Section 1.2.

Restricting the map $\pi_{\mathfrak{C}}$ to the subset of finite formal linear combinations $\mathfrak{R}[X] \subset \mathfrak{R}[[X]]$, one obtains the structure of a module over the monad $X \longmapsto \mathfrak{R}[X]$ on the underlying set of every left \mathfrak{R} -contramodule, which is the same as a left \mathfrak{R} -module structure. This construction defines a natural forgetful functor $\mathfrak{R}\text{-contra} \longrightarrow \mathfrak{R}\text{-mod}$ from the category of left \mathfrak{R} -contramodules to the category of left \mathfrak{R} -modules. The

monad $\mathbb{T}_{\mathfrak{R}}$ is additive, the category $\mathfrak{R}\text{-contra}$ is abelian, and the forgetful functor $\mathfrak{R}\text{-contra} \rightarrow \mathfrak{R}\text{-mod}$ is exact and preserves infinite products (but not coproducts).

For any set X , the free $\mathbb{T}_{\mathfrak{R}}$ -module $\mathbb{T}_{\mathfrak{R}}(X) = \mathfrak{R}[[X]]$ (with the contraaction map $\pi_{\mathfrak{R}[[X]]} = \phi_X$) is called the *free* left \mathfrak{R} -contramodule generated by X . Following the discussion in Section 1.2, for every left \mathfrak{R} -contramodule \mathfrak{C} , left \mathfrak{R} -contramodule morphisms $\mathfrak{R}[[X]] \rightarrow \mathfrak{C}$ are in bijective correspondence with maps of sets $X \rightarrow \mathfrak{C}$,

$$\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{C}) \cong \mathrm{Hom}_{\mathrm{Sets}}(X, \mathfrak{C}),$$

where we denote by $\mathrm{Hom}^{\mathfrak{R}}(\mathfrak{C}, \mathfrak{D})$ the group of morphisms between any two objects \mathfrak{C} and \mathfrak{D} in the category $\mathfrak{R}\text{-contra}$. There are enough projective objects in the abelian category $\mathfrak{R}\text{-contra}$; a left \mathfrak{R} -contramodule is projective if and only if it is a direct summand of a free left \mathfrak{R} -contramodule.

1.5. Discrete modules. Let R be a right linear topological ring. A right R -module N is said to be *discrete* if, for every element $x \in N$, the annihilator of x in R is an open right ideal. Equivalently, this means that the action map $N \times R \rightarrow N$ is continuous in the given topology on R and the discrete topology on N . The full subcategory of discrete right R -modules $\mathrm{discr}\text{-}R$ is closed under subobjects, quotient objects, and infinite direct sums in the abelian category of right R -modules $\mathrm{mod}\text{-}R$ (in other words, $\mathrm{discr}\text{-}R \subset \mathrm{mod}\text{-}R$ is a hereditary pretorsion class). It follows that $\mathrm{discr}\text{-}R$ is a locally finitely generated Grothendieck abelian category.

2. GENERALIZED TILTING THEORY

Let \mathbf{A} be an additive category with set-indexed coproducts, and let \mathbf{B} be an additive category with set-indexed products. For any object $T \in \mathbf{A}$ and any set X , we denote by $T^{(X)} \in \mathbf{A}$ the coproduct of X copies of T in \mathbf{A} . For any object $W \in \mathbf{B}$ and any set X , we denote by $W^X \in \mathbf{B}$ the product of X copies of W in \mathbf{B} .

Furthermore, we denote by $\mathrm{Add}(T) = \mathrm{Add}_{\mathbf{A}}(T) \subset \mathbf{A}$ the class of all direct summands of the coproducts $T^{(X)}$ of copies of the object T in the category \mathbf{A} . Similarly, we denote by $\mathrm{Prod}(W) = \mathrm{Prod}_{\mathbf{B}}(W) \subset \mathbf{B}$ the class of all direct summands of the products W^X of copies of the object W in \mathbf{B} .

Given an exact category \mathbf{E} (in Quillen's sense), we denote by $\mathbf{E}_{\mathrm{inj}}$ and $\mathbf{E}_{\mathrm{proj}} \subset \mathbf{E}$ the classes of all injective and projective objects in \mathbf{A} , respectively. In particular, this notation applies to abelian categories.

Let \mathbf{A} be an idempotent-complete additive category with set-indexed coproducts, and let $M \in \mathbf{A}$ be an object. In this section we recall the description of the category $\mathrm{Add}(M)$ as the category $\mathbf{B}_{\mathrm{proj}}$ of projective objects in a certain abelian category \mathbf{B} . This material first appeared in [26, Section 6] and [27, Section 1].

Remark 2.1. The latter two references are papers in tilting theory. So let us briefly explain the connection, which will also explain the title of this section and its first subsection, following below. In the infinitely generated tilting theory, one assigns to a cocomplete abelian category \mathbf{A} with an n -tilting object T another abelian category

\mathbf{B} , which is constructed as the heart of the tilting t-structure on the derived category $\mathbf{D}(\mathbf{A})$. One observes that the abelian category \mathbf{B} has enough projective objects, and the full subcategory of projective objects in \mathbf{B} is equivalent to the full subcategory $\mathbf{Add}(T) \subset \mathbf{A}$. (See Section 11 for a detailed discussion.) The next observation is that one does not need a tilting object to perform such a construction: for any object $M \in \mathbf{A}$, there exists a unique abelian category \mathbf{B} with enough projective objects such that $\mathbf{B}_{\text{proj}} \cong \mathbf{Add}(M)$. Hence the name “generalized tilting theory” which we give to this categorical construction and its basic properties.

2.1. Generalized tilting theory. Let \mathbf{A} be a category with coproducts and $M \in \mathbf{A}$ be an object. Consider the pair of adjoint functors

$$\Phi: \mathbf{Sets} \rightleftarrows \mathbf{A} : \Psi$$

defined as follows. For any set X , the object $\Phi(X) = M^{(X)}$ is the coproduct of X copies of M in \mathbf{A} . For any object $N \in \mathbf{A}$, the set $\Psi(N) = \text{Hom}_{\mathbf{A}}(M, N)$ is the set of all morphisms $M \rightarrow N$ in the category \mathbf{A} . The composition of the two adjoint functors $\mathbb{T}_M = \Psi \circ \Phi: \mathbf{Sets} \rightarrow \mathbf{Sets}$, taking a set X to the set $\mathbb{T}_M(X) = \text{Hom}_{\mathbf{A}}(M, M^{(X)})$, acquires a natural structure of a monad on the category of sets (see Section 1.2). According to [26, Proposition 6.2], the full subcategory formed by the objects $M^{(X)}$, $X \in \mathbf{Sets}$, in the category \mathbf{A} is equivalent to the full subcategory of free \mathbb{T}_M -modules $\mathbb{T}_M(X)$ in $\mathbb{T}\text{-mod}$.

Let \mathbf{B} be a cocomplete abelian category with a projective generator P . Then the related monad $\mathbb{T}_P: X \mapsto \text{Hom}_{\mathbf{B}}(P, P^{(X)})$ is additive, and the abelian category \mathbf{B} is equivalent to the abelian category of \mathbb{T}_P -modules [26, Corollary 6.3]:

$$(2.1) \quad \mathbf{B} \cong \mathbb{T}_P\text{-mod}.$$

The equivalence of categories (2.1) takes the projective generator $P \in \mathbf{B}$ to the free \mathbb{T}_P -module with one generator $\mathbb{T}_P(*)$.

Let \mathbf{A} be an idempotent-complete additive category with coproducts and $M \in \mathbf{A}$ be an object. Then $\mathbb{T}_M: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is an additive monad, and $\mathbf{B} = \mathbb{T}_M\text{-mod}$ is a complete, cocomplete abelian category with enough projective objects. The full subcategory of projective objects $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ is equivalent to the full subcategory $\mathbf{Add}(M) \subset \mathbf{A}$ [27, Theorem 1.1(a)], [28, Theorem 3.13]:

$$(2.2) \quad \mathbf{B} \supset \mathbf{B}_{\text{proj}} \cong \mathbf{Add}(M) \subset \mathbf{A}.$$

The equivalence of categories (2.2) takes the object $M \in \mathbf{Add}(M)$ to the free \mathbb{T}_M -module with one generator $P = \mathbb{T}_M(*) \in \mathbb{T}_M\text{-mod} = \mathbf{B}$, which is a projective generator of \mathbf{B} .

Assume that \mathbf{A} is a cocomplete additive category. Then the equivalence of full subcategories (2.2) extends naturally to a pair of adjoint functors between the ambient additive/abelian categories [27, Section 1]:

$$(2.3) \quad \Phi_M: \mathbf{B} \rightleftarrows \mathbf{A} : \Psi_M.$$

The right adjoint functor $\Psi_M: \mathbf{A} \rightarrow \mathbb{T}_M\text{-mod}$ takes an object $N \in \mathbf{A}$ to the set $\text{Hom}_{\mathbf{A}}(M, N)$ endowed with the \mathbb{T}_M -module structure provided by the map

$$\pi_{\Psi_M(N)}: \mathbb{T}_M(\text{Hom}_{\mathbf{A}}(M, N)) = \text{Hom}_{\mathbf{A}}(M, M^{(\text{Hom}_{\mathbf{A}}(M, N))}) \rightarrow \text{Hom}_{\mathbf{A}}(M, N)$$

of composition with the natural morphism $M^{(\text{Hom}_{\mathbf{A}}(M, N))} \rightarrow N$ in the category \mathbf{A} (cf. [26, Remark 6.4]). The left adjoint functor $\Phi_M: \mathbf{B} \rightarrow \mathbf{A}$ can be obtained as the extension of the fully faithful embedding $\mathbf{B}_{\text{proj}} \cong \text{Add}(M) \rightarrow \mathbf{A}$ to a right exact functor $\mathbf{B} \rightarrow \mathbf{A}$. The restrictions of the functors Φ_M and Ψ_M to the full subcategories $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ and $\text{Add}(M) \subset \mathbf{A}$ take these two full subcategories into each other, providing the equivalence $\mathbf{B}_{\text{proj}} \cong \text{Add}(M)$.

2.2. Contramodules in generalized tilting theory. For many additive categories \mathbf{A} with coproducts, the monads \mathbb{T}_M associated with objects $M \in \mathbf{A}$ have the form $\mathbb{T}_M \cong \mathbb{T}_{\mathfrak{R}}$ for certain complete, separated, right linear topological rings \mathfrak{R} . In particular, this is the case for the categories $\mathbf{A} = A\text{-mod}$ of modules over associative rings A .

The first related observation is that, for every monad \mathbb{T} on the category of sets, the set $\mathbb{T}(\ast)$ assigned by the functor \mathbb{T} to a one-element set \ast has a natural monoid structure. In fact, the set $\mathbb{T}(\ast) = \text{Hom}_{\mathbb{T}\text{-mod}}(\mathbb{T}(\ast), \mathbb{T}(\ast))$ is the set of \mathbb{T} -module endomorphisms of the free \mathbb{T} -module with one generator $\mathbb{T}(\ast)$. We will follow the convention that the multiplication in $\mathbb{T}(\ast)$ is opposite to the composition of endomorphisms (so the monoid $\mathbb{T}(\ast)$ acts in the object $\mathbb{T}(\ast) \in \mathbb{T}\text{-mod}$ on the right).

For every additive monad \mathbb{T} , the set $\mathbb{T}(\ast)$ has a natural structure of associative ring. In the case of the monad \mathbb{T}_M for an object $M \in \mathbf{A}$, the related ring $\mathbb{T}_M(\ast) = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ is the opposite ring to the ring of endomorphisms of the object M . In the case of the monad $\mathbb{T}_{\mathfrak{R}}$ for a topological ring \mathfrak{R} , the related ring is $\mathbb{T}_{\mathfrak{R}}(\ast) = \mathfrak{R}$. Thus, given an object $M \in \mathbf{A}$, in order to find a topological ring \mathfrak{R} for which the monad \mathbb{T}_M is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}$, one has to endow the endomorphism ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$ with an appropriate complete, separated right linear topology.

Additive categories \mathbf{A} with set-indexed coproducts in which the groups of morphisms $\text{Hom}_{\mathbf{A}}(M, N)$ carry topologies appropriate for the task are called *topologically agreeable categories* in [28]. In fact, it often happens that a given category \mathbf{A} can be endowed with several topologically agreeable structures, differing slightly from one another.

Examples 2.2. (1) Let A be an associative ring and $\mathbf{A} = A\text{-mod}$ be the category of left A -modules. Then, for $M, N \in \mathbf{A}$, the abelian group $\text{Hom}_A(M, N)$ can be endowed with what is known as the *finite topology*, in which annihilators of finite subsets (or equivalently, of finitely generated submodules) $E \subset M$ form a base of neighborhoods of zero in $\text{Hom}_A(M, N)$.

The finite topology on $\text{Hom}_A(M, N)$ is complete and separated; and the ring $\text{Hom}_A(M, M)$ is a *left* linear topological ring in the finite topology. So the ring $\mathfrak{R} = \text{Hom}_A(M, M)^{\text{op}}$ is a right linear topological ring. The monad $\mathbb{T}_M: X \mapsto \text{Hom}_A(M, M^{(X)})$ is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}: X \mapsto \mathfrak{R}[[X]]$. Thus the abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ is equivalent to $\mathfrak{R}\text{-contra}$ [26, Theorem 7.1].

(2) In the setting of (1), we say that a left A -module E is *weakly finitely generated* if, for any family of left A -modules $(N_x)_{x \in X}$, the natural map $\bigoplus_x \text{Hom}_A(E, N_x) \rightarrow \text{Hom}_A(E, \bigoplus_x N_x)$ is an isomorphism. Equivalently, this means that every A -module morphism $E \rightarrow \bigoplus_x N_x$ factorizes through the direct sum of the modules N_x over a finite subset of indices $x \in Z \subset X$, $|Z| < \infty$. Such modules E are known in the literature as “dually slender” or “small”.

For any left A -modules M and N , the *weakly finite topology* on the abelian group $\text{Hom}_A(M, N)$ has a base of neighborhoods of zero consisting of the annihilators of weakly finitely generated submodules $E \subset M$. The weakly finite topology on $\text{Hom}_A(M, N)$ is complete and separated, and once again, the ring $\text{Hom}_A(M, M)$ is a left linear topological ring in the weakly finite topology. Denoting by \mathfrak{R}' the ring $\text{Hom}_A(M, M)^{\text{op}}$ with the weakly finite topology on it, we once again obtain a complete, separated right linear topological ring. The monad $\mathbb{T}_{\mathfrak{R}'}: X \mapsto \mathfrak{R}'[[X]]$ is still isomorphic to the monad $\mathbb{T}_M: X \mapsto \text{Hom}_A(M, M^{(X)})$ [26, Theorem 9.9].

In fact, while the finite topology and the weakly finite topology on the endomorphism ring of a module may well differ, the sets $\mathfrak{R}[[X]]$ and $\mathfrak{R}'[[X]]$ are the same for any set X , as a family of A -module morphisms $r_x: M \rightarrow M$ converges to zero in the finite topology if and only if it converges to zero in the weakly finite one, and if and only if the morphism $r: M \rightarrow M^X$ with the components r_x factorizes through the submodule $M^{(X)} \subset M^X$. Thus the abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ can be alternatively described as the category $\mathfrak{R}'\text{-contra}$. (Cf. [28, Examples 3.10].)

(3) A left A -module M is said to be *self-small* if any A -module morphism $M \rightarrow M^{(X)}$ factorizes through the coproduct $M^{(Z)} \subset M^{(X)}$ of copies of M indexed over a finite subset $Z \subset X$. Equivalently, M is self-small if and only if the natural map of abelian groups $\bigoplus_{i=0}^{\infty} \text{Hom}_A(M, M) \rightarrow \text{Hom}_A(M, \bigoplus_{i=0}^{\infty} M)$ is an isomorphism.

For a self-small left A -module M , the ring $R = \text{Hom}_A(M, M)^{\text{op}}$ endowed with the *discrete* topology has the property that the monad $\mathbb{T}_M: X \mapsto \text{Hom}_A(M, M^{(X)})$ is isomorphic to the monad $\mathbb{T}_R: X \mapsto R[X]$. So the abelian category \mathbf{B} is equivalent to the category of left R -modules, $\mathbf{B} \cong R\text{-mod}$.

(4) Let N be a fixed left A -module. We say that a left A -module E is *N -small* if the natural map of abelian groups $\bigoplus_{i=0}^{\infty} \text{Hom}_A(E, N) \rightarrow \text{Hom}_A(E, \bigoplus_{i=0}^{\infty} N)$ is an isomorphism. Equivalently, E is N -small if and only if, for any set X , any A -module morphism $E \rightarrow N^{(X)}$ factorizes through the subcoproduct $N^{(Z)} \subset N^{(X)}$ indexed over a finite subset $Z \subset X$. An A -module E is weakly finitely generated or “small” (as defined in (2)) if and only if it is N -small for all left A -modules N . It is important for the present example that the class of all N -small A -modules is closed under quotients (while, e. g., the class of all self-small A -modules is *not*, generally speaking). In addition, the class of N -small A -modules is closed under extensions.

For any left A -modules M and N , the *N -small topology* on the abelian group $\text{Hom}_A(M, N)$ has a base of neighborhoods of zero consisting of the annihilators of N -small submodules $E \subset M$. The N -small topology on $\text{Hom}_A(M, N)$ is complete and separated. The ring $\text{Hom}_A(M, M)$ is a left linear topological ring in the M -small topology. Denoting by \mathfrak{R}'' the ring $\text{Hom}_A(M, M)^{\text{op}}$ with the M -small topology, we

obtain yet another complete, separated right linear topological ring structure for which the monad $\mathbb{T}_{\mathfrak{R}''}: X \mapsto \mathfrak{R}''[[X]]$ coincides with the monads $\mathbb{T}_{\mathfrak{R}}$ and $\mathbb{T}_{\mathfrak{R}}$ from (1) and (2). Consequently, the monad $\mathbb{T}_{\mathfrak{R}''}$ is also isomorphic to the monad \mathbb{T}_M , and the abelian category \mathbf{B} can be described as the category $\mathfrak{R}''\text{-contra}$.

Examples 2.3. Further examples of topologically agreeable additive/abelian categories include:

(1) all the locally finitely generated abelian categories \mathbf{A} (in particular, all the locally finitely presentable abelian categories), endowed with the finite topology [28, Example 3.7 (2)];

(2) all the locally weakly finitely generated abelian categories \mathbf{A} , endowed with the weakly finite topology [26, Section 9.2];

(3) all the additive categories \mathbf{A} with set-indexed coproducts admitting a *closed functor* $F: \mathbf{A} \rightarrow \mathbf{C}$ into a locally weakly finitely generated abelian category \mathbf{C} [26, Section 9.3], or more generally, into a topologically agreeable additive category \mathbf{C} [28, Example 3.9 (3)]. In particular, the additive/abelian categories of comodules over corings and semimodules over semialgebras belong to the class (3) [26, Section 10.3].

So, for any object M in an additive category \mathbf{A} satisfying (1), (2), or (3), the monad $\mathbb{T}_M: X \mapsto \text{Hom}_{\mathbf{A}}(M, M^{(X)})$ is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}: X \mapsto \mathfrak{R}[[X]]$ for a certain complete, separated right linear topology on the ring $\mathfrak{R} = \text{Hom}_{\mathbf{A}}(M, M)^{\text{op}}$. The abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ is equivalent to $\mathfrak{R}\text{-contra}$.

2.3. Accessible monads and locally presentable categories. We refer to the book [1] for the definitions and general discussion of accessible and locally presentable categories, and only recall here that a category is called *locally presentable* if it is accessible and cocomplete [1, Corollary 2.47]. A monad $\mathbb{T}: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is said to be *accessible* if its underlying functor \mathbb{T} is accessible, i. e., there exists a cardinal κ such that \mathbb{T} preserves κ -directed colimits.

The category $\mathbb{T}\text{-mod}$ is locally presentable if and only if the monad \mathbb{T} is accessible. For any accessible category \mathbf{A} with coproducts and an object $M \in \mathbf{A}$, the monad \mathbb{T}_M is accessible (so the category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ is locally presentable). For any complete, separated right linear topological ring \mathfrak{R} , the monad $\mathbb{T}_{\mathfrak{R}}$ is accessible and the category $\mathfrak{R}\text{-contra}$ is locally presentable. We refer to the paper [25, Introduction and Section 5] for the details.

3. SEVEN CLASSES OF TOPOLOGICAL RINGS

Let \mathfrak{A} be a complete, separated linear topological group. A closed subgroup $\mathfrak{K} \subset \mathfrak{A}$ is said to be *strongly closed* if the quotient group $\mathfrak{A}/\mathfrak{K}$ is complete in the quotient topology and, for every set X , the induced map of sets/abelian groups $\mathfrak{A}[[X]] \rightarrow (\mathfrak{A}/\mathfrak{K})[[X]]$ is surjective. We refer to [24, Sections 1.11–12] for a discussion of strongly closed subgroups in topological groups and strongly closed ideals in topological rings.

Let R be a separated topological ring. A subset $K \subset R$ is said to be *topologically left T -nilpotent* if, for every sequence of elements $a_1, a_2, a_3, \dots \in K$, the sequence of

products $a_1, a_1a_2, \dots, a_1a_2 \cdots a_n, \dots$ converges to zero in the topology of R . We refer to [28, Section 7] and [24, Section 5] for a discussion of topologically left T-nilpotent subsets and topologically left T-nilpotent ideals in right linear topological rings.

Let \mathfrak{R} be a complete, separated right linear topological ring. The following four classes of such topological rings \mathfrak{R} are considered in [24, Sections 10 and 12]:

- (a) the ring \mathfrak{R} is commutative; or
- (b) \mathfrak{R} has a countable base of neighborhoods of zero consisting of open two-sided ideals; or
- (c) \mathfrak{R} is a two-sided linear topological ring having only a finite number of classically semisimple (semisimple Artinian) discrete quotient rings; or
- (d) there is a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{K} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{K}$ is isomorphic, as a topological ring, to the product $\prod_{\delta \in \Delta} \mathfrak{T}_\delta$ of a family of two-sided linear topological rings \mathfrak{T}_δ , each of which satisfies one of the conditions (a), (b), or (c).

Note that all the topological rings satisfying (a), (b), or (c) must be two-sided linear, while a topological ring satisfying (d) can well be only right linear.

Furthermore, our discussion of the following two classes of (right linear) topological rings \mathfrak{R} is based on the results of the paper [28, Sections 12 and 13]:

- (e) \mathfrak{R} has a countable base of neighborhoods of zero; or
- (f) the abelian category $\mathbf{discr}\text{-}\mathfrak{R}$ is locally coherent.

We refer to the papers [29, 28] for the definition of a locally coherent abelian category. Note that (d) and (e) are two different generalizations of (b), while (d) is also a common generalization of (a), (b), and (c).

Finally, we consider the following common generalization of all the previous six conditions (a–f):

- (g) there is a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{K} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{K}$ is isomorphic, as a topological ring, to the product $\prod_{\delta \in \Delta} \mathfrak{T}_\delta$ of a family of right linear topological rings \mathfrak{T}_δ , each of which satisfies one of the conditions (a), (c), (e), or (f).

Lemma 3.1. (i) *Let $(\mathfrak{R}_\gamma)_{\gamma \in \Gamma}$ be a family of topological rings each of which satisfies one of the conditions (a), (b), (c), (d), (e), (f), or (g). Then the topological ring $\mathfrak{R} = \prod_{\gamma \in \Gamma} \mathfrak{R}_\gamma$ satisfies (g).*

(ii) *Let \mathfrak{R} be a complete, separated right linear topological ring, and let $\mathfrak{J} \subset \mathfrak{R}$ be a topologically left T-nilpotent strongly closed two-sided ideal. Assume that the topological quotient ring $\mathfrak{R}/\mathfrak{J}$ satisfies (g). Then the topological ring \mathfrak{R} satisfies (g).*

Proof. Part (i) is similar to [24, Lemma 12.6(a)]. Condition (b) is a particular case of (e), and therefore (d) is a particular case of (g). Conditions (a), (c), (e), and (f) are also particular cases of (g). Hence without loss of generality we can assume that \mathfrak{R}_γ satisfies (g) for every $\gamma \in \Gamma$.

Let $\mathfrak{K}_\gamma \subset \mathfrak{R}_\gamma$ be the related topologically left T-nilpotent strongly closed two-sided ideal. Then, in view of the discussion in [24, beginning of Section 7], $\mathfrak{K} = \prod_{\gamma} \mathfrak{K}_\gamma$ is a topologically left T-nilpotent strongly closed two-sided ideal in the topological

ring $\mathfrak{R} = \prod_{\gamma} \mathfrak{R}_{\gamma}$, and the topological quotient ring $\mathfrak{R}/\mathfrak{K} \cong \prod_{\gamma} \mathfrak{R}_{\gamma}/\mathfrak{K}_{\gamma}$ is isomorphic to the topological product of topological rings, each of which satisfies one of the conditions (a), (c), (e), or (f).

