CONTRAMODULES AND THEIR APPLICATIONS TO TILTING THEORY

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Introduction

Contramodules were first introduced by Eilenberg and Moore in 1965 alongside comodules over coalgebras or corings, but they achieved so little success that they were forgotten for three decades. From the first beginning of the 21st century, contramodules appeared again in literature thanks mainly to work by Leonid Positselski. The motivation towards their study was in particular for their many applications in algebraic geometry. They were studied for the purposes of the semi-infinite cohomology theory and the comodule-contramodule correspondence. Recently the notion of contramodules has been applied profitably in commutative algebra and tilting theory.

The aim of these three lectures is to advertise the theory of contramodules and to capture the interest of researchers and get them involved in applications of the theory to different contexts.

Generally, contramodules are sets with infinitary additive operations of the "arity" bounded by some cardinal. Typical examples are contramodules over complete topological rings. They provide a way of having an abelian category of non-topological modules with some completeness properties over a topological ring.

More precisely, let R be a commutative complete local ring with maximal ideal m. The category of infinitely generated m-adically complete R-modules is not abelian already for $R = k[[\epsilon]]$. The category of R-contramodules is the natural abelian category into which complete R-modules are embedded. In particular, when R is the ring of p-adic integers, the abelian category of R-contramodules is the $0, 1$-perpendicular category of $\mathbb{Z}[p^{-1}]$, i.e. the category of the abelian groups C such that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], C) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[p^{-1}], C)$. Similarly for contramodules over $R = k[[\epsilon]]$.

The $k[[\epsilon]]$-contramodules form a full subcategory of the category of $k[[\epsilon]]$-modules and even a full subcategory of the category of $k[\epsilon]$-modules. This subcategory contains all the $k[[\epsilon]]$-modules M, such that $M \cong \varprojlim_n M/\epsilon^n M$, i.e. separated and complete modules, but contains also some complete and non separated modules, i.e. the modules such that the natural map $M \to \varprojlim_n M/\epsilon^n M$ is surjective but not injective.

Analogous results hold for the adic completion of a commutative Noetherian ring with respect to an arbitrary ideal.

So surprisingly, in these cases (and some other cases) the natural forgetful functors from the categories of contramodules to the related categories of modules turn out be fully faithful. That is an infinitary additive operation is uniquely determined by its finite aspects.

1. FIRST LECTURE

1.1. **Infinite summation operations.** An elementary approach to the theory of contramodules can be achieved by introducing the notion of an s-power infinite summation operation on abelian groups.

Let C be an abelian group and s a symbol. An s-power infinite summation operation (s-power i.s.o.) on C is a map

$$\prod_{n\geq 0} C \longrightarrow C$$

$$(c_0, c_1, \dots, c_n, \dots) \longmapsto \sum_{n=0}^{\infty} s^n c_n,$$

satisfying three axioms:

(1) **(U)** Unitality: $\sum_{n=0}^{\infty} s^n c_n = c_0$ if $c_1 = c_2 = \dots, c_n = \dots = 0$.

(2) **(A)** Additivity: $\prod_{n\geq 0} C_n \longrightarrow C$ is a homomorphism of abelian groups, i.e.

$$\sum_{n=0}^{\infty} s^n (c_n + b_n) = \sum_{n=0}^{\infty} s^n c_n + \sum_{n=0}^{\infty} s^n b_n$$
$$\forall (c_0, c_1, \dots, c_n, \dots), (b_0, b_1, \dots, b_n, \dots) \in \prod_{n\geq 0} C.$$

(3) **(CA)** Contrassociativity:

$$\sum_{i=0}^{\infty} s^i \sum_{j=0}^{\infty} s^j (c_{ij}) = \sum_{n=0}^{\infty} s^n \sum_{i+j=n} c_{ij}.$$

An s-power i.s.o. on an abelian group C defines an abelian group endomorphism $s : C \to C$ by

$$sc = \sum_{n=0}^{\infty} s^n c_n, \qquad \text{where } c_1 = c \text{ and } c_i = 0 \text{ for } i \neq 1,$$

$$s^n c = \sum_{n=0}^{\infty} s^n c_n, \qquad \text{where } c_n = c \text{ and } c_i = 0 \text{ for } i \neq n.$$

Example 1.1.1. [12, Example 3.1 (1), (2)]

(1) For any abelian group C, the group of formal power series $C[[z]]$ is naturally endowed with a z-power i.s.o.

(2) The group of p-adic integers J_p is naturally endowed with a p-power i.s.o.

(3) For any set X the group $\prod_{x\in X} J_p$ is endowed with a p-power i.s.o.

(4) *The subgroup* $C = J_p[[X]] \subset \prod_{x\in X} J_p$ *consisting of families*

$$(a_x)_{x\in X}, a_x \in J_p$$

converging to zero in the topology of J_p *is preserved by the p-power i.s.o. on* $\prod_{x\in X} J_p$.

(5) *In all these cases, the infinite sum can be computed as the limit of finite partial sums in the adic topology of the group in question.*

(6) *The category of abelian groups with s-power i.s.o. and group homomorphisms preserving the infinite summation operations is an abelian category with products (indeed, kernels and cokernels inherit the s-power i.s.o. and for any family of groups* C_i *with s-power i.s.o. there is a natural s-power i.s.o. on* $\prod_i C_i$.)

Lemma 1.1.2. [12, Lemma 3.2] *An abelian group C endowed with an s-power i.s.o. has no non zero s-divisible subgroups, i.e. if* $D \leq C$ *and* $sD = D$ *then* $D = 0$.

Proof. Idea of the proof. Given a sequence $(c_0, c_1, \dots, c_n, \dots)$ of elements of C satisfying $c_n = sc_{n+1}$, for every $n \geq 0$, consider the element $\sum_{n=0}^{\infty} s^n c_n$ and

$$\sum_{n=0}^{\infty} s^n c_n = \sum_{n=0}^{\infty} s^n s c_{n+1} = \sum_{n=0}^{\infty} s^{n+1} c_{n+1} = \sum_{n=1}^{\infty} s^n c_n,$$

hence $c_0 = 0$. To make the calculation rigorous one should use the Contra-associativity axiom (CA). □

Theorem 1.1.3. *Let C be an abelian group. The following hold.*

(1) *(*[12, Theorem 3.3 (1)] *An s-power i.s.o. on an abelian group C is uniquely determined by the endomorphism* $s : C \to C$, *that is, given* $s : C \to C$ *there exists at most one s-power i.s.o. on C restricting to s.*

(2) *An endomorphism* $s : C \to C$ *can be extended to an s-power i.s.o. on C if and only if for any sequence* $(a_0, a_1, \dots, a_n, \dots)$ *of elements of C the infinite system of non-homogeneous equations*

$$(*) \quad b_n - sb_{n+1} = a_n, \qquad n \geq 0,$$

admits a unique solution $(b_0, b_1, \dots, b_n, \dots)$ *in C.*

(3) *An abelian group homomorphism* $f : C \to D$ *between two abelian groups C, D endowed with s-power i.s.o., preserves the s-power i.s.o. if and only if it commutes with the s-endomorphisms on C and D.*

Proof. (2) Let C be an abelian group with an s-power i.s.o.. By Lemma 1.1.2 C is s-reduced; thus, if a solution of the system $(*)$ exists, then it is unique.

For the existence, set

$$b_n := \sum_{i=0}^{\infty} s^i a_{n+i}$$

for all $n \geq 0$. The sequence $(b_n)_{n\geq 0}$ is a solution of the system:

$$b_n - sb_{n+1} = \sum_{i=0}^{\infty} s^i a_{n+i} - s\sum_{i=0}^{\infty} s^i a_{n+i+1} =$$

$$= \sum_{i=0}^{\infty} s^i a_{n+i} - \sum_{i=0}^{\infty} s^{i+1} a_{n+i+1} = \sum_{i=0}^{\infty} s^i a_{n+i} - \sum_{i=1}^{\infty} s^i a_{n+i} = a_n.$$

To make this calculation rigorous one should use axiom (CA) with the sequence of elements

$$a_{ij} := \begin{cases} a_{n+j+1} & \text{for } i = 1, j \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

For the converse, suppose now that the system $(*)$ is uniquely solvable in C for any sequence $(a_n)_{n\geq 0}$ of elements of C. Given such a sequence, take the unique solution $(b_n)_{n\geq 0}$ of the system and set

$$\sum_{n=0}^{\infty} s^n a_n := b_0.$$

This defines an s-power i.s.o. on C. One then has to check that the three axioms are satisfied.

We have proved not only (2), but also (1), since we see that once the endomorphism $s : C \to C$ is given, the s-power i.s.o. on C, can be recovered from the solution of the system $(*)$ which is unique.

To prove (3), note that any abelian group homomorphism $f : C \to D$ commuting with the endomorphisms s, takes solutions of the system $(*)$ in C to solutions in D and thus it preserves the s-power i.s.o. on C and D. □

From now on in this lecture R will denote a commutative ring with unit.

Definition 1.1.4. *Let R be a commutative ring, $s \in R$ a fixed element and C an R-module. Let $s : C \to C$ be multiplication by s. We say that C admits an s-power i.s.o. if it admits an s-power i.s.o. as an abelian group in a way that it is compatible with the action of s.*

If C admits an s-power i.s.o. then for $r \in R$, $r(\sum_{n=0}^{\infty} s^n c_n) = \sum_{n=0}^{\infty} s^n (rc_n)$.

If R is a commutative ring and $s \in R$ consider the localization $R[s^{-1}]$ and the inductive system

$$R \xrightarrow{s} R \xrightarrow{s} R \to \dots$$

where the maps are multiplications by s. Then $R[s^{-1}] \cong \varinjlim_{n\geq 0} R$, and $R[s^{-1}]$ admits a presentation

$$(\mathbf{a}) \quad 0 \to \bigoplus_{n=0}^{\infty} Rf_n \xrightarrow{\mu} \bigoplus_{n=0}^{\infty} Re_n \to R[s^{-1}] \to 0, \quad \mu(f_n) = e_n - se_{n+1}.$$

For every R-module C we get the following exact sequence.

$$(\mathbf{b}) \quad 0 \to \mathrm{Hom}_R(R[s^{-1}], C) \to \prod_{n=0}^{\infty} C \xrightarrow{\mathrm{Hom}(\mu,C)} \prod_{n=0}^{\infty} C \to \mathrm{Ext}^1_R(R[s^{-1}], C) \to 0,$$

$$\mathrm{Hom}(\mu, C)((b_n)_{n\geq 0}) = (b_n - sb_{n+1})_{n\geq 0}.$$

The following is an immediate consequence of the above discussion.

Theorem 1.1.5. *An R-module admits an s-power i.s.o. if and only if*

$$\mathrm{Hom}_R(R[s^{-1}], C) = 0 = \mathrm{Ext}^1_R(R[s^{-1}], C).$$

.

Proof. From the exact sequence (**b**) we infer that the system

$$(*) \quad b_n - sb_{n+1} = a_n, n \geq 0,$$

admits a solution $(b_0, b_1, \dots, b_n, \dots)$ in $\prod_{n=0}^{\infty} C$ if and only if

$$\mathrm{Ext}^1_R(R[s^{-1}], C) = 0.$$

It admits a unique solution if moreover $\mathrm{Hom}_R(R[s^{-1}], C) = 0$. The conclusion follows by Theorem 1.1.3. □

Definition 1.1.6. *Let R be a commutative ring and $s \in R$ a fixed element. An R-module C is an s-**contramodule** or s-contramodule R-module, if*

$$\mathrm{Hom}_R(R[s^{-1}], C) = 0 = \mathrm{Ext}^1_R(R[s^{-1}], C).$$

Denote by $R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}}$ the subcategory of s-contramodules.

Thus s-contramodules are the objects of the $0,1$-perpendicular category of $R[s^{-1}]$. Since the projective dimension of $R[s^{-1}]$ is at most one, the category of s-contramodules is closed under kernels, cokernels, extensions and products in $R\text{–}\mathsf{Mod}$, that is, it is exactly embedded in $R\text{–}\mathsf{Mod}$. Moreover, if

$$0 \to F_1 \xrightarrow{f} F_0 \to R[s^{-1}] \to 0$$

is a projective resolution of $R[s^{-1}]$, then M is an s-contramodule if and only if $\mathrm{Hom}_R(f, M)$ is a bijection.

Every R-module annihilated by s is clearly an s-contramodule.

1.2. Relations between s-contramodules and s-adic completion.

An R-module M over a commutative ring is s-torsion if for every $x \in M$ there is $n \in \mathbb{N}$ such that $s^n x = 0$. Thus M is s-torsion if and only if $R[s^{-1}] \otimes_R M = 0$ ($\mathrm{Tor}_1^R(R[s^{-1}], M) = 0$ always). This shows a duality between s-torsion modules and s-contramodules.

