RECOLLEMENTS FROM PARTIAL TILTING COMPLEXES

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ABSTRACT. We consider recollements of derived categories of differential graded algebras induced by self orthogonal compact objects obtaining a generalization of Rickard's Theorem. Specializing to the case of partial tilting modules over a ring, we extend the results on triangle equivalences proved in [B] and [BMT]. In the end we focus on the connection between recollements of derived categories of rings, bireflective subcategories and "generalized universal localizations".

1. INTRODUCTION

A recollement of triangulated categories is a diagram



where the six functors involved are the derived version of Grothendieck's functors. In particular, they are paired in two adjoint triples, i_* is fully faithful and \mathcal{T}'' is equivalent to a Verdier quotient of \mathcal{T} via j^* so that the straight arrows can be interpreted as an exact sequence of triangulated categories. The notion of recollements was introduced by Beilinson-Bernstein-Deligne [BBD] in a geometric context, where stratifications of varieties induce recollements of derived categories of constructible sheaves. The algebraic aspect of recollements has become more and more apparent.

Equivalence classes of recollements of triangulated categories are in bijection with torsion-torsion-free triples, that is triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full triangulated subcategories of the central term \mathcal{T} of a recollement, where $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are stable *t*-structures (see [N2, Section 9.2]).

This bijection is studied in details by Nicolas and Saorin in [NS] for recollements of derived categories of small flat differential graded categories (dg categories) and moreover, a parametrization of these recollements is given in terms of homological epimorphisms of dg categories. More precisely, the left end term of a recollement of the derived category $\mathcal{D}(B)$ of a small flat dg category B is the derived category of a dg category C linked to B via a

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homological epimorphism and such that the central term \mathcal{Y} of the torsion-torsion-free triple associated to the recollement is the essential image of the functor i_* .

In Section 2 we consider the particular case of the derived category $\mathcal{D}(B)$ of a k-flat differential graded algebra B and define explicitly a homological epimorphism of differential graded algebras (dg algebras) $B \to C$ associated to a torsion-torsion-free triple on $\mathcal{D}(B)$.

There is also a strong connection between recollements and tilting theory as shown by Koenig and Ageleri-Koenig-Liu in [K] and [AKL], by considering recollements of derived categories of rings.

In Section 3 we recall some instances of recollements of derived categories of differential graded algebras as the ones provided by Jorgensen in [J]. There, starting from results in [DG], [Mi] and [N], such recollements are characterized in terms of derived functors associated to two objects, one compact and the other self-compact. But in [J] there is no mention of the connection between the recollements involved and homological epimorphisms of dg algebras.

In Section 4 we specialize the situation to the case of self-orthogonal compact dg modules, which we call partial tilting. Our result (Theorem 4.3), can be viewed as a generalization of the Morita-type theorem proved by Rickard in [R]. In fact, if P is a partial tilting right dg module over a dg algebra B, we can use a quasi-isomorphism between the endomorphism ring A of P and the dg endomorphism of P, to show that the functor $P \bigotimes_{B}^{\mathbb{L}}$ – induces an equivalence between the quotient of $\mathcal{D}(B)$ modulo the full triangulated subcategory $\operatorname{Ker}(P \bigotimes_{B}^{\mathbb{L}} -)$ and the derived category $\mathcal{D}(A)$. Thus if P is moreover a tilting complex over a ring B with endomorphism ring A, then $\operatorname{Ker}(P \bigotimes_{B}^{\mathbb{L}} -)$ is zero and we recover Rickard's Theorem.

In Section 5 we consider applications to the case of classical partial tilting right modules T over a ring B, that is partial tilting complexes concentrated in degree zero. As examples of this case we start with a possibly infinitely generated left module $_{A}T$ over a ring A, which is self-orthogonal viewed as a complex concentrated in degree zero and finitely generating the ring A in $\mathcal{D}(A)$ (see Section 5 for the definitions). Under these assumptions, T, viewed as a right module over its endomorphism ring B, is a faithfully balanced classical partial tilting module and applying Proposition 5.2 we obtain a generalization of the result proved in [BMT] where the stronger assumption on $_{A}T$ to be a "good tilting module" was assumed. Moreover, this setting provides an instance of the situation considered in [Y].

In Section 6, given a classical partial tilting module T_B over a ring B, we look for conditions under which the class $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_B^{\mathbb{L}} -)$ is equivalent to the derived category of a ring S for which there is a homological ring epimorphism $\lambda \colon B \to S$. We show that this happens if and only if the perpendicular subcategory \mathcal{E} consisting of the left B-modules N such that $\operatorname{Tor}_i^B(T, N) = 0$ for every $i \geq 0$, is bireflective and every object of \mathcal{Y} is quasi-isomorphic to a complex with terms in \mathcal{E} . In Section 7 we show that, if such a ring S exists, it plays the role of a "generalized universal localization" of B with respect to a projective resolution of T_B and some properties of this localization are illustrated.

The situation considered in Sections 6 and 7 is a generalization of a recent article by Chen and Xi ([CX]). In fact, in [CX], completing the results proved in [B] for "good" 1-tilting modules T over a ring A, it is shown that the derived category $\mathcal{D}(B)$ of the endomorphism ring of T is the central term of a recollement with left term the derived category of a ring which is a universal localization in the sense of Schofield.

We note that in our approach, instead, we fix a ring B and obtain recollements of $\mathcal{D}(B)$ for every choice of classical partial tilting modules over B, while starting with an infinitely generated good tilting module ${}_{A}T$ over a ring A one obtains a recollement whose central term is the derived category of the endomorphism ring of ${}_{A}T$ and this ring might be very large and difficult to handle.

In Section 8 we consider examples of classical partial *n*-tilting modules T_B (with n > 1) over artin algebras B showing different possible behaviors of the associated perpendicular class \mathcal{E} . In some examples there exists a homological ring epimorphism $\lambda: B \to S$ such that the class $\mathcal{Y} = \text{Ker}(T \bigotimes_B^{\mathbb{L}} -)$ is triangle equivalent to $\mathcal{D}(S)$ (Example 1). But in these cases the modules T_B are not arising from good n -tilting modules.

Chen and Xi in [CX2] consider the case of a good *n*-tilting module ${}_{A}T$ (with n > 1) over a ring A with endomorphism ring B and investigate the problem to decide when the recollement of $\mathcal{D}(B)$ induced by T corresponds to a homological epimorphism. They prove some necessary sufficient conditions and show some counterexamples (see Example 4), but the problem in its full generality, remains open.

2. Preliminaries

In this section we recall some definitions and preliminary results that will be useful later on.

Let \mathcal{T} be a triangulated category admitting small coproducts (also called set indexed coproducts) and let \mathcal{C} be a class of objects in \mathcal{T} . Then:

- (1) Tria C denotes the smallest full triangulated subcategory of T containing C and closed under small coproducts,
- (2) tria C denotes the smallest full triangulated subcategory of T containing C and closed under finite coproducts and direct summands.

Moreover, indicating by [-] the shift functor, we define the following classes:

 $\mathcal{C}^{\perp} = \{ X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(C[n], X) = 0, \text{ for all } C \in \mathcal{C}, \text{ for all } n \in \mathbb{Z} \};$

$${}^{\perp}\mathcal{C} = \{ X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(X, C[n]) = 0, \text{ for all } C \in \mathcal{C}, \text{ for all } n \in \mathbb{Z} \}.$$

An object $X \in \mathcal{T}$ is called *self-orthogonal* if

 $\operatorname{Hom}_{\mathcal{T}}(X, X[n]) = 0$, for all $0 \neq n \in \mathbb{Z}$.

The object $X \in \mathcal{T}$ is called *compact* if the functor $\operatorname{Hom}_{\mathcal{T}}(X, -)$ commutes with small coproducts. M in \mathcal{T} is called *self-compact* if M is compact in Tria M.

