FINITE CHARACTER OF FINITELY STABLE DOMAINS

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Abstract. A commutative domain is finitely stable if every nonzero finitely generated ideal is stable, i.e. invertible over its endomorphism ring. A domain satisfies the *local stability property* provided that every locally stable ideal is stable.

We prove that a finitely stable domain satisfies the local stability property if and only if it has finite character, that is every nonzero ideal is contained in at most finitely many maximal ideals. This result allows to answer to the open problem of whether every Clifford regular domain is of finite character.

Introduction

An ideal of a commutative ring is stable if it is projective over its endomorphism ring and a commutative ring is said to be stable (finitely stable) if every regular (finitely generated) ideal is stable. The notion of stable ideals was first introduced in the case of noetherian rings and intensively studied by Lipman, Sally and Vasconcelos ([11], [16] and [17]. Now it plays an important role for arbitrary commutative rings. (See [13], [14], [15], [16] and [17].) In particular Olberding [14] proved that a stable commutative domain is of *finite character*, that is every nonzero ideal is contained in at most a finite number of maximal ideals; Rush [15] proved that the integral closure of a finitely stable domain is a Prüfer domain.

We say that a commutative domain satisfies the *local stability property* if every locally stable ideal is stable. Here a property is said to be satisfied locally if it holds for every localization at a maximal ideal (see [1] and [4] for more details on the subject).

The main result of this paper states that a finitely stable domain satisfies the local stability property if and only if it is of finite character (see Theorem 4.5).

The motivation of this investigation is to obtain more information on the class of Clifford regular domains, a class of domains properly intermediate between the classes of stable and finitely stable domains.

Recall that an integral domain R is said to be Clifford regular if the semigroup $\mathcal{S}(R)$ of the isomorphism classes of nonzero fractional ideals of R is a Clifford semigroup, that is every element of $\mathcal{S}(R)$ is von Neumann regular. The importance of a Clifford semigroup lies in the fact that it is a disjoint union of groups each one associated to an idempotent element

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of the semigroup and connected by bonding homomorphisms induced by multiplications by idempotent elements [5].

The study of Clifford regular domains was carried on by the author in [1] and [4] where a complete characterization of integrally closed and of noetherian Clifford regular domains was achieved. In both cases the domains turned out to be of finite character. In particular, it was shown that the class of integrally closed Clifford regular domains coincides with the class of Prüfer domains of finite character. Moreover, the idempotent elements and the constituent groups of the class semigroup of an integrally closed Clifford regular domain have been characterized by the author in [2] and [3].

Until now it has not been known whether every Clifford regular is of finite character. In this paper we show that the question has an affirmative answer. The result is obtained as a corollary of our main theorem. In fact, we proved in [4], that Clifford regular domains are finitely stable and that they satisfy the local stability property.

Firstly in [4] we showed that Clifford regular domains satisfy the *local invertibility property* , that is every locally invertible ideal is invertible and we posed the following conjecture whose interest goes beyond the problem of Clifford regularity of domains.

Conjecture 0.1. *If R is a Prüfer domain with the local invertibility property, then R is of finite character.*

The conjecture attracted the interest of many authors. Holland, Martinez, McGovern and Tesemma [9] have proved its validity by translating the problem into a statement on lattice ordered groups. Independently, almost at the same time, Halter-Koch [8] proved the conjecture using the language of ideal systems on cancellative commutative monoids.

Since the integrally closed Clifford regular domains are exactly the Prüfer domains of finite character ([4]) and the integral closure of a finitely stable domain is a Prüfer domain ([15]), it was natural to ask the question:

Question 0.2. ([4, Question 6.3]) *Let R be a finitely stable domain with the local stability property. Is R of finite character?*

The positive answer to this question (Theorem 4.5) allows us to conclude that a Clifford regular domain is of finite character and that the integral closure of a Clifford regular domain is again Clifford regular, hence it is a Prüfer domain of finite character.