Let $\mathcal{D}(R)$ denote the derived category of the ring R and $\otimes^{\mathbf{L}}_R$ denote the total left derived functor of the tensor product functor. Then:

Lemma 1.2.1. [12, Lemma 6.2]

Let $M^\bullet, N^\bullet, C^\bullet$ be complexes of R-modules.

1. *If either $H^n(M^\bullet)$ or $H^n(N^\bullet)$ are s-torsion, for all $n \in \mathbb{Z}$, then the $H^n(M^\bullet \otimes^{\mathbf{L}}{}_R N^\bullet)$ are s-torsion for all $n \in \mathbb{Z}$.*
2. *If either $H^n(M^\bullet)$ are s-torsion or $H^n(C^\bullet)$ are s-contramodules for all $n \in \mathbb{Z}$, then $\mathrm{Hom}_{\mathcal{D}(R)}(M^\bullet, C^\bullet[n])$ are s-contramodule, for all $n \in \mathbb{Z}$.*

An R-module C is said to be s-**complete** if the natural map to its s-adic completion

$$\lambda_{s,C} \colon C \to \varprojlim_{n \geq 1} C/s^n C,$$

is surjective. The R-module C is said to be s-**separated** if the map $\lambda_{s,C}$ is injective.

The module $\varprojlim_{n\geq 1} C/s^n C$ is denoted by $\Lambda_s(C)$.

Lemma 1.2.2. *Let C be an R-module. Then*

(1) [12, proof of Theorem 2.3] *C is s-complete if and only if for any sequence $(a_n)_{n\geq 0}$ of elements of C, the infinite system of linear equations:*

$$(**)\, s^n(b_n - sb_{n+1}) = s^n a_n, \quad n \geq 0$$

admits a solution $(b_n)_{n\geq 0}$ in C;

(2) C is s-separated if, and only if, for any pair of solutions $(b'_n)_{n\geq 0}$, $(b''_n)_{n\geq 0}$ of the system $(**)$ in C, one has $b'_0 = b''_0$.

(3) if C is an s-contramodule, then C is s-complete.

Proof. (1) Recall that:

$$\varprojlim C/s^nC = \big\{(\bar{c}_n)_{n\geq 1} \in \prod_{n\geq 1} \frac{C}{s^nC} \mid c_{n+1} \equiv c_n \bmod s^nC,\ \forall n \geq 1\big\}.$$

Asking $\lambda_{s,C}$ to be surjective means that, for any sequence of elements $(\bar{c}_n)_{n\geq 1}$ in $\varprojlim C/s^nC$, there exists $c \in C$ such that $c \equiv c_n \bmod s^nC$ for all $n \geq 1$.

Now, set $a_0 := c_1$ and for $n \geq 1$ choose elements $a_n \in C$ such that $c_{n+1} = c_n + s^n a_n$. Then for any $n \geq 0$ one has:

$$c_{n+1} = s^n a_n + s^{n-1}a_{n-1} + \cdots + sa_1 + a_0.$$

So surjectivity becomes equivalent to ask that for any sequence $(a_n)_{n\geq 0}$ of elements of C, there exists $c \in C$ such that $c - (s^n a_n + s^{n-1}a_{n-1} + \cdots + sa_1 + a_0) \in s^{n+1}C$.

Finally set $b_0 := c$. Then the statement above is equivalent to ask that for any sequence $(a_n)_{n\geq 0}$ of elements of C there exists a solution $(b_n)_{n\geq 0}$ to the system of linear equations

$$b_0 - s^{n+1}b_{n+1} = s^n a_n + \cdots + sa_1 + a_0, \quad n \geq 0$$

which in turn is equivalent to the system:

$$s^n(b_n - sb_{n+1}) = s^n a_n, \quad n \geq 0.$$

(2) If $\lambda_{s,C}$ is injective, then $b'_0 = b''_0$, since by the argument in (1) $\lambda_{s,C}(b'_0) = \lambda_{s,C}(b''_0)$. Conversely if $\lambda_{s,C}(b'_0) = \lambda_{s,C}(b''_0)$ for some $b'_0, b''_0 \in C$, then for any $n \geq 1$, $b'_0 - b''_0 \in s^nC$, i.e. $b'_0 - b''_0 = s^n b_n$. So $(b'_0 - b''_0, b_1, b_2, \dots)$ is a solution of the homogeneous version of the system $(**)$, hence by assumption it must be $b'_0 - b''_0 = 0$.

(3) $\mathrm{Ext}^1_R(R[s^{-1}], C) = 0$ implies that there is a solution of the system $(*)$, hence C is s-complete. □

Lemma 1.2.3. [12, Theorem 2.4]

(1) An s-separated, s-complete R-module C is an s-contramodule.

(2) An s-torsion free s-contramodule R-module C is s-separated and s-complete

Proof. (1) The category R–$\mathsf{Mod}_{\mathbf{s-ctr}}$ is closed under kernels and infinite direct products, hence under infinite projective limits. Moreover, for an R-module C, the quotient C/s^nC is always an s-contramodule for any $n \geq 1$. So an s-separated and s-complete R-module C is an s-contramodule, since $C \cong \varprojlim_{n\geq 1} C/s^nC$.

(2) If C is s-torsion free then the systems $(*)$ and $(**)$ are equivalent. If C is moreover an s-contramodule, then the system $(*)$ has a unique solution, by Theorem 1.1.5, hence Lemma 1.2.2 (2) is satisfied. □

There are examples of s-contramodules which are not s-separated.

Example 1.2.4. [12, Example 2.7 (1)] *For any prime $p \in \mathbb{Z}$ let J_p be the abelian group of p-adic integers and let C be the subgroup of $\prod_{n\geq 0} J_p$ consisting of all sequences converging to zero in the p-adic topology. Let $D \subseteq C$ be the subgroup of all sequences $(p^n v_n)_{n\geq 0}$, where $v_n \in J_p$ and let $E \subseteq D$ be the subgroup of all sequences $(p^n v_n)_{n\geq 0}$, where $(v_n)_{n\geq 0}$ converges to zero in the p-adic topology. All the three groups are p-contramodules, as they are p-separated and p-complete. Thus, the quotient C/E is a p-contramodule, too. However, it is not p-separated because $\bigcap_{n\geq 1} p^n(C/E) = D/E$.*

The above example allows to construct an abelian group with an s-power i.s.o which cannot be interpreted as any kind of limit of finite partial sums.

Example 1.2.5. [12, Example 3.1 (3)]
In the above notations, set $B = C/E$, and let $b_n = c_n + E$, $c_n = (u_m)_{m\geq 0}$ with $u_0 = 0, u_1 = 0, ..., u_n = 1, u_{n+1} = 0, \dots$. Then $p^n c_n \in E$, but $\sum_{n=0}^{\infty} p^n c_n \notin E$, because the sequence $1, p, p^2, \dots, p^n \dots$ does not have the form $p^n v_n$ with $v_n \to 0$ in J_p for $n \to \infty$.

Contrarily to the class of s-contramodules, the class of s-separated and s-complete R-modules is not abelian and it is not closed under cokernels (even under the cokernels of injective morphisms) in R–Mod, nor under extensions. Similarly, the class of s-complete R-modules does not have nice closure properties, since it is not closed under extensions, even though it is closed under quotients.

1.3. **The left adjoint functor Δ_s.**

In this subsection we will show that the category $R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}}$ is a *reflective* subcategory of R–Mod (i.e. the embedding functor has a left adjoint).

Notation 1.3.1. *Let R be a commutative ring, s an element of R and l_R the natural ring homomorphism $R \to R[s^{-1}]$.*

Denote by $K^\bullet(s)$ the two-term complex $R \xrightarrow{l_R} R[s^{-1}]$, concentrated in cohomological degrees -1 and 0.

Denote by $T^\bullet(s)$ the two term complex

$$\bigoplus_{n=0}^{\infty} R \xrightarrow{\psi_R^s} \bigoplus_{n=1}^{\infty} R,$$

where $\psi_R^s : (x_0, x_1, x_2, \dots) \longmapsto (x_1 - sx_0, x_2 - sx_1, \dots)$, concentrated in cohomological degrees 0 and 1.

Lemma 1.3.2. [12, Remark 6.5] *The complex $T^\bullet(s)[1]$ is a projective resolution of the complex $K^\bullet(s)$.*

Proof. A quasi isomorphism between $T^\bullet(s)[1]$ and $K^\bullet(s)$ is given by the following diagram

$$\begin{array}{ccc} \bigoplus_{n=0}^{\infty} R & \xrightarrow{\psi_R^s} & \bigoplus_{n=1}^{\infty} R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ R & \xrightarrow{\quad l_R \quad} & R[s^{-1}] \end{array}$$

where p is the projection on the first factor and f is defined by $f(y_1, y_2, \dots) := -\sum_{n\geq 1} y_n/s^n$. Then $f \circ \psi_R^s = l_R \circ p$ and (p, f) is a quasi isomorphism. □

We can now define the left adjoint functor to the embedding functor

$$R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}} \xrightarrow{\iota} R\text{–}\mathsf{Mod}.$$

Let $\mathcal{D}^b(R)$ denote the bounded derived category of R and $\mathcal{K}(R)$ the homotopy category of R.

Theorem 1.3.3. [12, Theorem 6.4] *Let R be a commutative ring and $s \in R$ be an element. For any R-module C, the following R-modules are naturally isomorphic:*

(1) The cokernel of the R-module morphism

$$\phi_C^s := \operatorname{Hom}_R(\psi_R^s, C) : \prod_{n\geq 1} C \to \prod_{n\geq 0} C$$

defined by $\phi_C^s((c_1, c_2, c_3, \dots)) := (-sc_1, c_1 - sc_2, c_2 - sc_3, \dots)$;

(2) the cokernel of the endomorphism of the R-module $C[[z]]$ of formal power series in one variable z with coefficients in C

$$(z - s) : C[[z]] \to C[[z]]$$

which is the difference of the endomorphism of multiplication by z and the endomorphism of multiplication by s (induced by the multiplication by s in C);

(3) the R-module $\operatorname{Hom}_{\mathcal{D}^b(R)}(K^\bullet(s), C[1]) = H_0(\operatorname{Hom}_R(T^\bullet(s), C))$ (denoted by $\operatorname{Ext}^1_R(K^\bullet(s), C)$).

Proof. (1) ⇔ (2). The natural R-module isomorphisms $\prod_{n\geq 1} C \cong C[[z]] \cong \prod_{n\geq 0} C$ identifies the R-module morphism ϕ_C^s with the endomorphism $(z - s)$.

(1)⇔ (3). We have

$$\operatorname{Hom}_{\mathcal{D}^b(R)}(K^\bullet(s), C[1]) = \operatorname{Hom}_{\mathcal{K}(R)}(T^\bullet(s), C)$$

since the complex $T^\bullet(s)[1]$ is a projective resolution of $K^\bullet(s)$. Now, a cochain map $T^\bullet(s) \to C$ is an R-linear map $p : T^0 \to C$, and is homotopic to zero if, and only if it factors through ψ_R^s

$$\begin{array}{ccccc} 0 & \longrightarrow & \bigoplus_{n=0}^{\infty} R & \xrightarrow{\psi_R^s} & \bigoplus_{n=1}^{\infty} R \;. \\ \downarrow & & {\scriptstyle p}\downarrow & \swarrow{\scriptstyle g} & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

In other words

$$\operatorname{Hom}_{\mathcal{K}(R)}(T^\bullet(s), C) \cong \operatorname{Coker} \operatorname{Hom}_R(\psi_R^s, C) = \operatorname{Coker} \phi_C^s.$$

□

We will denote by $\Delta_s(C)$ the R-module described in Theorem 1.3.3.

Now we have a functor: $\Delta_s : R\text{–}\mathsf{Mod} \to R\text{–}\mathsf{Mod}$. Indeed this is a functor, since it is built using direct products and cokernels of morphisms (both functorial).

Proposition 1.3.4. *For any R-module C, $\Delta_s(C)$ is an s-contramodule R-module.*

Proof. Using the characterization of $\Delta_s(C)$ as $\mathrm{Hom}_{\mathcal{D}(R)}(K^\bullet(s), C[1])$ from Theorem 1.3.3 (3), we have that $\Delta_s(C)$ is an s-contramodule by Lemma 1.2.1 (2), since $H^{-1}(K^\bullet(s))$ ($=\mathrm{Ker}\, l_R$) and $H^0(K^\bullet(s))$ ($=\mathrm{Coker}\, l_R$) are s-torsion R-modules. □

The following theorem is of key importance and it will be used to prove that Δ_s is left adjoint to the embedding functor.