Part (ii) is similar to [24, Lemma 12.6(b)]. Let $\mathfrak{K} \subset \mathfrak{R}/\mathfrak{J}$ be a two-sided ideal witnessing that the topological ring $\mathfrak{R}/\mathfrak{J}$ satisfies (g), and let $\mathfrak{H} \subset \mathfrak{R}$ be the full preimage of \mathfrak{K} under the topological ring homomorphism $\mathfrak{R} \rightarrow \mathfrak{R}/\mathfrak{J}$. Then the ideal \mathfrak{H} is strongly closed in \mathfrak{R} by [24, Lemma 1.4(b)] and topologically left T-nilpotent by [24, Lemma 5.3]. In view of the natural isomorphism of topological rings $\mathfrak{R}/\mathfrak{H} \cong (\mathfrak{R}/\mathfrak{J})/\mathfrak{K}$, the ideal $\mathfrak{H} \subset \mathfrak{R}$ witnesses that the topological ring \mathfrak{R} satisfies (g). \square

The following definition was given in the paper [28, Section 10]. A complete, separated right linear topological ring \mathfrak{R} is called *topologically left perfect* if there is a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{H} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ is isomorphic, as a topological ring, to the product $\mathfrak{S} = \prod_{\gamma \in \Gamma} \text{Hom}_{D_{\gamma}}(D_{\gamma}^{(\Upsilon_{\gamma})}, D_{\gamma}^{(\Upsilon_{\gamma})\text{op}})$ of the endomorphism rings of vector spaces over skew-fields (division rings) D_{γ} . Here Γ is a set, Υ_{γ} are nonempty sets, the endomorphism ring of the vector space $D_{\gamma}^{(\Upsilon_{\gamma})}$ is endowed with the finite topology, and the product of such endomorphism rings is endowed with the product topology. Right linear topological rings \mathfrak{S} of the above form are called *topologically semisimple* [28, Section 6].

Lemma 3.2. *All topologically left perfect topological rings \mathfrak{R} satisfy condition (g).*

Proof. The assertion holds because all topologically semisimple right linear topological rings \mathfrak{S} satisfy condition (f). Indeed, \mathfrak{S} is topologically semisimple if and only if the category of discrete right \mathfrak{S} -modules $\text{discr-}\mathfrak{S}$ is semisimple [28, Theorem 6.2 (2)]. Any semisimple Grothendieck abelian category is locally Noetherian (with simple objects forming a set of Noetherian generators); hence it is locally coherent. \square

Lemma 3.3. *Let \mathfrak{R} be a complete, separated right linear topological ring, and let $\mathfrak{K} \subset \mathfrak{R}$ be a topologically left T-nilpotent strongly closed two-sided ideal. Then the quotient ring $\mathfrak{R}/\mathfrak{K}$, endowed with the quotient topology, is topologically left perfect if and only if the topological ring \mathfrak{R} is.*

Proof. “If”: assume that \mathfrak{R} is topologically left perfect, and let $\mathfrak{H} \subset \mathfrak{R}$ be the related two-sided ideal, as per the definition. Then \mathfrak{H} is the (topological) Jacobson radical of \mathfrak{R} [28, Lemma 10.3], and any topologically left T-nilpotent ideal in \mathfrak{R} is contained in \mathfrak{H} [24, Lemma 6.6(a)]. Hence we have $\mathfrak{K} \subset \mathfrak{H}$. The two-sided ideal $\mathfrak{H}/\mathfrak{K} \subset \mathfrak{R}/\mathfrak{K}$ is topologically left T-nilpotent, since the ideal $\mathfrak{H} \subset \mathfrak{R}$ is. By [28, Lemma 1.4(c)], $\mathfrak{H}/\mathfrak{K}$ is strongly closed in $\mathfrak{R}/\mathfrak{K}$. Finally, we have an isomorphism of topological rings $(\mathfrak{R}/\mathfrak{K})/(\mathfrak{H}/\mathfrak{K}) \cong \mathfrak{R}/\mathfrak{H}$, and the topological ring $\mathfrak{R}/\mathfrak{H}$ is topologically semisimple by assumption. Hence the topological ring $\mathfrak{R}/\mathfrak{K}$ is topologically left perfect.

“Only if”: assuming that $\mathfrak{R}/\mathfrak{K}$ is topologically left perfect, an argument based on [28, Lemmas 1.4(b) and 5.3] and similar to the proof of Lemma 3.1(ii) shows that \mathfrak{R} is topologically left perfect as well. \square

Lemma 3.4. *The class of all topologically left perfect topological rings is closed under (infinite) topological products.*

Proof. The argument is based on the discussion in [24, beginning of Section 7] and similar to the proof of Lemma 3.1(i). \square

Theorem 3.5. *Let \mathfrak{R} be a complete, separated right linear topological ring. Then the topological ring \mathfrak{R} is topologically left perfect if and only if it satisfies one of the conditions (a), (b), (c), (d), (e), (f), or (g) and every descending chain of cyclic discrete right \mathfrak{R} -modules terminates.*

Proof. “Only if”: for any topologically left perfect topological ring \mathfrak{R} , any descending chain of cyclic discrete right \mathfrak{R} -modules terminates by [28, Theorem 14.4 (iv) \Rightarrow (v)], and condition (g) is satisfied by Lemma 3.2.

“If”: cases (a–c) are covered by [24, Theorem 10.1 (v) \Rightarrow (iv)], and case (d) is [24, Theorem 12.4 (v) \Rightarrow (iv)] (see also [28, Remark 14.6 and Corollary 14.7]). Case (e) is [28, Theorem 12.4 or 14.8], and case (f) is [28, Theorem 13.3 or 14.12].

To prove case (g), assume that $\mathfrak{K} \subset \mathfrak{R}$ is a topologically left T-nilpotent strongly closed two-sided ideal for which the topological ring $\mathfrak{R}/\mathfrak{K}$ is isomorphic to the topological product $\prod_{\delta \in \Delta} \mathfrak{T}_\delta$, where each topological ring \mathfrak{T}_δ satisfies one of the conditions (a), (c), (e), or (f). Then \mathfrak{T}_δ is a topological quotient ring of the topological ring \mathfrak{R} for every $\delta \in \Delta$. Hence $\text{discr-}\mathfrak{T}_\delta$ is the full subcategory in $\text{discr-}\mathfrak{R}$ consisting of all the modules annihilated by the kernel ideal of the surjective continuous ring homomorphism $\mathfrak{R} \rightarrow \mathfrak{T}_\delta$. Since every descending chain of cyclic discrete right \mathfrak{R} -modules terminates, so does every descending chain of cyclic discrete right \mathfrak{T}_δ -modules. According to the previous paragraph, in each of the cases (a), (c), (e), or (f) it follows that \mathfrak{T}_δ is a topologically left perfect topological ring. Using Lemmas 3.3 and 3.4, we can conclude that \mathfrak{R} is a topologically left perfect topological ring. \square

4. THE ENOCHS CONJECTURE

Throughout this paper, by “direct limits” in a category we mean inductive limits indexed by directed posets. Otherwise, these are known as the directed or filtered colimits. For any class of objects \mathbf{M} in a cocomplete category \mathbf{A} , we denote by $\varinjlim \mathbf{M} = \varinjlim^{\mathbf{A}} \mathbf{M} \subset \mathbf{A}$ the class of all direct limits of objects from \mathbf{M} in \mathbf{A} . This means the direct limits of diagrams $A: \Theta \rightarrow \mathbf{A}$ indexed by directed posets Θ and such that $A(\theta) \in \mathbf{M}$ for all $\theta \in \Theta$.

Let \mathbf{A} be a category and $\mathbf{L} \subset \mathbf{A}$ be a class of objects. A morphism $l: L \rightarrow C$ in \mathbf{A} is called an *L-precover* (of the object C) if $L \in \mathbf{L}$ and all the morphisms from objects of \mathbf{L} to the object C factorize through the morphism l in the category \mathbf{A} , that is, for every morphism $l': L' \rightarrow C$ with $L' \in \mathbf{L}$ there exists a morphism $f: L' \rightarrow L$ such that $l' = lf$. A morphism $l: L \rightarrow C$ in \mathbf{A} is called an *L-cover* if it is an L-precover and, for any endomorphism $e: L \rightarrow L$, the equation $le = l$ implies that e is an automorphism of L . We will say that a class of objects \mathbf{L} in a category \mathbf{A} is *precovering* if every object of \mathbf{A} has an L-precover. Similarly, the class \mathbf{L} is said to be *covering* if every object of \mathbf{A} has an L-cover.

Given another class of objects $E \subset A$, the definitions of an *E-preenvelope* and an *E-envelope* of an object $C \in A$ are dual to the above definitions of an *L-precover* and an *L-cover*. These notions are due to Enochs [14]; a detailed discussion of their properties in a relevant context can be found in the book [37].

Example 4.1. If A is an additive category with coproducts and $M \in A$ is an object, then the class of objects $\text{Add}(M) \subset A$ is precovering. Indeed, for any object $N \in A$, the obvious morphism $M^{\text{Hom}_A(M,N)} \rightarrow N$ is an $\text{Add}(M)$ -precover of N .

Example 4.2. Let B be an abelian category with enough projective objects and $L = B_{\text{proj}} \subset B$ be the class of all projective objects. Then a morphism $L \rightarrow C$ in B with $L \in L$ is an *L-precover* if and only if it is an epimorphism. So the class of all projective objects in an abelian category with enough projective objects is always precovering; but it is rarely covering, as we will see. A B_{proj} -cover in B is called a *projective cover*.

The first assertion of the following theorem is one of the main results of Bass' paper [6]. In fact, it is a part of the famous [6, Theorem P].

Theorem 4.3. *Let $B = R\text{-mod}$ be the category of modules over an associative ring, and let $L = R\text{-mod}_{\text{proj}} \subset R\text{-mod}$ be the class of projective left R -modules. Then the class L is covering in $R\text{-mod}$ if and only if L is closed under direct limits in $R\text{-mod}$.*

Moreover, if every countable direct limit of copies of the free left R -module R has a projective cover in $R\text{-mod}$, then all flat left R -modules are projective and all left R -modules have projective covers.

Proof. The first assertion is [6, Theorem P (2) \Leftrightarrow (5)]. The second assertion stems from the proof of the implication [6, Theorem P (5) \Rightarrow (6)], which only uses projectivity of the countable direct limits of copies of the R -module R . Such direct limits are now known as *Bass flat R -modules*. Associative rings R satisfying the equivalent conditions of [6, Theorem P] are called *left perfect*. So it is shown in [6] that a ring R is left perfect whenever all Bass flat left R -modules are projective.

A proof of the assertion that any flat module having a projective cover is projective can be found in [36, Section 36.3]. \square

The idea of the proof of the following result goes back to Enochs' paper [14, Theorems 2.1 and 3.1].

Theorem 4.4. *In a locally presentable category A , any precovering class closed under direct limits is covering.*

Proof. For module categories, this was established by Enochs in [14]. For Grothendieck abelian categories, a proof of this assertion can be found in [5, Theorem 1.2]; and for locally presentable categories, in [25, Theorem 2.7 or Corollary 4.17]. \square

It is easy to prove that, in any category A , any covering class $L \subset A$ is closed under retracts, and any precovering class that is closed under retracts is also closed under

coproducts (cf. [14, Proposition 2.1] or [37, Theorem 2.5.1]). Hence any covering class is closed under coproducts. The following inverse assertion to Theorem 4.4 (for module categories) is known as “the Enochs conjecture” (see [19, Section 5.4]; cf. [4, Section 5]).

Conjecture 4.5. *Let $\mathbf{A} = A\text{-mod}$ be the category of modules over an associative ring A , and let $\mathbf{L} \subset A\text{-mod}$ be a covering class. Then \mathbf{L} is closed under direct limits in $A\text{-mod}$.*

Far-reaching results confirming particular cases of the Enochs conjecture were obtained in the paper [4], based on the tools developed in [31]. (See also the preprint [9] for an alternative elementary proof of some of the results of [4].) The idea of our categorical approach to the Enochs conjecture is expressed in the following conjectural extension of Bass’ theorem.

Main Conjecture 4.6. *Let \mathbf{B} be a locally presentable abelian category with a projective generator P . Then the following conditions are equivalent.*

- (1) *the class \mathbf{B}_{proj} is covering in \mathbf{B} ;*
- (2) *any direct limit of projective objects has a projective cover in \mathbf{B} ;*
- (3) *any countable direct limit of copies of P has a projective cover in \mathbf{B} ;*
- (4) *any countable direct limit of copies of P is a projective object in \mathbf{B} ;*
- (5) *the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} .*

Notice that the implications (1) \implies (2) \implies (3) and (5) \implies (4) \implies (3) in the Main Conjecture are obvious, while the implication (5) \implies (1) holds by Example 4.2 and Theorem 4.4. The implications (3) \implies (2) \implies (1) \implies (5) and (3) \implies (4) \implies (5) are nontrivial (and unknown).

For the categories of contra modules over topological rings, some of the equivalences in Conjecture 4.6 are provided by the results of the paper [28].

Theorem 4.7. *Let \mathfrak{R} be a complete, separated right linear topological ring. Then the following equivalences of conditions in Main Conjecture 4.6 hold for the abelian category $\mathbf{B} = \mathfrak{R}\text{-contra}$ with the projective generator $P = \mathfrak{R}[[*]] = \mathfrak{R}$:*

$$(1) \iff (2) \iff (5) \quad \text{and} \quad (3) \iff (4).$$

Proof. The equivalences (1) \iff (2) \iff (5) are [28, Theorem 14.1 (ii) \Leftrightarrow (i') \Leftrightarrow (iii')]. The equivalence (3) \iff (4) is [28, Theorem 14.4 (i^b) \Leftrightarrow (iii^b)]. \square

A more refined version of Main Conjecture 4.6 in the particular case of contra modules over topological rings can be found in [28, Conjecture 14.3].

The following special cases of Main Conjecture 4.6 for the categories of contra modules over topological rings are provable with our methods.

Theorem 4.8. *Let \mathfrak{R} be a complete, separated right linear topological ring satisfying one of the conditions (a), (b), (c), (d), (e), (f), or (g). Then Main Conjecture 4.6 holds for the abelian category $\mathfrak{R}\text{-contra}$ with the projective generator \mathfrak{R} , i. e., the conditions (1), (2), (3), (4), and (5) are equivalent for $\mathbf{B} = \mathfrak{R}\text{-contra}$ and $P = \mathfrak{R}$.*

Proof. Follows from Theorem 3.5 and [28, Theorem 14.4]. \square

The next lemma, generalizing the “if” assertion of Lemma 3.3, is an application of projective covers to topological algebra.

Lemma 4.9. *The class of topologically left perfect complete, separated right linear topological rings is closed under the passages to topological quotient rings by strongly closed two-sided ideals.*

Proof. We use the following characterization: a right linear topological ring \mathfrak{R} is topologically left perfect if and only if all left \mathfrak{R} -contramodules have projective covers [28, Theorem 14.1 (ii) \Leftrightarrow (iv)]. Assume that \mathfrak{R} is topologically left perfect, $\mathfrak{J} \subset \mathfrak{R}$ is a strongly closed two-sided ideal, and $\mathfrak{T} = \mathfrak{R}/\mathfrak{J}$ is the topological quotient ring. Let us show that every left \mathfrak{T} -contramodule \mathfrak{C} has a projective cover. Using the contrarestriction of scalars [24, Section 1.9], one can consider \mathfrak{C} as a left \mathfrak{R} -contramodule. As such, \mathfrak{C} has a projective cover $p: \mathfrak{P} \rightarrow \mathfrak{C}$ in \mathfrak{R} -contra. Then the reduction construction of [24, Lemma 3.3] produces a projective cover of \mathfrak{C} in \mathfrak{T} -contra. \square

5. COVERS REDUCED TO PROJECTIVE COVERS

Let \mathbf{A} be an additive category and $f: A \rightarrow B$ be a morphism in \mathbf{A} . A morphism $k: K \rightarrow A$ is said to be a *weak kernel* of f if $fk = 0$ and for any object $C \in \mathbf{A}$ and any morphism $c: C \rightarrow A$ such that $fc = 0$ there exists a (not necessarily unique) morphism $h: C \rightarrow K$ such that $c = kh$. A morphism k is a kernel of f if and only if k is a weak kernel of f and k is a monomorphism.

Let $\mathbf{L} \subset \mathbf{A}$ be a precovering class of objects. We are interested in conditions under which \mathbf{L} is a covering class. First of all, if \mathbf{L} is covering, then \mathbf{L} is closed under direct summands in \mathbf{A} . If \mathbf{L} is precovering and closed under direct summands, then \mathbf{L} is closed under coproducts (see the discussion in the previous section). In particular, the full subcategory $\mathbf{L} \subset \mathbf{A}$ is additive.

Lemma 5.1. *Let \mathbf{A} be an additive category with weak kernels and $\mathbf{L} \subset \mathbf{A}$ be an additive full subcategory. Assume that the class of objects \mathbf{L} is precovering in \mathbf{A} . Then the category \mathbf{L} also has weak kernels.*

Proof. Let $f: L \rightarrow M$ be a morphism in \mathbf{L} and $a: A \rightarrow L$ be a weak kernel of f in \mathbf{A} . Let $p: K \rightarrow A$ be an \mathbf{L} -precover of the object $A \in \mathbf{A}$. Then the composition $k = ap: K \rightarrow L$ is a weak kernel of f in \mathbf{L} . \square

Lemma 5.2. *Let \mathbf{A} be an idempotent-complete additive category with weak kernels and $\mathbf{L} \subset \mathbf{A}$ be an additive full subcategory closed under direct summands. Assume that the class of objects \mathbf{L} is precovering in \mathbf{A} . Then there exists a unique abelian category \mathbf{B} with enough projectives such that the full subcategory of projective objects $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$ is equivalent to the full subcategory $\mathbf{L} \subset \mathbf{A}$.*

Proof. By Lemma 5.1, the category \mathbf{L} has weak kernels. Hence the category \mathbf{B} can be constructed as the category of finitely presented (or “coherent”) functors $\mathbf{L}^{\text{op}} \rightarrow \mathbf{Ab}$ [16, Corollary 1.5], [21, Lemma 2.2 and Proposition 2.3] (see also [27, proof of Theorem 1.1(a)] for a discussion with further references). \square

Proposition 5.3. *Let \mathbf{A} be an additive category with cokernels and weak kernels and $\mathbf{L} \subset \mathbf{A}$ be an additive full subcategory closed under direct summands. Assume that the class of objects \mathbf{L} is precovering in \mathbf{A} . Let \mathbf{B} be the abelian category from Lemma 5.2. Then the equivalence of full subcategories $\mathbf{B} \supset \mathbf{B}_{\text{proj}} \cong \mathbf{L} \subset \mathbf{A}$ can be extended, in a unique way, to a pair of adjoint functors*

$$\Phi_{\mathbf{L}} : \mathbf{B} \rightleftarrows \mathbf{A} : \Psi_{\mathbf{L}},$$

where the functor $\Phi_{\mathbf{L}}$ is the left adjoint and the functor $\Psi_{\mathbf{L}}$ is the right adjoint.

Proof. The inclusion functor $\mathbf{B}_{\text{proj}} \cong \mathbf{L} \rightarrow \mathbf{A}$ extends uniquely to a right exact functor $\Phi_{\mathbf{L}} : \mathbf{B} \rightarrow \mathbf{A}$. This suffices to prove uniqueness of the desired adjoint pair.

To construct the functor $\Psi_{\mathbf{L}}$, we assign to every object $N \in \mathbf{A}$ the functor $\text{Hom}_{\mathbf{A}}(-, N)|_{\mathbf{L}} : \mathbf{L}^{\text{op}} \rightarrow \mathbf{Ab}$. Let us check that the functor $\text{Hom}_{\mathbf{A}}(-, N)|_{\mathbf{L}}$ is finitely presented. Choose an \mathbf{L} -precover $l : L \rightarrow N$ of the object N . Let $a : A \rightarrow L$ be a weak kernel of the morphism l in the category \mathbf{A} , and let $p : K \rightarrow A$ be an \mathbf{L} -precover of the object $A \in \mathbf{A}$. Consider the morphism $m = pa : K \rightarrow L$ in the category \mathbf{L} . Then the functor $\text{Hom}_{\mathbf{A}}(-, N)|_{\mathbf{L}}$ is the cokernel of the morphism of representable functors $\text{Hom}_{\mathbf{L}}(-, m) : \text{Hom}_{\mathbf{L}}(-, K) \rightarrow \text{Hom}_{\mathbf{L}}(-, L)$.

So the functor $\Psi_{\mathbf{L}} : \mathbf{A} \rightarrow \mathbf{B}$ assigning to an object N the functor $\text{Hom}_{\mathbf{A}}(-, N)|_{\mathbf{L}}$ is well-defined. By the Yoneda lemma, a natural isomorphism $\text{Hom}_{\mathbf{B}}(\Psi_{\mathbf{L}}(M), \Psi_{\mathbf{L}}(N)) \cong \text{Hom}_{\mathbf{A}}(M, N)$ holds for all objects $M \in \mathbf{L}$ and $N \in \mathbf{A}$. Hence we have an adjunction isomorphism $\text{Hom}_{\mathbf{B}}(P, \Psi_{\mathbf{L}}(N)) \cong \text{Hom}_{\mathbf{A}}(\Phi_{\mathbf{L}}(P), N)$ for all objects $P \in \mathbf{B}_{\text{proj}}$ and $N \in \mathbf{A}$. The latter isomorphism extends uniquely to a functorial isomorphism $\text{Hom}_{\mathbf{B}}(B, \Psi_{\mathbf{L}}(N)) \cong \text{Hom}_{\mathbf{A}}(\Phi_{\mathbf{L}}(B), N)$ for all objects $B \in \mathbf{B}$ and $N \in \mathbf{A}$ by right exactness of the functor $\Phi_{\mathbf{L}}$. \square

Proposition 5.4. *Let \mathbf{A} and \mathbf{B} be two categories, and let $\Phi : \mathbf{B} \rightleftarrows \mathbf{A} : \Psi$ be a pair of adjoint functors, where Ψ is the right adjoint, such that the restrictions of Φ and Ψ are mutually inverse equivalences between a full subcategory $\mathbf{L} \subset \mathbf{A}$ and a full subcategory $\mathbf{P} \subset \mathbf{B}$. Then*

(a) *a morphism $l : L \rightarrow N$ in \mathbf{A} with $L \in \mathbf{L}$ is an \mathbf{L} -precover if and only if the morphism $\Psi(l) : \Psi(L) \rightarrow \Psi(N)$ is a \mathbf{P} -precover;*

(b) *a morphism $l : L \rightarrow N$ in \mathbf{A} with $L \in \mathbf{L}$ is an \mathbf{L} -cover if and only if the morphism $\Psi(l) : \Psi(L) \rightarrow \Psi(N)$ is a \mathbf{P} -cover;*

(c) *an object $N \in \mathbf{A}$ has an \mathbf{L} -precover if and only if the object $\Psi(N) \in \mathbf{B}$ has a \mathbf{P} -precover;*

(d) *an object $N \in \mathbf{A}$ has an \mathbf{L} -cover if and only if the object $\Psi(N) \in \mathbf{B}$ has a \mathbf{P} -cover.*

Proof. Part (a): given an object $P \in \mathbf{P}$, the map of sets

$$\text{Hom}_{\mathbf{A}}(\Phi(P), l) : \text{Hom}_{\mathbf{A}}(\Phi(P), L) \longrightarrow \text{Hom}_{\mathbf{A}}(\Phi(P), N)$$

is isomorphic to the map of sets

$$\mathrm{Hom}_{\mathbf{B}}(P, \Psi(l)): \mathrm{Hom}_{\mathbf{B}}(P, \Psi(L)) \longrightarrow \mathrm{Hom}_{\mathbf{B}}(P, \Psi(N)).$$

Hence former map is surjective if and only if the latter map is. Since one has $\Phi(P) \in \mathbf{L}$ for all $P \in \mathbf{P}$, and every object $L' \in \mathbf{L}$ is isomorphic to an object $\Phi(P)$ for some $P \in \mathbf{P}$, the assertion follows.