Differential graded algebras and differential graded modules. We review the notions of dg algebras, dg modules and of the derived category of dg modules. For more details see [Ke1], [Ke2], [Ke3] or [P].

Let k be a commutative ring. A differential graded algebra over k (dg k-algebra) is a \mathbb{Z} -graded k-algebra $B = \bigoplus_{p \in \mathbb{Z}} B^p$ endowed with a differential d

of degree one, such that:

$$d(ab) = d(a)b + (-1)^p ad(b)$$

for all $a \in B^p, b \in B$.

In particular, a ring is a dg \mathbb{Z} -algebra concentrated in degree 0.

Let B be a dg algebra over k with differential d_B . A differential graded (left) B-module (dg B-module) is a Z-graded (left) B-module $M = \bigoplus_{p \in \mathbb{Z}} M^p$ endowed with a differential d_M of degree 1 such that

$$d_M(bm) = bd_M(m) + (-1)^p d_B(b)m$$

for all $m \in M^p$, $b \in B$.

A morphism between dg *B*-modules is a morphism of the underlying graded *B*-modules, homogeneous of degree zero and commuting with the differentials. A morphism $f: M \to N$ of dg *B*-modules is said to be null-homotopic if there exists a morphism of graded modules $s: M \to N$ of degree -1 such that $f = sd_M + d_N s$.

In the sequel we will simply talk about a dg algebra without mentioning the ground ring k.

The left dg *B*-modules form an abelian category denoted by $\mathcal{C}(B)$. The homotopy category $\mathcal{H}(B)$ is the category with the same objects as $\mathcal{C}(B)$ and with morphisms the equivalence classes of morphisms in $\mathcal{C}(B)$ modulo the null-homotopic ones. The derived category $\mathcal{D}(B)$ is the localization of $\mathcal{H}(B)$ with respect to the quasi-isomorphisms, that is morphisms in $\mathcal{C}(B)$ inducing isomorphisms in homology. $\mathcal{H}(B)$ and $\mathcal{D}(B)$ are triangulated categories with shift functor which will be denoted by [-].

We denote by B^{op} the opposite dg algebra of B. Thus, dg right B-modules will be identified with left dg B^{op} -modules. Also $\mathcal{D}(B^{\text{op}})$ will denote the derived category of right dg B-modules.

We say that M is a dg B-A-bimodule if it is a left dg B-module and a left dg A^{op} -module, with compatible B and A^{op} module structure. In this case we also write $_BM_A$.

 $\mathcal{C}_{dg}(B)$ denotes the category of dg *B*-modules. If *M*, *N* are dg *B*-modules the morphism space $\operatorname{Hom}_{\mathcal{C}_{dg}(B)}(M, N)$ is the complex $\mathcal{H}om_B(M, N)$ with $[\mathcal{H}om_B(M, N)]^n = \operatorname{Hom}_B(M, N[n])$ (here $\operatorname{Hom}_B(M, N)$ denotes the group of morphisms of graded *B*-modules, homogeneous of degree zero) and differential defined, for each $f \in [\mathcal{H}om_B(M, N]^n)$, by

$$d(f) = d_N \circ f - (-1)^n f \circ d_M.$$

Thus, $\operatorname{H}^{n}(\operatorname{Hom}_{B}(M, N)) = \operatorname{Hom}_{\mathcal{H}(B)}(M, N[n])$. Observe that, if X is a dg B-module, then $\operatorname{Hom}_{B}(X, X)$ is a dg algebra called the dg endomorphism ring of X.

Thus, the morphism space $\operatorname{Hom}_B(M, N)$ of dg *B*-modules *M*, *N* in the category $\mathcal{C}(B)$ is $Z^0(\operatorname{Hom}_{\mathcal{C}_{dg}(B)}(M, N))$ and the morphism space in the homotopy category $\mathcal{H}(B)$ is $H^0(\operatorname{Hom}_{\mathcal{C}_{dg}(B)}(M, N))$.

Definition 2.1. A dg *B*-module is *acyclic* if it has zero homology and a dg *A*-module *P* (resp. *I*) is called \mathcal{H} -projective (resp. \mathcal{H} -injective) if $\operatorname{Hom}_{\mathcal{H}(B)}(P, N) = 0$ (resp. $\operatorname{Hom}_{\mathcal{H}(B)}(N, I) = 0$) for all acyclic dg *B*-modules *N*.

For every dg *B*-module *X* there is an \mathcal{H} -projective dg *B*-module pX and an \mathcal{H} -injective dg *B*-module iX such that *X* is quasi-isomorphic to pX and to iX and such that

$$\operatorname{Hom}_{\mathcal{D}(B)}(X,Y) = \operatorname{Hom}_{\mathcal{H}(B)}(pX,Y) = \operatorname{Hom}_{\mathcal{H}(B)}(X,iY).$$

Definition 2.2. (see e.g. [P]) Let M be a dg B-module. The functor $\mathcal{H}om_B(M, -)$ induces a total right derived functor $\mathbb{R}Hom_B(M, -): \mathcal{D}(B) \to \mathcal{D}(k)$ such that $\mathbb{R}Hom_B(M, N) = \mathcal{H}om_B(M, iN) = \mathcal{H}om_B(pM, N)$, for every dg B module N.

Moreover, if M is a B-A dg bimodule, then $\mathcal{H}om_B(M, -)$ induces a total derived functor $\mathbb{R}Hom_B(M, -): \mathcal{D}(B) \to \mathcal{D}(A)$ and $\mathbb{R}Hom_B(M, N) = \mathcal{H}om_B(M, iN) \in \mathcal{D}(A)$, for every dg B-module N.

If N is a dg B^{op} -module, then $N \otimes_B -$ induces a total left derived functor $N \otimes_B^{\mathbb{L}} -: \mathcal{D}(B) \to \mathcal{D}(k)$ such that $N \otimes_B^{\mathbb{L}} M = pN \otimes_B M = N \otimes_B pM$, for every dg B-module M.

If moreover, N is a dg A-B-bimodule, then $N \otimes_B -$ induces a derived functor $N \bigotimes_B^{\mathbb{L}} -: \mathcal{D}(B) \to \mathcal{D}(A)$ where $N \bigotimes_B^{\mathbb{L}} M = N \otimes_B pM$.

Definition 2.3. (see [Ke1, Sec 2.6]) A dg *B*-module *X* is called *perfect* if it is \mathcal{H} -projective and compact in $\mathcal{D}(B)$. The full subcategory of $\mathcal{H}(B)$ consisting of perfect dg *B*-modules is denoted by per *B*; it coincides with the subcategory tria *B* of $\mathcal{H}(B)$.

By Ravenel-Neeman's result, an object of $\mathcal{D}(B)$ is compact if and only if it is quasi-isomorphic to a perfect dg *B*-module.

If B is an ordinary algebra, then the perfect complexes are the bounded complexes with finitely generated projective terms and the category per B is also denoted by $\mathcal{H}_{p}^{b}(B)$.

Recollements and TTF. In this subsection we recall the notion of a recollement of triangulated categories and the correspondence with torsion-torsion-free triples. The concept of recollement was introduced by Beilinson, Bernstein and Deligne in [BBD] to study "exact sequences" of derived categories of constructible sheaves over geometric objects.

Definition 2.4. ([BBD]) Let $\mathcal{D}, \mathcal{D}'$ and \mathcal{D}'' be triangulated categories. \mathcal{D} is said to be a recollement of \mathcal{D}' and \mathcal{D}'' , expressed by the diagram



if there are six triangle functors satisfying the following conditions: (1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

- (2) i_*, j_* and $j_!$ are fully faithful functors;
- (3) $j^{!}i_{!} = 0$ (and thus also $i^{!}j_{*} = 0$ and $i^{*}j_{!} = 0$);

(4) for each object $C \in \mathcal{D}$, there are two triangles in \mathcal{D} :

$$i_!i'(C) \longrightarrow C \longrightarrow j_*j^*(C) \longrightarrow i_!i'(C)[1],$$

$$j_!j'(C) \longrightarrow C \longrightarrow i_*i^*(C) \longrightarrow j_!j'(C)[1].$$

In the sequel, if $F: \mathcal{D} \to \mathcal{D}'$ is a triangulated functor between two triangulated categories \mathcal{D} and \mathcal{D}' , we will denote by Im F the essential image of F in \mathcal{D}' .