Proposition 1.3.5. *For any R-module C, there exists a 5-term exact sequence of R-modules:*

$$(\mathbf{c})\; 0 \to \mathrm{Hom}_R(\mathrm{Coker}\, l_R, C) \to \mathrm{Hom}_R(R[s^{-1}], C) \to C \overset{\delta^s_C}{\to} \Delta_s(C) \to \mathrm{Ext}^1_R(R[s^{-1}], C) \to 0.$$

where l_R is the localization map $R \to R[s^{-1}]$.

Proof. Apply the triangulated functor $\mathrm{Hom}_{\mathcal{D}^b(R)}(-, C)$ to the distinguished triangle

$$R \to R[s^{-1}] \to K^\bullet(s) \to R[1]$$

in $\mathcal{D}^b(R\text{–}\mathsf{Mod})$ to get the long exact sequence of R-modules:

$$0 \to \mathrm{Hom}_{\mathcal{D}^b(R)}(K^\bullet(s), C) \to \mathrm{Hom}_{\mathcal{D}^b(R)}(R[s^{-1}], C) \to C \to$$

$$\to \mathrm{Hom}_{\mathcal{D}^b(R)}(K^\bullet(s)[-1], C) \cong \Delta_s(C) \to \mathrm{Ext}^1_R(R[s^{-1}], C) \to 0.$$

By Lemma 1.3.2, $\mathrm{Hom}_{\mathcal{D}^b(R)}(K^\bullet(s), C) = \mathrm{Hom}_{\mathcal{K}(R)}(T^\bullet(s)[1], C)$. A map of complexes $T^\bullet(s)[1] \to C$ is null-homotopic if and only if $f = 0$.

$$\begin{array}{ccccc} \bigoplus_{n\geq 0} R & \xrightarrow{\psi^s_R} & \bigoplus_{n\geq 1} R & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Thus:

$$\mathrm{Hom}_{\mathcal{K}(R)}(T^\bullet(s)[1], C) \cong \mathrm{Hom}_R(\mathrm{Coker}\, \psi^s_R, C) \cong \mathrm{Hom}_R(\mathrm{Coker}\, l_R, C).$$

□

Corollary 1.3.6. *Let C be an R-module. Then C is an s-contramodule if, and only if the adjunction morphism $C \overset{\delta^s_C}{\to} \Delta_s(C)$ is an isomorphism.*

Proof. Immediate from Proposition 1.3.4 and the exact sequence in Proposition1.3.5. □

Proposition 1.3.7. [12, Theorem 6.4] $\Delta_s : R\text{–}\mathsf{Mod} \to R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}}$ *is left adjoint to the embedding functor $R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}} \to R\text{–}\mathsf{Mod}$.*

Proof. We need to check that for any R-module C and any s-contramodule R-module D, there is a natural isomorphism:

$$\mathrm{Hom}_{R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}}}(\Delta_s(C), D) \cong \mathrm{Hom}_R(C, D)$$

functorial both in C and D. We show the isomorphism.

Let $f : C \to D$ be an R-module morphism and consider the R-module morphism $\delta^s_C : C \to \Delta_s(C)$ given by sequence (**c**). δ^s_C is an s-divisible R-module while $\mathrm{Coker}\, \delta^s_C \cong \mathrm{Ext}^1_R(R[s^{-1}], C)$ is an $R[s^{-1}]$-module.

The restriction of f to $\mathrm{Ker}\, \delta^s_C$ is zero, whence there is a map $\bar{f} : C/\mathrm{Ker}\, \delta^s_C \to D$. Considering an $R[s^{-1}]$-module presentation of $\mathrm{Coker}\, \delta^s_C$ one sees that $\mathrm{Ext}^1_R(\mathrm{Coker}\, \delta^s_C, D) = 0$, since $\mathrm{Ext}^i_R(R[s^{-1}], D) = 0$, $i = 0, 1$. Applying the functor $\mathrm{Hom}_R(-, D)$ to the exact sequence

$$0 \to C/\mathrm{Ker}\, \delta^s_C \to \Delta_s(C) \to \mathrm{Coker}\, \delta^s_C \to 0$$

we obtain an isomorphism $\mathrm{Hom}_R(\Delta_s(C), D) \cong \mathrm{Hom}_R(C/\mathrm{Ker}\, \delta^s_C, D)$. By the diagram

$$\begin{array}{ccccccc}
\mathrm{Ker}\, \delta^s_C & \longrightarrow & C & \xrightarrow{\delta^s_C} & \Delta_s(C) & \longrightarrow & \mathrm{Coker}\, \delta^s_C \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \\
 & & C/\mathrm{Ker}\, \delta^s_C & \xrightarrow{\bar{f}} & D & &
\end{array}$$

there is a unique map $g \in \mathrm{Hom}_R(\Delta_s(C), D)$ such that $g\delta^s_C = f$. □

Remark 1.3.8. *Δ_s is left adjoint to an exact functor, hence it sends projective modules to projective objects of R–$\mathsf{Mod}_{\mathbf{s-ctr}}$. Thus $\Delta_s(R) = P$ is a projective object of R–$\mathsf{Mod}_{\mathbf{s-ctr}}$ and it is moreover a generator since $\mathrm{Hom}_{R\text{–}\mathsf{Mod}_{\mathbf{s-ctr}}}(\Delta_s(R), C) \cong \mathrm{Hom}_R(R, C) \cong C$, for any object $C \in R$–$\mathsf{Mod}_{\mathbf{s-ctr}}$. Furthermore, a coproduct $P^{(X)}$ of copies of P in R–$\mathsf{Mod}_{\mathbf{s-ctr}}$ is computed as $\Delta_s(R^{(X)})$, since Δ_s preserves coproducts.*

We look for relations between the functor Δ_s and the completion functor Λ_s.

Let C be an R-module. Consider the inverse systems

$$(1) \qquad \cdots \longrightarrow C/s^3C \longrightarrow C/s^2C \longrightarrow C/sC$$

and

$$(2) \qquad \cdots \longrightarrow {}_{s^3}C \longrightarrow {}_{s^2}C \longrightarrow {}_{s}C,$$

where for any element $r \in R$ we denote by ${}_rC \subset C$ the submodule of all the elements of C annihilated by r. The transition map ${}_{s^{n+1}}C \longrightarrow {}_{s^n}C$ acts by the multiplication with s.

The next theorem shows the relation between the functor Δ_s and the completion functor Λ.

Theorem 1.3.9. [12, Lemma 6.7] *Let R be a commutative ring and $s \in R$ be an element. For any R-module C there is a natural short exact sequence of R-modules:*

$$0 \to \varprojlim^1_{n \geq 1} {}_{s^n}C \to \Delta_s(C) \to \varprojlim_{n \geq 1} C/s^nC = \Lambda_s(C) \to 0.$$

Proof. The complex $K^\bullet(s)[-1]$ is the inductive limit of the system of complexes $K^\bullet_n(s) = (R \xrightarrow{s^n} R)$. For $n \geq 1$, denote by $T^\bullet_n(s)$ the subcomplex of $T^\bullet(s)$

$$\bigoplus_{i=0}^{n-1} R \xrightarrow{\psi_s} \bigoplus_{i=1}^{n} R$$

The complex $T^\bullet_n(s)$ is quasi-isomorphic to $(R \xrightarrow{s^n} R)$ (in degrees $0, 1$).

The complexes $\operatorname{Hom}_R(T^\bullet_n(s), C)$ form a projective system of complexes with surjective structure maps. By standard arguments we obtain the short exact sequence of R-modules

$$0 \to \varprojlim^1_{n\geq 1} H_1(\operatorname{Hom}_R(T^\bullet_n(s), C)) \to H_0(\varprojlim_{n\geq 1} \operatorname{Hom}_R(T^\bullet_n(s), C)) \to \varprojlim_{n\geq 1} H_0(\operatorname{Hom}_R(T^\bullet_n(s), C)) \to 0$$

where:

- $H_1(\operatorname{Hom}_R(T^\bullet_n(s), C)) = H_1(\operatorname{Hom}_R(K^\bullet_n(s), C) = \operatorname{Ker}(C \xrightarrow{r^n} C) = {}_{s^n}C$;
- $H_0(\varprojlim_{n\geq 1} \operatorname{Hom}_R(T^\bullet_n(s), C)) = H_0(\operatorname{Hom}_R(\varinjlim_{n\geq 1} T^\bullet_n(s), C)) = H_0(\operatorname{Hom}_R(T^\bullet(s), C)) = \Delta_s(C)$;
- $H_0(\operatorname{Hom}_R(T^\bullet_n(s), C)) = H_0(\operatorname{Hom}_R(R \xrightarrow{s^n} R, C)) = \operatorname{Coker}(C \xrightarrow{s^n} C) = C/s^nC$.

□

A module C is said to be of bounded s-torsion if there exists $m \geq 1$ such that $s^n c = 0$ implies $s^m c = 0$ for every $n \geq 1$ and every $c \in C$.

Remark 1.3.10. Recall that an inverse system $\{M_n; f_{nk}, k \geq n\}$ of R-modules satisfies the trivial Mittag-Leffler condition if and only if for every $n \geq 1$ there is $k \geq n$ such that $M_k \xrightarrow{f_{nk}} M_n$ is the zero map.

Corollary 1.3.11. *Let R be a ring and $s \in R$ be an element. Let C be an R-module of bounded s-torsion. Then*

$$\varprojlim^1_{n\geq 1} {}_{s^n}C = 0, \qquad \Delta_s(C) \cong \Lambda_s(C).$$

Proof. If C has bounded s-torsion, the inverse system (2) satisfies the trivial Mittag-Leffler condition, hence $\varprojlim^1_n {}_{s^n}C = 0$. □

2. SECOND LECTURE

Given a ring R, one can define (left) R-modules in the following fancy way. For any set X, let $R[X]$ denote the set of all finite formal linear combinations of the elements of X with coefficients in R. We have the obvious embedding $X \to R[X]$, and the opening of parentheses map $R[R[X]] \to R[X]$ which makes the functor $X \to R[X]$ a monad on the category of sets (see the definition below). The R-modules are the algebras/modules over this monad.

Recall that a **monad** on the category of sets is a covariant functor

$$\mathbb{T}\colon \mathsf{Sets} \to \mathsf{Sets}$$

endowed with natural transformations $\epsilon\colon \mathrm{Id} \longrightarrow \mathbb{T}$ and multiplication $\phi\colon \mathbb{T}\circ \mathbb{T} \longrightarrow \mathbb{T}$ satisfying the equations of *associativity*

$$\mathbb{T}\circ\mathbb{T}\circ\mathbb{T} \rightrightarrows \mathbb{T}\circ\mathbb{T} \to \mathbb{T}, \quad \phi(\phi\circ\mathbb{T}) = \phi(\mathbb{T}\circ\phi)$$

unitality

$$\mathbb{T} \rightrightarrows \mathbb{T}\circ\mathbb{T} \to \mathbb{T}, \quad \phi(\mathbb{T}\circ\epsilon) = \phi(\epsilon\circ\mathbb{T}) = \mathrm{Id}_{\mathsf{Sets}}\,.$$

2.1. **Topological rings** ([9, Section 1]**).**

Let $\mathfrak{R}$ be a complete, separated topological ring with a base $\mathfrak{B}$ of neighbourhoods of zero formed by open right ideals $\mathfrak{I}$.