Part (b): given an endomorphism $e: L \longrightarrow L$, one has $le = l$ if and only if $\Psi(e)\Psi(l) = \Psi(l)$, since the map

$$\mathrm{Hom}_{\mathbf{A}}(L, N) \cong \mathrm{Hom}_{\mathbf{A}}(\Phi\Psi(L), N) \longrightarrow \mathrm{Hom}_{\mathbf{B}}(\Psi(L), \Psi(N))$$

is bijective. Since the map $\mathrm{Hom}_{\mathbf{A}}(L, L) \longrightarrow \mathrm{Hom}_{\mathbf{B}}(\Psi(L), \Psi(L))$ is bijective, too, the assertion follows in view of part (a).

Finally, part (a) implies (c), and part (b) implies (d), because any morphism $p: P \longrightarrow \Psi(N)$ in \mathbf{B} with $P \in \mathbf{P}$ has the form $p = \Psi(l)$ for a (uniquely defined) morphism $l: L = \Phi(P) \longrightarrow N$ in \mathbf{A} . \square

Proposition 5.3 describes one situation in which Proposition 5.4 is applicable. Let \mathbf{A} be an additive category with cokernels and (weak) kernels, and let $\mathbf{L} \subset \mathbf{A}$ be a precovering class closed under direct summands. Consider the related abelian category \mathbf{B} , and put $\mathbf{P} = \mathbf{B}_{\mathrm{proj}} \subset \mathbf{B}$. Then an object $N \in \mathbf{A}$ has an \mathbf{L} -cover if and only if the object $\Psi_{\mathbf{L}}(N) \in \mathbf{B}$ has a projective cover. Hence the class $\mathbf{L} \subset \mathbf{A}$ is covering if and only if all objects of the form $\Psi_{\mathbf{L}}(N)$, $N \in \mathbf{A}$, have projective covers in \mathbf{B} .

Another such situation is described in Section 2.1. Let \mathbf{A} be a cocomplete additive category and $M \in \mathbf{A}$ be an object. Consider the related abelian category $\mathbf{B} = \mathbf{T}_M\text{-mod}$, and put $\mathbf{L} = \mathbf{Add}(M) \subset \mathbf{A}$ and $\mathbf{P} = \mathbf{B}_{\mathrm{proj}} \subset \mathbf{B}$. By Proposition 5.4(d), an object $N \in \mathbf{A}$ has an $\mathbf{Add}(M)$ -cover if and only if the object $\Psi_M(N) \in \mathbf{B}$ has a projective cover. Once again, we conclude that the class $\mathbf{Add}(M) \subset \mathbf{A}$ is covering if and only if all objects of the form $\Psi_M(N)$, $N \in \mathbf{A}$, have projective covers in \mathbf{B} .

Remark 5.5. In both contexts above, the existence of cokernels in the category \mathbf{A} was used in order to extend the equivalence $\mathbf{B}_{\mathrm{proj}} \cong \mathbf{L} \subset \mathbf{A}$ to a right exact functor $\Phi: \mathbf{B} \longrightarrow \mathbf{A}$. However, looking into the proof of Proposition 5.4, one can observe that the functor Φ is never applied to any objects outside of the full subcategory $\mathbf{P} \subset \mathbf{B}$. So one can relax the assumptions of that proposition by requiring the functor Φ to be defined on the full subcategory $\mathbf{P} \subset \mathbf{B}$ only. For this reason, the assumption of existence of cokernels in the category \mathbf{A} can be replaced by the weaker assumption of idempotent-completeness. Then, in the first of the above two settings (based on Proposition 5.3), the existence of weak kernels in \mathbf{A} is sufficient; and in the second one (based on Section 2.1), it suffices to assume that \mathbf{A} has coproducts.

6. TELESCOPE HOM EXACTNESS CONDITION

In this section we introduce the most general setting in which we can show that Main Conjecture 4.6 implies some instances of the Enochs conjecture.

Definition 6.1. Let \mathbf{A} be an additive category with countable direct limits, and let $M \in \mathbf{A}$ be an object. Given a sequence of endomorphisms $f_1, f_2, f_3, \dots \in \text{Hom}_{\mathbf{A}}(M, M)$, we form the inductive system

$$M \xrightarrow{f_1} M \xrightarrow{f_2} M \xrightarrow{f_3} \dots$$

and consider the related telescope sequence

$$(6.1) \quad \coprod_{n=1}^{\infty} M \longrightarrow \coprod_{n=1}^{\infty} M \longrightarrow \varinjlim_{n \geq 1} M \longrightarrow 0.$$

The short sequence (6.1) is always right exact, i. e., the direct limit $\varinjlim_{n \geq 1} M$ is the cokernel of the morphism $\text{id} - \text{shift}: \coprod_{n=1}^{\infty} M \longrightarrow \coprod_{n=1}^{\infty} M$.

We will say that the object $M \in \mathbf{A}$ satisfies the *telescope Hom exactness condition (THEC)* if, for any sequence of endomorphisms $(f_n \in \text{Hom}_{\mathbf{A}}(M, M))_{n \geq 1}$ of the object M , the short sequence (6.1) remains right exact after applying the functor $\text{Hom}_{\mathbf{A}}(M, -)$, that is, the short sequence of abelian groups

$$(6.2) \quad \text{Hom}_{\mathbf{A}}(M, \coprod_{n=1}^{\infty} M) \longrightarrow \text{Hom}_{\mathbf{A}}(M, \coprod_{n=1}^{\infty} M) \longrightarrow \text{Hom}_{\mathbf{A}}(M, \varinjlim_{n \geq 1} M) \longrightarrow 0$$

is right exact.

Example 6.2. Let \mathbf{A} be an abelian category with exact functors of countable direct limit. Then the telescope sequence (6.1) is exact at its leftmost term, too,

$$0 \longrightarrow \coprod_{n=1}^{\infty} M \longrightarrow \coprod_{n=1}^{\infty} M \longrightarrow \varinjlim_{n \geq 1} M \longrightarrow 0,$$

as it is a countable direct limit of the split exact sequences

$$0 \longrightarrow \coprod_{i=1}^{n-1} M \longrightarrow \coprod_{i=1}^n M \longrightarrow M \longrightarrow 0.$$

In this case, the exactness of the short sequence of Hom groups (6.2) at the middle term is obvious, and the telescope Hom exactness condition simply means exactness of the sequence (6.2) at the rightmost term. In other words, this means that any morphism $M \longrightarrow \varinjlim_{n \geq 1} M$ in the category \mathbf{A} can be lifted to a morphism $M \longrightarrow \coprod_{n=1}^{\infty} M$. This is equivalent to the condition that the morphism $\coprod_{i=1}^{\infty} M \longrightarrow \varinjlim_{n \geq 1} M$ in (6.1) is an $\text{Add}(M)$ -precover.

Examples 6.3. (1) Let \mathbf{A} be an abelian category with exact countable direct limits. Then the telescope Hom exactness condition holds for any Σ -rigid (or Σ - Ext^1 -self-orthogonal) object $M \in \mathbf{A}$, that is, any object such that $\text{Ext}_{\mathbf{A}}^1(M, M^{(\omega)}) = 0$.

(2) More generally, if there is a notion of *purity* in the abelian category \mathbf{A} , then for any two object $M, N \in \mathbf{A}$ one can consider the group $\text{PExt}_{\mathbf{A}}^1(M, N)$ of equivalence classes of pure short exact sequences $0 \longrightarrow N \longrightarrow A \longrightarrow M \longrightarrow 0$. An object $M \in \mathbf{A}$ is called Σ -pure-rigid (or Σ -pure- Ext^1 -self-orthogonal) if $\text{PExt}_{\mathbf{A}}^1(M, M^{(\omega)}) = 0$.

For any meaningful notion of purity, one expects that split short exact sequences should be pure exact. It is also reasonable to assume that the class of pure short exact sequences in \mathbf{A} is closed under countable direct limits and pullbacks, among

other things. If this is the case, then any Σ -pure-rigid object in \mathbf{A} satisfies THEC. In particular, this applies to the module categories $\mathbf{A} = A\text{-mod}$ over associative rings A .

One specific notion of purity in abelian categories, called the *functor purity*, will be discussed below in Section 8. It has the above-mentioned properties.

Example 6.4. Let \mathbf{A} be an abelian category with exact countable direct limits and a class of pure short exact sequences satisfying the conditions of Example 6.3 (2). We will say that an object $M \in \mathbf{A}$ is ω -self-pure-projective if for any pure short exact sequence $0 \rightarrow K \rightarrow M^{(\omega)} \rightarrow L \rightarrow 0$ in \mathbf{A} the induced morphism of abelian groups $\text{Hom}_{\mathbf{A}}(M, M^{(\omega)}) \rightarrow \text{Hom}_{\mathbf{A}}(M, L)$ is surjective. Any ω -self-pure-projective object $M \in \mathbf{A}$ satisfies the telescope Hom exactness condition.

For the rest of this section, we are working with a fixed object M in a cocomplete additive category \mathbf{A} . We consider the related abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ and the pair of adjoint functors $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ and $\Phi: \mathbf{B} \rightarrow \mathbf{A}$, as in Section 2.1.

Furthermore, we denote by $\mathbf{G} \subset \mathbf{A}$ the full subcategory formed by all the objects $G \in \mathbf{A}$ for which the adjunction morphism $\Phi(\Psi(G)) \rightarrow G$ is an isomorphism, and by $\mathbf{H} \subset \mathbf{B}$ the full subcategory of all the objects $H \in \mathbf{B}$ for which the adjunction morphism $H \rightarrow \Psi(\Phi(H))$ is an isomorphism. One has $\Psi(\mathbf{G}) \subset \mathbf{H}$ and $\Phi(\mathbf{H}) \subset \mathbf{G}$, and the restrictions of the functors Ψ and Φ to the full subcategories \mathbf{G} and \mathbf{H} are mutually inverse equivalences between them [15, Theorem 1.1],

$$(6.3) \quad \Psi|_{\mathbf{G}}: \mathbf{G} \cong \mathbf{H} : \Phi|_{\mathbf{H}}.$$

By construction, we have $\text{Add}(M) \subset \mathbf{G}$ and $\mathbf{B}_{\text{proj}} \subset \mathbf{H}$, since $\Psi|_{\text{Add}(M)}: \text{Add}(M) \rightarrow \mathbf{B}_{\text{proj}}$ and $\Phi|_{\mathbf{B}_{\text{proj}}}: \mathbf{B}_{\text{proj}} \rightarrow \text{Add}(M)$ are mutually inverse equivalences.

Lemma 6.5. *Let \mathbf{A} be a cocomplete additive category, $M \in \mathbf{A}$ be an object, and $\mathbf{B} = \mathbb{T}_M\text{-mod}$ be the related abelian category. Suppose that the class of all projective objects in \mathbf{B} is closed under (arbitrary or countable) direct limits. Then the class of objects $\text{Add}(M) \subset \mathbf{A}$ is also closed under (arbitrary or countable, resp.) direct limits. More specifically, if every countable direct limit of copies of the projective generator $P = T_M(*)$ is projective in \mathbf{B} , then every countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$.*

Proof. Let Θ be a directed poset and $A: \Theta \rightarrow \mathbf{A}$ be a diagram such that the object $A(\theta)$ belongs to the class $\text{Add}(M)$ for all $\theta \in \Theta$. Applying the functor Ψ , we obtain a diagram $B = \Psi \circ A: \Theta \rightarrow \mathbf{B}$ such that $B(\theta)$ is a projective object in \mathbf{B} for all $\theta \in \Theta$. Applying the functor Φ to get back to the category \mathbf{A} , we come to the original diagram $A \cong \Phi \circ B$. Now the functor Φ , being a left adjoint, preserves all colimits, so the natural morphism $\varinjlim_{\theta \in \Theta} A(\theta) \cong \varinjlim_{\theta \in \Theta} \Phi(B(\theta)) \rightarrow \Phi(\varinjlim_{\theta \in \Theta} B(\theta))$ is an isomorphism in \mathbf{A} . Since $\varinjlim_{\theta \in \Theta} B(\theta)$ is a projective object in \mathbf{B} by assumption and $\Phi(\mathbf{B}_{\text{proj}}) = \text{Add}(M)$, the desired conclusion follows. \square

Proposition 6.6. *Let \mathbf{A} be a cocomplete additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Let $\mathbf{B} = \mathbb{T}_M\text{-mod}$ be the related cocomplete abelian category with a projective generator $P = \mathbb{T}_M(*) \in \mathbf{B}$ corresponding to the object $M \in \mathbf{A}$, and let $\mathbf{G} \subset \mathbf{A}$ and $\mathbf{H} \subset \mathbf{B}$ be the related two full subcategories.*

Then all countable direct limits of copies of the object M in \mathbf{A} belong to the class \mathbf{G} , and all the countable direct limits of copies of the object P in \mathbf{B} belong to the class \mathbf{H} . The functor Ψ preserves countable direct limits of copies of the object $M \in \mathbf{A}$ (taking them to countable direct limits of copies of the object $P \in \mathbf{B}$).

Proof. Let $M \xrightarrow{f_1} M \xrightarrow{f_2} M \xrightarrow{f_3} \dots$ be a countable inductive system of copies of the object M in \mathbf{A} . Then we have the right exact sequence (6.1) in the category \mathbf{A} and the right exact sequence (6.2) in the category of abelian groups.

Now, the abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ is endowed with a faithful exact forgetful functor $\mathbb{T}_M\text{-mod} \rightarrow \mathbf{Ab}$, and the composition of the functor Ψ with this forgetful functor is isomorphic to the functor $\text{Hom}_{\mathbf{A}}(M, -)$. It follows that the image of the sequence (6.1) under the functor Ψ is right exact in \mathbf{B} .

The functors Ψ and Φ restrict to mutually inverse equivalences between $\text{Add}(M) \subset \mathbf{A}$ and $\mathbf{B}_{\text{proj}} \subset \mathbf{B}$; so, in particular, they transform coproducts of objects from $\text{Add}(M)$ in \mathbf{A} to coproducts of projective objects in \mathbf{B} and vice versa. The short sequence

$$(6.4) \quad \coprod_{n=1}^{\infty} \Psi(M) \longrightarrow \coprod_{n=1}^{\infty} \Psi(M) \longrightarrow \varinjlim_{n \geq 1} \Psi(M) \longrightarrow 0$$

is right exact in \mathbf{B} ; and the natural morphism from the sequence (6.4) to the image of the sequence (6.1) under the functor Ψ is an isomorphism at the leftmost and the middle terms. Hence it is also an isomorphism at the rightmost terms, that is, the natural morphism $\varinjlim_n \Psi(M) \rightarrow \Psi(\varinjlim_n M)$ is an isomorphism.

The functor Φ , being a left adjoint, preserves all colimits. Since the adjunction morphism $\Phi\Psi(M) \rightarrow M$ is an isomorphism, it follows that the adjunction morphism $\Phi\Psi(\varinjlim_n M) \rightarrow \varinjlim_n M$ is an isomorphism, too. Thus $\varinjlim_n M \in \mathbf{G}$.

We have shown that countable direct limits of copies of the object M in \mathbf{A} belong to \mathbf{G} , and we have also seen that the functor Ψ transforms countable direct limits of copies of M in \mathbf{A} to countable direct limits of copies of P in \mathbf{B} . Therefore, countable direct limits of copies of P belong to $\Psi(\mathbf{G}) = \mathbf{H}$. \square

Corollary 6.7. *Let \mathbf{A} be a cocomplete additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Let $\mathbf{B} = \mathbb{T}_M\text{-mod}$ be the related cocomplete abelian category with a projective generator $P = \mathbb{T}_M(*) \in \mathbf{B}$ corresponding to the object $M \in \mathbf{A}$. Then the following two conditions are equivalent:*

- (1) any countable direct limit of copies of M has an $\text{Add}(M)$ -cover in \mathbf{A} ;
- (2) any countable direct limit of copies of P has a projective cover in \mathbf{B} .

The following three conditions are also equivalent to each other:

- (3) any countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$;
- (4) any countable direct limit of copies of P in \mathbf{B} is projective;
- (5) any countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$, and the related natural epimorphism $\coprod_{n=1}^{\infty} M \rightarrow \varinjlim_{n \geq 1} M$ (6.1) splits.

All the five conditions (1–5) are equivalent when \mathbf{A} is an abelian category with exact countable direct limits. Also, all the five conditions (1–5) are equivalent when $\mathbf{B} =$

\mathfrak{A} -contra is the category of left contramodules over a complete, separated, right linear topological ring \mathfrak{A} .

Proof. Both the equivalences (1) \iff (2) and (3) \iff (4) follow from Proposition 6.6 and the equivalence of categories (6.3). Since any epimorphism onto a projective object splits in \mathbf{B} , we also obtain the equivalence (4) \iff (5). Alternatively, the equivalence (3) \iff (5) follows directly from THEC.

When \mathbf{A} is an abelian category with exact countable direct limits, the equivalence (1) \iff (3) holds by (the proof of) [9, Theorem 4.4]. When $\mathbf{B} = \mathfrak{A}$ -contra, the equivalence (2) \iff (4) is provided by [9, Theorem 2.1 or Corollary 2.10]. \square

The following theorem is the main result of this section.

Theorem 6.8. *Let \mathbf{A} be a locally presentable additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Assume that Main Conjecture 4.6 holds for the locally presentable abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$. Then the following conditions are equivalent:*

- (1) *the class of objects $\text{Add}(M) \subset \mathbf{A}$ is covering;*
- (2) *any direct limit of objects from $\text{Add}(M)$ has an $\text{Add}(M)$ -cover in \mathbf{A} ;*
- (3) *any countable direct limit of copies of M has an $\text{Add}(M)$ -cover in \mathbf{A} ;*
- (4) *any countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$;*
- (5) *the class of objects $\text{Add}(M)$ is closed under direct limits in \mathbf{A} ;*
- (6) *the class \mathbf{B}_{proj} is covering in \mathbf{B} ;*
- (7) *any direct limit of projective objects has a projective cover in \mathbf{B} ;*
- (8) *any countable direct limit of copies of the projective generator $P = \mathbb{T}_M(*) \in \mathbf{B}$ has a projective cover in \mathbf{B} ;*
- (9) *any countable direct limit of copies of the projective generator $P = \mathbb{T}_M(*)$ is projective in \mathbf{B} ;*
- (10) *the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} .*

Proof. The implications (1) \implies (2) \implies (3) and (5) \implies (4) \implies (3) are obvious. The implication (5) \implies (1) holds by Example 4.1 and Theorem 4.4.

Conditions (6–10) are equivalent to each other by assumption. The implications (9) \implies (4) and (10) \implies (5) are provided by Lemma 6.5.

Finally, the conditions (3) and (8) are equivalent by Corollary 6.7 (1) \Leftrightarrow (2). \square

7. PERFECT DECOMPOSITIONS

The following definitions and terminology can be found in the manuscript [12].

Let \mathbf{A} be an additive category with set-indexed products and coproducts. Then the category \mathbf{A} is called *agreeable* if, for every family of objects $(N_x \in \mathbf{A})_{x \in X}$, the natural morphism from the coproduct to the product

$$\coprod_{x \in X} N_x \longrightarrow \prod_{x \in X} N_x$$

is a monomorphism in \mathbf{A} .

More generally, let \mathbf{A} be an additive category with coproducts (but not necessarily with products). Consider an object $M \in \mathbf{A}$ and a family of objects $(N_x \in \mathbf{A})_{x \in X}$. For every index $y \in X$, one has the natural coordinate projection morphism $\pi_y: \coprod_{x \in X} N_x \longrightarrow N_y$. Given a morphism $f: M \longrightarrow \coprod_{x \in X} N_x$, one can compose it with the morphism π_y , obtaining a morphism $\pi_y \circ f: M \longrightarrow N_y$. Consider the map of abelian groups

$$\eta: \text{Hom}_{\mathbf{A}}\left(M, \coprod_{x \in X} N_x\right) \longrightarrow \prod_{x \in X} \text{Hom}_{\mathbf{A}}(M, N_x)$$

assigning to a morphism f the collection of morphisms $f_x = \pi_x \circ f$, $x \in X$.

Following [12], we will say that the category \mathbf{A} is *agreeable* if the map η is injective for all objects M and families of objects $N_x \in \mathbf{A}$. When the category \mathbf{A} has products as well as coproducts, this definition is clearly equivalent to the previous one.

We will say that a family of morphisms $(f_x: M \rightarrow N_x)$ in an agreeable category \mathbf{A} is *summable* if there exists a morphism $f: M \longrightarrow \coprod_{x \in X} N_x$ such that $f_x = \pi_x \circ f$ for every $x \in X$. When $N_x = N$ is one and the same object for all $x \in X$, one can construct the *sum* $g = \sum_{x \in X} f_x$ of a summable family of morphisms $(f_x: M \rightarrow N)_{x \in X}$ as the composition $g = \Sigma \circ f$ of the morphism $f: M \longrightarrow N^{(X)}$ with the natural summation morphism $\Sigma: N^{(X)} \longrightarrow N$.

In this paper, we will not be dealing with the sums of summable families of morphisms. Instead, we will use the notion of a summable family in order to extend the classical concept of a module with *perfect decomposition* [3] to the categorical realm.

Let \mathbf{A} be an agreeable additive category and $(M_\xi \in \mathbf{A})_{\xi \in \Xi}$ be a family of objects. For any sequence of indices $\xi_1, \xi_2, \xi_3, \dots \in \Xi$ and any sequence of morphisms $f_i: M_{\xi_i} \longrightarrow M_{\xi_{i+1}}$ in \mathbf{A} , we consider the sequence of compositions

$$f_n f_{n-1} \cdots f_1: M_{\xi_1} \longrightarrow M_{\xi_{n+1}}, \quad n \geq 1.$$

The family of objects $(M_\xi)_{\xi \in \Xi}$ is said to be *locally T-nilpotent* if for every sequence of indices ξ_i and every sequence of *nonisomorphisms* $f_i: M_{\xi_i} \longrightarrow M_{\xi_{i+1}}$, the family of morphisms $(f_n f_{n-1} \cdots f_1)_{n \geq 1}$ is summable in \mathbf{A} .

In the case of a module category $\mathbf{A} = A\text{-mod}$, this reduces to the classical definition: a family of modules $(M_\xi)_{\xi \in \Xi}$ is *locally T-nilpotent* if for every sequence of indices ξ_i , every sequence of nonisomorphisms $f_i: M_{\xi_i} \longrightarrow M_{\xi_{i+1}}$ in $A\text{-mod}$, and every element $m \in M_{\xi_1}$, there exists an integer $n \geq 1$ such that $f_n f_{n-1} \cdots f_1(m) = 0$ in $M_{\xi_{n+1}}$.

An object M of an agreeable additive category \mathbf{A} is said to have a *perfect decomposition* if there exists a locally T-nilpotent family of objects $(M_\xi \in \mathbf{A})_{\xi \in \Xi}$ such that $M \cong \coprod_{\xi \in \Xi} M_\xi$. More generally, one can (and we will) drop the assumption that \mathbf{A} is agreeable and just assume that the full subcategory $\text{Add}(M) \subset \mathbf{A}$ is agreeable instead. Thus, let \mathbf{A} be an additive category with coproducts and let $M \in \mathbf{A}$ be an object. We will say that M has a perfect decomposition if the category $\text{Add}(M)$ is agreeable and there exists a locally T-nilpotent family of objects $(M_\xi \in \text{Add}(M))_{\xi \in \Xi}$ such that $M \cong \coprod_{\xi \in \Xi} M_\xi$.

The definition of a topologically left perfect topological ring, which was introduced in [28, Section 10] and reproduced above in Section 3, is the topological ring counterpart of the notion of an object with perfect decomposition. The following result obtained in the paper [28] illustrates the connection. In the case of module categories, the equivalence of conditions (i) and (iii) was established in [3, Theorem 1.4].

Theorem 7.1. *Let \mathbf{A} be an idempotent-complete additive category with coproducts and $M \in \mathbf{A}$ be an object. Assume that the monad $\mathbb{T}_M: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}$ for a complete, separated right linear topological ring \mathfrak{R} . Consider the following three properties:*

- (i) *the object $M \in \mathbf{A}$ has a perfect decomposition;*
- (ii) *the topological ring \mathfrak{R} is topologically left perfect;*
- (iii) *for any directed poset Θ and a diagram $A: \Theta \rightarrow \mathbf{Add}(M)$, the direct limit $\varinjlim_{\theta \in \Theta} A(\theta)$ exists in \mathbf{A} , belongs to $\mathbf{Add}(M)$, and the natural epimorphism $\coprod_{\theta \in \Theta} A(\theta) \rightarrow \varinjlim_{\theta \in \Theta} A(\theta)$ is split.*

Then the implications (i) \iff (ii) \implies (iii) hold.