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1. Preliminaries and Basic properties

Throughout R will denote a commutative domain and Q its field of quotients. For R-submodules A and B of Q, $A : B$ is defined as follows:

$$A : B = \{q \in Q \mid qB \subseteq A\}.$$

A fractional ideal F of R is an R-submodule of Q such that $R : F \neq 0$. A nonzero fractional ideal F of R is invertible if $F(R\colon F) = R$. By an overring of R is meant any ring between R and Q. If F is a fractional ideal of R, $F\colon F$ is the endomorphism ring $\mathrm{End}F$ of F.

Our local rings are not necessarily noetherian and $\mathrm{Max}(R)$ will denote the set of maximal ideals of R. Recall that two submodules X, Y of an R-module coincide if and only if $X_{\boldsymbol{m}} = Y_{\boldsymbol{m}}$, for every maximal ideal $\boldsymbol{m} \in \mathrm{Max}\, R$.

If I is a proper ideal of R, $\Omega_R(I)$ denotes the subset of $\mathrm{Max}(R)$ consisting of the maximal ideals of R containing I. Two proper ideals I, J of R are comaximal if $I + J = R$.

Definition 1.1. A domain R is of *finite character* if $\Omega_R(I)$ is a finite set for every nonzero proper ideal I of R.

We say that a nonzero element $x \in R$ is of finite character if $\Omega_R(xR)$ is finite.

We list some basic and well known properties of invertible ideals.

Lemma 1.2. *Let R be a commutative domain and let A be an R-submodule of Q. The following hold:*

(1) *If there is an R-submodule X of Q such that $AX = R$, then $X = R\colon A$ and A is an invertible fractional ideal of R.*
(2) *If A is an invertible fractional ideal of R and D is an overring of R, then AD is an invertible fractional ideal of D and $D\colon A = D\colon AD = (R\colon A)D$.*
(3) *If A is an invertible fractional ideal of R, then $\mathrm{End}(A) = A\colon A = R$.*

The notion of stable ideals is a generalization of the notion of invertible ideals.

Definition 1.3. An nonzero ideal of an integral domain is said to be *stable* provided that it is projective, or equivalently invertible, as an ideal of its endomorphism ring.

In order to deal easily with stable ideals of overrings, we will consider also the notion of stable fractional ideals, defined in the obvious way.

The following lemma states some easy but useful properties of stable fractional ideals.

Lemma 1.4. *Let R be a commutative domain and let A be a fractional ideal of R. The following hold:*

(1) *If there exist an R-submodule X of Q and an overring E of R such that $AX = E$, then AE is a stable fractional ideal of E with endomorphism ring E.*
(2) *If A is a stable fractional ideal of R with endomorphism ring E and D is an overring of E, then AD is an invertible fractional ideal of D and $D\colon AD = (E\colon A)D$.*
(3) *If A is a stable fractional ideal of R with endomorphism ring E and D is an overring of E, then*

$$E_S = A_S : A_S, \quad (A\colon A^2)_S = A_S\colon A_S^2, \quad (D : A)_S = D_S : A_S$$

for every multiplicative system S of R. In particular, A_S is a stable fractional ideal of R_S.

Proof. (1) Since $AEX = E$, Lemma 1.2 implies that AE is an invertible fractional ideal of E, hence $\mathrm{End}\, AE = E$.

(2) By assumption A is an invertible fractional ideal of E. Hence, the statement follows by Lemma 1.2 (2).

(3) Follows easily by the fact that $A = AE$ is a finitely generated fractional ideal of E. (see [4, Lemma 5.8]).

□

2. Finitely stable domains

Definition 2.1. A commutative domain R is said to be *stable (finitely stable)* if every nonzero (finitely generated) ideal of R is stable.

Note that an integral domain is stable (finitely stable) if and only if every nonzero (finitely generated) *fractional ideal* of R is stable. We recall some properties of finitely stable domains. $\overline{R}$ will denote the integral closure of a domain R.

Fact A [15, Proposition 2.1] The integral closure of a finitely stable domain is a Prüfer domain and every R-submodule of $\overline{R}$ containing R is an overring.

Fact B [14, Lemma 2.4, Corollary 2.5] Every overring of a semilocal finitely stable domain is semilocal and every overring of a finitely stable domain is again finitely stable.