- Given a set X, denote by $\mathfrak{R}[[X]]$ the set of all infinite formal linear combinations $\sum_{x\in X} r_x x$ of elements of X with the coefficients in $\mathfrak{R}$ such that the *X-indexed family of elements r_x converges to zero in the topology of* $\mathfrak{R}$. This means that the set $\mathfrak{R}[[X]] \subset \mathfrak{R}^X$ consists of all the infinite formal linear combinations $\sum_{x\in X} r_x x$ such that, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, $r_x \in \mathfrak{I}$ for *all but a finite number of indices $x \in X$*.
- In other words, $\mathfrak{R}[[X]] = \varprojlim_{\mathfrak{I}\in\mathfrak{B}} (\mathfrak{R}/\mathfrak{I})[X]$. $\mathfrak{R}[[X]]$ does not depend on the choice of the basis $\mathfrak{B}$.
- The map assigning to a set X the set $\mathfrak{R}[[X]]$ extends naturally to a covariant functor $\mathbb{T}_{\mathfrak{R}}$ from the category of sets to the category of sets. Given a map of sets $f\colon X \longrightarrow Y$, one defines the induced map $\mathfrak{R}[[f]]\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{R}[[Y]]$ by the rule $\sum_{x\in X} r_x x \longmapsto \sum_{y\in Y}\big(\sum_{f(x)=y} r_x\big)y$, where the sum of elements r_x in the parentheses is understood to be the limit of finite partial sums in the topology of $\mathfrak{R}$. Such a limit is unique and exists because the topological ring $\mathfrak{R}$ is separated and complete, while the family of elements $(r_x)_{x\in X}$, and consequently its subfamily indexed by all $x \in X$ with $f(x) = y$ for a fixed $y \in Y$, converges to zero in $\mathfrak{R}$.
- The functor $\mathbb{T}_{\mathfrak{R}}$ is endowed with natural transformations

  $$\epsilon\colon \mathrm{Id} \longrightarrow \mathbb{T}_{\mathfrak{R}},\ \phi\colon \mathbb{T}_{\mathfrak{R}}\circ\mathbb{T}_{\mathfrak{R}} \longrightarrow \mathbb{T}_{\mathfrak{R}}$$

  satisfying the monad equations.
  The monad unit $\epsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$ is the "point measure" map, assigning to an element $x_0 \in X$ the (finite) formal linear combination $\sum_{x\in X} r_x x \in \mathfrak{R}[[X]]$, where $r_{x_0} = 1$ and $r_x = 0$ for all $x \neq x_0$.
  The monad multiplication $\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ is the "opening of parentheses" defined by

  $$\sum_{y\in\mathfrak{R}[[X]]} r_y y \mapsto \sum_{x\in X}\left(\sum_{y\in\mathfrak{R}[[X]]} r_y r_{yx}\right)x, \qquad \text{where } y = \sum_{x\in X} r_{yx} x.$$

  The sum $\sum_{y\in\mathfrak{R}[[X]]} r_y r_{yx}$ converges in $\mathfrak{R}$, since the family r_y converges to zero and $\mathfrak{R}$ is complete.
  (For every open right ideal $\mathfrak{I}$, one has $r_y r_{yx} \in \mathfrak{I}$ whenever $r_y \in \mathfrak{I}$; and there is only a finite set of indices y with $r_y \notin \mathfrak{I}$, because $\sum_{y\in\mathfrak{R}[[X]]} r_y y \in \mathfrak{R}[[Y]]$, where $Y = \mathfrak{R}[[X]]$)

2.2. **$\mathfrak{R}$-contramodules ([9, Section 1].**
Recall that a module over a monad $\mathbb{T}\colon \mathsf{Sets} \to \mathsf{Sets}$ is a set C endowed with a map of sets $\pi_C\colon \mathbb{T}(C) \longrightarrow C$, called the *action map* satisfying the equations of associativity

$$\mathbb{T}(\mathbb{T}(C)) \rightrightarrows \mathbb{T}(C) \to C, \quad \pi_C \circ \phi_C = \pi_C \circ \mathbb{T}(\pi_C),$$

and unitality

$$C \to \mathbb{T}(C) \to C, \quad \pi_C \circ \epsilon_C = \mathrm{id}_C\,.$$

A left $\mathfrak{R}$-contramodule is a module over the monad $\mathbb{T}_{\mathfrak{R}}\colon X \longmapsto \mathfrak{R}[[X]]$ on the category of sets. This means that a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is a set endowed with a *left contraaction map* $\pi_{\mathfrak{C}}\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ satisfying the following conditions

$$\pi_{\mathfrak{C}} \circ \epsilon_{\mathfrak{C}} = \mathrm{id}_{\mathfrak{C}} : \quad \mathfrak{C} \xrightarrow{\epsilon_{\mathfrak{C}}} \mathfrak{R}[[\mathfrak{C}]] \xrightarrow{\pi_{\mathfrak{C}}} \mathfrak{C}$$

$$\pi_{\mathfrak{C}} \circ \phi_{\mathfrak{C}} = \pi_{\mathfrak{C}} \circ \mathfrak{R}[[\pi_{\mathfrak{C}}]] : \quad \mathfrak{R}[[\mathfrak{R}[[\mathfrak{C}]]]] \underset{\mathfrak{R}[[\pi_{\mathfrak{C}}]]}{\overset{\phi_{\mathfrak{C}}}{\rightrightarrows}} \mathfrak{R}[[\mathfrak{C}]] \xrightarrow{\pi_{\mathfrak{C}}} \mathfrak{C}$$

In other words, a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ can be defined as a set endowed with the following infinite summation operations.

For any family of elements r_α converging to zero in $\mathfrak{R}$ and any family of elements $c_\alpha \in \mathfrak{C}$ there is a well-defined element $\sum_\alpha r_\alpha c_\alpha \in \mathfrak{C}$ satisfying

(a) *contraassociativity*

$$\sum_\alpha r_\alpha \sum_\beta r_{\alpha\beta} c_{\alpha\beta} = \sum_{\alpha,\beta} (r_\alpha r_{\alpha\beta}) c_{\alpha\beta} \text{ if } r_\alpha \to 0 \text{ and } r_{\alpha\beta} \to 0 \ \text{ in } \ \mathfrak{R}, \forall \alpha.$$

(b) the *contraunitality*:

$$\sum_{\alpha \in A} r_\alpha c_\alpha = c_{\alpha_0}, \text{ if } A = \{\alpha_0\} \text{ and } \ r_{\alpha_0} = 1,$$

(c) *distributivity*

$$\sum_{\alpha,\beta} r_{\alpha\beta} c_\alpha = \sum_\alpha \left(\sum_\beta r_{\alpha\beta} \right) c_\alpha \quad \text{ if } r_{\alpha\beta} \to 0 \in \mathfrak{R}.$$

(d) The finite and infinite operations are compatible.

A morphism of $\mathfrak{R}$-contramodules $\mathfrak{C}$ and $\mathfrak{D}$ is a map f such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{R}[[\mathfrak{C}]] & \xrightarrow{\mathfrak{R}[[f]]} & \mathfrak{R}[[\mathfrak{D}]] \\ \downarrow{\scriptstyle \pi_{\mathfrak{C}}} & & \downarrow{\scriptstyle \pi_{\mathfrak{D}}} \\ \mathfrak{C} & \xrightarrow[f]{} & \mathfrak{D} \end{array}$$

We denote the category of left $\mathfrak{R}$-contramodules by $\mathfrak{R}$–contra.

2.3. **The category** $\mathfrak{R}$–contra
([9, Section1], [16, Section 1, 5], [17, Section 6.4], [14, Section 1]).

(1) For any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, let $\mathfrak{R}[\mathfrak{C}]$ be the set of all finite formal linear combinations. The composition

$$\mathfrak{R}[\mathfrak{C}] \longrightarrow \mathfrak{R}[[\mathfrak{C}]] \xrightarrow{\pi_{\mathfrak{C}}} \mathfrak{C}$$

defines a natural structure of a left $\mathfrak{R}$*-module on* $\mathfrak{C}$.
   • In particular, it means that all left $\mathfrak{R}$-contramodules, which were originally defined as only sets endowed with a contraaction map, are actually abelian groups, even $\mathfrak{R}$-modules.

(2) Using the above identities (a)–(d) , one can define the $\mathfrak{R}$-contramodule structure on the kernel and cokernel of an $\mathfrak{R}$-contramodule morphism taken in the category of $\mathfrak{R}$-modules. Hence $\mathfrak{R}$–contra is an abelian category and the forgetful functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow \mathfrak{R}$–$\mathsf{Mod}$ is an exact functor.

(3) For any set X the set $\mathfrak{R}[[X]]$ has a natural structure of an $\mathfrak{R}$-contramodule: The monad multiplication $\phi_X \colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ plays the rôle of the contraaction map.

(4) Let $\mathbb{T}_{\mathfrak{R}}$ be the functor sending a set X to $\mathfrak{R}[[X]]$. Then $\mathbb{T}_{\mathfrak{R}}$ is left adjoint to the forgetful functor from $\mathfrak{R}$–contra to the category of sets.

$$\mathsf{Sets} \underset{\text{Forget}}{\overset{\mathbb{T}_{\mathfrak{R}}}{\rightleftarrows}} \mathfrak{R}\text{–}\mathsf{contra}$$

   Hence, contramodules of the form $\mathfrak{R}[[X]]$ are the free $\mathfrak{R}$-contramodules. They are projective objects in the abelian category $\mathfrak{R}$–contra, there are enough of them, and hence every projective $\mathfrak{R}$-contramodule is a direct summand of a free $\mathfrak{R}$-contramodule.

(5) For any collection of sets X_α, the free contramodule $\mathfrak{R}[[\coprod_\alpha X_\alpha]]$ generated by the disjoint union of X_α is the direct sum of the free contramodules $\mathfrak{R}[[X_\alpha]]$ in the category $\mathfrak{R}$–contra. This allows to compute, the direct sum of $\mathfrak{R}$-contramodules, by presenting them as cokernels of morphisms of free contramodules and using the fact that infinite direct sums commute with cokernels. So infinite direct sums exist in $\mathfrak{R}$–contra.

(6) The category $\mathfrak{R}$–contra is cocomplete, with a projective generator $\mathfrak{R} = \mathbb{T}_{\mathfrak{R}}(*)$, hence it is also complete. The forgetful functor $\mathfrak{R}$–$\mathsf{contra} \longrightarrow \mathfrak{R}$–$\mathsf{Mod}$ preserves infinite products (but not coproducts).

(7) The category $\mathfrak{R}$–contra has the additional property that for every family of projective objects $P_\alpha \in \mathfrak{R}$–$\mathsf{contra}$, the natural morphism $\coprod_\alpha P_\alpha \to \prod_\alpha P_\alpha$ is a monomorphism. This is because the property can be checked for free $\mathfrak{R}$-contramodules.

(8) The monad $\mathbb{T}_{\mathfrak{R}}$ is additive meaning that the category of $\mathbb{T}_{\mathfrak{R}}$-modules is an additive category (see [14, Lemma 1.1]).

(9) (See below for the terminology) Let λ^+ be the successor cardinal of the cardinality of a base of neighbourhoods of zero in $\mathfrak{R}$. The monad $\mathbb{T}_{\mathfrak{R}}$ is λ^+-accessible, meaning that it preserves λ^+-filtered colimits.

So the abelian category $\mathfrak{R}$–contra is locally λ^+-presentable with a natural λ^+-presentable projective generator, which is the free left $\mathfrak{R}$-contramodule with one generator $\mathfrak{R} = \mathfrak{R}[[*]]$.

Recall:
• Let κ be a regular cardinal. A poset is κ-directed if every subset of cardinality smaller than κ has an upper bound. A colimit of a diagram indexed by a κ-directed poset is called κ-directed colimit.
• An object C of a category $\mathcal{C}$ is called κ-presentable if $\mathrm{Hom}_{\mathcal{C}}(C, -)$ preserves κ-directed colimits.
• A category is called *locally κ-presentable* if it is cocomplete and has a set $\mathcal{A}$ of κ-presentable objects such that every object is a κ-directed colimit of objects from $\mathcal{A}$.

Proposition 2.3.1. [16, Section 1], [17, Section 6.4] *Every cocomplete abelian category $\mathcal{B}$ with a projective generator P is equivalent to the category of modules over the monad*

$$\mathbb{T}_P \colon \mathsf{Sets} \to \mathsf{Sets}; \quad X \mapsto \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)}).$$

Proof. (shortly) First we describe the monad multiplication

$$\mathbb{T}_P \circ \mathbb{T}_P(X) = \mathrm{Hom}_{\mathcal{B}}(P, P^{(\mathbb{T}(X))}) \to \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)}) = \mathbb{T}_P(X).$$

Note that $\mathrm{Hom}_{\mathcal{B}}(P^{(Y)}, P^{(X)})$ is computed as $\mathrm{Hom}_{\mathcal{B}}(P, P^{(X)})^Y = \mathbb{T}_P(X)^Y$. Let $f \colon Y \to \mathbb{T}_P(X)$ be viewed as a map $h \colon P^{(Y)} \to P^{(X)}$. Then

$$\mathrm{Hom}_{\mathcal{B}}(P, h) \colon \mathrm{Hom}_{\mathcal{B}}(P, P^{(Y)}) \to \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)}).$$

Letting $Y = \mathbb{T}_P(X)$ and f the identity we get the monad multiplication. The equivalence is given by assigning to every $N \in \mathcal{B}$ the set $\mathrm{Hom}_{\mathcal{B}}(P, N)$with its natural structure as a $\mathbb{T}_P$-module.
Indeed, every $t \in \mathbb{T}_P(X) = \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)})$ induces an X-ary operation on $\mathrm{Hom}_{\mathcal{B}}(P, N)$:

$$\mathrm{Hom}_{\mathcal{B}}(P, N)^X = \mathrm{Hom}_{\mathcal{B}}(P^{(X)}, N) \to \mathrm{Hom}_{\mathcal{B}}(P, N),$$
$$(f \colon P^{(X)} \to N) \mapsto f \circ t.$$

□

- The functors of infinite direct sum are not exact in $\mathfrak{R}$–contra in general; they are not exact already for the ring $\mathfrak{R} = k[[z, t]]$ of formal power series in two variables over a field k.
- When $\mathfrak{R}$–contra has global homological dimension at most 1, the infinite direct sums in $\mathfrak{R}$–contra are exact.