If \mathbf{A} is a cocomplete abelian category with exact direct limits, then all the three conditions (i–iii) are equivalent. If $\mathbf{A} = \mathfrak{R}\text{-contra}$ is the category of contra-modules over a complete, separated right linear topological ring and $M = \mathfrak{R}$ is the free contra-module, then all the three conditions (i–iii) are equivalent as well.

Proof. (i) \iff (ii) By the definition, an object $M \in \mathbf{A}$ having a perfect decomposition means that M has a perfect decomposition as an object of the category $\mathbf{Add}(M)$. Following (2.2), the category $\mathbf{Add}(M)$ is equivalent to $\mathbb{T}_M\text{-mod}_{\text{proj}}$; so an isomorphism of monads $\mathbb{T}_M \cong \mathbb{T}_{\mathfrak{R}}$ implies that $\mathbf{Add}(M)$ is equivalent to the category of projective left \mathfrak{R} -contra-modules $\mathfrak{R}\text{-contra}_{\text{proj}}$. This equivalence of categories takes the object $M \in \mathbf{Add}(M)$ to the object $\mathfrak{R} \in \mathfrak{R}\text{-contra}_{\text{proj}}$. According to [28, Remark 3.11], the category $\mathfrak{R}\text{-contra}_{\text{proj}}$ is topologically agreeable. Now the equivalence of the two conditions (i) \iff (ii) is provided by [28, Theorem 10.4].

(i) \implies (iii) By [28, Theorem 10.2], condition (i) implies (in fact, is equivalent to) the category $\mathbf{Add}(M)$ having split direct limits in the sense of [28, Section 9]. According to [28, Lemma 9.2(b)], it follows that the direct limits of diagrams in $\mathbf{Add}(M)$ exist in \mathbf{A} and belong to $\mathbf{Add}(M)$; and by [28, Lemma 9.1 (1) \implies (3)], the natural epimorphisms $\coprod_{\theta \in \Theta} A(\theta) \rightarrow \varinjlim_{\theta \in \Theta} A(\theta)$ are split.

(iii) \implies (i) By [28, Theorem 10.2], in order to prove (i) it suffices to check that the category $\mathbf{Add}(M)$ has split direct limits. Now in the case of an abelian category \mathbf{A} with exact direct limits, the desired implication is provided by [28, Corollary 9.3 (3) \implies (0)]. In the case when $\mathbf{A} = \mathfrak{R}\text{-contra}$ and $M = \mathfrak{R}$, [28, Proposition 9.6] is applicable. \square

Countable direct limits of copies of the free \mathfrak{R} -contra-module with one generator $\mathfrak{R} = \mathfrak{R}[[*]]$ in $\mathfrak{R}\text{-contra}$ are called *Bass flat \mathfrak{R} -contra-modules* [24, Section 4].

Corollary 7.2. *Let \mathbf{A} be a cocomplete additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Assume that the monad $\mathbb{T}_M: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is isomorphic to the*

monad $\mathbb{T}_{\mathfrak{R}}$ for a complete, separated right linear topological ring \mathfrak{R} . Consider the following ten properties:

- (1) the object $M \in \mathbf{A}$ has a perfect decomposition;
- (2) the topological ring \mathfrak{R} is topologically left perfect;
- (3) the class $\mathfrak{R}\text{-contra}_{\text{proj}}$ is closed under direct limits in $\mathfrak{R}\text{-contra}$;
- (4) the class of objects $\text{Add}(M)$ is closed under direct limits in \mathbf{A} ;
- (5) every countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$;
- (6) all Bass flat left \mathfrak{R} -contramodules are projective;
- (7) all Bass flat left \mathfrak{R} -contramodules have projective covers in $\mathfrak{R}\text{-contra}$;
- (8) every countable direct limit of copies of M has an $\text{Add}(M)$ -cover in \mathbf{A} ;
- (9) all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;
- (10) all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.

Then the following implications hold:

$$(1) \iff (2) \iff (3) \implies (4) \implies (5) \iff (6) \iff (7) \iff (8) \implies (9) \implies (10).$$

If the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d), then all the conditions (1–10) are equivalent. If the topological ring \mathfrak{R} satisfies one of the conditions (e), (f), or (g), then the nine conditions (1–9) are equivalent.

Proof. (1) \iff (2) is Theorem 7.1(i) \Leftrightarrow (ii).

(2) \iff (3) is [28, Theorem 14.1 (iv) \Leftrightarrow (iii')].

(1) \implies (4) is Theorem 7.1(i) \Rightarrow (iii); (3) \implies (4) is Lemma 6.5.

The implications (4) \implies (5) \implies (8) and (3) \implies (6) \implies (7) are obvious.

The equivalences (5) \iff (6) \iff (7) \iff (8) are provided by Corollary 6.7.

(7) \implies (10) is [24, Corollary 4.7]; (6) \implies (10) is [24, Corollary 4.5].

(6) \implies (9) is [24, Proposition 4.3 and Lemma 6.3]; (9) \implies (10) is explained in [24, proof of Theorem 10.1] (see also [28, Theorem 14.4]).

The last two assertions of the corollary follow from Corollary 7.3 below. \square

Left modules M over an associative ring A for which there exists a topological ring \mathfrak{R} satisfying (e) such that the monad \mathbb{T}_M is isomorphic to $\mathbb{T}_{\mathfrak{R}}$ are discussed under the name of *weakly countably generated modules* in the paper [9, Section 7.2].

The next corollary, covering the assertions of Theorem 0.1 from the introduction, is our main result in the setting of Sections 4–7.

Corollary 7.3. *Let \mathbf{A} be a locally presentable additive category and $M \in \mathbf{A}$ be an object satisfying THEC. Assume that the monad $\mathbb{T}_M: \mathbf{Sets} \rightarrow \mathbf{Sets}$ is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}$ for a complete, separated right linear topological ring \mathfrak{R} satisfying one of the conditions (a), (b), (c), or (d). Let $\mathbf{B} = \mathbb{T}_M\text{-mod} \cong \mathfrak{R}\text{-contra}$ be the related abelian category of contramodules. Then the following conditions are equivalent:*

- (1) the class of objects $\text{Add}(M) \subset \mathbf{A}$ is covering;
- (2) any direct limit of objects from $\text{Add}(M)$ has an $\text{Add}(M)$ -cover in \mathbf{A} ;
- (3) every countable direct limit of copies of M has an $\text{Add}(M)$ -cover in \mathbf{A} ;
- (4) every countable direct limit of copies of M in \mathbf{A} belongs to $\text{Add}(M)$;
- (5) the class of objects $\text{Add}(M)$ is closed under direct limits in \mathbf{A} ;

- (6) the class \mathbf{B}_{proj} is covering in \mathbf{B} ;
- (7) any direct limit of projective objects has a projective cover in \mathbf{B} ;
- (8) every countable direct limit of copies of the projective generator $\mathfrak{R} \in \mathbf{B}$ has a projective cover in \mathbf{B} ;
- (9) every countable direct limit of copies of the projective generator $\mathfrak{R} \in \mathbf{B}$ is a projective object in \mathbf{B} ;
- (10) the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} ;
- (11) the object $M \in \mathbf{A}$ has a perfect decomposition;
- (12) the topological ring \mathfrak{R} is topologically left perfect;
- (13) all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;
- (14) there is a topologically left T -nilpotent strongly closed two-sided ideal $\mathfrak{H} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is isomorphic, as a topological ring, to a product of simple Artinian discrete rings endowed with the product topology;
- (15) all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.

Replacing the assumption of one of the conditions (a–d) with that of one of the conditions (e), (f), or (g), the thirteen conditions (1–13) are equivalent.

Proof. The conditions (1–10) are equivalent to each other by Theorem 6.8, whose applicability follows from any one of the conditions (a), (b), (c), (d), (e), (f), or (g) by Theorem 4.8. Under any one of the conditions (a), (b), (c), or (d), the conditions (6–10) and (13–15) are equivalent to each other by [24, Theorem 12.4], while the conditions (6–10) and (12–13) are equivalent to each other by [28, Corollary 14.7]. Assuming (e), (f), or (g), the conditions (6–10) and (12–13) are equivalent to each other by Theorems 4.8 and 3.5, and [28, Theorem 14.4]. The equivalence (11) \iff (12) holds by Theorem 7.1 (i) \iff (ii).

This suffices to prove the corollary; but alternatively, here is a direct proof of the equivalence (12) \iff (14) under the assumption of condition (d). By the Artin–Wedderburn classification of simple Artinian rings, (14) implies (12) unconditionally. In fact, the only difference between (14) and (12) is that the sets Υ_γ can be infinite in (12); the class of topological rings \mathfrak{S} in (14) is obtained by such class in (12) by imposing the condition that all the sets Υ_γ are finite (cf. [28, Remark 14.6]).

Let \mathfrak{R} be a topologically left perfect topological ring satisfying (d). Let $\mathfrak{H} \subset \mathfrak{R}$ be the ideal from the definition of a topologically left perfect topological ring, and $\mathfrak{K} \subset \mathfrak{R}$ be the ideal from condition (d). Then the argument from the proof of Lemma 3.3 (based on [28, Lemma 10.3] and [24, Lemma 6.6(a)]) shows that $\mathfrak{K} \subset \mathfrak{H}$. So the topological ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is a quotient ring of the topological ring $\mathfrak{R}/\mathfrak{K}$. Hence for every $\gamma \in \Gamma$ the topological ring $\mathfrak{S}_\gamma = \text{Hom}_{D_\gamma}(D_\gamma^{(\Upsilon_\gamma)}, D_\gamma^{(\Upsilon_\gamma)\text{op}})$ is also a quotient ring of the topological ring $\mathfrak{R}/\mathfrak{K}$.

The ring $\mathfrak{R}/\mathfrak{K} \cong \prod_{\delta \in \Delta} \mathfrak{T}_\delta = \mathfrak{T}$, on the other hand, is the product of two-sided linear topological rings \mathfrak{T}_δ , so \mathfrak{T} is a two-sided linear topological ring, too. As any topological quotient ring of a two-sided linear topological ring is two-sided linear, the ring \mathfrak{S}_γ must be two-sided linear, i. e., it has a base of neighborhoods of zero

consisting of two-sided ideals. Since, in fact, there are no nonzero proper open two-sided ideals in \mathfrak{S}_γ , it follows that \mathfrak{S}_γ must be discrete, which happens exactly when the set Υ_γ is finite. \square

8. FUNCTOR PURITY IN ABELIAN CATEGORIES

Let A be an associative ring. A short exact sequence of left A -modules $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ is said to be *pure* if the map of abelian groups $N \otimes_A K \rightarrow N \otimes_A M$ is injective for every right A -module N , or equivalently, if the map $\text{Hom}_A(E, M) \rightarrow \text{Hom}_A(E, L)$ is surjective for every finitely presented left A -module E . A short exact sequence of left A -modules is pure if and only if it is a direct limit of split short exact sequences of left A -modules [19, Lemma 2.19].

The aim of this section is to suggest a simple way to extend the notion of purity to arbitrary cocomplete abelian categories. We will use it in the next Section 9.

Let \mathbf{A} be a cocomplete abelian category. We will say that a monomorphism $f: K \rightarrow M$ is *pure* (or *functor pure*) in \mathbf{A} if for every cocomplete abelian category \mathbf{V} with exact direct limit functors, and any additive functor $F: \mathbf{A} \rightarrow \mathbf{V}$ preserving all colimits (that is, a right exact covariant functor preserving coproducts), the morphism $F(f): F(K) \rightarrow F(M)$ is a monomorphism in \mathbf{V} . If this is the case, the object K is said to be a *(functor) pure subobject* of the object $M \in \mathbf{A}$.

A short exact sequence $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ in \mathbf{A} is called *(functor) pure* if the monomorphism $K \rightarrow M$ is pure, or equivalently, if the short sequence $0 \rightarrow F(K) \rightarrow F(M) \rightarrow F(L) \rightarrow 0$ is exact in \mathbf{V} for every functor $F: \mathbf{A} \rightarrow \mathbf{V}$ as above. The morphism $M \rightarrow L$ is then said to be a *(functor) pure epimorphism*, and the object L a *pure quotient* of M . A long exact sequence M^\bullet in \mathbf{A} is said to be *pure* if it is obtained by splicing pure short exact sequences in \mathbf{A} , or equivalently, if the complex $F(M^\bullet)$ is exact in \mathbf{V} for every abelian category \mathbf{V} with exact direct limits and any colimit-preserving functor $F: \mathbf{A} \rightarrow \mathbf{V}$.

Lemma 8.1. *Let $\mathbf{A} = A\text{-mod}$ be the abelian category of left modules over an associative ring A . Then a monomorphism (or a short exact sequence, or a long exact sequence) in $A\text{-mod}$ is functor pure if and only if it is pure in the conventional sense of the word (as in [19]).*

Proof. A functor $A\text{-mod} \rightarrow \mathbf{Ab}$ from the category of left A -modules to the category of abelian groups \mathbf{Ab} preserves colimits if and only if it is isomorphic to the functor of tensor product $M \mapsto N \otimes_A M$ with a certain right A -module N [35, Theorem 1]. (Colimit-preserving functors $A\text{-mod} \rightarrow \mathbf{V}$ can be similarly described as the functors of tensor product with an object in \mathbf{V} endowed with a right action of the ring A .) So any functor pure exact sequence in $A\text{-mod}$ remains exact after taking the tensor product with any right A -module N , i. e., it is pure exact in the conventional sense.

Conversely, any pure short exact sequence of left A -modules is a direct limit of split short exact sequences. Hence its image under any colimit-preserving functor (and

more generally, under any direct limit-preserving additive functor) $F: \mathbf{A}\text{-mod} \rightarrow \mathbf{V}$, taking values in an abelian category \mathbf{V} with exact direct limits, is exact. \square

Lemma 8.2. *In any cocomplete abelian category \mathbf{A} , the class of functor pure monomorphisms is closed under pushouts and compositions. The class of functor pure epimorphisms is closed under pullbacks and compositions.*

Proof. Essentially, the lemma claims that the category \mathbf{A} with the class of all pure short exact sequences is a Quillen exact category. To prove such an assertion, it suffices to check that the class of pure monomorphisms is closed under pushouts and compositions, and the class of pure epimorphisms is closed under pullbacks. Closedness of the class of pure epimorphisms with respect to compositions will then follow [20, Section A.1].

Let $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ be a pure short exact sequence in \mathbf{A} and $L' \rightarrow L$ be a morphism. Since \mathbf{A} is an abelian category, the pullback sequence $0 \rightarrow K \rightarrow M' \rightarrow L' \rightarrow 0$ is exact. To show that the epimorphism $M' \rightarrow L'$ is pure, it suffices to check that the monomorphism $K \rightarrow M'$ is pure. Indeed, the composition $K \rightarrow M' \rightarrow M$ is a pure monomorphism. Since for any colimit-preserving functor $F: \mathbf{A} \rightarrow \mathbf{V}$ the morphism $F(K) \rightarrow F(M)$ is a monomorphism, the morphism $F(K) \rightarrow F(M')$ is a monomorphism, too.

Let $K \rightarrow K''$ be a morphism in \mathbf{A} and $0 \rightarrow K'' \rightarrow M'' \rightarrow L \rightarrow 0$ be the pushout sequence. Once again, since \mathbf{A} is abelian, the pushout sequence is exact. Any colimit-preserving functor $F: \mathbf{A} \rightarrow \mathbf{V}$ preserves pushouts; so $F(K) \rightarrow F(M) \rightarrow F(M'')$, $F(K) \rightarrow F(K'') \rightarrow F(M'')$ is a pushout square. Since the morphism $F(K) \rightarrow F(M)$ is a monomorphism, the morphism $F(K'') \rightarrow F(M'')$ is a monomorphism, too.

The assertion that the composition of any two pure monomorphisms is a pure monomorphism is obvious. \square

Example 8.3. Let \mathbf{A} be a cocomplete abelian category with exact countable direct limits. Then, for any sequence of objects and morphisms $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \dots$ in \mathbf{A} , the short sequence

$$(8.1) \quad 0 \rightarrow \coprod_{n=1}^{\infty} A_n \longrightarrow \coprod_{n=1}^{\infty} A_n \longrightarrow \varinjlim_{n \geq 1} A_n \longrightarrow 0.$$

is pure exact. Indeed, the sequence (8.1) is exact as the countable direct limit of split exact sequences $0 \rightarrow \coprod_{i=1}^{n-1} A_i \rightarrow \coprod_{i=1}^n A_i \rightarrow A_n \rightarrow 0$. The image of (8.1) under a colimit-preserving functor $F: \mathbf{A} \rightarrow \mathbf{V}$ is the similar short sequence for the inductive system $F(A_1) \rightarrow F(A_2) \rightarrow F(A_3) \rightarrow \dots$ in the category \mathbf{V} , which is exact whenever countable direct limits are exact in \mathbf{V} .

Example 8.4. Let \mathbf{A} be a cocomplete abelian category with exact direct limits. Let Θ be a directed poset and $A: \Theta \rightarrow \mathbf{A}$ be a Θ -indexed diagram in \mathbf{A} . Then the augmented bar-complex

$$(8.2) \quad \dots \longrightarrow \coprod_{\theta_0 \leq \theta_1 \leq \theta_2} A(\theta_0) \longrightarrow \coprod_{\theta_0 \leq \theta_1} A(\theta_0) \longrightarrow \coprod_{\theta_0} A(\theta_0) \longrightarrow \varinjlim_{\theta \in \Theta} A(\theta) \longrightarrow 0$$

is pure exact in \mathbf{A} . Indeed, the complex (8.2) is the direct limit (over $\delta \in \Theta$) of the similar bar-complexes related to the subposets $\Theta_\delta = \{\theta \in \Theta: \theta \leq \delta\} \subset \Theta$ and the subdiagrams $A|_{\Theta_\delta}$ of A . The bar-complex of any diagram indexed by a poset with a greatest element is easily seen to be contractible (by the explicit contracting homotopy given by the morphisms taking the summand $A(\theta_0)$ indexed by $\theta_0 \leq \cdots \leq \theta_n$ to the summand $A(\theta_0)$ indexed by $\theta_0 \leq \cdots \leq \theta_n \leq \delta$).

This proves exactness of (8.2). To prove the pure exactness, one observes that the image of (8.2) under a colimit-preserving functor $F: \mathbf{A} \rightarrow \mathbf{V}$ is the similar augmented bar-complex for the diagram $F \circ A: \Theta \rightarrow \mathbf{V}$ in the category \mathbf{V} , which is exact for the same reason explained above whenever direct limits are exact in \mathbf{V} .

In addition, we have shown that all the objects of cycles in the bar-complex (8.2) are direct limits (over the poset Θ) of direct sums of copies of the objects $A(\theta)$, $\theta \in \Theta$. Indeed, one easily observes that all the objects of cycles in the bar-complexes related to the subposets $\Theta_\delta \subset \Theta$ are direct sums of copies of the objects $A(\theta)$.

9. SELF-PURE-PROJECTIVE AND \varinjlim -PURE-RIGID OBJECTS

The aim of this section is to prove the analogues of such results as Proposition 6.6, Corollary 6.7, and the related equivalence of properties in Corollary 7.2 for uncountable direct limits, under appropriate assumptions.

Let \mathbf{A} be a cocomplete abelian category. We use the notion of (functor) purity defined in Section 8.

An object $M \in \mathbf{A}$ is said to be *pure-split* if every pure monomorphism $K \rightarrow M$ is split in \mathbf{A} . One says that an object $T \in \mathbf{A}$ is *Σ -pure-split* if all the objects M from the class $\text{Add}(T) \subset \mathbf{A}$ are pure-split in \mathbf{A} .

An object $Q \in \mathbf{A}$ is said to be *pure-projective* if, for any pure short exact sequence $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ in the category \mathbf{A} , the short sequence of abelian groups $0 \rightarrow \text{Hom}_{\mathbf{A}}(Q, K) \rightarrow \text{Hom}_{\mathbf{A}}(Q, M) \rightarrow \text{Hom}_{\mathbf{A}}(Q, L) \rightarrow 0$ is exact. In other words, an object of \mathbf{A} is pure-projective if it is projective with respect to the pure exact structure on \mathbf{A} .

We will say that an object $M \in \mathbf{A}$ is *self-pure-projective* if, for any pure short exact sequence $0 \rightarrow K \rightarrow M' \rightarrow L \rightarrow 0$ in \mathbf{A} with $M' \in \text{Add}(M)$, the short sequence of abelian groups $0 \rightarrow \text{Hom}_{\mathbf{A}}(M, K) \rightarrow \text{Hom}_{\mathbf{A}}(M, M') \rightarrow \text{Hom}_{\mathbf{A}}(M, L) \rightarrow 0$ is exact. The following examples mention classes of objects that are known to be self-pure-projective, showing that self-pure-projective objects and, in particular, self-pure-projective modules are not uncommon.

Examples 9.1. The following objects in a cocomplete abelian category \mathbf{A} are self-pure-projective:

- (1) all pure-projective objects;
- (2) all Σ -pure-split objects;
- (3) all the objects belonging to $\text{Add}(M)$, if $M \in \mathbf{A}$ is a self-pure-projective object.

Examples 9.2. (1) Let \mathbf{L} and $\mathbf{E} \subset \mathbf{A}$ be two classes of objects such that $\text{Ext}_{\mathbf{A}}^1(L, E) = 0$ for all $L \in \mathbf{L}$ and $E \in \mathbf{E}$ (cf. Section 10 below). Assume that the class $\mathbf{E} \subset \mathbf{A}$ is closed under coproducts and pure subobjects. Then all objects in the intersection $\mathbf{L} \cap \mathbf{E} \subset \mathbf{A}$ are self-pure-projective.

Indeed, $M \in \mathbf{L} \cap \mathbf{E}$ and $M' \in \text{Add}(M)$ implies $M' \in \mathbf{E}$; and if a (pure) subobject K of M' also belongs to \mathbf{E} , then $\text{Ext}_{\mathbf{A}}^1(M, K) = 0$. Consequently $\text{Hom}_{\mathbf{A}}(M, M') \rightarrow \text{Hom}_{\mathbf{A}}(M, M'/K)$ is a surjective map.

(2) In particular, let $\mathbf{A} = A\text{-mod}$ be the category of modules over an associative ring A . Then any n -tilting left A -module (cf. Section 11 below) is self-pure-projective. Indeed, any n -tilting class \mathbf{E} in $A\text{-mod}$ is definable, which implies, in particular, that it is closed under direct sums and pure submodules [19, Definition 6.8 and Corollary 13.42].

Remarks 9.3. (1) A pair of classes of objects (\mathbf{E}, \mathbf{L}) in abelian category \mathbf{A} is said to be a *cotorsion pair* if both the classes \mathbf{L} and \mathbf{E} are maximal with respect to the property that $\text{Ext}_{\mathbf{A}}^1(L, E) = 0$ for all $L \in \mathbf{L}$ and $E \in \mathbf{E}$ (see Section 10). Notice that if \mathbf{A} is a complete, cocomplete abelian category with exact direct limits and (\mathbf{L}, \mathbf{E}) is a cotorsion pair in \mathbf{A} such that the class $\mathbf{E} \subset \mathbf{A}$ is closed under pure subobjects, then the class \mathbf{E} is also closed under coproducts in \mathbf{A} . Indeed, the right class \mathbf{E} in a cotorsion pair (\mathbf{L}, \mathbf{E}) is always closed under products in \mathbf{A} [13, Appendix A]. For any family of objects $A_\alpha \in \mathbf{A}$, the natural morphism $\coprod_{\alpha} A_\alpha \rightarrow \prod_{\alpha} A_\alpha$ is a direct limit of split monomorphisms, hence $\coprod_{\alpha} A_\alpha$ is a pure subobject of $\prod_{\alpha} A_\alpha$. It follows that all the objects in the class $\mathbf{L} \cap \mathbf{E}$ are self-pure-projective.

(2) Let (\mathbf{L}, \mathbf{E}) be a cotorsion pair in the category of left modules over an associative ring A . In this context, if the class \mathbf{E} is closed under direct limits in $R\text{-mod}$, then it is definable [31, Theorem 6.1]. If the cotorsion pair (\mathbf{L}, \mathbf{E}) is hereditary and the class \mathbf{E} is closed under unions of well-ordered chains in $A\text{-mod}$, then the class \mathbf{E} is definable as well [32, Theorem 3.5]. In both cases, the class $\mathbf{E} \subset A\text{-mod}$ is closed under (direct sums and) pure submodules, and it follows that all the A -modules in the class $\mathbf{L} \cap \mathbf{E}$ are self-pure-projective.