Other properties are illustrated by the next two lemmas.

Lemma 2.2. *Let R be a commutative domain. The following hold:*

1. *R is a finitely stable domain if and only if every localization of R at a maximal ideal is finitely stable.*
2. *If R is a semilocal finitely stable domain and I is a nonzero stable ideal of R with endomorphism ring E, then $I = aE$ for some element $0 \neq a \in I$ and $I^2 = aI$.*

Proof. (1) If R is finitely stable, then it is locally finitely stable by Lemma 1.4 (3). Conversely if I is a nonzero finitely generated ideal of R and E is its endomorphism ring, then for every maximal ideal $\boldsymbol{m}$ of R, $E_{\boldsymbol{m}}$ is the endomorphism ring of $I_{\boldsymbol{m}}$ and by checking locally we get that $I(E\colon I) = E$.

(2) This is an easy generalization of [14, Lemma 3.1] noting that I is an invertible ideal of the semilocal domain E, hence I is a principal ideal of E. □

If (P) is any property, we say that a fractional ideal F of R satisfies (P) locally if each localization $FR_{\boldsymbol{m}}$ of F at a maximal ideal $\boldsymbol{m}$ of R satisfies (P).

Lemma 2.3. *Let R be a finitely stable domain and let I be a nonzero locally stable ideal of R. The following hold*

1. $\mathrm{End} I = \mathrm{End} I^2$.
2. *If I contains an element of finite character, then I is stable.*

Proof. (1) Let $q \in \mathrm{End} I$; then $qI \subseteq I$, hence $qI^2 \subseteq I^2$ and $q \in \mathrm{End} I^2$. Conversely, assume $qI^2 \subseteq I^2$. To prove that $qI \subseteq I$ it is enough to show that $qI_{\boldsymbol{m}} \subseteq I_{\boldsymbol{m}}$ for every maximal ideal $\boldsymbol{m} \in \mathrm{Max}\, R$. By Fact B, $R_{\boldsymbol{m}}$ is a local finitely stable domain and by Lemma 2.2(2), $I^2_{\boldsymbol{m}} = a_{\boldsymbol{m}} I_{\boldsymbol{m}}$, for some nonzero element $a_{\boldsymbol{m}} \in I$. Thus, $qI^2_{\boldsymbol{m}} \subseteq I^2_{\boldsymbol{m}}$ implies $qI_{\boldsymbol{m}} \subseteq I_{\boldsymbol{m}}$.

(2) Let $0 \neq x \in I$ be such that x is contained in at most a finite number of maximal ideals of R, say $\Omega_R(xR) = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_n\}$. By Lemma 2.2(2), for each $i = 1, 2, \dots, n$, we can choose $0 \neq a_i \in I$ such that $I^2_{\boldsymbol{m}_i} = a_i I_{\boldsymbol{m}_i}$. Let $A = x^2R + \sum_{1 \leq i \leq n} a_i I$. By checking locally we show that $A = I^2$. In fact, $A \subseteq I^2$ and $A_{\boldsymbol{m}_i} = I^2_{\boldsymbol{m}_i}$ for every $i = 1, 2, \dots, n$. If $\boldsymbol{n}$ is a maximal ideal of R and $\boldsymbol{n} \notin \Omega_R(xR)$, then $x^2 \notin \boldsymbol{n}$ and $I \nsubseteq \boldsymbol{n}$, hence $R_{\boldsymbol{n}} = A_{\boldsymbol{n}} \subseteq I^2_{\boldsymbol{n}} = R_{\boldsymbol{n}}$ and so $A = I^2$. Let $B = xR + \sum_{1 \leq i \leq n} a_i R$, then $B \subseteq I$ and $A \subseteq BI \subseteq I^2$, so $BI = I^2$. If $qB \subseteq B$, then $qI^2 \subseteq I^2$, hence $\mathrm{End}B \subseteq \mathrm{End}I$, by (1).