2.4. The category of I-contramodules for a finitely generated ideal I of a commutative ring R ([12, Section 7]).

Definition 2.4.1. [12, Section 7] *Let R be a commutative ring and $I \le R$ be the ideal generated by a finite set of elements $s_1, \dots, s_m \in R$. An R-module C is said to be an I-**contramodule** if $\mathrm{Hom}_R(R[s_j^{-1}], C) = 0 = \mathrm{Ext}^1_R(R[s_j^{-1}], C)$ for all $1 \le j \le m$.*

This property does not depend on the chosen set of generators, but only on the ideal I.

Properties of the category of I-contramodules
Denote by R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ the full subcategory of R–Mod consisting of I-contramodule R-modules.

(i) I-contramodules are the objects of the $0,1$-perpendicular category of $\oplus_{j=1}^{m} R[s_j^{-1}]$. Since the projective dimension of $R[s_j^{-1}]$ is at most one for every j, the category of I-contramodules is closed under kernels, cokernels, extensions and products in R–Mod, i.e. it is exactly embedded in R–Mod.

(ii) Every R-module annihilated by I^n for some $n \geq 1$ is an I-contramodule.

(iii) Let $E = \oplus_{j=1}^{m} R[s_j^{-1}]$, and let $f\colon U^{-1} \to U^0$ be a two terms free resolution of E. Then R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ coincides with the full subcategory $f^{\perp}$, consisting of the R-modules C such that $\mathrm{Hom}_R(f, C)$ is an isomorphism.

Theorem 2.4.2. [12, Theorem 7.2], [14, Example 2.2 (1)] *Let R be a commutative ring and $I \leq R$ be the ideal generated by a finite set of elements $s_1, \dots, s_m \in R$. The following hold:*

(1) The exact embedding functor R–$\mathsf{Mod}_{\mathbf{I-ctr}} \to R$–$\mathsf{Mod}$ has a left adjoint functor $\Delta_I\colon R$–$\mathsf{Mod}_{\mathbf{I-ctr}} \to R$–$\mathsf{Mod}$ given by $\Delta_{s_m} \dots \Delta_{s_2}\Delta_{s_1}$ (Δ_{s_i} from First Lecture).

(2) $\Delta_I(R) = P$ is a projective generator of R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ and the coproduct of X copies of P in R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is given by $\Delta_I(R[X])$.

(3) The abelian category R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is equivalent to the category of modules over the additive monad $\mathbb{T}_I$ assigning to every set X the underlying set of the R-module $\Delta_I(R[X])$.

Proof. (1) is a generalization of the 1-element case: Δ_s is left adjoint to the exact embedding R–$\mathsf{Mod}_{\mathbf{s-ctr}} \to R$–$\mathsf{Mod}$.

(2) Follows from the fact that Δ_I is left adjoint to an exact functor (see also Remark 1.3.8).

(3) Let $\mathcal{B} = R$–$\mathsf{Mod}_{\mathbf{I-ctr}}$. By (1) and (2) $\mathcal{B}$ is a cocomplete category with a projective generator $P = \Delta_I(R)$. By Proposition 2.3.1, $\mathcal{B}$ is equivalent to the category of modules over the monad $\mathbb{T}_P$.
Now $\mathbb{T}_P(X) = \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)}) \cong \mathrm{Hom}_R(R, \Delta_I(R[X])) \cong \Delta_I(R[X])$, thus $\mathbb{T}_P \cong \mathbb{T}_I$. □

The functor Δ_I and the I-adic completion functor.
Fix a finitely generated ideal I of a commutative ring R.

Notation 2.4.3.

(1) For every R-module C denote by $\Lambda_I(C) = \varprojlim_n C/I^nC$ the I-adic completion of C. By [12, Lemma 5.7 and Theorem 5.8] $\Lambda_I(C)$ is an I-contramodule and it is separated and complete in the I-adic topology which coincides with the projective limit topology.

(2) Denote by $\mathfrak{R} = \varprojlim_n R/I^n$ the I-adic completion of the ring R endowed with the I-adic topology.

We generalize the complex $T^\bullet(s)$, quasi isomorphic to $R \to R[s^{-1}]$ and its subcomplex $T_n^\bullet(s)$, quasi isomorphic to $R \xrightarrow{s^n} R$ from the first lecture, by setting

(3)
(a) $T^\bullet(s_1,\dots,s_m) = T^\bullet(s_1)\otimes_R \cdots \otimes_R T^\bullet(s_m)$
in cohomological degrees $0,\dots,m$, quasi isomorphic to
$(R \to R[s_1^{-1}])\otimes_R \cdots \otimes (R \to R[s_m^{-1}])$, and
(b) $T_n^\bullet(s_1,\dots,s_m) = T_n^\bullet(s_1)\otimes_R \cdots \otimes_R T_n^\bullet(s_m)$,

(4) then, $T^\bullet(s_1,\dots,s_m) = \varinjlim_{n\geq 1} T_n^\bullet(s_1,\dots,s_m)$, and

$$\Delta_I(C) = H^0(\mathrm{Hom}_R(T^\bullet(s_1,\dots,s_m),C).$$

Proposition 2.4.4. [12, Lemma7.5] *In Notations 2.4.3 there is a natural short exact sequence of R-modules:*

$$0 \to \varprojlim^1_{n\geq 1} H_1(\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C) \to \Delta_I(C) \to \Lambda_I(C) \to 0.$$

In particular, for every R-module C, there is a natural surjection

$$\Delta_I(C) \to \Lambda_I(C).$$

Proof. The complexes $\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)$ form a countable projective system with surjective maps in each degree. Hence there is a short exact sequence

$$0 \to \varprojlim^1_{n\geq 1} H_1(\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \to H_0(\varprojlim_{n\geq 1} \mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \to$$
$$\to \varprojlim_{n\geq 1} H_0(\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \to 0.$$

Now we have:

- $H_0(\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \cong C/(s_1^n,\dots,s_m^n)C$, hence $\varprojlim_{n\geq 1} H_0(\mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \cong \Lambda_I(C)$,
- $H_0(\varprojlim_{n\geq 1} \mathrm{Hom}_R(T_n^\bullet(s_1,\dots,s_m),C)) \cong H_0(\mathrm{Hom}_R(T^\bullet(s_1,\dots,s_m),C)) \cong \Delta_I(C)$.

□

Proposition 2.4.5. [14, Example 2.2 (2)] *In Notations 2.4.3, the forgetful functor* $\mathfrak{R}$–$\mathsf{contra} \to R$–Mod *is fully faithful and its image is contained in the full subcategory of I-contramodule R-modules.*

Proof. For the claim about the image it suffices to check that the free $\mathfrak{R}$-contramodules are I-contramodule R-modules, as every $\mathfrak{R}$-contramodule is the cokernel of a morphism of free $\mathfrak{R}$-contramodules and R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is closed under cokernels. For any set X, the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ coincides with $\varprojlim_{n\geq 1} R/I^n[X]$ which is an I-contramodule by Notation 2.4.3 (1).

Fully faithfulness: the abelian category $\mathfrak{R}$–contra is the category of modules over the monad $\mathbb{T}_{\mathfrak{R}}\colon X \to \mathfrak{R}[[X]]$ and

$$\mathfrak{R}[[X]] = \varprojlim_{n\geq 1}(R/I^nR)[X] = \Lambda_I(R[X]),$$

while the abelian category R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is the category of modules over the monad $X \to \Delta_I(R[X])$, by Theorem 2.4.2 (3).

The functor $\mathfrak{R}$–$\mathsf{contra} \to R$–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is induced by the morphism of monads $\Delta_I(R[X]) \to \Lambda_I(R[X])$, and surjectivity of this map for every set X implies that the forgetful functor is fully faithful.

The idea is that the forgetful functor can be seen as a "restriction functor" by means of the surjection $\Delta_I(R[X]) \to \Lambda_I(R[X])$. □

Consequence: The abelian category $\mathfrak{R}$–contra is a full subcategory of the abelian category R–$\mathsf{Mod}_{\mathbf{I-ctr}}$.

2.5. When is $\mathfrak{R}$–contra equivalent to R–$\mathsf{Mod}_{\mathbf{I-ctr}}$?

Proposition 2.5.1. [14, Proposition 2.1] *The forgetful functor $\mathfrak{R}$–contra $\to$ R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is an equivalence of abelian categories if and only if the natural morphism*

$$\Delta_I(R[X]) \to \Lambda_I(R[X])$$

is an isomorphism for every set X.

By Proposition 2.4.4, the kernel of the natural morphism in the above statement is

$$\varprojlim\nolimits^1_{n\geq 1} H_1(\mathrm{Hom}_R(T^\bullet_n(s_1,\dots,s_m),R[X]),$$

and a sufficient condition for the kernel to be zero is that I is **weakly proregular**.

We follow the notations in [20]. For any sequence $\mathbf{s} = (s_1, s_2, \dots, s_m)$ of elements of R denote by $K^\bullet(\mathbf{s})$ the Koszul complex $K^\bullet(s_1, s_2, \dots, s_m)$ in degrees $-m, \dots, 0$. Moreover, for every $n \geq 1$, write $\mathbf{s}^n = (s_1^n, s_2^n, \dots, s_m^n)$ and denote by $K^\bullet(\mathbf{s}^n)$ the Koszul complex $K^\bullet(s_1^n, s_2^n, \dots, s_m^n)$. Then for every $k \geq n$, there are morphisms $K^\bullet(\mathbf{s}^k) \to K^\bullet(\mathbf{s}^n)$.

Definition 2.5.2. [19], [20] *An ideal $I = (s_1, \dots, s_m)$ is* **weakly proregular** *if the inverse systems of the Koszul cohomology modules $\{H^i(K^\bullet(\mathbf{s}^n))\}_{n\geq 1}$ are pro-zero for every $i = -m, \dots, -1$ meaning that they satisfy the trivial Mittag-Leffler condition* (see Remark 1.3.10).

- If R is noetherian every finitely generated ideal is weakly proregular.
- If $I = (s)$ then I is weakly proregular if and only if the s-torsion of R is bounded, i.e. there is $n \geq 1$ such that the s-torsion submodule of R is annihilated by s^n.

By the remarks above, we conclude that

Proposition 2.5.3. *The forgetful functor $\mathfrak{R}$–contra $\to$ R–$\mathsf{Mod}_{\mathbf{I-ctr}}$ is an equivalence of abelian categories for any weakly proregular finitely generated ideal I in a commutative ring R.*

2.6. The case of a multiplicative subset S of R with p.dim $R_S \leq 1$ ([13], [14, Section 2]).

Let R be a commutative ring and S a multiplicative subset of R such that the projective dimension (p.dim) of the localization R_S is at most one.

An R-module M is S-torsion if for every $x \in M$ there is $s \in S$ such that $xs = 0$ (i.e. $M \otimes_R R_S = 0$) and it is S-divisible if $Ms = M$ for every $s \in S$.

Definition 2.6.1. *An R-module C is said to be an* S**-contramodule** *if*

$$\mathrm{Hom}_R(R_S, C) = 0 = \mathrm{Ext}^1_R(R_S, C).$$

Denote by R–$\mathsf{Mod}_{\mathbf{S-ctr}}$ the full subcategory of R–Mod consisting of S-contramodule R-modules.

- Let $K_S^\bullet$ be the complex $R \overset{l_S}{\to} R_S$ in cohomological degrees $-1, 0$.
- For every R-module C, let $\Delta_S(C) = \mathrm{Hom}_{\mathcal{D}^b(R)}(K_S^\bullet, C[1])$.
- A generalization of Lemma 1.2.1 shows that $\Delta_S(C)$ is an S-contramodule, since the modules $H^n(K^\bullet)$ are S-torsion for every $n \in \mathbb{Z}$.

Proposition 2.6.2. [14, Example 2.4]

(1) The category R–$\mathsf{Mod}_{\mathbf{S-ctr}}$ is an abelian category exactly embedded in R–Mod.
(2) The functor Δ_S is left adjoint to the embedding functor R–$\mathsf{Mod}_{\mathbf{S-ctr}} \to R$–$\mathsf{Mod}$, hence $\Delta_S(R)$ is a projective generator of R–$\mathsf{Mod}_{\mathbf{S-ctr}}$.
(3) The abelian category R–$\mathsf{Mod}_{\mathbf{S-ctr}}$ is equivalent to the category of modules over the additive monad $\mathbb{T}$ assigning $\Delta_S(R[X])$ to every set X.

Proof. (1) follows from the assumption p.dim $R_S \leq 1$.