Let \mathbf{A} be a cocomplete abelian category. We will say that an object $M \in \mathbf{A}$ is *\varinjlim -pure-rigid* if $\text{PExt}_{\mathbf{A}}^1(M, N) = 0$ for any object $N \in \varinjlim^{\mathbf{A}} \text{Add}(M)$. Here $\text{PExt}_{\mathbf{A}}^1(-, -)$ denotes the group Ext^1 in the functor pure exact structure on the category \mathbf{A} (cf. Example 6.3(2)) and the notation $\varinjlim^{\mathbf{A}} M$ was defined in the beginning of Section 4. Any \varinjlim -pure-rigid object is Σ -pure-rigid by definition.

Example 9.4. Let \mathbf{L} and $\mathbf{E} \subset \mathbf{A}$ be two classes of objects such that $\text{PExt}_{\mathbf{A}}^1(L, E) = 0$ for all $L \in \mathbf{L}$ and $E \in \mathbf{E}$. Assume that the class $\mathbf{E} \subset \mathbf{A}$ is closed under direct limits. Then all objects in the intersection $\mathbf{L} \cap \mathbf{E}$ are \varinjlim -pure-rigid.

Let \mathbf{A} be a cocomplete abelian category and $M \in \mathbf{A}$ be an object. As in Section 2.1, we consider the related abelian category $\mathbf{B} = \mathbb{T}_M\text{-mod}$ and the pair of adjoint functors $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ and $\Phi: \mathbf{B} \rightarrow \mathbf{A}$. As in Section 6, we also consider the related pair of full subcategories $\mathbf{G} \subset \mathbf{A}$ and $\mathbf{H} \subset \mathbf{B}$.

The following proposition is the uncountable version of Proposition 6.6.

Proposition 9.5. *Let \mathbf{A} be a cocomplete abelian category with exact direct limits, $M \in \mathbf{A}$ be an object that is either self-pure-projective or \varinjlim -pure-rigid, $\mathbf{B} = \mathbb{T}_M\text{-mod}$ be the related abelian category, and $\mathbf{G} \subset \mathbf{A}$ and $\mathbf{H} \subset \mathbf{B}$ be the related two full subcategories. Then one has $\varinjlim^{\mathbf{A}} \text{Add}(M) \subset \mathbf{G}$ and $\varinjlim^{\mathbf{B}} \mathbf{B}_{\text{proj}} \subset \mathbf{H}$. The functor Ψ preserves direct limits of objects from $\text{Add}(M)$ in \mathbf{A} (taking them to direct limits of the corresponding projective objects in \mathbf{B}).*

Proof. Let Θ be a directed poset and $A: \Theta \rightarrow \mathbf{A}$ be a diagram in \mathbf{A} with $A(\theta) \in \text{Add}(M)$ for all $\theta \in \Theta$. Then the augmented bar-complex (8.2) is pure exact in \mathbf{A} , and the objects of cycles in the complex (8.2) are direct limits of objects from $\text{Add}(M)$ (see Example 8.4). As all the terms of this complex, except perhaps the rightmost one, belong to $\text{Add}(M)$ and the object M is either self-pure-projective or \varinjlim -pure-rigid, it follows that the functor $\text{Hom}_{\mathbf{A}}(M, -)$ takes the complex (8.2) to an exact sequence of abelian groups. As in the proof of Proposition 6.6, we conclude that the functor Ψ transforms the complex (8.2) into an exact complex in \mathbf{B} .

On the other hand, for any cocomplete abelian category \mathbf{B} , any poset Θ , and any diagram $B: \Theta \rightarrow \mathbf{B}$, the augmented bar-complex (9.1)

$$\cdots \longrightarrow \coprod_{\theta_0 \leq \theta_1 \leq \theta_2} B(\theta_0) \longrightarrow \coprod_{\theta_0 \leq \theta_1} B(\theta_0) \longrightarrow \coprod_{\theta_0} B(\theta_0) \longrightarrow \varinjlim_{\theta \in \Theta} B(\theta) \longrightarrow 0$$

is exact, at least, at its rightmost term.

In the situation at hand, put $B = \Psi \circ A: \Theta \rightarrow \mathbf{B}$. Then the natural morphism from the complex (9.1) to the image of the complex (8.2) under Ψ is an isomorphism at all the terms, except perhaps the rightmost one. It follows that this morphism of complexes is an isomorphism at the rightmost terms, too; that is, the natural morphism $\varinjlim_{\theta \in \Theta} \Psi(A(\theta)) \rightarrow \Psi(\varinjlim_{\theta \in \Theta} A(\theta))$ is an isomorphism.

The argument finishes in the same way as the proof of Proposition 6.6. \square

The next corollary is an uncountable version of Corollary 6.7.

Corollary 9.6. *Let \mathbf{A} be a cocomplete abelian category with exact direct limits and $M \in \mathbf{A}$ be an object that is either self-pure-projective or \varinjlim -pure-rigid. Let $\mathbf{B} = \mathbb{T}_M\text{-mod}$ be the related abelian category. Then the following conditions are equivalent:*

- (1) *all the objects from $\varinjlim \text{Add}(M)$ have $\text{Add}(M)$ -covers in \mathbf{A} ;*
- (2) *all the objects from $\varinjlim \mathbf{B}_{\text{proj}}$ have projective covers in \mathbf{B} ;*
- (3) *the class of objects $\text{Add}(M) \subset \mathbf{A}$ is closed under direct limits;*
- (4) *the class of all projective objects in \mathbf{B} is closed under direct limits;*
- (5) *the object $M \in \mathbf{A}$ satisfies the condition (iii) of Theorem 7.1.*

Proof. Both the equivalences (1) \iff (2) and (3) \iff (4) follow from Proposition 9.5 and the equivalence of categories (6.3). Since any epimorphism onto a projective object splits in \mathbf{B} , we also obtain the equivalence (4) \iff (5). Alternatively, the equivalence (3) \iff (5) follows from self-pure-projectivity/ \varinjlim -pure-rigidity of M and the properties of the augmented bar-complex mentioned in Example 8.4.

Finally, the equivalence (1) \iff (3) is [9, Corollary 7.2] (for \varinjlim -pure-rigid objects M) or a particular case of [9, Theorem 4.4] (for self-pure-projective objects M). \square

In particular, in the assumptions of Corollary 9.6, the two properties (3) and (4) in Corollary 7.2 are equivalent.

10. COVERS IN HEREDITARY COTORSION PAIRS

In this section we discuss \mathbf{L} -covers in an abelian category \mathbf{A} with a hereditary cotorsion pair (\mathbf{L}, \mathbf{E}) , aiming to gradually pass from Theorem 0.1 of the introduction to Theorem 0.2.

Let us recall the relevant definitions. Let \mathbf{A} be an abelian category, and let \mathbf{L} and $\mathbf{E} \subset \mathbf{A}$ be two classes of objects. We denote by $\mathbf{L}^{\perp 1} \subset \mathbf{A}$ the class of all objects $X \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^1(L, X) = 0$ for all $L \in \mathbf{L}$, and by ${}^{\perp 1}\mathbf{E} \subset \mathbf{A}$ the class of all objects $Y \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^1(Y, E) = 0$ for all $E \in \mathbf{E}$. The pair of classes of objects (\mathbf{L}, \mathbf{E}) in \mathbf{A} is called a *cotorsion pair* (or a *cotorsion theory*) if $\mathbf{E} = \mathbf{L}^{\perp 1}$ and $\mathbf{L} = {}^{\perp 1}\mathbf{E}$. A cotorsion pair (\mathbf{L}, \mathbf{E}) is called *hereditary* if $\text{Ext}_{\mathbf{A}}^n(L, E) = 0$ for all $L \in \mathbf{L}$, $E \in \mathbf{E}$, and $n \geq 1$. These concepts go back to Salce [30].

An epimorphism $l: L \rightarrow C$ in \mathbf{A} is called a *special \mathbf{L} -precover* if $L \in \mathbf{L}$ and $\ker(l) \in \mathbf{L}^{\perp 1}$. A monomorphism $b: B \rightarrow E$ in \mathbf{A} is called a *special \mathbf{E} -preenvelope* if $E \in \mathbf{E}$ and $\text{coker}(b) \in {}^{\perp 1}\mathbf{E}$. The following lemma summarizes the properties of precovers, special precovers, and covers.

Lemma 10.1. *Let \mathbf{L} be a class of objects in an abelian category \mathbf{A} . Then the following assertions hold true:*

- (a) *Any special \mathbf{L} -precover is an \mathbf{L} -precover.*
- (b) *If the class \mathbf{L} is closed under extensions in \mathbf{A} , then the kernel of any \mathbf{L} -cover belongs to $\mathbf{L}^{\perp 1}$. In particular, any epic \mathbf{L} -cover is special in this case.*
- (c) *Let $l: L \rightarrow C$ be an \mathbf{L} -cover, and let $l': L' \rightarrow C$ be an \mathbf{L} -precover. Then there exists a split epimorphism $f: L' \rightarrow L$ forming a commutative triangle diagram with the morphisms l and l' . The kernel K of the morphism f is a direct summand of L' contained in $\ker(l') \subset L'$. So one has $L' \cong L \oplus K$ and $\ker(l') \cong \ker(l) \oplus K$.*
- (d) *Assume that an object $C \in \mathbf{A}$ has an \mathbf{L} -cover, and let $l': L' \rightarrow C$ be an \mathbf{L} -precover. Then the morphism l' is an \mathbf{L} -cover if and only if the object L' has no nonzero direct summands contained in $\ker(l')$.*

Proof. Part (a) is [37, Proposition 2.1.3 or 2.1.4]. Part (b) is known as Wakamatsu lemma; this is [37, Lemma 2.1.1 or 2.1.2]. Part (c) is [37, Proposition 1.2.2 or Theorem 1.2.7], and part (d) is [37, Corollary 1.2.3 or 1.2.8]. \square

Let (\mathbf{L}, \mathbf{E}) be a cotorsion pair in \mathbf{A} . If $c: L \rightarrow C$ is an epimorphism in \mathbf{A} with $L \in \mathbf{L}$ and the object $\ker(c) \in \mathbf{A}$ has a special \mathbf{E} -preenvelope, then the object C has a special \mathbf{L} -precover. If $b: B \rightarrow E$ is a monomorphism in \mathbf{A} with $E \in \mathbf{E}$ and the object $\text{coker}(b) \in \mathbf{A}$ has a special \mathbf{L} -precover, then the object B has a special \mathbf{E} -preenvelope.

In particular, if there are enough injective and projective objects in \mathbf{A} , then, given a cotorsion pair (\mathbf{L}, \mathbf{E}) in \mathbf{A} , every object of \mathbf{A} has a special \mathbf{L} -precover if and only if every object of \mathbf{A} has a special \mathbf{E} -preenvelope. These results are known as Salce lemmas [30]. A cotorsion pair (\mathbf{L}, \mathbf{E}) in \mathbf{A} is called *complete* if every object of \mathbf{A} has a special \mathbf{L} -precover and a special \mathbf{E} -preenvelope.

Lemma 10.2. *Let (\mathbf{L}, \mathbf{E}) be a complete cotorsion pair in an abelian category \mathbf{A} , and let $E \in \mathbf{E} \subset \mathbf{A}$ be an object. Then a morphism $l: L \rightarrow E$ in \mathbf{A} is an \mathbf{L} -cover if and only if it is an $\mathbf{L} \cap \mathbf{E}$ -cover.*

Proof. Since the cotorsion pair (\mathbf{L}, \mathbf{E}) is complete in \mathbf{A} , every object of \mathbf{A} has a special \mathbf{L} -precover, which is, in particular, an epic \mathbf{L} -precover. It follows that all the \mathbf{L} -precovers in \mathbf{A} are epic.

Assume that l is an \mathbf{L} -cover. Then, by Lemma 10.1(b), the morphism l is a special \mathbf{L} -precover; so its kernel belongs to \mathbf{E} . Since the class \mathbf{E} is closed under extensions in \mathbf{A} , it follows that $L \in \mathbf{L} \cap \mathbf{E}$. Therefore, l is an $\mathbf{L} \cap \mathbf{E}$ -cover.

Assume that l is an $\mathbf{L} \cap \mathbf{E}$ -cover. Let $l': L' \rightarrow E$ be a special \mathbf{L} -precover of the object E in \mathbf{A} . Following the above argument, we have $L' \in \mathbf{L} \cap \mathbf{E}$; so l' is also an $\mathbf{L} \cap \mathbf{E}$ -precover of E . According to Lemma 10.1(c) applied to the class of objects $\mathbf{L} \cap \mathbf{E} \subset \mathbf{A}$, the object $\ker(l)$ is a direct summand of $\ker(l')$. Hence $\ker(l) \in \mathbf{E}$. So l is a special \mathbf{L} -precover of E in \mathbf{A} . In particular, by Lemma 10.1(a), l is an \mathbf{L} -precover. Since l is an $\mathbf{L} \cap \mathbf{E}$ -cover, we can conclude that l is an \mathbf{L} -cover. \square

Lemma 10.3. *Let (\mathbf{L}, \mathbf{E}) be a hereditary complete cotorsion pair in an abelian category \mathbf{A} . Assume that every object of \mathbf{E} has an \mathbf{L} -cover in \mathbf{A} . Then every object of \mathbf{A} has an \mathbf{L} -cover.*

Proof. Let A be an object in \mathbf{A} . By assumption, A has a special \mathbf{E} -preenvelope $a: A \rightarrow E$. Set $L = \text{coker}(a)$; then we have a short exact sequence $0 \rightarrow A \rightarrow E \rightarrow L \rightarrow 0$ in \mathbf{A} with $E \in \mathbf{E}$ and $L \in \mathbf{L}$. By assumption, the object E has an \mathbf{L} -cover $m: M \rightarrow E$ in \mathbf{A} . Set $F = \ker(m)$; by Lemma 10.1(b), we have $F \in \mathbf{E}$. Let K denote the kernel of the composition of epimorphisms $M \rightarrow E \rightarrow L$; then we have $K \in \mathbf{L}$, since $M, L \in \mathbf{L}$ and the cotorsion pair (\mathbf{L}, \mathbf{E}) is assumed to be hereditary.

We have constructed a commutative diagram of four short exact sequences

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \xrightarrow{a} & E & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow k & & \uparrow m & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \\
& & F & \equiv & F & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array}$$

The morphism $k: K \rightarrow A$ is an epimorphism with the kernel $F \in \mathbf{E}$, so it is a special \mathbf{L} -precover. Let us show that it is an \mathbf{L} -cover. Let $h: K \rightarrow K$ be an endomorphism such that $kh = k$. Consider a pushout of the short exact sequence $0 \rightarrow K \rightarrow M \rightarrow L \rightarrow 0$ by the morphism h and denote it by $0 \rightarrow K \rightarrow N \rightarrow L \rightarrow 0$. We have $N \in \mathbf{L}$, since $K, L \in \mathbf{L}$ and the class \mathbf{L} is closed under extensions in \mathbf{A} . In view of the universal property of the pushout, we have a commutative diagram of two morphisms of short exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{a} & E & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow k & & \uparrow n & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow h & & \uparrow s & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L \longrightarrow 0
\end{array}$$

with $kh = k$ and $ns = m$. Since the morphism $m: M \rightarrow E$ is an \mathbf{L} -cover and $N \in \mathbf{L}$, there exists a morphism $r': N \rightarrow M$ such that $mr' = n$. Moreover, one has $mr's = ns = m$, hence $r's: M \rightarrow M$ is automorphism. Setting $r = (r's)^{-1}r': N \rightarrow M$, we have $rs = \text{id}_M$ and $mr = m(r's)^{-1}r' = mr' = n$.

It follows from the latter equality that the morphism $r: N \rightarrow M$ forms a commutative triangle diagram with the epimorphisms $N \rightarrow L$ and $M \rightarrow L$. Passing to the kernels of these two epimorphisms, we obtain a morphism $g: K \rightarrow K$ such that $gh = \text{id}_K$. We have constructed a commutative diagram of two morphisms of

short exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow g & & \uparrow r & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow h & & \uparrow s & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L \longrightarrow 0
\end{array}$$

whose composition is the identity endomorphism of the short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$.

Thus we have shown that any endomorphism $h: K \longrightarrow K$ such that $kh = k$ is a (split) monomorphism. Furthermore, there is a commutative diagram of two morphisms of short exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{a} & E & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow k & & \uparrow m & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L \longrightarrow 0 \\
& & \uparrow g & & \uparrow r & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L \longrightarrow 0
\end{array}$$

where $kg = k$, because $mr = n$ (indeed, since a is a monomorphism, it suffices to show that $akg = ak$, which follows from the equality $mr = n$ and the commutativity of the left squares of our diagrams).

Therefore, the morphism $g: K \longrightarrow K$ is a (split) monomorphism, too, and we can conclude that both g and h are isomorphisms. \square

Corollary 10.4. *Let (\mathbf{L}, \mathbf{E}) be a hereditary complete cotorsion pair in an abelian category \mathbf{A} . Then the following three conditions are equivalent:*

- (1) every object of \mathbf{A} has an \mathbf{L} -cover;
- (2) every object of \mathbf{E} has an \mathbf{L} -cover in \mathbf{A} ;
- (3) every object of \mathbf{E} has an $\mathbf{L} \cap \mathbf{E}$ -cover.

Proof. (1) \iff (2) is Lemma 10.3; (2) \iff (3) is Lemma 10.2. \square

11. THE TILTING-COTILTING CORRESPONDENCE

Let \mathbf{A} be a complete, cocomplete abelian category with a fixed injective cogenerator $J \in \mathbf{A}$. So there are enough injective objects in the category \mathbf{A} , and the class of all injective objects is $\mathbf{A}_{\text{inj}} = \mathbf{Prod}(J) \subset \mathbf{A}$.

Let $n \geq 0$ be an integer, and let $T \in \mathbf{A}$ be an object satisfying the following two conditions:

- (i) the projective dimension of T (as an object of \mathbf{A}) does not exceed n , that is $\text{Ext}_{\mathbf{A}}^i(T, A) = 0$ for all $A \in \mathbf{A}$ and $i > n$; and
- (ii) for any set X , one has $\text{Ext}_{\mathbf{A}}^i(T, T^{(X)}) = 0$ for all $i > 0$.

Denote by $\mathbf{E} \subset \mathbf{A}$ the class of all objects $E \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^i(T, E) = 0$ for all $i > 0$. Notice that, by the definition, one has $\mathbf{A}_{\text{inj}} = \text{Prod}_{\mathbf{A}}(J) \subset \mathbf{E}$ and, by the condition (ii), $\text{Add}_{\mathbf{A}}(T) \subset \mathbf{E}$.

Furthermore, for each integer $m \geq 0$, denote by $\mathbf{L}_m \subset \mathbf{A}$ the class of all objects $L \in \mathbf{A}$ for which there exists an exact sequence of the form

$$0 \longrightarrow L \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^m \longrightarrow 0$$

in the category \mathbf{A} with the objects $T^m \in \text{Add}(T)$. By the definition, $\text{Add}(T) = \mathbf{L}_0 \subset \mathbf{L}_1 \subset \mathbf{L}_2 \subset \cdots \subset \mathbf{A}$. According to [26, Lemma 3.2], one has $\mathbf{L}_n = \mathbf{L}_{n+1} = \mathbf{L}_{n+2} = \cdots$ (so we set $\mathbf{L} = \mathbf{L}_n$) and $\mathbf{L} \cap \mathbf{E} = \text{Add}(T) \subset \mathbf{A}$.

According to [26, Theorem 3.4], every object of \mathbf{E} is a quotient of an object from $\text{Add}(T)$ in \mathbf{A} if and only if every object of \mathbf{A} is a quotient of an object from \mathbf{L} . If this is the case, we say that the object $T \in \mathbf{A}$ is *n-tilting*. For an *n-tilting* object T , the pair of classes of objects (\mathbf{L}, \mathbf{E}) in \mathbf{A} is a hereditary complete cotorsion pair, called the *n-tilting cotorsion pair* associated with T .

Let \mathbf{B} be a complete, cocomplete abelian category with a fixed projective generator $P \in \mathbf{B}$. So there are enough projective objects in \mathbf{B} , and one has $\mathbf{B}_{\text{proj}} = \text{Add}(P) \subset \mathbf{B}$.

The definition of an *n-cotilting object* $W \in \mathbf{B}$ is dual to the above definition of an *n-tilting object*. In other words, an object $W \in \mathbf{B}$ is said to be *n-cotilting* if the object W^{op} is *n-tilting* in the abelian category \mathbf{B}^{op} opposite to \mathbf{B} .

Specifically, this means, first of all, that the two conditions dual to (i) and (ii) have to be satisfied:

- (i*) the injective dimension of W (as an object of \mathbf{B}) does not exceed n , that is $\text{Ext}_{\mathbf{B}}^i(B, W) = 0$ for all $B \in \mathbf{B}$ and $i > n$; and
- (ii*) for any set X , one has $\text{Ext}_{\mathbf{B}}^i(W^X, W) = 0$ for all $i > 0$.

On top of that, denoting by $\mathbf{F} \subset \mathbf{B}$ the class of all objects $F \in \mathbf{B}$ such that $\text{Ext}_{\mathbf{B}}^i(F, W) = 0$ for all $i > 0$, it is required that every object of \mathbf{F} should be a subobject of an object from $\text{Prod}(W)$ in \mathbf{B} .

The following theorem from [26] describes the phenomenon of *n-tilting-cotilting correspondence*.

Theorem 11.1. *There is a bijective correspondence between (the equivalence classes of) complete, cocomplete abelian categories \mathbf{A} with an injective cogenerator J and an *n-tilting object* $T \in \mathbf{A}$, and (the equivalence classes of) complete, cocomplete abelian categories \mathbf{B} with a projective generator P and an *n-cotilting object* $W \in \mathbf{B}$. The abelian categories \mathbf{A} and \mathbf{B} corresponding to each other under this correspondence are connected by the following structures:*

- (a) *there is a pair of adjoint functors between \mathbf{A} and \mathbf{B} , with a left adjoint functor $\Phi: \mathbf{B} \longrightarrow \mathbf{A}$ and a right adjoint functor $\Psi: \mathbf{A} \longrightarrow \mathbf{B}$;*
- (b) *one has $\Phi(\mathbf{F}) \subset \mathbf{E}$ and $\Psi(\mathbf{E}) \subset \mathbf{F}$; the restrictions of the functors Φ and Ψ are mutually inverse equivalences between the full subcategories $\mathbf{E} \subset \mathbf{A}$ and $\mathbf{F} \subset \mathbf{B}$;*

(c) the full subcategory $E \subset A$ is closed under extensions and the cokernels of monomorphisms, while the full subcategory $F \subset B$ is closed under extensions and the kernels of epimorphisms; hence they inherit exact category structures (in Quillen's sense) from their ambient abelian categories; the equivalence of categories $E \cong F$ provided by the functors Φ and Ψ is an equivalence of exact categories; in other words, the functor Φ preserves exactness of short exact sequences of objects from F , and the functor Ψ preserves exactness of short exact sequences of objects from E ;

(d) both the full subcategories $E \subset A$ and $F \subset B$ are closed under both the products and coproducts in their ambient abelian categories; the functor $\Phi: B \rightarrow A$ preserves the products (and coproducts) of objects from F , while the functor $\Psi: A \rightarrow B$ preserves the (products and) coproducts of objects from E ;

(e) under the equivalence of exact categories $E \cong F$, the injective cogenerator $J \in E \subset A$ corresponds to the n -cotilting object $W \in F \subset B$, and the n -tilting object $T \in E \subset A$ corresponds to the projective generator $P \in F \subset B$;

(f) there are enough projective and injective objects in the exact category $E \cong F$; the full subcategories of projectives and injectives in E are $E_{\text{proj}} = \text{Add}(T)$ and $E_{\text{inj}} = A_{\text{inj}} = \text{Prod}(J)$, while the full subcategories of projectives and injectives in F are $F_{\text{proj}} = B_{\text{proj}} = \text{Add}(P)$ and $F_{\text{inj}} = \text{Prod}(W)$;

(g) the equivalence of exact categories $A \supset E \cong F \subset B$ can be extended to a triangulated equivalence between the derived categories $D^*(A) \cong D^*(B)$, which exists for any conventional derived category symbol $\star = \mathbf{b}, +, -, \text{ or } \emptyset$.

Proof. The bijective correspondence is constructed in [26, Corollary 4.12] (based on [26, Theorems 4.10 and 4.11]), and the assertions (e–f) are a part of that construction (cf. [26, Proposition 2.6 and Theorem 3.4]). The adjoint functors Φ and Ψ are described in [26, beginning of Section 5], and parts (b–c) are also explained there. Part (d) is [26, Lemma 5.3 and Remark 5.4]. Part (g) is [26, Proposition 4.2 and/or Corollary 5.6]. \square

The following characterization of the n -tilting-cotilting correspondence situations will be useful in Section 14. It may also be of an independent interest.