Let $E = \mathrm{End}I$. Since B is a finitely generated ideal of R, B is an invertible ideal of $\mathrm{End}B$ and BE is an invertible ideal of E, by Lemma 1.2 (2). We have $E\colon I = I\colon I^2 = (I\colon I)\colon B = E\colon B = E\colon BE$, so $I(E\colon I) = I(E\colon B) \supseteq B(E\colon BE) = E$ and thus $I(E\colon I) = E$, that is I is a stable ideal of R. $\square$

We finish this section by noticing that there might be non-finitely stable domains whose integral closure is finitely stable, or equivalently a Prüfer domain. We show that this may happen even in the case of local noetherian domains.

Example 1. Let k be a field and let $R = k[[x^3, x^5]]$. R is a noetherian local domain with non-stable maximal ideal $\boldsymbol{m}$. In fact, $\boldsymbol{m}$ is not invertible over $\mathrm{End}(\boldsymbol{m}) = k[[x^3, x^5, x^7]]$. The integral closure of R is the valuation domain $k[[x]]$.

3. Local invertibility and local stability properties

We are interested in globalizing two types of local properties.

Definition 3.1. An integral domain R satisfies the *local invertibility property* if any *locally invertible ideal* of R is invertible.

An integral domain R satisfies the *local stability property* if any *locally stable ideal* of R is stable.

Note that if a domain R has the local invertibility property (local stability property), then every locally invertible (stable) *fractional ideal* of R is invertible (stable).

We now state and prove a result comparing the two local properties.

Lemma 3.2. *If an integral domain R satisfies the local stability property, then it also satisfies the local invertibility property.*

Proof. Let I be a locally invertible ideal of R, then I is locally stable and for every maximal ideal $\boldsymbol{m} \in \mathrm{Max}\, R$, $I_{\boldsymbol{m}}\colon I_{\boldsymbol{m}} = R_{\boldsymbol{m}}$, by Lemma 1.2 (3). By assumption I is a stable ideal of R, hence by Lemma 1.4 (3), $(I\colon I)_{\boldsymbol{m}} = I_{\boldsymbol{m}}\colon I_{\boldsymbol{m}} = R_{\boldsymbol{m}}$. Thus, $\mathrm{End}I = R$ and I is an invertible ideal of R. $\square$

In Section 4 we will consider the question to decide whether a finitely stable domain with the local invertibility property satisfies also the local stability property.

We recall now the relations between Clifford regularity and the above local properties. First we consider the integrally closed case.

Proposition 3.3. *Let R be an integrally closed domain. The following are equivalent:*

(1) *R is a Prüfer domain satisfying the local invertibility property.*
(2) *R is a Prüfer domain of finite character.*
(3) *R is Clifford regular.*
(4) *R is a Prüfer domain satisfying the local stability property.*

Proof. (1) ⇒ (2) follows by the validity of Conjecture 0.1 proved in [9], [12] and [8].

(2) ⇒ (3) By [1, Theorem 2.14].

(3) ⇒ (4) [4, Lemmas 4.1 and 5.7].

(4) ⇒ (1) By Lemma 3.2. □

Moreover, in [4] we proved the following.

Proposition 3.4. *The following hold.*

(1) *A Clifford regular domain is finitely stable and satisfies the local stability property and hence the local invertibility property.*
(2) *A noetherian domain is Clifford regular if and only if it is a stable domain. Hence, by* [14, Theorem 3.3] *a noetherian Clifford regular domain is of finite character.*

The following question, generalizing Conjecture 0.1 was posed for the class of finitely stable domains.

Question 3.5. *(*[4, Question 6.3]*) Let R be a finitely stable domain with the local stability property. Is R of finite character?*

Our aim is to answer to Question 3.5, but first of all we note that every finitely stable domain of finite character has the local stability property. In fact, as a consequence of Lemmas 2.3 and 3.2 we obtain immediately:

Proposition 3.6. *Let R be a finitely stable domain of finite character. Then R has satisfies the local stability property and hence also the local invertibility property.*

Note that there many examples of finitely stable domains which are not of finite character. Any Prüfer domain not of finite character is one of those, for instance any almost Dedekind domain, which is not Dedekind (see [6, Theorem 37.2]).

4. The finite character

For a commutative domain R we consider a particular subset of the set $\operatorname{Max} R$ and we outline some properties of this subset.