We will make frequent use of the following well known properties of recollements (see [BBD, Proposition 1.4.5].

Lemma 2.5. Given a recollement as in Definition 2.4 the following hold true:

- (1) Im $i_* = \text{Ker} j^*$.
- (2) $\mathcal{D}/\mathrm{Ker}j^*$ is triangle equivalent to \mathcal{D}' .

Definition 2.6. Two recollements defined by the data $(\mathcal{D}, \mathcal{D}', \mathcal{D}'', i^*, i_*, i^!, j^!, j_!, j_*)$ and $(\mathcal{D}, \mathcal{T}', \mathcal{T}'', i'^*, i'_*, i'^!, j'^!, j'_*, j'_*)$ are said to be equivalent if the following equality between essential images holds :

 $(\operatorname{Im} (j_{!}), \operatorname{Im} (i_{*}), \operatorname{Im} (j_{*})) = (\operatorname{Im} (j'_{!}), \operatorname{Im} (i'_{*}), \operatorname{Im} (j'_{*})).$

Let \mathcal{D} be a triangulated category.

Definition 2.7. A torsion pair in \mathcal{D} is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of \mathcal{D} closed under isomorphisms satisfying the following conditions:

- (1) $\operatorname{Hom}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) = 0;$
- (2) $X[1] \subseteq \mathcal{X}$ and $Y[-1] \subseteq \mathcal{Y}$ for each $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
- (3) for each object $C \in \mathcal{D}$, there is a triangle

$$X_C \longrightarrow C \longrightarrow Y_C \longrightarrow X_C[1]$$

in \mathcal{D} with $X_C \in \mathcal{X}$ and $Y_C \in \mathcal{Y}$.

In this case \mathcal{X} is called a torsion class and \mathcal{Y} a torsion free class.

Definition 2.8. A torsion-torsion-free triple (TTF triple) in \mathcal{D} is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of subcategories \mathcal{X}, \mathcal{Y} and \mathcal{Z} of \mathcal{D} such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs in \mathcal{D} . In this case \mathcal{X}, \mathcal{Y} , and \mathcal{Z} are triangulated subcategories of \mathcal{D} .

We recall the following important result:

Lemma 2.9. ([N2, Section 9.2], [NS, Section 2.1]) Let \mathcal{D} be a triangulated category. There is a bijection between:

(1) TTF triples $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in $\mathcal{D}(A)$;

(2) equivalence classes of recollements for $\mathcal{D}(A)$.

In particular, given a recollement as in Definition 2.4, the triple

 $(\operatorname{Im} j_{!}, \operatorname{Im} i_{*}, \operatorname{Im} j_{*})$ is a TTF triple in \mathcal{D} .

Homological epimorphisms of dg algebras. In this subsection we recall the correspondence between bireflective subcategories of module categories and ring epimorphisms. Then we introduce the definition of homological epimorphisms of dg algebras and we state the correspondence between them, TTF triples and recollements.

It is well known (see e.g. [GL, Theorem 4.4]) that a ring homomorphism $f: R \longrightarrow S$ between two rings R and S is a ring epimorphism if and only if the multiplication map $S \otimes_R S \longrightarrow S$ is an isomorphism of S-bimodules or, equivalently, if the restriction of scalars functor $f_*: S$ -Mod $\rightarrow R$ -Mod is fully faithful. Two ring epimorphisms $f: R \longrightarrow S$ and $g: R \longrightarrow S'$ are said

to be equivalent if there exists an isomorphism of rings $h: S \longrightarrow S'$ such that hf = g.

We recall now the well known bijection existing between equivalence classes of ring epimorphisms and bireflective subcategories of module categories.

Definition 2.10. Let \mathcal{E} be a full subcategory of R-Mod. A morphism $f: M \longrightarrow E$, with E in \mathcal{E} , is called an \mathcal{E} -reflection if for every map $g: M \longrightarrow E'$, with E' in \mathcal{E} , there is a unique map $h: E \longrightarrow E'$ such that hf = g. A subcategory \mathcal{E} of R-Mod is said to be reflective if every R-module X admits an \mathcal{E} -reflection. The definition of coreflective subcategory is given dually. A subcategory that is both reflective and coreflective is called *bireflective*.

Remark 1. It is clear that a full subcategory \mathcal{E} of *R*-Mod is reflective if and only if the inclusion functor $i : \mathcal{E} \longrightarrow R$ -Mod admits a left adjoint. Dually, a subcategory \mathcal{X} is coreflecting if and only if the inclusion functor $j : \mathcal{X} \longrightarrow R$ -Mod admits a right adjoint.

Lemma 2.11. ([GL] and [GP]) Let \mathcal{E} be a full subcategory of R-Mod. The following assertions are equivalent:

1) \mathcal{E} is a bireflective subcategory of R-Mod;

2) \mathcal{E} is closed under isomorphic images, direct sums, direct products, kernels and cokernels;

3) there is a ring epimorphism $f : R \longrightarrow S$ such that \mathcal{E} is the essential image of the restriction of scalars functor $f_* : S$ -Mod $\rightarrow R$ -Mod.

In particular there is a bijection between the bireflective subcategory of R-Mod and the equivalence classes of ring epimorphisms starting from R. Moreover the map $f: R \longrightarrow S$ as in 3) is an \mathcal{E} -reflection.

A ring epimorphism $f : R \longrightarrow S$ is said to be *homological* if $S \bigotimes_R S = S$ in $\mathcal{D}(S)$ or, equivalently, if the restriction of scalars functor $f_* : \mathcal{D}(S) \to \mathcal{D}(R)$ is fully faithful (see [GL, Section 4]).

Two homological epimorphisms of rings $f: R \longrightarrow S$ and $g: R \longrightarrow S'$ are said to be equivalent if there exists an isomorphism of rings $h: S \longrightarrow S'$ such that hf = g. The concept of homological epimorphism of rings can be "naturally" generalized to the setting of dg algebras ([P]) and to the more general setting of dg categories ([NS]). Here we give the definition of homological epimorphism of dg algebras and its characterization at the level of derived categories.

Definition 2.12. [P, Theorem 3.9] Let $F: C \longrightarrow D$ be a morphism between two dg algebras C and D. Then F is called a *homological epimorphism of* dg algebras if the the canonical map $D \overset{\mathbb{L}}{\otimes}_C D \rightarrow D$ is an isomorphism, or equivalently if the induced functor $F_*: \mathcal{D}(D) \longrightarrow \mathcal{D}(C)$ is fully faithful.

Definition 2.13. Two homological epimorphisms of dg algebras $F: C \longrightarrow D$ and $G: C \longrightarrow D'$ are said to be equivalent if there exists an isomorphism of dg k-algebras $H: D \longrightarrow D'$ such that HF = G.

Remark 2. From the definitions it is clear that a homological epimorphism of rings is exactly a homological epimorphism of dg \mathbb{Z} -algebras concentrated in degree 0.

In [NS, Theorem 5] it is proved that for a flat small dg category \mathcal{B} there are bijections between equivalence classes of recollements of $\mathcal{D}(\mathcal{B})$, TTF triples on $\mathcal{D}(\mathcal{B})$ and equivalence classes of homological epimorphisms of dg categories $F: \mathcal{B} \to \mathcal{C}$.

Moreover, in [NS, Lemma 5] it is observed that every derived category of a small dg category is triangle equivalent to the derived category of a small flat dg category. To achieve this result one uses construction of a model structure on the category of all small dg categories defined by Tabuada (see [T].)