Proposition 11.2. *Let A be a complete, cocomplete abelian category with an injective cogenerator J , and let B be a complete, cocomplete abelian category with a projective generator P . Suppose that there is a derived equivalence $D^{\mathbf{b}}(A) \cong D^{\mathbf{b}}(B)$ taking the object $J \in A$ to an object $W \in B \subset D^{\mathbf{b}}(B)$ and the object $P \in B$ to an object $T \in A \subset D^{\mathbf{b}}(A)$. Then, for any integer $n \geq 0$, the following conditions are equivalent:*

- (I) *the projective dimension of the object T in the category A does not exceed n ;*
- (II) *the injective dimension of the object W in the category B does not exceed n ;*
- (III) *the standard t -structures on the derived categories $D^{\mathbf{b}}(A)$ and $D^{\mathbf{b}}(B)$, viewed as two t -structures on the same triangulated category D using the triangulated equivalence $D^{\mathbf{b}}(A) \cong D^{\mathbf{b}}(B)$, satisfy the inclusion $D^{\mathbf{b}, \leq 0}(A) \subset D^{\mathbf{b}, \leq n}(B)$, or equivalently, $D^{\mathbf{b}, \geq n}(B) \subset D^{\mathbf{b}, \geq 0}(A)$.*

If any one of these conditions is satisfied, then the object $T \in A$ is n -tilting; the object $W \in B$ is n -cotilting; and moreover, the abelian category A with the injective

cogenerator J and the n -tilting object T and the abelian category \mathbf{B} with the projective generator P and the n -cotilting object W are connected by the n -tilting-cotilting correspondence. The n -tilting class $\mathbf{E} \subset \mathbf{A}$ is the intersection $\mathbf{A} \cap \mathbf{B} \subset \mathbf{D} = \mathbf{D}^b(\mathbf{A}) = \mathbf{D}^b(\mathbf{B})$ viewed as a full subcategory in \mathbf{A} , and the n -cotilting class $\mathbf{F} \subset \mathbf{B}$ is the same intersection $\mathbf{B} \cap \mathbf{A} \subset \mathbf{D}$ viewed as a full subcategory in \mathbf{B} (hence the equivalence of exact categories $\mathbf{E} \cong \mathbf{F}$). The functor $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ assigns to an object $A \in \mathbf{A}$ the degree-zero cohomology of the related complex in $\mathbf{D}^b(\mathbf{B})$, and the functor $\Phi: \mathbf{B} \rightarrow \mathbf{A}$ assigns to an object $B \in \mathbf{B}$ the degree-zero cohomology of the related complex in $\mathbf{D}^b(\mathbf{A})$, that is, $\Psi(A) = H_{\mathbf{B}}^0(A)$ and $\Phi(B) = H_{\mathbf{A}}^0(B)$.

Proof. This is essentially the material of [26, Sections 2 and 4] (the description of the functors Φ and Ψ can be found in the beginning of [26, Section 5]). So we only give a brief sketch of the argument.

Notice, first of all, that the inclusions $\mathbf{D}^{b, \leq 0}(\mathbf{B}) \subset \mathbf{D}^{b, \leq 0}(\mathbf{A})$ and $\mathbf{D}^{b, \geq 0}(\mathbf{A}) \subset \mathbf{D}^{b, \geq 0}(\mathbf{B})$ always hold in our assumptions, because an object $Z \in \mathbf{D}$ belongs to $\mathbf{D}^{b, \geq 0}(\mathbf{B})$ if and only if $\text{Hom}_{\mathbf{D}}(P, Z[i]) = 0$ for all $i < 0$, while one has $\text{Hom}_{\mathbf{D}}(S, Z[i]) = 0$ for all $Z \in \mathbf{D}^{b, \geq 0}(\mathbf{A})$, all $i < 0$, and all $S \in \mathbf{A}$ (in particular, for $S = T$).

In the same way one shows that the two inclusions in (III) (which are obviously equivalent to each other) are equivalent to (I) on the one hand and to (II) on the other hand, (I) \iff (III) \iff (II). Indeed, an object $Z \in \mathbf{D}$ belongs to $\mathbf{D}^{b, \leq n}(\mathbf{B})$ if and only if $\text{Hom}_{\mathbf{D}}(P, Z[i]) = 0$ for all $i > n$, while the projective dimension of T in \mathbf{A} does not exceed n if and only if $\text{Hom}_{\mathbf{D}}(T, Z[i]) = 0$ for all $Z \in \mathbf{D}^{b, \leq 0}(\mathbf{A})$ and all $i > n$. The argument for W is similar.

The inclusion $\mathbf{A} \rightarrow \mathbf{D}^b(\mathbf{A})$ preserves coproducts, because the coproduct functors are exact in \mathbf{A} ; and the inclusion $\mathbf{B} \rightarrow \mathbf{D}^b(\mathbf{B})$ preserves products, because the product functors are exact in \mathbf{B} . Furthermore, we have $\mathbf{A} \cap \mathbf{B} = \mathbf{A} \cap \mathbf{D}^{b, \leq 0}(\mathbf{B}) \subset \mathbf{D}$, since $\mathbf{B} = \mathbf{D}^{b, \leq 0}(\mathbf{B}) \cap \mathbf{D}^{b, \geq 0}(\mathbf{B})$ and $\mathbf{A} \subset \mathbf{D}^{b, \geq 0}(\mathbf{A}) \subset \mathbf{D}^{b, \geq 0}(\mathbf{B})$. The full subcategory $\mathbf{D}^{b, \leq 0}(\mathbf{B})$ is closed under coproducts in \mathbf{D} (those coproducts that exist in \mathbf{D}), because the left part of any t-structure is closed under coproducts. Hence the full subcategory $\mathbf{A} \cap \mathbf{B}$ is closed under coproducts in \mathbf{D} , and consequently in \mathbf{A} and \mathbf{B} . Similarly, the full subcategory $\mathbf{A} \cap \mathbf{B}$ is closed under products in \mathbf{D} , and consequently in \mathbf{A} and \mathbf{B} . So the products and coproducts of objects of \mathbf{E} computed in \mathbf{A} agree with the products and coproducts of objects of \mathbf{F} computed in \mathbf{B} . (Cf. [26, Lemma 5.3 and Remark 5.4].)

Now we can see that $\text{Ext}_{\mathbf{A}}^i(T, T^{(X)}) = \text{Hom}_{\mathbf{D}^b(\mathbf{A})}(T, T^{(X)}[i]) = \text{Hom}_{\mathbf{D}^b(\mathbf{B})}(P, P^{(X)}[i]) = 0$ for all $i > 0$, and similarly $\text{Ext}_{\mathbf{B}}^i(W^X, W) = 0$ for all $i > 0$ and all sets X . This proves the n -tilting axiom (ii) for T and the n -cotilting axiom (ii*) for W ; while the axioms (i) and (i*) are provided by the conditions (I) and (II). It remains to apply [26, Proposition 2.5 and Corollary 4.4(b)] in order to conclude that the object $T \in \mathbf{A}$ is n -tilting and the object $W \in \mathbf{B}$ is n -cotilting. It is also clear from the construction of the n -tilting-cotilting correspondence in [26, Theorems 4.10–4.11 and Corollary 4.12] that the triples (\mathbf{A}, J, T) and (\mathbf{B}, P, W) are connected by such. \square

Remark 11.3. Given a complete, cocomplete abelian category \mathbf{A} with an injective cogenerator and an n -tilting object T , the related abelian category \mathbf{B} can be described as the category $\mathbf{B} = \mathbb{T}_T\text{-mod}$ of modules over the monad $\mathbb{T}_T: X \mapsto \text{Hom}_{\mathbf{A}}(T, T^{(X)})$.

The functors Φ and Ψ from Section 2.1 can be identified with the functors Φ and Ψ from Theorem 11.1 in this case [26, Remark 6.6].

Dually, given a complete, cocomplete abelian category \mathbf{B} with a projective generator and an n -cotilting object W , the related abelian category \mathbf{A} can be described as the opposite category $\mathbf{A} = \mathbb{T}_{W^{\text{op}}}\text{-mod}^{\text{op}}$ to the category of modules over the monad $\mathbb{T}_{W^{\text{op}}}: X \mapsto \text{Hom}_{\mathbf{B}}(W^X, W)$ (cf. [27, Section 1]).

Examples 11.4. Suppose that there is an associative ring A such that the abelian category \mathbf{A} can be embedded into $A\text{-mod}$ as a full subcategory closed under coproducts. So, in particular, the n -tilting object $T \in \mathbf{A}$ can be viewed as a left A -module. Then it follows from [26, Theorem 7.1 or 9.9] that the abelian category \mathbf{B} can be described as the category of left contra**modules** $\mathfrak{R}\text{-contra}$ over the topological ring $\mathfrak{R} = \text{Hom}_A(T, T)^{\text{op}}$ from Examples 2.2 (1), (2) or (4). Further examples of classes of abelian categories \mathbf{A} for which the category \mathbf{B} admits such a description are discussed in [26, Sections 9–10] and [28, Section 3] (see Examples 2.3).

12. DIRECT LIMITS IN CATEGORICAL TILTING THEORY

In this section we discuss the properties of direct limits in the n -tilting-cotilting correspondence context. We start with the case of the direct limits indexed by the poset of natural numbers.

Lemma 12.1. *In the context of the n -tilting-cotilting correspondence, assume that countable direct limits are exact in the abelian category \mathbf{A} . Then both the full subcategories $\mathbf{E} \subset \mathbf{A}$ and $\mathbf{F} \subset \mathbf{B}$ are closed under countable direct limits in their ambient abelian categories, and the functor $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ preserves countable direct limits of objects from \mathbf{E} . Furthermore, for any sequence of objects and morphisms $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow \cdots$ with $F_i \in \mathbf{F}$, the short sequence $0 \rightarrow \coprod_{i=1}^{\infty} F_i \rightarrow \coprod_{i=1}^{\infty} F_i \rightarrow \varinjlim_{i \geq 1} F_i \rightarrow 0$ with the map $\text{id} - \text{shift}: \coprod_i F_i \rightarrow \coprod_i F_i$ is exact in \mathbf{B} . The functors of countable direct limit are exact in the exact category \mathbf{F} .*

Proof. The argument resembles the proof of Proposition 6.6. For any sequence of objects and morphisms $B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow \cdots$ in an abelian category \mathbf{B} with countable coproducts, the short sequence $\coprod_{i=1}^{\infty} B_i \rightarrow \coprod_{i=1}^{\infty} B_i \rightarrow \varinjlim_{i \geq 1} B_i \rightarrow 0$ is right exact in \mathbf{B} . Moreover, for any sequence of objects and morphisms $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow \cdots$ in an abelian category \mathbf{A} with exact countable direct limits, the short sequence $0 \rightarrow \coprod_{i=1}^{\infty} A_i \rightarrow \coprod_{i=1}^{\infty} A_i \rightarrow \varinjlim_{i \geq 1} A_i \rightarrow 0$ is exact in \mathbf{A} (see Example 8.3). In particular, for any sequence of objects and morphisms $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$ with $E_i \in \mathbf{E}$, the short sequence $0 \rightarrow \coprod_i E_i \rightarrow \coprod_i E_i \rightarrow \varinjlim_i E_i \rightarrow 0$ is exact in \mathbf{A} . Hence it follows that $\varinjlim_i E_i \in \mathbf{E}$, because the full subcategory $\mathbf{E} \subset \mathbf{A}$ is closed under coproducts and the cokernels of monomorphisms.

The functor Φ , being a left adjoint, preserves all colimits. Thus, for any sequence of objects and morphisms $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow \cdots$ in \mathbf{F} , the short sequence $0 \rightarrow \Phi(\coprod_i F_i) \rightarrow \Phi(\coprod_i F_i) \rightarrow \Phi(\varinjlim_i F_i) \rightarrow 0$, being isomorphic to the short

sequence $0 \rightarrow \coprod_i \Phi(F_i) \rightarrow \coprod_i \Phi(F_i) \rightarrow \varinjlim \Phi(F_i) \rightarrow 0$, is exact in \mathbf{A} . This is a short exact sequence in \mathbf{A} with all the three terms belonging to \mathbf{E} , so the functor Ψ transforms it into a short exact sequence in \mathbf{B} with all the three terms belonging to \mathbf{F} . We have a natural (adjunction) morphism from the right exact sequence $\coprod_i F_i \rightarrow \coprod_i F_i \rightarrow \varinjlim F_i \rightarrow 0$ to the exact sequence $0 \rightarrow \Psi\Phi(\coprod_i F_i) \rightarrow \Psi\Phi(\varinjlim F_i) \rightarrow \Psi\Phi(\varinjlim \Phi(F_i)) \rightarrow 0$, which is an isomorphism at the first two terms, and therefore at the third term, too. Hence the object $\varinjlim F_i \cong \Psi(\varinjlim \Phi(F_i))$ belongs to \mathbf{F} and the short sequence $0 \rightarrow \coprod_i F_i \rightarrow \coprod_i F_i \rightarrow \varinjlim F_i \rightarrow 0$ is exact. Since the coproduct functors are exact in \mathbf{F} (because they are exact in \mathbf{E}) and the cokernel of an admissible monomorphism is an exact functor, it follows that the functors of countable direct limit are exact in \mathbf{F} . The functor $\Psi|_{\mathbf{E}}: \mathbf{E} \rightarrow \mathbf{B}$ preserves countable direct limits, because both the equivalence of categories $\mathbf{E} \cong \mathbf{F}$ and the inclusion functor $\mathbf{F} \rightarrow \mathbf{B}$ do. This proves all the assertions of the lemma. \square

Corollary 12.2. *In the context of the n -tilting-cotilting correspondence, assume that countable direct limits are exact in the abelian category \mathbf{A} . Then the following three conditions are equivalent:*

- (i) *the full subcategory \mathbf{L} is closed under countable direct limits in \mathbf{A} ;*
- (ii) *the class of objects $\mathbf{Add}(T)$ is closed under countable direct limits in \mathbf{A} ;*
- (iii) *the class of all projective objects \mathbf{B}_{proj} is closed under countable direct limits in \mathbf{B} .*

Proof. (i) \implies (ii) According to Lemma 12.1, the class \mathbf{E} is closed under countable direct limits in \mathbf{A} . Hence, if the class \mathbf{L} is closed under countable direct limits, too, then so is the class $\mathbf{L} \cap \mathbf{E} = \mathbf{Add}(T)$.

(ii) \iff (iii) By the same lemma, the equivalence of categories $\mathbf{E} \cong \mathbf{F}$ transforms countable direct limits of objects from \mathbf{E} computed in \mathbf{A} to countable direct limits of objects from \mathbf{F} computed in \mathbf{B} . Thus the class $\mathbf{B}_{\text{proj}} = \Psi(\mathbf{Add}(T)) \subset \mathbf{F}$ is closed under countable direct limits in \mathbf{B} if and only if the class $\mathbf{Add}(T) \subset \mathbf{E}$ is closed under countable direct limits in \mathbf{A} .

(ii) \implies (i) Given an object $L \in \mathbf{L}$, an exact sequence $0 \rightarrow L \rightarrow T^0 \rightarrow \dots \rightarrow T^n \rightarrow 0$ with $T^j \in \mathbf{Add}(T)$ can be constructed in the following way. Let $L \rightarrow E$ be a special \mathbf{E} -preenvelope of L ; then we have a short exact sequence $0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$ with $E \in \mathbf{E}$ and $M \in \mathbf{L}$. Since the class \mathbf{L} is closed under extensions in \mathbf{A} , we have $E \in \mathbf{L} \cap \mathbf{E} = \mathbf{Add}(T)$. Set $T^0 = E$ and $M^1 = M$, and let $M^1 \rightarrow T^1$ be a special \mathbf{E} -preenvelope of M^1 , etc. Proceeding in this way, one obtains an exact sequence $0 \rightarrow L \rightarrow T^0 \rightarrow T^1 \rightarrow \dots \rightarrow T^{n-1} \rightarrow M^n \rightarrow 0$ with $M^n \in \mathbf{L}$; and one also has $M^n \in \mathbf{E}$ by cohomological dimension shifting, since the projective dimension of T does not exceed n . It remains to set $T^n = M^n$. Conversely, in any exact sequence $0 \rightarrow L \rightarrow T^0 \rightarrow T^1 \rightarrow \dots \rightarrow T^n \rightarrow 0$ with $L \in \mathbf{L}$ and $T^j \in \mathbf{Add}(T)$, the objects of cocycles belong to \mathbf{L} , since the class \mathbf{L} , being the left class in a hereditary cotorsion pair, is closed under the kernels of epimorphisms.

Now, for any two objects A' and $A'' \in \mathbf{A}$, their special \mathbf{E} -preenvelopes $A' \rightarrow E'$ and $A'' \rightarrow E''$, and a morphism $A' \rightarrow A''$, there is a morphism $E' \rightarrow E''$ forming

a commutative triangle diagram with the composition $A' \longrightarrow A'' \longrightarrow E''$. Using this observation, for any sequence of objects and morphisms $L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow \cdots$ in \mathbf{L} and any exact sequences $0 \longrightarrow L_i \longrightarrow T_i^0 \longrightarrow \cdots \longrightarrow T_i^n \longrightarrow 0$ with $T_i^j \in \text{Add}(T)$, one can extend the sequence of morphisms $\cdots \longrightarrow L_i \longrightarrow L_{i+1} \longrightarrow \cdots$ to a sequence of morphisms of exact sequences $\cdots \longrightarrow (0 \rightarrow L_i \rightarrow T_i^0 \rightarrow \cdots \rightarrow T_i^n \rightarrow 0) \longrightarrow (L_{i+1} \rightarrow T_{i+1}^0 \rightarrow \cdots \rightarrow T_{i+1}^n \rightarrow 0) \longrightarrow \cdots$. Passing to the direct limit, we obtain an exact sequence

$$0 \longrightarrow \varinjlim_{i \geq 1} L_i \longrightarrow \varinjlim_{i \geq 1} T_i^0 \longrightarrow \cdots \longrightarrow \varinjlim_{i \geq 1} T_i^n \longrightarrow 0$$

in the abelian category \mathbf{A} . Since $\varinjlim_i T_i^j \in \text{Add}(T)$ for all $j = 0, \dots, n$, it follows that $\varinjlim_i L_i \in \mathbf{L}$ by the definition. \square

The following proposition provides a generalization to uncountable direct limits.

Proposition 12.3. *In the context of the n -tilting-cotilting correspondence, assume that direct limits are exact in the abelian category \mathbf{A} . Then both the full subcategories \mathbf{E} and \mathbf{F} are closed under direct limits in their ambient abelian categories \mathbf{A} and \mathbf{B} , and the functor $\Psi: \mathbf{A} \longrightarrow \mathbf{B}$ preserves direct limits of objects from \mathbf{E} . The functors of direct limit are exact in the exact category \mathbf{F} .*

Proof. The argument resembles the proof of Proposition 9.5. Let $E: \Theta \longrightarrow \mathbf{E}$ be a diagram in the exact category \mathbf{E} indexed by a directed poset Θ . Then the augmented bar-complex (8.2) (from Example 8.4) for the diagram E is an unbounded resolution of an object of \mathbf{A} by objects of \mathbf{E} (since the full subcategory $\mathbf{E} \subset \mathbf{A}$ is closed under coproducts). Since the full subcategory $\mathbf{E} \subset \mathbf{A}$ is defined as the class of all objects $E \in \mathbf{A}$ such that $\text{Ext}_{\mathbf{A}}^i(T, E) = 0$ for all $i > 0$, and the tilting object $T \in \mathbf{A}$ has finite projective dimension, a simple cohomological dimension shifting argument shows that $\varinjlim_{\theta \in \Theta} E(\theta) \in \mathbf{E}$. Moreover, all the objects of cycles of the exact complex (8.2) for the diagram E also belong to \mathbf{E} . So this complex is exact in the exact category \mathbf{E} .

Applying the functor Ψ to the augmented bar-complex for the diagram E , we get an exact complex in the category \mathbf{F} , which coincides, except possibly at his rightmost term, with the augmented bar-complex (9.1) for the diagram $\Psi \circ E$ in \mathbf{B} (because both the equivalence of categories $\mathbf{E} \cong \mathbf{F}$ and the inclusion functor $\mathbf{F} \longrightarrow \mathbf{B}$ preserve coproducts). Since the bar-complex of any diagram in a cocomplete abelian category is exact at its rightmost term, it follows that the natural morphism $\varinjlim_{\theta} \Psi(E(\theta)) \longrightarrow \Psi(\varinjlim_{\theta} E(\theta))$ is an isomorphism and $\varinjlim_{\theta} \Psi(E(\theta)) \in \mathbf{F}$. As any diagram in \mathbf{F} can be obtained by applying the functor Ψ to a diagram in \mathbf{E} , we can conclude that the full subcategory $\mathbf{F} \subset \mathbf{B}$ is also closed under direct limits, and the bar-complexes (9.1) computing such direct limits in \mathbf{F} are exact. Exactness of the direct limit functors in \mathbf{F} easily follows. \square

Corollary 12.4. *In the context of the n -tilting-cotilting correspondence, assume that direct limits are exact in the abelian category \mathbf{A} . Then the following three conditions are equivalent:*

- (i) *the full subcategory \mathbf{L} is closed under direct limits in \mathbf{A} ;*

- (ii) the class of objects $\text{Add}(T)$ is closed under direct limits in \mathbf{A} ;
- (iii) the class of all projective objects \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} .

Proof. Provable in the same way as Corollary 12.2, using Proposition 12.3 in place of Lemma 12.1. Let us just say a few words about the implication (ii) \implies (i).

In view of [1, Sections 1.6–1.7], it suffices to show that \mathbf{L} is closed under the direct limits of well-ordered chains in \mathbf{A} (in fact, it suffices to consider direct limits indexed by regular cardinals). Let us prove that \mathbf{L} is closed under λ -indexed direct limits for any ordinal λ .

Let $(L_i \rightarrow L_j)_{0 \leq i < j < \lambda}$ be a λ -indexed diagram in \mathbf{L} . Proceeding by transfinite induction in $0 \leq i < \lambda$, we construct a λ -indexed diagram of exact sequences $0 \rightarrow L_i \rightarrow T_i^0 \rightarrow \cdots \rightarrow T_i^n \rightarrow 0$ with $T_i^k \in \text{Add}(T)$, connected by morphisms of exact sequences for all $0 \leq i < j < \lambda$.

The case $i = 0$ is clear. Assume that the desired directed diagram of exact sequences has been constructed for $0 \leq i < j < \alpha$, where $0 < \alpha < \lambda$ is some ordinal. Then we have an exact sequence

$$(12.1) \quad 0 \longrightarrow \varinjlim_{i < \alpha} L_i \longrightarrow \varinjlim_{i < \alpha} T_i^0 \longrightarrow \cdots \longrightarrow \varinjlim_{i < \alpha} T_i^n \longrightarrow 0$$

in \mathbf{A} with $\varinjlim_{i < \alpha} T_i^k \in \text{Add}(T)$ by (ii) for all $0 \leq k \leq n$, hence $\varinjlim_{i < \alpha} L_i \in \mathbf{L}$. Starting from the natural morphism $\varinjlim_{i < \alpha} L_i \rightarrow L_\alpha$ and arguing as in the proof of Corollary 12.2, we construct a morphism from the exact sequence (12.1) to an exact sequence $0 \rightarrow L_\alpha \rightarrow T_\alpha^0 \rightarrow \cdots \rightarrow T_\alpha^n \rightarrow 0$ with $T_\alpha^k \in \text{Add}(T)$.

Having obtained the desired λ -indexed diagram of exact sequences, it remains to say that, in the exact sequence $0 \rightarrow \varinjlim_{i < \lambda} L_i \rightarrow \varinjlim_{i < \lambda} T_i^0 \rightarrow \cdots \rightarrow \varinjlim_{i < \lambda} T_i^n \rightarrow 0$ in \mathbf{A} , the objects $\varinjlim_{i < \lambda} T_i^k$ belong to $\text{Add}(T)$ by (ii) for all $0 \leq k \leq n$. Hence $\varinjlim_{i < \lambda} L_i \in \mathbf{L}$, so (i) holds. \square

13. WHEN IS THE LEFT TILTING CLASS COVERING?

In this section we prove Theorem 0.2 from the introduction. As in the previous sections, we start with weaker assumptions and then gradually strengthen them.

Proposition 13.1. *In the context of the n -tilting-cotilting correspondence, the following four conditions are equivalent:*

- (1) the class \mathbf{L} is covering in \mathbf{A} ;
- (2) every object of \mathbf{E} has an \mathbf{L} -cover in \mathbf{A} ;
- (3) the class $\text{Add}(T)$ is covering in \mathbf{E} ;
- (4) the class \mathbf{B}_{proj} is covering in \mathbf{F} .