Definition 4.1. Denote by $\mathcal{T}(R)$ the set of maximal ideals $\boldsymbol{m}$ of a domain R for which there exists a finitely generated ideal with the property that $\boldsymbol{m}$ is the only maximal ideal containing it.

Lemma 4.2. *Let I be a finitely generated ideal of a domain R. The following hold true.*

(1) *Assume that $\Omega_R(I)$ is finite. Then, for every $\boldsymbol{m} \in \Omega_R(I)$ there is a finitely generated ideal J containing I such that $\Omega_R(J) = \{\boldsymbol{m}\}$, hence $\Omega_R(I) \subseteq \mathcal{T}(R)$.*
(2) *If $\Omega_R(I)$ contains two distinct maximal ideals, then I is contained in two finitely generated comaximal ideals of R.*

(3) *If $\Omega_R(I) \cap \mathcal{T}(R) = \emptyset$, then for every finitely generated proper ideal $J \geq I$ $\Omega_R(J)$ is infinite.*

Proof. (1) Let $\Omega_R(I) = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_n\}$ and for each $i = 1, 2, \dots, n$ let $x_i \in \boldsymbol{m}_i \setminus \cup_{j \neq i} \boldsymbol{m}_j$. Then, $\Omega_R(I + x_i R) = \{\boldsymbol{m}_i\}$, hence $I + x_i R$ satisfies condition (1).

(2) Let $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \Omega_R(I)$, $\boldsymbol{m}_1 \neq \boldsymbol{m}_2$. Choose $x_1 \in \boldsymbol{m}_1$, $x_2 \in \boldsymbol{m}_2$ such that $1 = x_1 + x_2$. Then, $J_1 = I + x_1 R$ and $J_2 = I + x_2 R$ are comaximal finitely generated ideals containing I.

(3) Assume that there exists a finitely generated proper ideal $J \geq I$ such that $\Omega_R(J)$ is finite. Then, by part (1) $\Omega_R(J) \subseteq \mathcal{T}(R)$, hence $\Omega_R(I) \cap \mathcal{T}(R) \neq \emptyset$, a contradiction. □

In [4] we gave a partial answer to Question 3.5 by proving that if R is a finitely stable domain satisfying the local stability property, then every nonzero ideal of R is contained in at most a finite number of maximal ideals of $\mathcal{T}(R)$.

We look now for conditions equivalent to the finite character property.

Proposition 4.3. *Let R be a finitely stable domain with the local stability property. The following are equivalent:*

(1) *For every nonzero finitely generated proper ideal I of R, $\Omega_R(I) \subseteq \mathcal{T}(R)$.*
(2) *For every nonzero finitely generated proper ideal I of R, $\Omega_R(I) \cap \mathcal{T}(R) \neq \emptyset$.*
(3) *R has finite character.*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Clearly, it is enough to show that every nonzero element of R is of finite character. Let $0 \neq x \in R$ and assume by way of contradiction that $\Omega_R(xR)$ is infinite. By [4, Proposition 5.9]. $\Omega_R(xR) \cap \mathcal{T}(R)$ is finite, say $\Omega_R(xR) \cap \mathcal{T}(R) = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \dots, \boldsymbol{m}_n\}$. Let $\boldsymbol{m} \in \Omega_R(xR)$, $\boldsymbol{m} \neq \boldsymbol{m}_i$, for every $i = 1, 2, \dots, n$ and choose $y \in \boldsymbol{m} \setminus \cup_{1 \leq i \leq n} \boldsymbol{m}_i$. Let $J = xR + yR$. Then, $\Omega_R(J) \subseteq \Omega_R(xR)$ and $\boldsymbol{m}_i \notin \Omega_R(J)$, for every $i = 1, 2, \dots, n$, hence $\Omega_R(J) \cap \mathcal{T}(R) = \emptyset$, a contradiction.

(3) $\Rightarrow$ (1). Obvious from Lemma 4.2 (1). □

The following result is a crucial step towards the finite character property.