We now state [NS, Theorem 4] for the case of a flat dg k-algebra and we give a proof, since in this case the construction of the homological epimorphism becomes more explicit and it will also be used later on in Theorem 6.1.

Lemma 2.14. Let B be a dg algebra flat as a k-module and $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ be a TTF triple in $\mathcal{D}(B)$. Then there is a dg algebra C and a homological epimorphism $F: B \to C$ such that \mathcal{Y} is the essential image of the restriction of scalars functor $F_*: \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$.

Proof. Since $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ is a TTF triple in $\mathcal{D}(B)$ there exists a triangle

(2) $X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1], \text{ with } X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}.$

where φ_B is the unit morphism of the adjunction. Without loss of generality, we may assume that Y is an \mathcal{H} -injective left dg B-module and that φ_B is a morphism in $\mathcal{C}(B)$.

Let $E = \mathbb{R}\text{Hom}_B(Y, Y) = \mathcal{H}om_B(Y, Y)$, then ${}_BY_E$ is a dg B-E-bimodule. Applying the functor $\mathbb{R}\text{Hom}_B(-, Y)$ to the triangle (2) we obtain a triangle in the derived category $\mathcal{D}(E^{\text{op}})$:

 $\mathbb{R}\mathrm{Hom}_B(X[1],Y) \longrightarrow \mathbb{R}\mathrm{Hom}_B(Y,Y) \xrightarrow{\beta} \mathbb{R}\mathrm{Hom}_B(B,Y) \longrightarrow \mathbb{R}\mathrm{Hom}_B(X,Y).$

where $\beta = \mathbb{R}\text{Hom}_B(\varphi_B, Y) = \varphi_*$. Since $X \in \mathcal{X}, Y \in \mathcal{Y}$ and Y is \mathcal{H} -injective, we have, for each $i, n \in \mathbb{Z}$:

$$\operatorname{H}^{n}\mathbb{R}\operatorname{Hom}_{B}(X[i],Y) \cong \operatorname{Hom}_{\mathcal{D}(B)}(X[i],Y[n]) = 0$$

Therefore we deduce that β is a quasi-isomorphism, so we have

(3)
$$E = \mathbb{R}\mathrm{Hom}_B(Y,Y) \stackrel{\beta}{\simeq} \mathbb{R}\mathrm{Hom}_B(B,Y) \stackrel{\gamma}{\simeq} Y \text{ in } \mathcal{D}(E^{\mathrm{op}}).$$

Let $\xi: Y \to Y'$ be a quasi isomorphism of dg *B-E*-bimodules such that Y' is an \mathcal{H} -injective resolution of Y as a dg *B-E*-bimodule. Since B is assumed to be k-flat, we have that the restriction functor from dg *B-E*-bimodules to dg *E*-modules preserves \mathcal{H} -injectivity. In fact, its left adjoint $B \otimes_k -$ preserves acyclicity. Then, Y'_E is an \mathcal{H} -injective right dg *E*-module. Consider the dg algebra $C = \mathcal{H}om_{E^{\mathrm{op}}}(BY'_E, BY'_E) = \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}(BY'_E, BY'_E)$ and a morphism of dg algebras defined by:

$$F \colon B \longrightarrow C$$
$$b \longmapsto F(b) \colon y' \longmapsto (-1)^{|b||y'|} by'$$

where $|\cdot|$ denotes the degree.

Since Y'_E is \mathcal{H} -injective we have quasi-isomorphisms:

$$C = \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}(BY'_{E}, BY'_{E}) \xrightarrow{\xi_{*}} \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}(BY_{E}, BY'_{E}) \xrightarrow{\beta_{*}} \mathbb{R}\mathrm{Hom}_{E^{\mathrm{op}}}(E, BY'_{E}) \cong Y'_{*}$$

We regard C as a dg B-B-bimodule with the action induced by F, so F is also a morphism of dg B-B-bimodules and the morphism $\beta_* \circ \xi_* \colon C \to Y'$ is a quasi-isomorphism of left dg B-modules; moreover, $\xi \circ \varphi_B = \beta_* \circ \xi_* \circ F$. Now define the morphism $\varepsilon := \xi^{-1} \circ \beta_* \circ \xi_* \colon C \to Y$ in $\mathcal{D}(B)$. Then ε is a quasi isomorphism of left dg B-modules such that $\varepsilon \circ F = \varphi$ and we get an isomorphism of triangles:

$$\begin{array}{cccc} X \longrightarrow B \xrightarrow{F} C \longrightarrow X[1] \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1] \end{array}$$

Consider the restriction of scalars functor $F_* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$. F_* is a triangulated functor admitting a right adjoint, hence it commutes with small coproducts. Moreover, $F_*(C) =_B C \cong Y' \cong Y \in \mathcal{Y}$, hence $F_*(\text{Tria } C) = F_*(\mathcal{D}(C))$ is a subcategory of \mathcal{Y} , closed under coproducts and containing the generator $_BY$. Now we notice that, F being a morphism of dg B-B-bimodules, one has a triangle of B-B bimodules:

Consider the adjunction:

$$\mathcal{D}(C) \xrightarrow[F^*]{C \otimes -B}{\mathbb{Z} \longrightarrow \mathcal{D}(B)} \mathcal{D}(B)$$

and let $M \in \mathcal{X}$ and N in $\mathcal{D}(C)$, then

$$\operatorname{Hom}_{\mathcal{D}(C)}(C \underset{B}{\overset{\mathbb{L}}{\otimes}} M, N) \cong \operatorname{Hom}_{\mathcal{D}(B)}(M, F^{*}(N)) = 0$$

since $F^*(N) \in \mathcal{Y}$. Then $C \bigotimes_B^{\mathbb{L}} M = 0$ for each $M \in \mathcal{X}$. Hence, applying the functor $C \bigotimes_B^{\mathbb{L}} -$ to the triangle (4), we obtain

$$C \bigotimes_{B}^{\mathbb{L}} B \cong C \bigotimes_{B}^{\mathbb{L}} C,$$

which shows that F is a homological epimorphism of dg algebras. In particular, Im F_* is a triangulated subcategory of \mathcal{Y} , hence Im $F_* = \mathcal{Y}$, by the principle of infinite dévissage.

3. Recollements from compact objects

In this section we consider recollements between dg algebras induced by compact objects. Our approach follows the exposition in [J] which generalizes to dg algebras the situation considered in [DG] for derived categories of rings. We collect in the next lemma some self-explanatory facts.

Lemma 3.1. Let B be a dg algebra and let Q be an \mathcal{H} -projective left dg B-module compact in $\mathcal{D}(B)$ (i.e. Q is a perfect left dg B-module (2.3)). Consider the dg endomorphism ring D of Q, that is $D = \mathcal{H}om_B(Q,Q)$; then Q becomes a dg B-D-bimodule. Let $P = Q^* = \mathbb{R}Hom_B(Q,B)$, then P is an \mathcal{H} -projective right dg B-module and compact in $\mathcal{D}(B^{op})$; moreover P is a dg D-B-bimodule. The following hold true:

(1) ([DG, Sec 2.5] or [J, Sec 2.1]) The functors

$$H = \mathbb{R}\mathrm{Hom}_B(Q, -), \ G = P \bigotimes_B^{\mathbb{L}} -: \mathcal{D}(B) \to \mathcal{D}(D).$$

are isomorphic.

(2) The functor $\mathcal{H}om_B(-, B)$ induces an equivalence

$$\mathcal{H}om_B(-,B)$$
: per $B \to \text{per } B^{\text{op}}$

with inverse $\mathcal{H}om_{B^{OP}}(-, B)$. Thus, $P^* = \mathbb{R}Hom_B(P, B)$ is isomorphic to Q.