(2) From the triangle $R \to R_S \to K_S^\bullet \to R[1]$ we get the exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Coker}\, l_S, C) \cong \mathrm{Hom}_{\mathcal{D}^b(R)}(K_S^\bullet, C) \to \mathrm{Hom}_R(R_S, C) \to C \overset{\delta_C^S}{\to}$$

$$\overset{\delta_C^S}{\to} \mathrm{Hom}_{\mathcal{D}^b(R)}(K_S^\bullet, C[1]) = \Delta_S(C) \to \mathrm{Ext}_R^1(R_S, C) \to 0,$$

where $\mathrm{Ker}\, \delta_C^S$ is an S-divisible R-module while $\mathrm{Coker}\, \delta_C^S \cong \mathrm{Ext}_R^1(R_S, C)$ is an R_S-module.

Then the proof continues arguing similarly to the case of the localization $R[s^{-1}]$ at a single element $s \in R$ (see Proposition 1.3.7).

(3) We argue as in Theorem 2.4.2. Let $\mathcal{B} = R$–$\mathsf{Mod}_{\mathbf{S-ctr}}$. By (2) $\mathcal{B}$ is a cocomplete category with a projective generator $P = \Delta_S(R)$, hence by Proposition 2.3.1 $\mathcal{B}$ is equivalent to category of modules over the monad $\mathbb{T}_P \colon X \to \mathrm{Hom}_{\mathcal{B}}(P, P^{(X)})$.

Now $\mathrm{Hom}_{\mathcal{B}}(P, P^{(X)}) \cong \mathrm{Hom}_R(R, \Delta_S(R[X])) \cong \Delta_S(R[X])$. □

Denote by $\mathfrak{R}$ the ring $\varprojlim_{s \in S} R/sR$, the S-completion of the ring R.

(i) $\mathfrak{R}$ endowed with the **projective limit topology** is a complete, separated topological commutative ring.
(ii) For every R-module C, let $\Lambda_S(C) = \varprojlim_{s \in S} C/sC$ and let $\lambda_S(C) \colon C \to \varprojlim_{s \in S} C/sC$ the canonical map. The following hold:
  (1) $\Lambda_S(C)$ is an S-contramodule (use that C/sC is an S-contramodule for every $s \in S$ and the closure properties of R–$\mathsf{Mod}_{\mathbf{S-ctr}}$).
  (2) For every R module C there is a unique R-module morphism $\beta_S(C) \colon \Delta_S(C) \to \Lambda_S(C)$ forming a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\delta_C^S} & \Delta_S(C) \\ & \searrow^{\lambda_C^S} & \downarrow{\scriptstyle \beta_S(C)} \\ & & \Lambda_S(C) \end{array}$$

(analogously to the 1-element case $s \in R$, see the proof of Proposition 1.3.7 and [13, Lemma 2.1 (b)]).

Proposition 2.6.3. [13, Theorem 2.5] *The natural morphism*

$$\beta_S(C)\colon \Delta_S(C) \to \Lambda_S(C)$$

is an isomorphism provided that the S-torsion of C is bounded, i.e. the S-torsion submodule of C is annihilated by some element $s \in S$.

Theorem 2.6.4. [14, Example 2.4]

(1) The image of the forgetful functor $\mathfrak{R}$–contra $\to R$–Mod is contained in the full subcategory R–$\mathsf{Mod}_{\mathbf{S-ctr}}$.

(2) The forgetful functor $\mathfrak{R}$–contra $\to R$–$\mathsf{Mod}_{\mathbf{S-ctr}}$ is an equivalence of categories if and only if the natural morphism $\Delta_S(R[X]) \cong \Lambda_S(R[X])$ is an isomorphism for every set X.

Proof. (1) It suffices to check that the free $\mathfrak{R}$-contramodules are S-contramodule R-modules, since every $\mathfrak{R}$-contramodule is the cokernel of a morphism of free $\mathfrak{R}$-contramodules and R–$\mathsf{Mod}_{\mathbf{S-ctr}}$ is closed under cokernels.

For any set X, the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]] = \varprojlim_{s\in S} R/sR[X]$ is an S-contramodule R-module (by (ii) (1).)

(2) Follows from [14, Proposition 2.1]. □

Combining Proposition 2.6.3 with Theorem 2.6.4 (2) we obtain:

Theorem 2.6.5. *If the S-torsion of R is bounded, the forgetful functor $\mathfrak{R}$–contra $\to R$–Mod is an equivalence of categories.*

3. Third Lecture

3.1. n-tilting objects.

Notation 3.1.1. *Let $\mathcal{A}$ be an abelian category with coproducts, and $\mathcal{B}$ be an abelian category with products. For any object $T \in \mathcal{A}$ we denote by $\mathsf{Add}(T) = \mathsf{Add}_{\mathcal{A}}(T) \subset \mathcal{A}$ the class of all direct summands of the coproducts $T^{(X)}$ of copies of T in $\mathcal{A}$. For any object $W \in \mathcal{B}$ we denote by $\mathsf{Prod}(W) = \mathsf{Prod}_{\mathcal{B}}(W) \subset \mathcal{B}$ the class of all direct summands of the products W^X of copies of W in $\mathcal{B}$.*

In the sequel $\mathcal{A}$ will be a complete, cocomplete abelian category with an injective cogenerator.

Definition 3.1.2. [17, Pages 5-6] *Let $n \geq 0$. An object $T \in \mathcal{A}$ is an* **n-tilting** *object if*

(i) the projective dimension of T is at most n, that is $\mathrm{Ext}^i_{\mathcal{A}}(T, A) = 0$ for all $A \in \mathcal{A}$ and $i > n$;

(ii) $\mathrm{Ext}^i_{\mathcal{A}}(T, T^{(X)}) = 0$ for all $i > 0$, for any set X;

(iii) every $X^\bullet \in \mathcal{D}(\mathcal{A})$ such that $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(T, X[i]) = 0$ for all $i \in \mathbb{Z}$ is acyclic.

Assume that T satisfies (i) and (ii), let

$$\mathcal{E} = \{E \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(T, E) = 0, \text{ for all } i > 0\},$$

then, $\mathcal{A}_{\mathsf{inj}}$, the class of injective objects of $\mathcal{A}$, is contained in $\mathcal{E}$ and by the condition (ii), $\mathsf{Add}_{\mathcal{A}}(T) \subset \mathcal{E}$.

For each integer $m \geq 0$ and every n-tilting object T, let

$$\mathcal{L}_m = \{L \in \mathcal{A} \mid \exists \,\text{an exact sequence}\, 0 \to L \to T^0 \to T^1 \to \cdots \to T^m \to 0\},$$

with the objects $T^i \in \mathsf{Add}(T)$, for every $i = 0, \ldots, m$. The following hold ([17, Lemma 2.2])

- $\mathsf{Add}(T) = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots$ and $\mathcal{L}_n = \mathcal{L}_{n+1} = \mathcal{L}_{n+2} = \cdots$.
- Set $\mathcal{L} = \mathcal{L}_n$,
  then $\mathrm{Ext}^i_{\mathcal{A}}(L, E) = 0$ for every $i > 0$, for every $L \in \mathcal{L}$, $E \in \mathcal{E}$.
- $\mathcal{L} \cap \mathcal{E} = \mathsf{Add}(T) \subset \mathcal{A}$.

Theorem 3.1.3. *Assume that an object $T \in \mathcal{A}$ satisfies (i) and (ii) of Definition 3.1.2. Then T satisfies also (iii) if and only if every object of $\mathcal{E}$ is a quotient of an object from $\mathsf{Add}(T)$ in $\mathcal{A}$ if and only if every object of $\mathcal{A}$ is a quotient of an object from $\mathcal{L}$. (see* [17, Theorem 2.4]*).*

Assume that T is an n-tilting object in $\mathcal{A}$. Then

(1) [17, Theorem 2.4] *The pair of classes $(\mathcal{L}, \mathcal{E})$ in $\mathcal{A}$ is a hereditary complete cotorsion pair called the n-**tilting cotorsion pair** associated with T.*
In particular, $\mathcal{E}$ is a coresolving class in $\mathcal{A}$, meaning that it is closed under summands, extensions and cokernels of monomorphisms in $\mathcal{A}$.

(2) [17, Lemma 4.1] *$\mathcal{E}$ consists of the objects E such that there is an exact sequence:*

$$T^{(I_n)} \to \ldots T^{(I_1)} \to E \to 0,$$

for some sets $I_1, \ldots, I_n$.

(3) [17, Lemma 4.1] *$\mathcal{E}$ is closed under coproducts.*

Remark 3.1.4. *If $\mathcal{A}$ is the category A–Mod for an associative ring A, an n-tilting object T is exactly an n-tilting module, i.e. it satisfies (i) and (ii) and the ring A has a finite coresolution in $\mathsf{Add}T$:*

$$0 \to A \to T^0 \to \ldots\ T^n \to 0.$$

Theorem 3.1.5. [17, Theorem 1.3 and Corollary 1.4] *Let $T \in \mathcal{A}$ be an n-tilting object. Then the pair of full subcategories*

$$^T\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathcal{D}(\mathcal{A}) \mid \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(T, X^\bullet[i]) = 0, \textit{ for all } i > 0\},$$

$$^T\mathcal{D}^{\geq 0} = \{X^\bullet \in \mathcal{D}(\mathcal{A}) \mid \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(T, X^\bullet[i]) = 0, \textit{ for all } i < 0\},$$

*is a t-structure on the unbounded derived category $\mathcal{D}(\mathcal{A})$, called the **tilting t-structure**.*

Moreover, the pair $(^T\mathcal{D}^{b,\leq 0}, {}^T\mathcal{D}^{b,\geq 0})$, restriction of the tilting t-structure to the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of $\mathcal{A}$, is a t-structure in $\mathcal{D}^b(\mathcal{A})$.

Proposition 3.1.6. [17, Proposition 1.5] *Assume that T satisfies (i) and (ii) of Definition 3.1.2 and that $(^T\mathcal{D}^{b,\leq 0}, {}^T\mathcal{D}^{b,\geq 0})$ is a t-structure on $\mathcal{D}^b(\mathcal{A})$. Then T satisfies also condition (iii).*

Proposition 3.1.7. [17, Proposition 1.6] *Let*

$$\mathcal{B} = {}^T\mathcal{D}^{\leq 0} \cap {}^T\mathcal{D}^{\geq 0} = \{X^\bullet \in \mathcal{D}(\mathcal{A}) \mid \mathrm{Hom}_{\mathcal{A}}(T, X^\bullet[i]) = 0, \forall i \neq 0\}$$

*be the **heart** of the tilting t-structure.*

(1) The n-tilting object $T \in \mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$ belongs to $\mathcal{B}$ and is a projective generator of $\mathcal{B}$. $\mathcal{B}$ has coproducts and the projective objects of $\mathcal{B}$ are the summands of coproducts of copies of T.

(2) The subcategory $\mathcal{E} \subseteq \mathcal{A}$ can be described as $\mathcal{A} \cap \mathcal{B}$, the intersection of the hearts of the standard t-structure and the n-tilting t-structure on $\mathcal{D}(\mathcal{A})$.

3.2. n-**cotilting objects.**

Definition 3.2.1. [17, Pag 18] *Let $\mathcal{B}$ be a complete, cocomplete abelian category with a projective generator $P \in \mathcal{B}$. Let $n \geq 0$. An object $W \in \mathcal{B}$ is an n-**cotilting** object if W^{op} is n-tilting in the abelian category $\mathcal{B}^{op}$, that is:*

(i) the injective dimension of W is at most n, that is $\mathrm{Ext}^i_{\mathcal{B}}(B, W) = 0$ for all $B \in \mathcal{B}$ and $i > n$;*

(ii) $\mathrm{Ext}^i_{\mathcal{B}}(W^X, W) = 0$ for all $i > 0$, for any set X;*

(iii) every $Y^\bullet \in \mathcal{D}(\mathcal{B})$ such that $\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Y, W[i]) = 0$ for all $i \in \mathbb{Z}$ is acyclic.*

All the statements about an n-tilting object in $\mathcal{A}$ dualize for an n-cotilting object in $\mathcal{B}$. In particular:

Assume that W is an n-cotilting object in $\mathcal{B}$, and let

$$\mathcal{F} = \{F \in \mathcal{B} \mid \mathrm{Ext}^i_{\mathcal{B}}(F, W) = 0, \text{ for all } i > 0, \}$$

- $\mathcal{B}_{\mathsf{proj}}$, the class of projective objects of $\mathcal{B}$, is contained in $\mathcal{F}$ and by condition (ii^*), $\mathsf{Prod}_{\mathcal{B}}(W) \subset \mathcal{F}$.
- $\mathcal{F}$ consists of the objects $F \in \mathcal{B}$ such that there is a exact sequence
$$0 \to F \to W^{I_1} \to \cdots \to W^{I_n},$$
for some sets $I_1, \ldots, I_n$.
- There is a class $\mathcal{R}$ such that the pair $(\mathcal{F}, \mathcal{R})$ in $\mathcal{B}$ is a hereditary complete cotorsion pair, called the n-**cotilting cotorsion pair** associated to W.
In particular, $\mathcal{F}$ is a resolving class in $\mathcal{B}$, meaning that it is closed under summands, extensions and kernels of epimorphisms in $\mathcal{B}$.
- The class $\mathcal{F}$ is closed under under products.