Proposition 4.4. *Let R be a commutative domain with the local stability property. Then, every stable proper ideal of R is contained in at most a finite number of pairwise comaximal stable ideals of R.*

Proof. Let I be a proper stable ideal of R and assume, by way of contradiction, that there is an infinite set $\{J_n \mid n \in \mathbb{N}\}$ of pairwise comaximal stable ideals of R each one containing I. For every $n \in \mathbb{N}$, let E_n be the endomorphism ring of J_n, so that $J_n(E_n : J_n) = E_n$.

(*) If $\boldsymbol{m}$ is a maximal ideal of R not containing J_n, then $E_n R_{\boldsymbol{m}} = R_{\boldsymbol{m}}$ and $(J_n : J_n^2) R_{\boldsymbol{m}} = (E_n : J_n) R_{\boldsymbol{m}}$, by Lemma 1.4 (3).

Let $B = \sum_{n \in \mathbb{N}} (E_n : J_n)$. We first note that B is a fractional ideal of R. In fact, for every $n \in \mathbb{N}$ we have $J_n^2 (E_n : J_n) = J_n$ and $I^2 \subseteq J_n^2$, hence $I^2 B \subseteq R$. We claim that B is locally stable. Let $\boldsymbol{m}$ be a maximal ideal of R. If $J_n \nsubseteq \boldsymbol{m}$ for every $n \in \mathbb{N}$, then by (*), $BR_{\boldsymbol{m}} = R_{\boldsymbol{m}}$. Assume that there is an $n \in \mathbb{N}$ such that $J_n \subseteq \boldsymbol{m}$, then $J_k \nsubseteq \boldsymbol{m}$ for each $k \neq n$, since J_n and J_k are comaximal. Thus $BR_{\boldsymbol{m}} = (E_n \colon J_n)R_{\boldsymbol{m}}$. By Lemma 1.4 (3), we get that $BR_{\boldsymbol{m}} = (E_nR_{\boldsymbol{m}} \colon J_nR_{\boldsymbol{m}})$ is a stable fractional ideal of $R_{\boldsymbol{m}}$ with endomorphism ring $E_nR_{\boldsymbol{m}}$.

By the assumption on R, B is stable. Checking locally we show that the endomorphism ring of B is $E = \sum_{n\in\mathbb{N}} E_n$. First note that for every maximal ideal $\boldsymbol{m}$ of R, $(B\colon B)R_{\boldsymbol{m}} = BR_{\boldsymbol{m}} \colon BR_{\boldsymbol{m}}$, by Lemma 1.4 (3). Let now $\boldsymbol{m}$ be a maximal ideal of R not containing J_n for every $n \in \mathbb{N}$, then as noted above, $ER_{\boldsymbol{m}} = R_{\boldsymbol{m}}$ and $BR_{\boldsymbol{m}} = R_{\boldsymbol{m}}$, so $ER_{\boldsymbol{m}} = (B\colon B)R_{\boldsymbol{m}}$. If there is an $n \in \mathbb{N}$ such that $J_n \subseteq \boldsymbol{m}$, then $ER_{\boldsymbol{m}} = E_nR_{\boldsymbol{m}}$ and $BR_{\boldsymbol{m}} \colon BR_{\boldsymbol{m}}$ coincides with $E_nR_{\boldsymbol{m}}$, since it is the endomorphism ring of the invertible $E_nR_{\boldsymbol{m}}$-ideal $(E_n \colon J_n)R_{\boldsymbol{m}}$.

Thus B is a finitely generated fractional ideal of E, so

$$B = BE = (E_1 \colon J_1)E + (E_2 \colon J_2)E + \cdots + (E_k \colon J_k)E,$$

for some $k \in \mathbb{N}$. Hence, by Lemma 1.4 (2) we have

$$E\colon B = J_1E \cap J_2E \cdots \cap J_kE$$

and for every $n \in \mathbb{N}$

$$(E_n \colon J_n)E \subseteq (E_1 \colon J_1)E + (E_2 \colon J_2)E + \cdots + (E_k \colon J_k)E.$$