(3) From (2) it follows that the functor $\mathcal{H}om_B(-, B) \colon \mathcal{C}(B) \to \mathcal{C}(B^{\mathrm{op}})$ induces a quasi-isomorphism between the dg algebras $\mathcal{H}om_B(Q,Q)$ and $\mathcal{H}om_{B^{\mathrm{op}}}(P, P)$. Thus we can identify $\mathcal{H}om_{B^{\mathrm{op}}}(P, P)$ with D.

Setup 3.2. In the notations of Lemma 3.1, we set $\mathcal{Y} := \operatorname{Ker}(P \bigotimes_{B}^{\mathbb{L}} -)$. It is well known that the inclusion functor $inc : \mathcal{Y} \longrightarrow \mathcal{D}(B)$ admits both left and right adjoints $L, R : \mathcal{D}(B) \longrightarrow \mathcal{Y}$ and that \mathcal{Y} is generated by L(B). Moreover, L(B) is self-compact, since the inclusion functor preserves coproducts. (See also, [NS, Lemma 2.3].)

The following result appears already in the literature in different forms. In particular, statement (1) can be found in papers by Dwyer Greenless [DG, Sec. 2], Miyachi [Mi, Proposition], Jørgensen [J, Proposition 3.2] and [Y, Theorem 1].

Lemma 3.3. In the notations of Lemma 3.1 and Setup 3.2 the following hold true:

(1) There is a recollement



(2) (Tria Q, Q^{\perp} , Im $\mathbb{R}\text{Hom}_D(P, -)$) is a TTF triple in $\mathcal{D}(B)$.

Proof. (1) We give an alternative proof following the arguments used by Yang in the proof of [Y, Theorem 1]. We first show that the functor $j_! = Q \bigotimes_{D}^{\mathbb{L}} -$ is fully faithful.

By construction we have that $Q \bigotimes_{D}^{\mathbb{L}}$ – induces an equivalence between tria $D \to \text{tria } Q$. In other words the pair $(D, _BQ_D)$ is a standard lift (see [Ke2, Sec.7]). The functor j_1 commutes with set index coproducts, its restriction to tria D is fully faithful and $j_1(D) = Q$ is a compact object. Thus by [Ke2, Lemma 4.2 b] we conclude that j_1 is fully faithful, since D is a compact generator of $\mathcal{D}(D)$.

So the functor $\mathbb{R}\text{Hom}_B(Q, -) \cong (P \bigotimes_D^{\mathbb{L}} -)$ has a fully faithful left adjoint and a right adjoint $\mathbb{R}\text{Hom}_D(P, -)$. By [Mi, Proposition 2.7], the functor $\mathbb{R}\text{Hom}_D(P, -)$ is fully faithful, so the right part of the diagram in the statement can be completed to a recollement with left term the kernel of the functor $\mathbb{R}\text{Hom}_B(Q, -)$, which coincides with the category ${}_BQ^{\perp}$, since Q is a compact object.

(2) To prove the statement it is enough to show that Tria Q is the essential image of the functor $j_! = Q \bigotimes_{D}^{\mathbb{L}} -$. This follows from the facts that the fully faithful functor $Q \bigotimes_{D}^{\mathbb{L}} -$ is a triangle functor which commutes with coproducts and sends the compact generator D of the category $\mathcal{D}(D)$ to the object Q of $\mathcal{D}(B)$, hence its image is Tria Q.

Remark 3. In the notation of Setup 3.2, let $E := \mathbb{R}\text{Hom}_B(L(B), L(B))$. Then by Keller's theorem ([Ke4, Theorem 8.7]) there is a derived equivalence $\mathcal{D}(E) \simeq \mathcal{Y}$ which can be illustrated by the following diagram:



Combining the above remark with Lemmas 2.14 and 3.3 we state in the next proposition the main result of this section. The second part of the statement can be viewed as a generalization of [J, Theorem 3.3], since it characterizes the left term of the recollement as the derived category of a dg algebra obtained by a homological epimorphism.

Proposition 3.4. In the notations of Lemma 3.1 and Setup 3.2 there is a recollement



If B is moreover a k-flat dg algebra, there is a homological epimorphism of dg algebras $F: B \to C$ and a recollement:



such that the essential image of F_* is \mathcal{Y} .

In particular, if $B \in \text{tria } Q$, then \mathcal{Y} vanishes and the functor $\mathbb{R}\text{Hom}_D(P, -)$ induces an equivalence between $\mathcal{D}(D)$ and $\mathcal{D}(B)$ with inverse $P \bigotimes_{B}^{\mathbb{L}} -$.

4. PARTIAL TILTING DG MODULES

In this section we specialize the situation illustrated by Proposition 3.4 to the case of self-orthogonal perfect dg modules.

Our next result, Theorem 4.3, can be viewed as a generalization of the Morita-type theorem proved by Rickard in [R] in the sense that we consider partial tilting dg modules instead of tilting complexes.

Note that some generalizations were obtained also by Koenig in [K] in the case of bounded derived categories of rings.

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Definition 4.1. Let B be a dg algebra. A right (left) dg B-module P is called *partial tilting* if it is perfect and self orthogonal, i.e.

 $\operatorname{Hom}_{\mathcal{D}(B^{\operatorname{op}})}(P, P[n]) = 0 \ (\operatorname{Hom}_{\mathcal{D}(B)}(P, P[n]) = 0), \ \text{ for every } 0 \neq n \in \mathbb{Z}.$

A right (left) dg *B*-module *P* is called *tilting* if it is partial tilting and $B^{\text{op}} \in \text{tria } P \ (B \in \text{tria } P).$

By Lemma 3.1 we have that if ${}_{B}Q$ is a partial tilting left dg B-module, then $P = \mathbb{R}\operatorname{Hom}_{B}(Q, B)$ is a partial tilting right dg B-module and $P^{*} = \mathbb{R}\operatorname{Hom}_{B^{\operatorname{op}}}(P, B)$ is isomorphic to Q. Moreover, $D = \mathbb{R}\operatorname{Hom}_{B}({}_{B}Q, {}_{B}Q) \cong \mathbb{R}\operatorname{Hom}_{B^{\operatorname{op}}}(P_{B}, P_{B})$.

Stalk algebras 4.2. Let P be a partial tilting right dg B module. Let $D = \mathbb{R}\operatorname{Hom}_{B^{\operatorname{op}}}(P_B, P_B)$ and $A = \operatorname{Hom}_{\mathcal{D}(B^{\operatorname{op}})}(P, P)$.

Then, $\mathrm{H}^n(D) \cong \mathrm{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(P, P[n]) = 0$, for every $0 \neq n \in \mathbb{Z}$, hence the dg algebra D has homology concentrated in degree zero and $\mathrm{H}^0(D) \cong A$. Thus, by [Ke4, Sec. 8.4] there is a triangle equivalence $\rho \colon \mathcal{D}(D) \to \mathcal{D}(A)$. For later purposes we give explicitly the functors defining this equivalence and its inverse.

Let $\tau_{\leq 0}$ be the truncation functor and consider the subalgebra

$$D_{-} := \tau_{\leq 0}(D) = \cdots \to D^{-2} \to D^{-1} \to Z^{0}(D) \to 0 \to \dots$$

Then the inclusion $f: D_- \to D$ and $\pi: D_- \to \mathrm{H}^0(D) = A$ are quasiisomorphisms of dg algebras, inducing equivalences f_* and π_* between the corresponding derived categories. Thus we have the following diagrams:



So $\rho = (A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ f_{*}$ and its inverse is $\rho^{-1} = (D \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ \pi_{*}$. Note that $f_{*} \cong_{D_{-}} D \bigotimes_{\otimes}^{\mathbb{L}} -$ and $\pi_{*} \cong_{D_{-}} A \bigotimes_{\otimes}^{\mathbb{L}} -$.

As a special case of Proposition 3.4 we obtain a recollement where the right term is the derived category of a ring.