Theorem 3.2.2. [17, Theorem 3.3 and Corollary 3.4] *Let $\mathcal{B}$ be a complete, cocomplete abelian category with a projective generator $P \in \mathcal{B}$ and an n-cotilting object W. Then the pair of full subcategories*

$${}^W\mathcal{D}^{\leq 0} = \{Y^\bullet \in \mathcal{D}(\mathcal{B}) \mid \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Y^\bullet, W[i]) = 0, \textit{ for all } i < 0\},$$

$${}^W\mathcal{D}^{\geq 0} = \{Y^\bullet \in \mathcal{D}(\mathcal{B}) \mid \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Y^\bullet, W[i]) = 0, \textit{ for all } i > 0\},$$

is a t-structure on the unbounded derived category $\mathcal{D}(\mathcal{B})$, called the **cotilting t-structure**.

Moreover, the pair $({}^W\mathcal{D}^{b,\leq 0}, {}^W\mathcal{D}^{b,\geq 0})$, restriction of the cotilting t-structure to the bounded derived category $\mathcal{D}^b(\mathcal{B})$ of $\mathcal{B}$ is a t-structure in $\mathcal{D}^b(\mathcal{B})$.

Proposition 3.2.3. [17, Proposition 3.8] *Let W be an n-cotilting object in $\mathcal{B}$ and $\mathcal{A} = {}^W\mathcal{D}^{\leq 0} \cap {}^W\mathcal{D}^{\geq 0}$ be the* **heart** *of the cotilting t-structure on $\mathcal{D}(\mathcal{B})$.*

(1) $W \in \mathcal{B} \subseteq \mathcal{D}(\mathcal{B})$ belongs to $\mathcal{A}$ and is an injective cogenerator of $\mathcal{A}$. $\mathcal{A}$ has products and the injective objects of $\mathcal{A}$ are the summands of products of copies of W.

(2) The subcategory $\mathcal{F} \subseteq \mathcal{B}$ can be described as $\mathcal{B} \cap \mathcal{A}$, the intersection of the hearts of the standard t-structure and the n-cotilting t-structure on $\mathcal{D}(\mathcal{B})$.

3.3. **The Tilting-Cotilting Correspondence.**

Theorem 3.3.1. [17, Theorem 3.10 and Theorem 3.11] *Let $\mathcal{A}$ be a complete, cocomplete abelian category with an injective cogenerator W and an n-tilting object T and let $\mathcal{B} = {}^T\mathcal{D}^{\leq 0} \cap {}^T\mathcal{D}^{\geq 0}$ be the heart of the tilting t-structure.*

(1) The object $W \in \mathcal{B} \subseteq \mathcal{D}(\mathcal{A})$ is an n-cotilting object in the abelian category $\mathcal{B}$.

Let $\mathcal{B}$ be a complete, cocomplete abelian category with a projective generator T and an n-cotilting object W and let $\mathcal{A} = {}^W\mathcal{D}^{\leq 0} \cap {}^W\mathcal{D}^{\geq 0}$ be the heart of the cotilting t-structure.

(2) The object $T \in \mathcal{A} \subseteq \mathcal{D}(\mathcal{B})$ is an n-tilting object in the abelian category $\mathcal{A}$.

The previous results can be summarized as follows.

Theorem 3.3.2. [17, Theorem 3.10 and 3.11], [3, Theorem 12.1] *There is a bijective correspondence between complete, cocomplete abelian categories $\mathcal{A}$ with an injective cogenerator J and an n-tilting object $T \in \mathcal{A}$, and complete, cocomplete abelian categories $\mathcal{B}$ with a projective generator P and an n-cotilting object $W \in \mathcal{B}$.*

(1) The correspondence is given by a pair of adjoint functors (Φ, Ψ)

$$\Phi \colon \mathcal{B} \longrightarrow \mathcal{A}; \quad \Psi \colon \mathcal{A} \to \mathcal{B},$$

obtained respectively from the truncation functors with respect to the cotilting t-structure on $\mathcal{D}(\mathcal{B})$ and the tilting t-structure on $\mathcal{D}(\mathcal{A})$.

(2) The restrictions of Φ and Ψ between $\mathcal{F} \subset \mathcal{B}$ and $\mathcal{E} \subset \mathcal{A}$ are mutually inverse equivalences, as exact categories.

(3) Under the equivalence $\mathcal{E} \cong \mathcal{F}$, the injective cogenerator $J \in \mathcal{E} \subset \mathcal{A}$ corresponds to the n-cotilting object $W \in \mathcal{F} \subset \mathcal{B}$, and the n-tilting object $T \in \mathcal{E} \subset \mathcal{A}$ corresponds to the projective generator $P \in \mathcal{F} \subset \mathcal{B}$.

(4) [17, Theorem 4.5] *The exact embedding $\mathcal{E} \hookrightarrow \mathcal{A}$ induces a triangle equivalence $\mathcal{D}(\mathcal{E}) \cong \mathcal{D}(\mathcal{A})$ and the exact embedding $\mathcal{F} \hookrightarrow \mathcal{B}$ induces a triangle equivalence $\mathcal{D}(\mathcal{F}) \cong \mathcal{D}(\mathcal{B})$. Hence there is a triangle equivalence $\mathcal{D}(\mathcal{A}) \cong \mathcal{D}(\mathcal{B})$.*

3.4. **Applications to full subcategories of a module category.**

Proposition 3.4.1. [17, Theorem 7.1] *Let R be an associative ring and M a left R-module. The category $\mathsf{Add}(M)$ is equivalent to the category of projective left $\mathfrak{R}$-contramodules where $\mathfrak{R}$ is the complete, separated topological ring $\mathrm{Hom}_R(M, M)^{op}$ with a basis of neighbourhoods of zero formed by right ideals.*

Proof. Consider the ring $\mathrm{Hom}_R(M, M)$ with the finite topology, that is the topology in which the base of neighbourhoods of zero is formed by the annihilator ideals $\mathrm{Ann}(F)$ of the finitely generated R-submodules $F \subseteq M$.

$\mathrm{Ann}(F) \cong \mathrm{Hom}_R(M/F, M)$ and $\mathrm{Hom}_R(M, M)/\mathrm{Ann}(F)$ is the set of all morphisms $F \to M$ that can be extended to morphisms $M \to M$.

A morphism $f\colon M \to M$ is given by a compatible system of R-module morphisms $F \to M$, for all the finitely generated submodules $F \subseteq M$, hence we have an isomorphism

$$\mathrm{Hom}_R(M, M) \cong \varprojlim_{F \subseteq M} \frac{\mathrm{Hom}_R(M, M)}{\mathrm{Ann}(F)}.$$

Let $\mathfrak{R} = \mathrm{Hom}_R(M, M)^{op}$. $\mathfrak{R}$ is a complete, separated topological ring with a base of the topology formed by open right ideals.

We have the two monads

$$\mathbb{T}_M \colon \mathsf{Sets} \to \mathsf{Sets}, \quad X \mapsto \mathrm{Hom}_R(M, M^{(X)})$$

$$\mathbb{T}_{\mathfrak{R}} \colon \mathsf{Sets} \to \mathsf{Sets}, \quad X \mapsto \mathfrak{R}[[X]],$$

where $\mathrm{Hom}_R(M, M^{(X)}) \subset \mathrm{Hom}_R(M, M)^X$ and $\mathfrak{R}[[X]] \subset \mathfrak{R}^X$.
We claim that $\mathbb{T}_M \cong \mathbb{T}_{\mathfrak{R}}$, that is $\mathrm{Hom}_R(M, M^{(X)}) = \mathfrak{R}[[X]]$. In fact, an X-indexed family of morphisms $g_x\colon M \to M$ corresponds to a morphism $M \to M^{(X)}$ if and only if for every finitely generated submodule F of M, $g_x(F) = 0$ for all but finitely many indices $x \in X$, that is, if and only if it converges to zero in the topology of $\mathrm{Hom}_R(M, M)$, if and only if $\{g_x\}_{x\in X} \in \mathfrak{R}[[X]]$.

The functor

$$\mathrm{Hom}_R(M, -)\colon R\text{–}\mathsf{Mod} \to \mathfrak{R}\text{–}\mathsf{contra}$$

sends $M^{(X)}$ to the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]]$ and induces an equivalence between $\mathsf{Add}M$ and the full subcategory of projective $\mathfrak{R}$-contramodules. □

Remark 3.4.2. More explicitly: For any associative ring R and R-modules M, N, the group $\mathrm{Hom}_R(M, N)$ has a natural structure of a left $\mathfrak{R}$-contramodule over the topological ring $\mathfrak{R} = \mathrm{Hom}_R(M, M)^{op}$ described as follows.

For every set X, let r_x be a family of elements of $\mathfrak{R}$ converging to zero and corresponding to a family $g_x \in \mathrm{Hom}_R(M, M)$. For every family $f_x \in \mathrm{Hom}_R(M, N)$ and $m \in M$ define $\sum_{x\in X} r_x f_x \in \mathrm{Hom}_R(M, N)$ by:

$$\left(\sum_{x\in X} r_x f_x\right)(m) = \sum_{x\in X} f_x(g_x(m)),$$

the sum on the right-hand side is finite since the family g_x converges to zero.

Theorem 3.4.3. [17, Corollary 7.2 and Corollary 7.4] *Let R be an associative ring and $\mathcal{A} \subseteq R$–Mod be a full subcategory closed under coproducts. Suppose that $\mathcal{A}$ is abelian with products and an injective cogenerator. Then, for any n-tilting object $T \in \mathcal{A}$, the abelian category $\mathcal{B}$, that is the heart of the n-tilting t-structure on $\mathcal{D}(\mathcal{A})$ associated with T is equivalent to the abelian category of left contramodules $\mathfrak{R}$–contra over the topological ring $\mathfrak{R} = \mathrm{Hom}_{\mathcal{A}}(T, T)^{op}$:*

$$\mathcal{B} \cong \mathfrak{R}\text{–}\mathsf{contra}.$$

In particular, there is a derived equivalence:

$$\mathbb{R}\Psi\colon \mathcal{D}(\mathcal{A}) \leftrightarrows \mathcal{D}(\mathfrak{R}\text{–}\mathsf{contra})\colon \mathbb{L}\Phi.$$

Proof. The abelian categories $\mathcal{B}$ and $\mathfrak{R}$–contra have enough projectives. The category of projective objects in $\mathcal{B}$ is equivalent to $\mathsf{Add}(T) \subseteq \mathcal{A} \subseteq R$–Mod by Proposition 3.1.7. An abelian category with enough projectives is determined by its full subcategory of projective objects and by (the proof of) Proposition 3.4.1 $\mathsf{Add}(T)$ is equivalent to the category of projective objects in $\mathfrak{R}$–contra. Hence the derived equivalence follows by Theorem 3.3.2 (4). □

3.5. **Example** [3, Sections 17 and 19]**.** Let $f: R \longrightarrow S$ be an injective ring epimorphism such that $\mathrm{Tor}_1^R(S,S)=0$, the flat dimension (f.dim) of S as a right R-module is at most 1 and the projective dimension (p.dim) of S as a left R-module is at most 1. There is a short exact sequence

$$0 \to R \to S \to S/R = K \to 0$$

where K and S are R-R-bimodules. It is known (see [1]) that the left R-module $S \oplus K$ is a 1-tilting left R-module.

Consider the full subcategory $\mathcal{A}$ of R–Mod consisting of the left R-modules M such that

$$S \otimes_R M = 0 = \mathrm{Tor}_1^R(S,M).$$

Under our homological assumptions we have:

- $\mathcal{A}$ is closed under kernels, cokernels, extensions, and coproducts in R–Mod, hence it is exactly embedded in R–Mod.
- [3, Proposition 17.1] The functor $\Gamma_S = \mathrm{Tor}_1^R(K,-)$ is right adjoint to the embedding functor $\mathcal{A} \to R$–Mod.
- $X \in \mathcal{A}$ if an only if $X \cong \mathrm{Tor}_1^R(K,X)$.
- [3, Proposition 17.4] $\mathcal{A}$ is a Grothendieck category; if I is an injective cogenerator of R–Mod, then $\Gamma_S(I)$ is an injective cogenerator of $\mathcal{A}$.

Remark 3.5.1. [3, Remark 16.9] Note that $\mathcal{A}$ is not a torsion class in R–Mod. It is contained in the torsion class $\mathcal{T}$ consisting of the left R-modules M such that $S \otimes_R M = 0$ and the two classes $\mathcal{A}$ and $\mathcal{T}$ coincide if and only if S is flat as a right R-module. In this case the torsion class is also hereditary.