Furthermore, assume that countable direct limits are exact in the abelian category \mathbf{A} . Then the following six conditions (5–10) are equivalent:

- (5) any countable direct limit of copies of the tilting object T has an \mathbf{L} -cover in \mathbf{A} ;
- (6) any countable direct limit of copies of the object T has an $\text{Add}(T)$ -cover in \mathbf{A} ;

- (7) any countable direct limit of copies of the projective generator P has a projective cover in \mathbf{B} ;
- (8) any countable direct limit of copies of the tilting object T in \mathbf{A} belongs to \mathbf{L} ;
- (9) any countable direct limit of copies of the object T in \mathbf{A} belongs to $\mathbf{Add}(T)$;
- (10) any countable direct limit of copies of the projective generator P in \mathbf{B} is projective.

Moreover, let us assume that countable direct limits are exact in \mathbf{A} and that \mathbf{B} is the abelian category of left contramodules over a complete, separated right linear topological ring \mathfrak{R} . Consider the following six properties:

- (11) the object $T \in \mathbf{A}$ has a perfect decomposition;
- (12) the topological ring \mathfrak{R} is topologically left perfect;
- (13) the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} ;
- (14) the class \mathbf{B}_{proj} is covering in \mathbf{B} ;
- (15) all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;
- (16) all the discrete quotient rings of \mathfrak{R} are left perfect.

Then the following implications hold:

$$(11) \iff (12) \iff (13) \iff (14) \implies (4) \implies (7) \implies (15) \implies (16).$$

If the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d), then all the conditions (1–16) are equivalent to each other. If the topological ring \mathfrak{R} satisfies one of the conditions (e), (f), or (g) of Section 3, then the fifteen conditions (1–15) are equivalent to each other.

Proof. (1) \iff (2) \iff (3) is Corollary 10.4.

(3) \iff (4) holds in view of the equivalence of categories $\mathbf{E} \cong \mathbf{F}$ taking the class $\mathbf{Add}(T) \subset \mathbf{E}$ to the class $\mathbf{B}_{\text{proj}} = \mathbf{F}_{\text{proj}} \subset \mathbf{F}$ (see Theorem 11.1(b,f)).

(5) \iff (6) By Lemma 12.1, any countable direct limit of copies of the object T in \mathbf{A} belongs to \mathbf{E} . So Lemma 10.2 applies.

(6) \iff (7) The equivalence of categories $\mathbf{E} \cong \mathbf{F}$ identifies the class of objects $\mathbf{Add}(T) \subset \mathbf{E}$ with the class $\mathbf{B}_{\text{proj}} \subset \mathbf{F}$. By Lemma 12.1, it also identifies countable direct limits of copies of the object T in \mathbf{A} with countable direct limits of copies of the object P in \mathbf{B} .

(8) \iff (9) holds, since any countable direct limit of copies of T in \mathbf{A} belongs to \mathbf{E} .

(9) \iff (10) is similar to Corollary 12.2 (ii) \Leftrightarrow (iii).

(6) \iff (7) \iff (9) \iff (10) An n -tilting object $T \in \mathbf{A}$ satisfies THEC by Example 6.3 (1), so Corollary 6.7 is applicable.

The implications (14) \implies (4) \implies (7) and (13) \implies (10) are obvious. So are the implications (2) \implies (5) and (3) \implies (6), in view of Lemma 12.1.

(11) \iff (12) \iff (13) is Corollary 7.2 (1) \Leftrightarrow (2) \Leftrightarrow (3).

(13) \iff (14) is [28, Theorem 14.1 (iii') \Leftrightarrow (ii)].

(10) \implies (15) \implies (16) is Corollary 7.2 (6) \Rightarrow (9) \Rightarrow (10).

This proves all the assertions of the proposition except the last two (in which one of the conditions (a), (b), (c), (d), (e), (f), or (g) is assumed). Now we assume (d) (which is a common generalization of (a), (b), and (c)) and prove the related implications.

(16) \implies (14) If all the discrete quotient rings of \mathfrak{R} are left perfect and (d) is satisfied, then all left \mathfrak{R} -contramodules have projective covers by Corollary 7.3 (15) \implies (6) or [24, Theorem 12.4 (vi) \implies (ii)].

(16) \implies (13) Follows from Corollary 7.3 (15) \implies (10) or [24, Theorem 12.4 (vi) \implies (iii)] (since the direct limits of projective contramodules are always flat).

(16) \implies (12) is Corollary 7.3 (15) \implies (12).

Finally, assuming that one of the conditions (e), (f), or (g) holds, all the conditions (11–15) are equivalent by Corollary 7.3 (6) \Leftrightarrow (10) \Leftrightarrow (11) \Leftrightarrow (12) \Leftrightarrow (13). \square

Theorem 13.2. *In the context of the n -tilting-cotilting correspondence, assume that \mathbf{A} is a Grothendieck abelian category. Then the following conditions are equivalent:*

- (1) *the class \mathbf{L} is covering in \mathbf{A} ;*
- (2) *any direct limit of objects from $\mathbf{Add}(T)$ has an \mathbf{L} -cover in \mathbf{A} ;*
- (3) *the class \mathbf{L} is closed under direct limits in \mathbf{A} ;*
- (4) *the class $\mathbf{Add}(T)$ is covering in \mathbf{A} ;*
- (5) *any direct limit of objects from $\mathbf{Add}(T)$ has an $\mathbf{Add}(T)$ -cover in \mathbf{A} ;*
- (6) *the class $\mathbf{Add}(T)$ is closed under direct limits in \mathbf{A} ;*
- (7) *the class \mathbf{B}_{proj} is covering in \mathbf{B} ;*
- (8) *any direct limit of projective objects has a projective cover in \mathbf{B} ;*
- (9) *the class \mathbf{B}_{proj} is closed under direct limits in \mathbf{B} .*

Furthermore, assume that \mathbf{B} is the abelian category of left contramodules over a complete, separated right linear topological associative ring \mathfrak{R} . Consider the following four properties:

- (10) *the object $T \in \mathbf{A}$ has a perfect decomposition;*
- (11) *the topological ring \mathfrak{R} is topologically left perfect;*
- (12) *all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;*
- (13) *all the discrete quotient rings of \mathfrak{R} are left perfect.*

Then the following implications hold:

$$(9) \iff (10) \iff (11) \implies (12) \implies (13).$$

If the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d), then all the conditions (1–13) are equivalent to each other. If the topological ring \mathfrak{R} satisfies one of the conditions (e), (f), or (g) of Section 3, then the twelve conditions (1–12) are equivalent to each other.

Proof. The implications (1) \implies (2), (4) \implies (5), and (7) \implies (8) are obvious (as are the implications (3) \implies (2), (6) \implies (5), and (9) \implies (8)).

The equivalences (3) \iff (6) \iff (9) hold by Corollary 12.4.

(3) \implies (1) holds by Theorem 4.4, since the class \mathbf{L} is (special) precovering in \mathbf{A} .

(6) \implies (4) is Example 4.1 and Theorem 4.4.

(9) \implies (7) is Example 4.2 and Theorem 4.4. Notice that Theorem 4.4 requires the category \mathbf{A} to be locally presentable for its applicability. An abelian category is locally presentable *and* has exact direct limit functors if and only if it is Grothendieck; that is why we assume that \mathbf{A} is a Grothendieck category in the present theorem.

(5) \iff (6) is a particular case of [9, Application 7.3].

(5) \iff (6) \iff (8) \iff (9) The object $M \in \mathbf{A}$ is \varinjlim -pure-rigid by Example 9.4, since the n -tilting class $\mathbf{E} \subset \mathbf{A}$ is closed under direct limits by Proposition 12.3. Therefore, Corollary 9.6 is applicable.

(2) \iff (5) is Lemma 10.2.

This proves the first assertion of the theorem (see also [9, Remark 7.4] for a brief summary of this argument). The remaining implications are provided by Corollary 7.3 (10–15) as well as by Proposition 13.1 (11–16). \square

Proof of Theorem 0.2. Follows from Proposition 13.1 and Theorem 13.2. \square

14. INJECTIVE RING EPIMORPHISMS OF PROJECTIVE DIMENSION 1

In this section we discuss a certain tilting-cotilting correspondence situation associated with an injective homological ring epimorphism satisfying additional conditions on the flat and projective dimension.

We recall that a *ring epimorphism* $u: R \rightarrow U$ is a homomorphism of associative rings such that the multiplication map $U \otimes_R U \rightarrow U$ is an isomorphism of U - U -bimodules. We refer to the book [34, Section XI.1] for background information on ring epimorphisms, and to the paper [8] for more advanced recent results. A ring epimorphism u is said to be *homological* if $\mathrm{Tor}_i^R(U, U) = 0$ for all $i \geq 1$.

The two-term complex of R - R -bimodules $K^\bullet = (R \rightarrow U)$ plays a key role in the theory developed in [8]. In the present paper, we deal with *injective* ring epimorphisms, i. e., ring epimorphisms u such that the map u is injective. In this case, the two-term complex of R - R -bimodules K^\bullet is naturally isomorphic to the quotient bimodule U/R . So we set $K = U/R$ and use K in lieu of K^\bullet .

We will denote by $\mathrm{pd}_R E$ the projective dimension of a left R -module E and by $\mathrm{fd} E_R$ the flat dimension of a right R -module E . For any injective homological ring epimorphism $u: R \rightarrow U$ such that $\mathrm{pd}_R U \leq 1$, the left R -module $U \oplus K$ is 1-tilting [2, Theorem 3.5]. In this section we discuss a different tilting-cotilting correspondence situation, in which $\mathbf{A} \subset R\text{-mod}$ is a certain abelian subcategory.

Let $u: R \rightarrow U$ be an injective homological ring epimorphism. A left R -module A is said to be *u -torsionfree* if it is an R -submodule of a left U -module, or equivalently, if the R -module morphism $u \otimes_R \mathrm{id}_A: A \rightarrow U \otimes_R A$ is injective. The class of u -torsionfree left R -modules is closed under submodules, direct sums, and products. Any left R -module A has a unique maximal u -torsionfree quotient module, which can be constructed as the image of the R -module morphism $A \rightarrow U \otimes_R A$. When $\mathrm{fd} U_R \leq 1$, the class of u -torsionfree R -modules is also closed under extensions in $R\text{-mod}$ [8, Lemma 2.7(a)].

A left R -module B is said to be *u -divisible* if it is a quotient R -module of a left U -module, or equivalently, if the R -module morphism $\mathrm{Hom}_R(u, \mathrm{id}_B): \mathrm{Hom}_R(U, B) \rightarrow B$ is surjective. (See [8, Remarks 1.2] for a terminological discussion.) The class of all u -divisible left R -modules is closed under epimorphic images, direct sums, and

products. Any left R -module B has a unique maximal u -divisible submodule, which can be constructed as the image of the R -module morphism $\text{Hom}_R(U, B) \rightarrow B$. When $\text{pd}_R U \leq 1$, the class of u -divisible R -modules is also closed under extensions in $R\text{-mod}$ [8, Lemma 2.7(b)].

A left R -module M is called a u -comodule (or a *left u -comodule*) if $U \otimes_R M = 0 = \text{Tor}_1^R(U, M)$. Assuming that $\text{fd } U_R \leq 1$, the full subcategory $R\text{-mod}_{u\text{-co}}$ of left u -comodules is closed under kernels, cokernels, extensions, and direct sums in $R\text{-mod}$ [18, Proposition 1.1]; so $R\text{-mod}_{u\text{-co}}$ is an abelian category and the embedding functor $R\text{-mod}_{u\text{-co}} \rightarrow R\text{-mod}$ is exact. The embedding functor $R\text{-mod}_{u\text{-co}} \rightarrow R\text{-mod}$ has a right adjoint (“coreflector”) $\Gamma_u: R\text{-mod} \rightarrow R\text{-mod}_{u\text{-co}}$, computable as $\Gamma_u(A) = \text{Tor}_1^R(K, A)$ for all $A \in R\text{-mod}$. The category $R\text{-mod}_{u\text{-co}}$ is a Grothendieck abelian category with an injective cogenerator $\Gamma_u(J)$, where J is any chosen injective cogenerator of $R\text{-mod}$ [8, Proposition 3.1 and Corollary 3.6].

A left R -module C is called a u -contramodule (or a *left u -contramodule*) if $\text{Hom}_R(U, C) = 0 = \text{Ext}_R^1(U, C)$. Assuming that $\text{pd}_R U \leq 1$, the full subcategory $R\text{-mod}_{u\text{-contra}}$ of left u -contramodules is closed under kernels, cokernels, extensions, and direct products in $R\text{-mod}$ [18, Proposition 1.1]; so $R\text{-mod}_{u\text{-contra}}$ is an abelian category and the embedding functor $R\text{-mod}_{u\text{-contra}} \rightarrow R\text{-mod}$ is exact. The embedding functor $R\text{-mod}_{u\text{-contra}} \rightarrow R\text{-mod}$ has a left adjoint (“reflector”) $\Delta_u: R\text{-mod} \rightarrow R\text{-mod}_{u\text{-contra}}$, computable as $\Delta_u(B) = \text{Ext}_R^1(K, B)$ for all $B \in R\text{-mod}$. The category $R\text{-mod}_{u\text{-contra}}$ is a locally presentable abelian category with a projective generator $\Delta_u(R) \in R\text{-mod}_{u\text{-contra}}$ [8, Proposition 3.2 and Lemma 3.7].

The following two theorems are the main results of this section.

Theorem 14.1. *Let $u: R \rightarrow U$ be an injective homological ring epimorphism. Assume that $\text{fd } U_R \leq 1$ and $\text{pd}_R U \leq 1$. Then the two abelian categories $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ and $\mathbf{B} = R\text{-mod}_{u\text{-contra}}$ are connected by the 1-tilting-cotilting correspondence in the following way. The injective cogenerator is $J = \Gamma_u(\text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \in \mathbf{A}$, and the 1-tilting object is $T = K \in \mathbf{A}$. The projective generator is $P = \Delta_u(R) \in \mathbf{B}$, and the 1-cotilting object is $W = \text{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z}) \in \mathbf{B}$. The functor $\Psi: \mathbf{A} \rightarrow \mathbf{B}$ is $\Psi = \text{Hom}_R(K, -)$, and the functor $\Phi: \mathbf{B} \rightarrow \mathbf{A}$ is $\Phi = K \otimes_R -$. The 1-tilting class $\mathbf{E} \subset \mathbf{A}$ is the class of all u -divisible u -comodule left R -modules, and the 1-cotilting class $\mathbf{F} \subset \mathbf{B}$ is the class of all u -torsionfree u -contramodule left R -modules. The equivalence of exact categories $\mathbf{E} \cong \mathbf{F}$ is the first Matlis category equivalence of [8, Theorem 1.3].*

Consider the topological ring $\mathfrak{R} = \text{Hom}_R(K, K)^{\text{op}}$ opposite to the ring of endomorphisms of the left R -module K , and endow it with the finite topology, as defined in Example 2.2 (1). Then the right action of the ring R in the R - R -bimodule K induces a homomorphism of associative rings $R \rightarrow \mathfrak{R}$. We are interested in the composition of the forgetful functor $\mathfrak{R}\text{-contra} \rightarrow \mathfrak{R}\text{-mod}$ defined in Section 1.4 with the obvious functor of restriction of scalars $\mathfrak{R}\text{-mod} \rightarrow R\text{-mod}$.

Theorem 14.2. *Let $u: R \rightarrow U$ be an injective homological ring epimorphism. Assume that $\text{pd}_R U \leq 1$. Then the forgetful functor $\mathfrak{R}\text{-contra} \rightarrow R\text{-mod}$ is fully*

faithful, and its essential image coincides with the full subcategory of u -contramodule left R -modules $R\text{-mod}_{u\text{-contra}} \subset R\text{-mod}$. So we have an equivalence of abelian categories $\mathfrak{A}\text{-contra} \cong R\text{-mod}_{u\text{-contra}}$.

Proof of Theorems 14.1 and 14.2. We discuss the proofs of the two theorems simultaneously, because they are closely related (even though the assumptions in Theorem 14.1 are slightly more restrictive than in Theorem 14.2).

The argument is largely based in the following result, which is a particular case of [11, Corollary 4.4] or [8, Corollary 7.3].

Theorem 14.3. *Let $u: R \rightarrow U$ be an injective homological ring epimorphism. such that $\text{fd}U_R \leq 1$ and $\text{pd}_R U \leq 1$. Then, for any derived category symbol $\star = \mathbf{b}, +, -, \text{ or } \emptyset$, there is a triangulated equivalence between the derived categories of the abelian categories of left u -comodules and left u -contramodules,*

$$(14.1) \quad \mathbf{D}^\star(R\text{-mod}_{u\text{-co}}) \cong \mathbf{D}^\star(R\text{-mod}_{u\text{-contra}}).$$

Proof. The additional assumptions of [11, Corollary 4.4] or [8, Corollary 7.3] hold for all injective ring epimorphisms by [8, Example 7.4]. \square

Theorem 14.1 is simplest obtained by applying Proposition 11.2 (for $n = 1$) to the derived equivalence (14.1) (for $\star = \mathbf{b}$). To be more precise, one needs to know a bit about how the derived equivalence (14.1) is constructed. In the proof of [8, Corollary 7.3], the triangulated equivalence is obtained from the *recollement* of [8, Section 6], and it needs to be shifted by [1] before it becomes a tilting derived equivalence. The triangulated equivalence in [8, Corollary 6.2] is provided by the functors $\mathbb{R}\text{Hom}_R(K^\bullet[-1], -)$ and $K^\bullet[-1] \otimes_R^{\mathbb{L}} -$, while in our present context one has to consider the equivalence provided by the functors $\mathbb{R}\text{Hom}_R(K, -)$ and $K \otimes_R^{\mathbb{L}} -$.

Now one observes that the R - R -bimodule K is both a left and a right u -comodule, and consequently $\text{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z})$ is a left u -contramodule. Furthermore, one can compute that $\mathbb{R}\text{Hom}_R(K, K) = \text{Hom}_R(K, K) = \text{Ext}_R^1(K, R) = \Delta_u(R) = P$, since $\text{Ext}_R^1(K, K) = \text{Ext}_R^2(K, R) = 0$. Similarly, $\mathbb{R}\text{Hom}_R(K, J) = \text{Hom}_R(K, J) = \text{Hom}_R(K, \text{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z}) = W$, since $\text{Ext}_R^1(K, J) = \text{Ext}_A^1(K, J) = 0$ (as $\mathbf{A} = R\text{-mod}_{u\text{-co}} \subset R\text{-mod}$ is a full subcategory closed under extensions). Finally, any one of the conditions (I-III) of Proposition 11.2 is easily verified. The descriptions of the classes $\mathbf{E} \subset \mathbf{A}$ and $\mathbf{F} \subset \mathbf{B}$ follow from [8, Lemma 2.7]. This finishes the proof of Theorem 14.1.

Alternatively, one can check that $K \in R\text{-mod}_{u\text{-co}}$ is a 1-tilting object in the way similar to the argument in [26, Example 5.7]. Following Examples 11.4, the abelian category \mathbf{B} corresponding to this tilting object in the abelian category $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ can be described as $\mathbf{B} = \mathfrak{A}\text{-contra}$. The functor Ψ is then still computed as $\Psi = \text{Hom}_R(K, -)$ [26, Remark 6.6], while the left adjoint functor Φ is the functor of so-called *contratensor product* $\Phi = K \odot_{\mathfrak{A}} -$ with the discrete right \mathfrak{A} -module K [26, formula (20)] (which is the same thing as the tensor product $K \otimes_R -$ provided that the forgetful functor $\mathfrak{A}\text{-contra} \rightarrow R\text{-mod}$ is fully faithful, cf. [26, Lemma 7.11]). Comparing this approach to the previous one yields $\mathfrak{A}\text{-contra} \cong \mathbf{B} \cong R\text{-mod}_{u\text{-contra}}$, that is the assertion of Theorem 14.2 (in the assumptions of Theorem 14.1).

A direct proof of Theorem 14.2 (in full generality) can be given based on [22, Proposition 2.1]. For any set X , we have to construct a natural isomorphism of left R -modules $\Delta_u(R[X]) \cong \mathfrak{R}[[X]]$. Indeed,

$$\Delta_u(R[X]) = \text{Ext}_R^1(K, R[X]) \cong \text{Hom}_R(K, K[X]) \cong \mathfrak{R}[[X]]$$

by [26, proof of Theorem 7.1].

Let us spell out this argument a bit more explicitly. There are enough projective objects of the form $P = \Delta_u(R[X])$ in $R\text{-mod}_{u\text{-ctra}}$, and these are precisely the images of the free \mathfrak{R} -contramodules $\mathfrak{R}[[X]]$ under the forgetful functor. To show that the whole image of the forgetful functor $\mathfrak{R}\text{-contra} \rightarrow R\text{-mod}$ lies inside $R\text{-mod}_{u\text{-ctra}}$, observe that the forgetful functor preserves cokernels, the full subcategory $R\text{-mod}_{u\text{-ctra}} \subset R\text{-mod}$ is closed under cokernels, and every left \mathfrak{R} -contramodule is the cokernel of a morphism of free left \mathfrak{R} -contramodules.

As an abelian category with enough projective objects is determined by its full subcategory of projective objects, in order to prove that the functor $\mathfrak{R}\text{-contra} \rightarrow R\text{-mod}_{u\text{-ctra}}$ is an equivalence of categories it suffices to show that it is an equivalence in restriction to the full subcategories of projective objects. In other words, we have to check that the natural map $\text{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]) \rightarrow \text{Hom}_R(\mathfrak{R}[[X]], \mathfrak{R}[[Y]])$ is isomorphism for all sets X and Y . Indeed, we have

$$\text{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]) \cong \mathfrak{R}[[Y]]^X \cong \text{Hom}_R(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]),$$

where the second isomorphism holds because, by [8, Theorem 1.3],

$$\text{Hom}_R(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]) \cong \text{Hom}_R(K[X], K[Y]) \cong \text{Hom}_R(K, K[Y])^X \cong \mathfrak{R}[[Y]]^X$$

as $K[X]$ is a u -divisible left u -comodule and $\text{Hom}_R(K, K[X]) \cong \mathfrak{R}[[X]]$.

The proof of Theorems 14.1 and 14.2 is finished. \square

Remark 14.4. The above “alternative” argument follows the lines of the exposition in [26, Section 8] (see, in particular, [26, formulas (21–23)]). However, the assumptions in [26] presume existence of a left linear topological ring \mathfrak{A} such that \mathbf{A} is the category of discrete left \mathfrak{A} -modules, or in other words, a hereditary pretorsion class in $\mathfrak{A}\text{-mod}$. In the context of the present section, \mathbf{A} is the full abelian subcategory of left u -comodules in $R\text{-mod}$, which is not necessarily a pretorsion class (see the discussion in [8, Section 5] and the examples in [8, Section 8]).

Nevertheless, the arguments in the beginning of [26, Section 8] are still valid in our present context. The key observation is that, for any associative ring S , any R - S -bimodule E whose underlying left R -module is a u -comodule, and any left S -module C , the left R -module $E \otimes_R C$ is a left u -comodule. This follows easily from the fact that the full subcategory of left u -comodules is closed under cokernels and direct sums in $R\text{-mod}$. So the functor $\Phi = K \otimes_R - : R\text{-mod}_{u\text{-ctra}} \rightarrow R\text{-mod}_{u\text{-co}}$ is well-defined. A similar observation holds for the contratensor product in place of the tensor product; so the functor $\Phi = K \odot_{\mathfrak{R}} - : \mathfrak{R}\text{-contra} \rightarrow R\text{-mod}_{u\text{-co}}$ is well-defined, too.

15. COVERS AND DIRECT LIMITS FOR INJECTIVE RING EPIMORPHISM

In this final section, we discuss the covering and direct limit closedness properties of the tilting objects $U \oplus K \in R\text{-mod}$ and $K \in R\text{-mod}_{u\text{-co}}$ in connection with the perfectness properties of the related rings.

Let $u: R \rightarrow U$ be an injective homological ring epimorphism. Assuming that $\text{pd}_R U \leq 1$, denote by (\mathbf{N}, \mathbf{G}) the 1-tilting cotorsion pair in $R\text{-mod}$ associated with the 1-tilting left R -module $U \oplus K$. Assuming that $\text{fd} U_R \leq 1$ and $\text{pd}_R U \leq 1$, we also have the 1-tilting cotorsion pair (\mathbf{L}, \mathbf{E}) in the abelian category $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ associated with the 1-tilting object K .

Lemma 15.1. (a) $\mathbf{G} \subset R\text{-mod}$ is the class of all u -divisible left R -modules.