Theorem 4.3. Let B be a dg algebra and let P be a partial tilting right dg B-module. Let $A = \operatorname{Hom}_{\mathcal{D}(B^{\operatorname{op}})}(P, P)$, $Q = \mathbb{R}\operatorname{Hom}_{B^{\operatorname{op}}}(P, B)$. Then there exists a dg algebra E and a recollement:



where, letting $D = \mathbb{R}\mathrm{Hom}_{B^{\mathrm{op}}}(P, P)$ there is a triangle equivalence $\rho \colon \mathcal{D}(D) \to$ $\mathcal{D}(A)$ such that:

- (1) $j_! = (Q \bigotimes_D^{\mathbb{L}} -) \circ \rho^{-1}.$
- (2) $j^* = \rho \circ (P \bigotimes_{B}^{\mathbb{L}} -).$
- (3) $j_* = \mathbb{R}\operatorname{Hom}_D(P, -) \circ \rho^{-1}.$ (4) $\mathcal{D}(A)$ is triangle equivalent to $\mathcal{D}(B)/\operatorname{Ker}(j^*).$

In particular, if P is a tilting right dq B-module, then \mathcal{Y} vanishes and

$$\rho \circ (P \bigotimes_{B}^{\mathbb{L}} -) \colon \mathcal{D}(B) \to \mathcal{D}(A)$$

is a triangle equivalence with inverse $\mathbb{R}Hom_D(P, -) \circ \rho^{-1}$. Moreover, if B is k-flat there exists a homological epimorphism of dq algebras $F: B \longrightarrow C$ such that the above recollement becomes:



Proof. By Lemma 3.1 we can identify P with $\mathbb{R}\text{Hom}_B(Q, B)$ and $\mathbb{R}\text{Hom}_{B^{\text{op}}}(P, P)$ with \mathbb{R} Hom_B(Q, Q).

The existence of an equivalence $\rho: \mathcal{D}(D) \to \mathcal{D}(A)$ is ensured by Stalk algebras 4.2. An application of Lemma 3.3 (4) proves the statement. If Bis k-flat we use Proposition 3.4 to conclude. \square

Remark 4. Rickard's Theorem states that if B is a flat k-algebra over a commutative ring k and P_B is a tilting complex of right B-modules with endomorphism ring A, then there is a complex ${}_{A}X_{B}$, with terms that are A-B bimodules, isomorphic to P_B in $\mathcal{D}(B)$, and such that $X \bigotimes_{B}^{\mathbb{L}} -: \mathcal{D}(B) \to \mathcal{D}(A)$ is an equivalence with inverse the functor $\mathbb{R}Hom_A(X, -)$. Equivalences of this form are called *standard equivalences* (see [Ke1, Sec. 1.4]). It is still an open problem to decide if all triangle equivalences between derived categories of rings (or dg algebras) are isomorphic to standard equivalence (see [Ke4, Sec. 6.1]).

In the same assumptions as in Rickard's Theorem, but without any flatness condition on B, our Theorem 4.3 provides an equivalence between $\mathcal{D}(B)$ and $\mathcal{D}(A)$. An analysis of the way in which this equivalence is constructed, shows that it is induced by the composite derived functor $A \bigotimes_{D_{-}}^{\mathbb{L}} (D_{-}P \bigotimes_{B}^{\mathbb{L}} -)$ where $D_{-} = \tau_{\leq 0}(D)$ and P is viewed as a dg D_{-} -B-bimodule.

5. TILTING AND PARTIAL TILTING MODULES

The notion of tilting modules goes back to works by Bernstein, Gel'fand and Ponomarev, Brenner and Butler, Happel and Ringel, Auslander, Platzeck and Reiten [BGP, BB, HR, APR], and it has first been considered in the case

of finitely generated modules of projective dimension at most one over artin algebras and later generalized to arbitrary rings and to possibly infinitely generated modules of finite projective dimension (see [AC], [CF], [CT]).

We recall the definition of (partial) tilting modules over a ring R by using the canonical embedding of the category of R-modules into the derived category $\mathcal{D}(R)$ and we restate the various definitions using the terminology of derived categories theory.

Definition 5.1. Let R be a ring and T an R-module. Consider the following conditions on T viewed as an object of $\mathcal{D}(R)$ under the canonical embedding:

- (T1) T is isomorphic to a bounded complex with projective terms;
- (T1') T is a compact object of $\mathcal{D}(R)$;
- (T2) T is orthogonal to coproducts of copies of T, that is $\operatorname{Hom}_{\mathcal{D}(R)}(T, T^{(\alpha)}[n]) = 0$ for every $0 \neq n \in \mathbb{Z}$ and every set α .
- (T2') T is self orthogonal, that is $\operatorname{Hom}_{\mathcal{D}(R)}(T, T[n]) = 0$ for every $0 \neq n \in \mathbb{Z}$.
- (T3) $R \in \text{Tria } T$.
- (T3') $R \in \text{tria } T$.

If the projective dimension of T is at most n, then T is called a *classical* n-*tilting* module, if it satisfies (T1'), (T2') and (T3') and a *classical partial* n-*tilting* module if it satisfies (T1') and (T2'). T is called an n-*tilting* module (possibly infinitely generated), if it satisfies (T1), (T2) and (T3) and it is called a *good* n-*tilting* module if it satisfies (T1), (T2) and (T3').

Classical tilting modules were introduced mainly to generalize Morita theory. They provide equivalences between suitable subcategories of module categories (see [BB], [Miy]).

In [H] and [CPS] it was shown that a classical *n*-tilting module over a ring A with endomorphism ring B induces a triangle equivalence between $\mathcal{D}(A)$ and $\mathcal{D}(B)$.

Infinitely generated tilting modules do not provide equivalences between derived categories of rings, but the first named author proved in [B] that, if T is a good 1-tilting module over a ring A with endomorphism ring B, then the total left derived functor $T \bigotimes_{B}^{\mathbb{L}}$ – induces an equivalence between

 $\mathcal{D}(B)/\operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$ and $\mathcal{D}(A)$. This result has been generalized in [BMT] to the case of good *n*-tilting modules.

Recently Chen and Xi [CX] completed the result proved in [B] for good 1-tilting modules T over a ring A, by showing that the derived category of the endomorphism ring B of T is the central term of a recollement with right term $\mathcal{D}(A)$ and left term the derived category of a ring C for which there is a homological ring epimorphism $\lambda : B \to C$ where moreover C is a universal localization of a suitable morphism between finitely generated projective B-modules.

The disadvantage of starting with an infinitely generated *n*-tilting module $_{A}T$ over a ring A, is that a good *n*-tilting module T' equivalent to $_{A}T$ is obtained as a summand of a possibly infinite direct sum of copies of T and this procedure produces a very large endomorphism ring B of T'. So the

recollement induced by T' concerns the derived category of a ring which is hardly under control.

In our approach, instead, we can fix a ring B and obtain recollements of $\mathcal{D}(B)$ for every choice of classical partial tilting modules. Here we note that if a (possible infinitely generated) module ${}_{A}T$ over A satisfies conditions (T2') and (T3'), then, by [Miy, Proposition 1.4 (2)], T is a partial classical ntilting module over its endomorphism ring B and, moreover, $\operatorname{End}_{B}(T) \cong A$. In particular, we can apply the results of Section 4. This allows to obtain the same conclusion as in [BMT], but with weaker hypotheses on the module ${}_{A}T$, namely without asking that it is a good n-tilting A-module, but only that it satisfies conditions (T2') and (T3').

More precisely a direct consequence of Theorem 4.3 yields the following proposition which can be viewed as a generalization of [BMT, Theorem 2.2].