Proposition 3.5.2. [3, Theorem 19.1] *The object K is a 1-tilting object of the category $\mathcal{A}$.*

Proof. $K \in \mathcal{A}$ and the left R-module $S \oplus K$ is a 1-tilting module in R–Mod, hence K satisfies conditions (i) and (ii) in $\mathcal{A}$. The tilting class in R–Mod corresponding to the 1-tilting module $S \oplus K$ is the class of S-divisible R-modules, that is the class generated by S. Let $\mathcal{E} = \mathrm{Ker}(\mathrm{Ext}^1_{\mathcal{A}}(K,-))$. An object $X \in \mathcal{A}$ is in $\mathcal{E}$ if and only if X is S-divisible. There is a left approximation $0 \to Y \to S^{(\alpha)} \oplus K^{(\alpha)} \to X \to 0$ of X in R–Mod with Y an S-divisible left R-module.

Applying the functor $\mathrm{Tor}_1^R(K,-)$ to the above sequence we get

$$\mathrm{Tor}_1^R(K, S^{(\alpha)} \oplus K^{(\alpha)}) \cong K^{(\alpha)} \to \mathrm{Tor}_1^R(K,X) \cong X \to K \otimes_R Y = 0$$

where the last term vanishes since Y is S-divisible and $K \otimes S = 0$.

Hence K satisfies condition (iii) by the characterization in Theorem 3.1.3. □

Consider the full subcategory $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$ of $R\text{–}\mathsf{Mod}$ consisting of the left R-modules C such that

$$\operatorname{Hom}_R(S, C) = 0 = \operatorname{Ext}^1_R(S, C).$$

- By the assumption $\operatorname{p.dim}_R S \leq 1$, the full subcategory $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$ is closed under kernels, cokernels extensions and products in $R\text{–}\mathsf{Mod}$, hence it is exactly embedded in $R\text{–}\mathsf{Mod}$.
- For every R-module C the module $\operatorname{Ext}^1_R(K, C)$ is an S-contramodule. This can be proved analogously to Lemma 1.2.1 (2) using that

  $$S \otimes_R K = 0 = \operatorname{Tor}^R_1(S, K).$$

  (See also [3, Lemma 16.7 (c)].)

Proposition 3.5.3. [3, Proposition 17.2] *The functor*

$$\Delta_S = \operatorname{Ext}^1_R(K, -)\colon R\text{–}\mathsf{Mod} \longrightarrow R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$$

is left adjoint to the embedding functor $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}} \longrightarrow R\text{–}\mathsf{Mod}$. *In particular,* $\Delta_S(R) = \operatorname{Ext}^1_R(K, R)$ *is a projective generator of* $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$.

Proof. The natural isomorphism $\operatorname{Hom}_{R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}}(\Delta_S(C), D) \cong \operatorname{Hom}_R(C, D)$, for every R-module C and S-contramodule D is proved similarly to the case of one element localization $R[s^{-1}]$(see Proposition 1.3.7). Indeed, there is an exact sequence

$$\operatorname{Hom}_R(S, C) \to C \overset{\delta_S}{\to} \operatorname{Ext}^1_R(K, C) \to \operatorname{Ext}^1_R(S, C) \to 0$$

where $\operatorname{Ker} \delta_S(C)$ is S-divisible and $\operatorname{Ext}^1_R(S, C)$ is an S-module. □

Theorem 3.5.4. [3, Theorems 19.1, 19.2] *Let* $\mathcal{A}$ *and* K *be as above. Let* $\mathfrak{R}$ *be the topological ring* $\operatorname{Hom}_R(K, K)^{\mathrm{op}}$ *with the finite topology. The following hold:*

1. *The forgetful functor* $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{Mod}$ *is fully faithful, and its essential image coincides with the full subcategory* $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$ *of* S*-contramodules. In particular,* $\mathfrak{R}\text{–}\mathsf{contra} \cong R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$.
2. *The categories* $\mathcal{A}$ *and* $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$ *are connected by the* 1*-tilting correspondence:*

$$\operatorname{Hom}_R(K, -)\colon \mathcal{A} \leftrightarrows R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}\colon K \otimes_R -.$$

Proof. (1) The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{Mod}$ sends $\mathfrak{R}[[X]]$ to $\Delta_S(R^{(X)})$ (see the next paragraph) and it preserves cokernels. The full subcategory $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}} \subset R\text{–}\mathsf{Mod}$ is closed under cokernels, and every left $\mathfrak{R}$-contramodule is the cokernel of a morphism of free left $\mathfrak{R}$-contramodules, hence the image of the forgetful functor is contained in $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$.

To show that $\mathfrak{R}\text{–}\mathsf{contra}$ is equivalent to $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$ it is enough to show the equivalence between their subcategories of projective objects. By [14, Proposition 2.1] this holds provided that there is a natural isomorphism $\Delta_S(R^{(X)}) \cong \mathfrak{R}[[X]]$ for every set X. Now, from the exact sequence

$$0 \to R^{(X)} \to S^{(X)} \to K^{(X)} \to 0,$$

we have natural isomorphisms $\mathfrak{R}[[X]] \cong \operatorname{Hom}_R(K, K^{(X)}) \cong \operatorname{Ext}^1_R(K, R^{(X)}) = \Delta_S(R^{(X)})$.

(2) By Theorem 3.3.2 (1) there is a correspondence

$$\Psi\colon \mathcal{A} \leftrightarrows \mathcal{B}\colon \Phi$$

where $\mathcal{B}$ is the heart of the tilting t-structure on $\mathcal{D}(\mathcal{A})$ induced by K. By Theorem 3.4.3, $\mathcal{B} \cong \mathfrak{R}\text{–}\mathsf{contra}$, hence by part (1) we get the correspondence between $\mathcal{A}$ and $R\text{–}\mathsf{Mod}_{S\text{–}\mathsf{contra}}$. By [17, Corollary 7.4] the functor Ψ can be computed as $\mathrm{Hom}_R(K,-)$ and Φ is given by $K \otimes_R -$, since the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{Mod}$ is fully faithful (see [17, Lemma 7.9].) □

3.6. **Good tilting modules.**

(1) If ${}_RT$ is a finitely generated n-tilting module with endomorphism ring S, then it is well known that there is a derived equivalence:

$$T \otimes^{\mathbb{L}}_S -\colon \mathcal{D}(S) \rightleftarrows \mathcal{D}(R)\colon \mathbb{R}\,\mathrm{Hom}_R(T,-).$$

(2) An infinitely generated n-tilting module ${}_RT$ is said to be **good** if the ring R has an $\mathsf{Add}T$-coresolution $0 \to R \to T^0 \to \cdots \to T^n \to 0$ where T^i are summands of finite direct sums of copies of T.
(For every n-tilting module T there is a set Y such that $T^{(Y)}$ is a good n-tilting module giving rise to the same tilting class).

(3) [4] Let ${}_RT$ be a good n-tilting module with endomorphism ring S. There is an adjunction

$$T \otimes^{\mathbb{L}}_S -\colon \mathcal{D}(S) \rightleftarrows \mathcal{D}(R)\colon \mathbb{R}\,\mathrm{Hom}_R(T,-),$$

where $\mathbb{R}\,\mathrm{Hom}_R(T,-)$ is fully faithful and induces a derived equivalence:

$$\mathcal{D}(R) \cong \frac{\mathcal{D}(S)}{\mathrm{Ker}(T \otimes^{\mathbb{L}}_S -)}$$

where $\mathcal{D}(S)/\,\mathrm{Ker}(T \otimes^{\mathbb{L}}_S -)$ is a Verdier quotient of $\mathcal{D}(S)$.

(4) In Proposition 3.6.4 we will see that the above derived equivalence can be stated in terms of contramodules.

Definition 3.6.1. *An object M in a category with coproducts is said to be* κ-**small** *if every morphism $M \to M^{(X)}$ factors through $M^{(Z)} \to M^{(X)}$ for a subset Z of X of cardinality strictly smaller than κ.*

Theorem 3.6.2. [17, Theorem 6.6] *Let $\mathcal{B}$ be a cocomplete abelian category with a κ-small projective generator P. Let Y be a set of cardinality λ such that $\lambda^+ \geq \kappa$. Let $Q = P^{(Y)}$ and $S = \mathrm{Hom}_{\mathcal{B}}(Q,Q)^{\mathrm{op}}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{B}}(Q,-)\colon \mathcal{B} \to S\text{–}\mathsf{Mod}$$

is fully faithful and admits a left adjoint functor.

Proposition 3.6.3. [17, Theorem 7.10] *Let R be an associative ring, M a left R-module, and Y be a set of cardinality greater or equal to the minimal cardinality of a set of generators of M. Consider $L = M^{(Y)}$. Let $\mathfrak{R} = \mathrm{Hom}_R(M,M)^{\mathrm{op}}$ and $\mathfrak{S} = \mathrm{Hom}_R(L,L)^{\mathrm{op}}$.*

Then $\mathsf{Add}M \cong \mathfrak{R}_{\mathsf{proj}}$, $\mathsf{Add}L \cong \mathfrak{S}_{\mathsf{proj}}$, hence $\mathfrak{R}\text{–}\mathsf{contra} \cong \mathfrak{S}\text{–}\mathsf{contra}$ (since $\mathsf{Add}M = \mathsf{Add}L$).

Moreover, the forgetful functor

$$\mathfrak{S}\text{–}\mathsf{contra} \to \mathfrak{S}\text{–}\mathsf{Mod}$$

is fully faithful.

Proof. The functor $\mathrm{Hom}_R(M,-)\colon R\text{–}\mathsf{Mod}\to\mathfrak{R}\text{–}\mathsf{contra}$ sends $L=M^{(Y)}$ to $\mathfrak{R}[[Y]]$ (see the proof of Proposition 3.4.1). The annihilators of the finitely generated submodules of M give a basis for the topology of $\mathfrak{R}$, hence, by the choice of Y, $\mathfrak{R}$ is a κ-small projective generator of $\mathfrak{R}$–contra.

Theorem 3.6.2 shows that

$$\mathrm{Hom}_{\mathfrak{R}\text{–}\mathsf{contra}}(\mathfrak{R}[[Y]],-)\colon \mathfrak{R}\text{–}\mathsf{contra}\to\mathfrak{S}\text{–}\mathsf{Mod}$$

is fully faithful. □

Application [17, Corollary 7.11] Let T be an n-tilting left R-module and $\mathfrak{R}=\mathrm{Hom}_R(T,T)^{\mathrm{op}}$. Let $\mathcal{B}$ be the heart of the tilting t-structure induced by T on $\mathcal{D}(R)$ and Y a set as in Proposition 3.6.3, then

$$\mathcal{B}\cong\mathfrak{R}\text{–}\mathsf{contra}\cong\mathfrak{S}\text{–}\mathsf{contra}$$

and the adjunction

$$\Phi\colon\mathcal{B}\longrightarrow R\text{–}\mathsf{Mod};\quad \Psi\colon R\text{–}\mathsf{Mod}\to\mathcal{B},$$

from the Tilting-Cotilting correspondence is given by the restriction of the adjunction

$$T^{(Y)}\otimes_{\mathfrak{S}}-\colon\mathfrak{S}\text{–}\mathsf{Mod}\rightleftarrows R\text{–}\mathsf{Mod}\colon\ \mathrm{Hom}_R(T^{(Y)},-).$$

As a consequence of the previous discussion we obtain:

Proposition 3.6.4. [17, Proposition 7.13] *If $T\in R$–Mod is a good n-tilting module and $\mathfrak{S}=\mathrm{Hom}_R(T,T)^{\mathrm{op}}$, then the forgetful functor*

$$\mathfrak{S}\text{–}\mathsf{contra}\to\mathfrak{S}\text{–}\mathsf{Mod}$$

induces a fully faithful functor $\mathcal{D}(\mathfrak{S}\text{–}\mathsf{contra})\to\mathcal{D}(\mathfrak{S})$. The functor

$$\mathbb{R}\,\mathrm{Hom}_R(T,-)\colon\mathcal{D}(R)\to\mathcal{D}(\mathfrak{S})$$

is fully faithful and the adjunction

$$T\otimes^{\mathbb{L}}_{\mathfrak{S}}-\colon\mathcal{D}(\mathfrak{S})\rightleftarrows\mathcal{D}(R)\colon\mathbb{R}\,\mathrm{Hom}_R(T,-)$$

restricts to the equivalence

$$T\otimes^{\mathbb{L}}_{\mathfrak{S}}-\colon\mathcal{D}(\mathfrak{S}\text{–}\mathsf{contra})\rightleftarrows\mathcal{D}(R)\colon\mathbb{R}\,\mathrm{Hom}_R(T,-).$$

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