(b) $\mathbf{E} = \mathbf{A} \cap \mathbf{G}$ is the class of all u -divisible left u -comodules.

(c) One has $\mathbf{L} = \mathbf{A} \cap \mathbf{N}$.

Proof. By the definition, for a 1-tilting left R -module $U \oplus K$ we have $\mathbf{G} = \{U \oplus K\}^{\perp_1} \subset R\text{-mod}$, and it is clear from the short exact sequence of left R -modules $0 \rightarrow R \rightarrow U \rightarrow K \rightarrow 0$ that $\mathbf{G} = \{K\}^{\perp_1} \subset R\text{-mod}$. Similarly, $\mathbf{E} \subset \mathbf{A}$ is the right $\text{Ext}_{\mathbf{A}}^1$ -orthogonal class to the 1-tilting object $K \in \mathbf{A}$. Now part (a) is [2, Theorem 3.5 (4)] or [8, Lemma 2.7(b)]. Part (b) is a part of Theorem 14.1 (essentially, it holds because the functors Ext_R^1 and $\text{Ext}_{\mathbf{A}}^1$ agree).

To prove part (c), we observe that the definitions of the class \mathbf{N} as the left Ext_R^1 -orthogonal class to \mathbf{G} in $R\text{-mod}$ and the class \mathbf{L} as the left $\text{Ext}_{\mathbf{A}}^1$ -orthogonal class to \mathbf{E} in \mathbf{A} together with the inclusion $\mathbf{E} \subset \mathbf{G}$ imply the inclusion $\mathbf{L} \supset \mathbf{A} \cap \mathbf{N}$. On the other hand, the definitions of \mathbf{N} as the class of all finitely $\text{Add}(U \oplus K)$ -coresolved objects in $R\text{-mod}$ and \mathbf{L} as the class of all finitely $\text{Add}(K)$ -coresolved objects in \mathbf{A} (see the beginning of Section 11 or [26, Theorem 3.4]) imply the inverse inclusion $\mathbf{L} \subset \mathbf{A} \cap \mathbf{N}$. \square

Let us start with the 1-tilting object $K \in R\text{-mod}_{u\text{-co}}$. Recall that \mathfrak{R} denotes the topological ring $\text{Hom}_R(K, K)^{\text{op}}$ with the finite topology (see Section 14). We keep the notation \mathbf{F} for the 1-cotilting class in the abelian category $R\text{-mod}_{u\text{-contra}} = \mathbf{B} = \mathfrak{R}\text{-contra}$ (so the exact category \mathbf{F} is equivalent to $\mathbf{E} = \mathbf{A} \cap \mathbf{G}$).

Theorem 15.2. Assume that $\text{fd} U_R \leq 1$ and $\text{pd}_R U \leq 1$. Then the following sixteen conditions are equivalent:

- (1) every left R -module has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (2) every module from \mathbf{G} has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (3) every module from \mathbf{A} has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (4) every module from $\mathbf{A} \cap \mathbf{G}$ has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (5) any direct limit of modules from $\text{Add}(K)$ has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (6) the class of modules $\mathbf{A} \cap \mathbf{N}$ is closed under direct limits;
- (7) every left R -module has an $\text{Add}(K)$ -cover;
- (8) every module from \mathbf{G} has an $\text{Add}(K)$ -cover;
- (9) every module from \mathbf{A} has an $\text{Add}(K)$ -cover;
- (10) every module from $\mathbf{A} \cap \mathbf{G}$ has an $\text{Add}(K)$ -cover;

- (11) any direct limit of modules from $\text{Add}(K)$ has an $\text{Add}(K)$ -cover;
- (12) the class of modules $\text{Add}(K)$ is closed under direct limits;
- (13) every object of \mathbf{B} has a projective cover;
- (14) the class of projective objects in \mathbf{B} is closed under direct limits;
- (15) the topological ring \mathfrak{R} is topologically perfect;
- (16) the left R -module K has a perfect decomposition.

Proof. Notice first of all that the direct limits in \mathbf{A} and $R\text{-mod}$ agree (since \mathbf{A} is closed under direct limits in $R\text{-mod}$). The implications (1) \implies (2) \implies (4), (3) \implies (4) \implies (5) and (7) \implies (8) \implies (10), (9) \implies (10) \implies (11) are obvious.

The implication (3) \implies (1) holds because the embedding functor $\mathbf{A} \rightarrow R\text{-mod}$ has a right adjoint Γ_u (in other words, \mathbf{A} is coreflective in $R\text{-mod}$). Given a left R -module C , let $L \rightarrow \Gamma_u(C)$ be an $\mathbf{A} \cap \mathbf{N}$ -cover of the module $\Gamma_u(C) \in \mathbf{A}$; then the composition $L \rightarrow \Gamma_u(C) \rightarrow C$ is an $\mathbf{A} \cap \mathbf{N}$ -cover of C .

To check the implication (8) \implies (7), recall that \mathbf{G} is the class of all u -divisible left R -modules and $\text{Add}(K) \subset \mathbf{G}$. Every left R -module C has a unique maximal u -divisible R -submodule $h(C)$. Let $M \rightarrow h(C)$ be an $\text{Add}(K)$ -cover of $h(C)$; then the composition $M \rightarrow h(C) \rightarrow C$ is an $\text{Add}(K)$ -cover of C .

The implication (10) \implies (8) follows from [8, Lemma 3.3(a)]. Let C be a u -divisible left R -module; then the left R -module $\Gamma_u(C)$ belongs to $\mathbf{A} \cap \mathbf{G}$. If $M \rightarrow \Gamma_u(C)$ is an $\text{Add}(K)$ -cover of $\Gamma_u(C)$, then the composition $M \rightarrow \Gamma_u(C) \rightarrow C$ is an $\text{Add}(K)$ -cover of C .

The equivalence of the three conditions (3), (4), and (10) is a particular case of the equivalence of conditions (1–3) in Proposition 13.1. Finally, all the conditions (3), (5), (6), (9), and (11–16) are equivalent by Theorem 13.2. Notice that $\mathbf{A} = R\text{-mod}_{u\text{-co}}$ is a Grothendieck abelian category by [8, Corollary 3.6].

One can also observe that the class $\text{Add}(K)$ is always precovering in $R\text{-mod}$ by Example 4.1; and the class $\mathbf{A} \cap \mathbf{N}$ is precovering in $R\text{-mod}$ because \mathbf{A} is coreflective in $R\text{-mod}$ and $\mathbf{A} \cap \mathbf{N}$ is special precovering in \mathbf{A} . Hence the implications (6) \implies (1) and (12) \implies (7) hold by Enochs' theorem (see Theorem 4.4). \square

We recall from [24, Section 10] that a topological ring \mathfrak{R} is said to be *left pro-perfect* if it is separated and complete, two-sided linear, and all the discrete quotient rings of \mathfrak{R} are left perfect.

Theorem 15.3. *Let $u: R \rightarrow U$ be an injective homological ring epimorphism. Assume that $\text{fd}U_R \leq 1$ and $\text{pd}_R U \leq 1$, and assume further that the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d) of Section 3. Then the conditions in Theorem 15.2 are equivalent to the following ones:*

- (1) any countable direct limit of copies of the left R -module K has an $\mathbf{A} \cap \mathbf{N}$ -cover;
- (2) the class of left R -modules $\mathbf{A} \cap \mathbf{N}$ is closed under countable direct limits;
- (3) any countable direct limit of copies of the R -module K has an $\text{Add}(K)$ -cover;
- (4) the class of left R -modules $\text{Add}(K)$ is closed under countable direct limits;
- (5) any countable direct limit of copies of the projective generator $P = \mathfrak{R}$ has a projective cover in \mathbf{B} ;

- (6) the class of objects \mathbf{B}_{proj} is closed under countable direct limits in \mathbf{B} ;
- (7) all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;
- (8) all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.

In particular, if the ring R is commutative and $\text{pd}_R U \leq 1$, then the eight conditions (1–8) are equivalent to each other and to the conditions in Theorem 15.2. The condition (8) can be rephrased by saying that the topological ring \mathfrak{R} is pro-perfect in this case. Replacing the assumption of one of the conditions (a–d) with that of one of the conditions (e), (f), or (g), the seven conditions (1–7) are equivalent to each other and to all the conditions in Theorem 15.2.

Proof. The conditions (2), (4), and (6) are equivalent to each other by Corollary 12.2.

In the assumption of any one of the conditions (a–d), all the conditions (5–8) are equivalent to each other and to the conditions in Theorem 15.2 (13–15) by [24, Theorem 12.4]. In the assumption of any one of the conditions (a–g), all the conditions (1), (3), and (5–7) are equivalent to each other and to the conditions in Theorem 15.2 (3–4, 10, 13–16) by Proposition 13.1.

Alternatively, all the conditions (3–7) are equivalent to each other and to the conditions in Theorem 15.2 (7–16) by Corollary 7.3. Notice that the left R -module K is always self-pure-projective by Examples 9.1 (3) and 9.2 (2), as a direct summand of a 1-tilting left R -module $U \oplus K$. Besides, K is also Σ -rigid, of course; so it satisfies THEC by Example 6.3 (1).

If the ring R is commutative, then so is the ring \mathfrak{R} by [8, Lemma 4.1]. So condition (a) is satisfied. (It is worth recalling that $\text{pd}_R U \leq 1$ implies $\text{fd}_R U = 0$ for commutative rings R , by [8, Theorem 5.2].) \square

Now let us discuss the 1-tilting left R -module $U \oplus K$. We denote by \mathfrak{S} the topological ring $\text{Hom}_R(U \oplus K, U \oplus K)^{\text{op}}$ with the finite topology, and denote by $\mathbf{H} \subset \mathfrak{S}$ –contra the 1-cotilting class associated with the 1-cotilting left \mathfrak{S} -contramodule $\text{Hom}_{\mathbb{Z}}(U \oplus K, \mathbb{Q}/\mathbb{Z})$. So the exact category \mathbf{H} is equivalent to \mathbf{G} .

Lemma 15.4. (i) *The topological ring \mathfrak{S} is topologically left perfect if and only if the ring U is left perfect and the topological ring \mathfrak{R} is topologically left perfect.*

(ii) *All the discrete quotient rings of the topological ring \mathfrak{S} are left perfect if and only if the ring U is left perfect and all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.*

(iii) *If the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d) of Section 3, then the topological ring \mathfrak{S} satisfies condition (d).*

(iv) *If the topological ring \mathfrak{R} satisfies one of the conditions (e), (f), or (g) of Section 3, then the topological ring \mathfrak{S} satisfies condition (g).*

Proof. We have $\text{Hom}_R(U, U)^{\text{op}} = U$, $\text{Hom}_R(K, K)^{\text{op}} = \mathfrak{R}$, and $\text{Hom}_R(U/R, U) = 0$. So \mathfrak{S} is the matrix ring (cf. [24, Example 12.1])

$$\begin{pmatrix} U & \mathfrak{R} \\ 0 & \mathfrak{R} \end{pmatrix}$$

where $\mathfrak{K} = \text{Hom}_R(U, U/R)$ is a nilpotent strongly closed two-sided ideal in \mathfrak{S} (obviously, $\mathfrak{K}^2 = 0$ in \mathfrak{S}). Now we have $\mathfrak{S}/\mathfrak{K} = U \times \mathfrak{R}$, so part (ii) of the lemma follows from [24, Lemma 12.3]. Similarly, part (i) follows from Lemmas 3.3, 3.4, and 4.9. Furthermore, the discrete ring U trivially satisfies the condition (b) of Section 3. Hence it remains to apply [24, Lemma 12.6] in order to prove part (iii) of the lemma; and part (iv) is a particular case of Lemma 3.1. \square

Theorem 15.5. *Let $u: R \rightarrow U$ be an injective homological ring epimorphism such that $\text{pd}_R U \leq 1$. Then the following thirteen conditions are equivalent:*

- (1) *all left R -modules have \mathbf{N} -covers;*
- (2) *any countable direct limit of copies of the R -module $U \oplus K$ has an \mathbf{N} -cover;*
- (3) *the class of left R -modules \mathbf{N} is closed under (countable) direct limits;*
- (4) *all left R -modules have $\text{Add}(U \oplus K)$ -covers;*
- (5) *any countable direct limit of copies of the R -module $U \oplus K$ has an $\text{Add}(U \oplus K)$ -cover;*
- (6) *the class of left R -modules $\text{Add}(U \oplus K)$ is closed under (countable) direct limits;*
- (7) *the left R -module $U \oplus K$ is Σ -pure-split;*
- (8) *the left R -module $U \oplus K$ has a perfect decomposition;*
- (9) *all the objects of \mathfrak{S} -contra have projective covers;*
- (10) *any countable direct limit of copies of the free left \mathfrak{S} -contramodule \mathfrak{S} has a projective cover in \mathfrak{S} -contra;*
- (11) *the class of all projective left \mathfrak{S} -contramodules is closed under (countable) direct limits in \mathfrak{S} -contra;*
- (12) *the topological ring \mathfrak{S} is topologically left perfect;*
- (13) *the ring U is left perfect and the topological ring \mathfrak{R} is topologically left perfect.*

Furthermore, consider the next four properties:

- (14) *all descending chains of cyclic discrete right \mathfrak{S} -modules terminate;*
- (15) *the ring U is left perfect and all descending chains of cyclic discrete right \mathfrak{R} -modules terminate;*
- (16) *all the discrete quotient rings of the topological ring \mathfrak{S} are left perfect;*
- (17) *the ring U is left perfect and all the discrete quotient rings of the topological ring \mathfrak{R} are left perfect.*

Then the following implications hold:

$$(13) \implies (14) \implies (15) \implies (16) \iff (17).$$

If the topological ring \mathfrak{R} satisfies one of the conditions (e), (f), or (g) of Section 3, then all the conditions (1–14) are equivalent to each other. If the topological ring \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d), then all the conditions (1–17) are equivalent to each other. In particular, if the ring R is commutative, then the seventeen conditions (1–17) are equivalent.

Proof. The condition (3) (for uncountable direct limits) is equivalent to (7) by [19, Proposition 13.55]. All the eight conditions (1–8) are equivalent to each other by [4, Theorem 3.6, Theorem 5.2, and Corollary 5.5].

The conditions (3), (6), and (11) are equivalent to each other, for countable direct limits, by Corollary 12.2, and for uncountable ones, by Corollary 12.4. The conditions (2), (5), and (10) are equivalent to each other by Proposition 13.1 (5) \Leftrightarrow (6) \Leftrightarrow (7). All the conditions (1), (4), (8), (9), and (12), and the uncountable versions of (3), (6), (11) are equivalent to each other by Theorem 13.2.

The implications (12) \implies (14) \implies (16) and (13) \implies (15) \implies (17) hold by [28, Theorem 14.4 (iv) \Rightarrow (v) \Rightarrow (vi)]. The equivalences (12) \Leftrightarrow (13) and (16) \Leftrightarrow (17) hold by Lemma 15.4(i–ii). The implication (14) \implies (15) is easy (cf. the discussion in the proof of Theorem 3.5, case (g)).

If \mathfrak{R} satisfies one of the conditions (a), (b), (c), or (d), then all the conditions (9–12), (14), and (16) are equivalent to each other by Lemma 15.4(iii) and [24, Theorem 12.4]. In the assumption of any one of the conditions (a–g), all the conditions (1), (2), (5), (8–12), and (14), are equivalent to each other by Lemma 15.4(iv) and Proposition 13.1. This also establishes the equivalence of the countable and uncountable versions of the condition (11).

Alternatively, all the conditions (4–6), (8–12), and (14) are equivalent to each other by Corollary 7.3. Notice that the left R -module $U \oplus K$ satisfies THEC by Example 6.3 (1) (it is also self-pure-projective by Example 9.2 (2)).

The last assertion of the theorem follows from [8, Lemma 4.1]. □

Example 15.6. Let R be a commutative ring and $S \subset R$ be a multiplicative subset consisting of regular elements. Denote the multiplicative subset of all regular elements in R by $S \subset S_{reg} \subset R$. Set $U = S^{-1}R$; then the localization map $u: R \rightarrow U$ is an injective flat epimorphism of commutative rings. The topological ring $\mathfrak{R} = \text{Hom}_R(U/R, U/R)$ is naturally topologically isomorphic to the S -completion $\varprojlim_{s \in S} R/sR$ of the ring R (viewed as the topological ring in the projective limit topology), which was discussed in [24, Example 11.2].

Assume that $\text{pd}_R S^{-1}R \leq 1$, and set $K = U/R$. Then the homomorphism of commutative rings $R \rightarrow S^{-1}R = U$ satisfies the assumptions of Theorems 15.3 and 15.5. By Theorem 15.3, the class of R -modules $\mathbf{A} \cap \mathbf{N}$ is covering (if and only if the class $\text{Add}(K) \subset R\text{-mod}$ is covering and) if and only if the ring R/sR is perfect for every $s \in S$. By Theorem 15.5, the class of R -modules \mathbf{N} is covering (if and only if the class $\text{Add}(U \oplus K) \subset R\text{-mod}$ is covering and) if and only if two conditions hold: the ring R/sR is perfect for every $s \in S$, and the ring $S^{-1}R$ is perfect.

The latter two conditions are equivalent to the following two: one has $S^{-1}R = S_{reg}^{-1}R$, and the ring R is almost perfect (in the sense of the paper [17]). It is worth noticing that the condition that all the rings R/sR are perfect already implies $\text{pd}_R S^{-1}R \leq 1$ [17, Lemma 3.4], [7, Theorem 6.13].

For example, let $R = \mathbb{Z}$ be the ring of integers, p be a prime number, and $S = \{1, p, p^2, p^3, \dots\} \subset R$ be the multiplicative subset in \mathbb{Z} generated by p . Then the class of abelian groups $\mathbf{A} \cap \mathbf{N} \subset \mathbf{Ab}$ is covering, but the class $\mathbf{N} \subset \mathbf{Ab}$ is not. Alternatively,

let $S' \subset \mathbb{Z}$ be the multiplicative subset of all integers *not* divisible by p . Then, once again, the related class $\mathbf{A} \cap \mathbf{N}' \subset \mathbf{Ab}$ is covering, but the class of abelian groups $\mathbf{N}' \subset \mathbf{Ab}$ is not.

REFERENCES

- [1] J. Adámek, J. Rosický. Locally presentable and accessible categories. London Math. Society Lecture Note Series 189, Cambridge University Press, 1994.
- [2] L. Angeleri Hügel, J. Sánchez. Tilting modules arising from ring epimorphisms. *Algebras and Represent. Theory* **14**, #2, p. 217–246, 2011. [arXiv:0804.1313](#) [math.RT]
- [3] L. Angeleri Hügel, M. Saorín. Modules with perfect decompositions. *Math. Scand.* **98**, #1, p. 19–43, 2006.
- [4] L. Angeleri Hügel, J. Šaroch, J. Trlifaj. Approximations and Mittag–Leffler conditions—the applications. *Israel Journ. of Math.* **226**, #2, p. 757–780, 2018. [arXiv:1612.01140](#) [math.RA]
- [5] R. El Bashir. Covers and directed colimits. *Algebras and Represent. Theory* **9**, #5, p. 423–430, 2006.
- [6] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. of the Amer. Math. Soc.* **95**, #3, p. 466–488, 1960.
- [7] S. Bazzoni, L. Positselski. S -almost perfect commutative rings. *Journ. of Algebra* **532**, p. 323–356, 2019. [arXiv:1801.04820](#) [math.AC]
- [8] S. Bazzoni, L. Positselski. Matlis category equivalences for a ring epimorphism. *Journ. of Pure and Appl. Algebra* **224**, #10, article ID 106398, 25 pp. [arXiv:1907.04973](#) [math.RA]
- [9] S. Bazzoni, L. Positselski, J. Šťovíček. Projective covers of flat contra-modules. Electronic preprint [arXiv:1911.11720](#) [math.RA].
- [10] T. Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Phil. Soc.* **129**, #3, p. 447–475, 2000.
- [11] H.-X. Chen, C.-C. Xi. Derived decompositions of abelian categories I. Electronic preprint [arXiv:1804.10759](#) [math.RT].
- [12] A. L. S. Corner. On the exchange property in additive categories. Unpublished manuscript, 1973, 60 pp.
- [13] P. Čoupek, J. Šťovíček. Cotilting sheaves on Noetherian schemes. *Math. Zeitschrift*, published online at <https://doi.org/10.1007/s00209-019-02404-8> in November 2019. [arXiv:1707.01677](#) [math.AG]
- [14] E. Enochs. Injective and flat covers, envelopes and resolvents. *Israel Journ. of Math.* **39**, #3, p. 189–209, 1981.
- [15] A. Frankild, P. Jørgensen. Foxby equivalence, complete modules, and torsion modules. *Journ. of Pure and Appl. Algebra* **174**, #2, p. 135–147, 2002.
- [16] P. Freyd. Representations in abelian categories. Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer, 1966, p. 95–120.
- [17] L. Fuchs, L. Salce. Almost perfect commutative rings. *Journ. of Pure and Appl. Algebra* **222**, #12, p. 4223–4238, 2018.
- [18] W. Geigle, H. Lenzing. Perpendicular categories with applications to representations and sheaves. *Journ. of Algebra* **144**, #2, p. 273–343, 1991.
- [19] R. Göbel, J. Trlifaj. Approximations and endomorphism algebras of modules. Second revised and extended edition. De Gruyter, 2012.
- [20] B. Keller. Chain complexes and stable categories. *Manuscripta Math.* **67**, #4, p. 379–417, 1990.
- [21] H. Krause. Functors on locally finitely presented additive categories. *Colloquium Math.* **75**, #1, p. 105–132, 1998.

- [22] L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint [arXiv:1705.04960](https://arxiv.org/abs/1705.04960) [math.CT].
- [23] L. Positselski. Flat ring epimorphisms of countable type. *Glasgow Math. Journ.* **62**, #2, p. 383–439, 2020. [arXiv:1808.00937](https://arxiv.org/abs/1808.00937) [math.RA]
- [24] L. Positselski. Contramodules over pro-perfect topological rings. Electronic preprint [arXiv:1807.10671](https://arxiv.org/abs/1807.10671) [math.CT].
- [25] L. Positselski, J. Rosický. Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories. *Journ. of Algebra* **483**, p. 83–128, 2017. [arXiv:1512.08119](https://arxiv.org/abs/1512.08119) [math.CT]
- [26] L. Positselski, J. Šťovíček. The tilting-cotilting correspondence. *Internat. Math. Research Notices*, published online at <https://doi.org/10.1093/imrn/rnz116> in July 2019. [arXiv:1710.02230](https://arxiv.org/abs/1710.02230) [math.CT]
- [27] L. Positselski, J. Šťovíček. ∞ -tilting theory. *Pacific Journ. of Math.* **301**, #1, p. 297–334, 2019. [arXiv:1711.06169](https://arxiv.org/abs/1711.06169) [math.CT]
- [28] L. Positselski, J. Šťovíček. Topologically semisimple and topologically perfect topological rings. Electronic preprint [arXiv:1909.12203](https://arxiv.org/abs/1909.12203) [math.CT].
- [29] J.-E. Roos. Locally Noetherian categories and generalized strictly linearly compact rings. Applications. *Category Theory, Homology Theory, and their Applications, II*, Springer, Berlin, 1969, p. 197–277.
- [30] L. Salce. Cotorsion theories for abelian groups. *Symposia Math.* **XXIII**, Academic Press, London–New York, 1979, p. 11–32.
- [31] J. Šaroch. Approximations and Mittag–Leffler conditions—the tools. *Israel Journ. of Math.* **226**, #2, p. 737–756, 2018. [arXiv:1612.01138](https://arxiv.org/abs/1612.01138) [math.RA]
- [32] J. Šaroch, J. Šťovíček. The countable telescope conjecture for module categories. *Advances in Math.* **219**, #3, p. 1002–1036, 2008. [arXiv:0801.3936](https://arxiv.org/abs/0801.3936) [math.RA]
- [33] J. Šťovíček. Exact model categories, approximation theory, and cohomology of quasi-coherent sheaves. In: *Advances in Representation Theory of Algebras*, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, p. 297–367, 2013. [arXiv:1301.5206](https://arxiv.org/abs/1301.5206) [math.CT]
- [34] B. Stenström. Rings of quotients. An introduction to methods of ring theory. Springer-Verlag, Berlin–Heidelberg–New York, 1975.
- [35] C. E. Watts. Intrinsic characterizations of some additive functors. *Proc. of the Amer. Math. Soc.* **11**, #1, p. 5–8, 1960.
- [36] R. Wisbauer. Foundations of Module and Ring Theory: A Handbook for Study and Research. Gordon and Breach Science Publishers, Reading, 1991.
- [37] J. Xu. Flat covers of modules. *Lecture Notes in Math.* **1634**, Springer, 1996.

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