Proposition 5.2. Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Keeping the notations in Setup 3.2 there is a dg algebra E and a recollement



where:

- (1) $j_* = \mathbb{R}\text{Hom}_A(T, -)$ is fully faithful;
- (2) $\mathcal{D}(A)$ is triangle equivalent to $\mathcal{D}(B)/\mathrm{Ker} (T \bigotimes_{B}^{\mathbb{L}} -).$

Moreover, if B is k-flat, there is a homological epimorphism of dg algebras $F: B \to C$ and the recollement above becomes



Proof. Let P be a projective resolution of the module T in Mod-B. Then P is a partial tilting complex of $\mathcal{D}(B)$ so that we may apply Theorem 4.3 which states that there is a triangle equivalence $\rho: \mathcal{D}(D) \to \mathcal{D}(A)$ where $D = \mathbb{R}\operatorname{Hom}_{B^{op}}(P, P)$.

As shown in Stalk algebras 4.2 we have:

$$\rho = (A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ f_*.$$

where $f_*: \mathcal{D}(D) \to \mathcal{D}(D_-)$ is the restriction of scalar functors induced by the quasi-isomorphism of dg algebras $f: D_- \to D$ To conclude the proof we must show that

(a)
$$\rho \circ (P \bigotimes_{B}^{\mathbb{L}} -) \cong T \bigotimes_{B}^{\mathbb{L}} -,$$

(b) $\mathbb{R}\operatorname{Hom}_{D}(P, -) \circ \rho^{-1} \cong \mathbb{R}\operatorname{Hom}_{A}(T, -).$

We first prove (a).

Let $\sigma: P_B \to T_B$ be a morphism of complexes inducing a quasi-isomorphism in D(B). From the dg algebra morphisms $f: D_- \to D$ and $\pi: D_- \to A$ we have that P and T are left dg D_- -modules. Checking the action of the dg algebra D_- on P and T we see that σ is a morphism of dg D_- -modules. Thus, σ is a quasi isomorphism between P and T as dg D_-B -bimodules.

This implies that the functors $P \bigotimes_{B}^{\mathbb{L}} - \text{and } T \bigotimes_{B}^{\mathbb{L}} - \text{from } \mathcal{D}(B)$ to $\mathcal{D}(D_{-})$ are isomorphic (see [Ke2, Lemma 6.1 b]). Consequently, , in the notations of Stalk algebras 4.2, we have:

$$j^* = \rho \circ (P \bigotimes_{B}^{\mathbb{L}} -) = (A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ f_* \circ (P \bigotimes_{B}^{\mathbb{L}} -) \cong (A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ (D_{-} P \bigotimes_{B}^{\mathbb{L}} -) \cong$$
$$\cong ((A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ (D_{-} T \bigotimes_{B}^{\mathbb{L}} -) \cong (A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ \pi_* \circ (A T \bigotimes_{B}^{\mathbb{L}} -).$$

Since $(A \bigotimes_{D_{-}}^{\mathbb{L}} -) \circ \pi_*$ is isomorphic to the identity of $\mathcal{D}(A)$, we conclude that $j^* \cong_A T \bigotimes_{D_{-}}^{\mathbb{L}} -.$

Next, from the uniqueness of right adjoints up to isomorphisms, we also get

$$\mathbb{R}\mathrm{Hom}_D(P,-) \circ \rho^{-1} \cong \mathbb{R}\mathrm{Hom}_A(T,-).$$

Note 5.3. In the assumption of Proposition 5.2 if we let $Q = \mathbb{R}\text{Hom}_{B^{op}}(T, B)$, then, by lemma 3.1 (1) we have $\mathbb{R}\text{Hom}_B(Q, -) \cong P \bigotimes_B^{\mathbb{L}} -$, hence also

$$f_* \circ \mathbb{R}\mathrm{Hom}_B(Q, -) \cong f_* \circ (P \bigotimes_B^{\mathbb{L}} -) \cong {}_AT \bigotimes_B^{\mathbb{L}} -.$$

6. The case of homological epimorphism of rings

As said in the introduction, the situation in which a compactly generated triangulated category is a recollement of triangulated categories compactly generated by a single object has been studied by many authors ([K], [NS, Corollary 3.4] and [AKL, Sec. 1.5]).

We are interested in studying the case in which the subcategory $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$ in Proposition 5.2 is equivalent to the derived category of a ring via a homological ring epimorphism.

The problem is related to the notion of bireflective and perpendicular categories (see Definition 2.10 and Lemma 2.11.)

Let T_B be a classical partial *n*-tilting module over a ring *B* with endomorphism ring *A*. Consider the canonical embedding of *B*-Mod in $\mathcal{D}(B)$ and let \mathcal{E} be the full subcategory of *B*-Mod defined by:

$$\mathcal{E} = \{ N \in B \text{-Mod} \mid N \in \mathcal{Y} \} = \{ N \in B \text{-Mod} \mid T \bigotimes_{B}^{\mathbb{L}} N = 0 \}$$

Then, $\mathcal{E} = \{ N \in B \text{-Mod} \mid \text{Tor}_i^B(T, N) = 0 \text{ for all } i \ge 0 \}.$

Remark 5. Note that \mathcal{E} is closed under extensions, direct sums and direct products (since T is a classical partial tilting module). So \mathcal{E} is bireflective if and only if it is closed under kernel and/or cokernels.

Theorem 6.1. Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$, L the left adjoint of the inclusion inc: $\mathcal{Y} \to \mathcal{D}(B)$ and \mathcal{E} the subcategory of B-Mod defined above.

Then the following conditions are equivalent:

- (1) $\operatorname{H}^{i}(L(B)) = 0$ for every $0 \neq i \in \mathbb{Z}$.
- (2) there is a ring S and a homological ring epimorphism $\lambda: B \to S$ inducing a recollement:



- (3) Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in \mathcal{E} and \mathcal{E} is a bireflective subcategory of B-Mod.
- (4) Every $N \in \mathcal{Y}$ is quasi-isomorphic to a complex with terms in \mathcal{E} and the homologies of N belong to \mathcal{E} .

Proof. Note that the equivalence between (1) and (2) was somehow known to topologists, as shown for instance in [D].

 $(1) \Rightarrow (2)$ Let Y = L(B). First note that, by adjunction, we have Hom_{$\mathcal{D}(B)$} $(Y, Y[i]) \cong \text{Hom}_{\mathcal{D}(B)}(B, Y[i]) \cong \text{H}^{i}(Y)$. Thus, by [Ke4, Theorem 8.7], condition (1) implies that the dg algebra $E = \mathbb{R}\text{Hom}_{B}(Y, Y)$ has homology concentrated in degree zero and $\text{H}^{0}(E) \cong \text{Hom}_{\mathcal{D}(B)}(Y, Y)$.

Consider a triangle

(5)
$$X \longrightarrow B \xrightarrow{\varphi_B} Y \longrightarrow X[1], \text{ with } X \in^{\perp} \mathcal{Y}.$$

where φ_B is the unit of the adjunction morphism and set $S = \operatorname{Hom}_{\mathcal{D}(B)}(Y, Y)$. As in [AKL, Proposition 1.7], define a ring homomorphism $\lambda \colon B \to S$ by $\lambda(b) = L(\dot{b})$, where \dot{b} denotes the right multiplication by b on B. We have ${}_BS = \operatorname{Hom}_{\mathcal{D}(B)}(Y,Y) \cong \operatorname{Hom}_{\mathcal{D}(B)}(B,Y) \cong \operatorname{H}^0(Y) \cong Y$. So we have a quasiisomorphism $\varepsilon \colon BS \to_B Y$ and from the definition one sees that $\varepsilon \circ \lambda = \varphi_B$. Thus we have an isomorphism of triangles:



Now we can continue arguing as in the last part of the proof of Lemma 2.14 to conclude that λ is a homological epimorphism and that \mathcal{Y} is the essential image of λ_* . So condition (2) follows.

(2) \Rightarrow (3) The subcategory $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$ is the essential image of the functor λ_* , hence the image of S-Mod under λ_* is the category \mathcal{E} . Every object in \mathcal{Y} is quasi-isomorphic to a complex with S-modules terms, hence in \mathcal{E} . Moreover, since λ is an epimorphism of rings, the differentials are S-module morphisms. Hence, Lemma 2.11 tells us that \mathcal{E} is bireflective.