Thus, for every $n \in N$

$$E\colon B = J_1E \cap J_2E \cdots \cap J_kE \subseteq J_nE.$$

Let $n > k$ and let $\boldsymbol{m}$ be a maximal ideal of R containing J_n. For every $1 \le i \le k$, J_i is not contained in $\boldsymbol{m}$, hence $J_iER_{\boldsymbol{m}} = ER_{\boldsymbol{m}} = E_nR_{\boldsymbol{m}}$, so that $E_nR_{\boldsymbol{m}} \subseteq J_nE_nR_{\boldsymbol{m}}$, a contradiction, since $J_nR_{\boldsymbol{m}}$ is a proper stable ideal of $R_{\boldsymbol{m}}$ and thus also a proper ideal of $E_nR_{\boldsymbol{m}}$. □

Theorem 4.5. *Let R be a finitely stable domain. Then R has the local stability property if and only it is of finite character.*

Proof. The sufficiency follows by Proposition 3.6.

For the necessary condition note that by Proposition 4.3, it is enough to show that every nonzero finitely generated proper ideal I is contained in a maximal ideal $\boldsymbol{m} \in \mathcal{T}(R)$. Assume by way of contradiction that $\Omega_R(I) \cap \mathcal{T}(R) = \emptyset$. Then, by Lemma 4.2, $\Omega_R(J)$ is infinite for every finitely generated proper ideal J containing I. Thus every finitely generated ideal containing I is contained in two comaximal finitely generated ideals, by Lemma 4.2 (2).

Arguing as in the proof of [12, Theorem 5], it is possible to define by induction a countable set of pairwise comaximal finitely generated ideals of R containing I in the following way. For each $1 < n \in \mathbb{N}$, let I_n, J_n be two comaximal finitely generated ideals containing I_{n-1}. Then, it is easy to show, by induction, that for every $1 \le k < n$, J_k, J_n are comaximal. In fact, $I_1 \subseteq J_2$ and $I_1 + J_1 = R$ imply $J_2 + J_1 = R$. Assume the statement true for every $m < n$ and let $k < n$. We have $I_k \subseteq I_{n-1} \subseteq J_n$ and $I_k + J_k = R$, thus $J_k + J_n = R$.

This contradicts Proposition 4.4, hence R has finite character. □

We have seen in Lemma 3.2 that the local stability property implies the local invertibility property and that the two conditions are equivalent for Prüfer domains (Proposition 3.3). We are not able to prove that for finitely stable domains they are equivalent. So we ask the following question:

Question 4.6. *If R is a finitely stable domain, does the local invertibility property imply the local stability property?*

In view of Theorem 4.5 the above question is equivalent to asking whether a finitely stable domain with the local invertibility property has finite character.

We list now some consequences of Theorem 4.5. They provide answers to some questions posed in [4].

Theorem 4.7. *Let R be a Clifford regular domain. Then R is of finite character.*

Proof. By Proposition 3.4 a Clifford regular domain is finitely stable and satisfies the local stability property. Hence the conclusion follows by Theorem 4.5. □

The next corollary answers [4, Question 6.4].

Corollary 4.8. *Let R be a Clifford regular domain. The integral closure $\overline{R}$ of R is Clifford regular. In particular, $\overline{R}$ is a Prüfer domain of finite character.*

Proof. A Clifford regular domain is finitely stable, by Proposition 3.4, hence $\overline{R}$ is a Prüfer domain, by Fact A. By Proposition 3.3 it is enough to show that $\overline{R}$ is of finite character. Let $x = a/b$ be a nonzero element of $\overline{R}$, with $a, b \in R$. Then, $\Omega_{\overline{R}}(x\overline{R}) \subseteq \Omega_{\overline{R}}(a\overline{R})$. For every maximal ideal $\overline{\boldsymbol{m}}$ of $\Omega_{\overline{R}}(x\overline{R})$, $\overline{\boldsymbol{m}} \cap R = \boldsymbol{m}$ is a maximal ideal of R containing a. By Theorem 4.7, $\Omega_R(aR)$ is finite and by Fact B there are only finitely many maximal ideals of $\overline{R}$ lying above $\boldsymbol{m}$, thus we conclude that $\overline{R}$ is of finite character. □

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