 $(3) \Rightarrow (4)$ This follows from the fact that \mathcal{E} is closed under kernels and cokernels.

(4) \Rightarrow (1) We first show that condition (4) implies that \mathcal{E} is bireflective. Indeed, let $E_0 \xrightarrow{f} E_1$ be a morphism in \mathcal{E} . Then, the complex $E' = \ldots 0 \rightarrow E_0 \xrightarrow{f} E_1 \rightarrow 0 \rightarrow \ldots$ has $(T \bigotimes_B -)$ -acyclic terms so $T \bigotimes_B^{\mathbb{L}} E' = T \bigotimes_B E' = 0$. By (4) the kernel and the cokernel of f belong to \mathcal{E} . Thus \mathcal{E} is bireflective by Remark 5. By Lemma 2.11 there is a ring S and a ring epimorphism $\lambda \colon R \to S$ such that $\mathcal{E} = \lambda_*(S$ -Mod) where $\lambda_* \colon S$ -Mod $\to B$ -Mod is the restriction functor.

We show now that $L(B) \cong \lambda_*(S)$.

For this aim we follows the arguments used in [CX, Proposition 3.6]. Let Y_0 be a complex in \mathcal{Y} with terms in \mathcal{E} and quasi-isomorphic to L(B). Let $B \xrightarrow{\varphi} Y_0$ be the unit adjunction morphism associated to the adjoint pair (L, j). Since S viewed as a left B-module belongs to \mathcal{Y} we have that $\operatorname{Hom}_{\mathcal{Y}}(Y_0, S) \cong \operatorname{Hom}_{\mathcal{D}(B)}(B, S)$, so there is a unique morphism $f: Y_0 \to S$ such that $\lambda = f \circ \varphi$.

We have $\operatorname{Hom}_{\mathcal{H}(B)}(S, Y_0) \cong \operatorname{H}^0(\operatorname{Hom}_B(S, Y_0))$ and, since $\lambda \colon B \to S$ is a ring epimorphism, $\operatorname{Hom}_B(S, Y_0) = \operatorname{Hom}_S(S, Y_0)$, and the terms of Y_0 are *S*-modules. Thus, $\operatorname{Hom}_{\mathcal{H}(B)}(S, Y_0) \cong \operatorname{H}^0(Y_0) \cong \operatorname{Hom}_{\mathcal{H}(B)}(B, Y_0)$. Now, every morphism in $\operatorname{Hom}_{\mathcal{D}(B)}(S, Y_0)$ is the image under the canonical quotient functor of a morphism in $\operatorname{Hom}_{\mathcal{H}(B)}(S, Y_0)$, hence going through the construction of the above isomorphisms, we conclude that there is $g \in \operatorname{Hom}_{\mathcal{D}(B)}(S, Y_0)$ such that $g \circ \lambda = \varphi$. Consequently, $g \circ f \circ \varphi = \varphi$ and $\lambda = f \circ g \circ \lambda$. Since λ is an \mathcal{E} -reflection of B and φ is the unit morphism of the adjunction, we conclude that $f \circ g = id_S$ and $g \circ f = id_{Y_0}$. So $S \cong Y_0 \cong L(B)$, hence (1) follows.

Remark 6. Note that if condition (2) of Proposition 6.1 holds, then there is a homological ring epimorphism $\lambda: B \to S$ even without the assumption of flatness on B. The key point is the existence of a quasi-isomorphim between the ring S and the left adjoint of B

We add another property related to the situation considered above.

Proposition 6.2. In the notations of Theorem 6.1 consider the following conditions:

- (a) a complex of $\mathcal{D}(B)$ belongs to \mathcal{Y} if and only if all its homologies belong to \mathcal{E} .
- (b) \mathcal{E} is bireflective.
- (c) There is a ring R and a ring epimorphism $\mu: B \to R$ such that ${}_{B}R \in \mathcal{E}$ and \mathcal{Y} is contained in the essential image of the restriction functor $\mu_*: \mathcal{D}(R) \to \mathcal{D}(B)$.

Then (a) implies (b) and (a) together with (c) is equivalent to any one of the conditions in Theorem 6.1.

In particular, if $_{A}T$ is a good n-tilting module with endomorphism ring B, then (a) is equivalent to any one of the conditions in Theorem 6.1.

Proof. Assume that condition (a) holds. Arguing as in the first part of the proof of $(4) \Rightarrow (1)$ in Theorem 6.1, we see that \mathcal{E} is bireflective.

Condition (c) imply $\mu_*(R\text{-Mod}) \subseteq \mathcal{E}$, hence every complex in \mathcal{Y} is quasiisomorphic to a complex with terms in \mathcal{E} . Thus, assuming both (a) and (c), we have that condition (4) in Theorem 6.1 is satisfied.

Conversely, if condition (2) of Theorem 6.1 is satisfied, then by [AKL, Lemma 4.6] (a) holds; moreover, (c) is satisfied by choosing the ring epimorphism $\lambda: B \to S$.

To prove the last statement it is enough to show that, if ${}_{A}T$ is a good *n*-tilting module then condition (c) holds. This follows as in the proof of [CX, Proposition 4.6], which is stated for the case of 1-good tilting module, but the argument used there works also in case of higher projective dimension.

7. Generalized Universal localizations

Chen and Xi in [CX] consider the case of a good 1-tilting module ${}_{A}T$ with endomorphism ring B. In particular T_B becomes a classical partial 1-tilting module over B. They show that the left end term of a recollement as in the statement of Theorem 6.1, is the derived category of a universal localization of B. Indeed if $0 \to P_1 \xrightarrow{f} P_0 \to T_B \to 0$ is a projective resolution of T as right B-module, then $\lambda \colon B \to S$ is the universal localization, in the sense of Cohn and Schofield, of B at the morphism f. This means that $f \bigotimes_B S$ is an isomorphism and S satisfies a universal property with respect to this condition, that is for any ring homomorphism $\mu \colon B \to S'$ such that $f \bigotimes_B S'$ is an isomorphism, there is a unique ring homomorphism $\nu \colon S \to S'$ such that $\nu \circ \lambda = \mu$.

Note that $f \underset{B}{\otimes} S$ is an isomorphism if and only is the complex

$$\cdots \to 0 \to P_1 \underset{B}{\otimes} S \xrightarrow{f \otimes_B S} P_0 \underset{B}{\otimes} S \to 0 \to \dots$$

is acyclic.

Inspired by the above interpretation of universal localization, there is a natural way to generalize the notion of universal localization as follows. We give the definition, which was first introduced by Krause under the name *homological localization*.

Definition 7.1. (See [Kr, Section 15]) Let *B* be a ring and Σ a set of perfect complexes $P \in \mathcal{H}(B)$. A ring *S* is a *generalized universal localization* of *B* at the set Σ if:

- (1) there is a ring homomorphism $\lambda \colon B \to S$ such that $P \bigotimes_B S$ is acyclic;
- (2) for every ring homomorphism $\mu: B \to R$ such that $P \bigotimes_B R$ is acyclic, there exists a unique ring homomorphism $\nu: S \to R$ such that $\nu \circ \lambda = \mu$.

Lemma 7.2. If $\lambda: B \to S$ is a generalized universal localization of B at a set Σ of compact objects P of $\mathcal{D}(B)$, then λ is a ring epimorphism.

Proof. Let $\delta \colon S \to R$ be a ring homomorphism. Then, for every $P \in \Sigma$ we have:

$$P \underset{B}{\otimes} R = (P \underset{B}{\otimes} S) \underset{S}{\otimes} R.$$

Now $P \underset{B}{\otimes} S$ is an acyclic and bounded complex whose terms are projective right *S*-modules, then the complex $(P \underset{B}{\otimes} S) \underset{S}{\otimes} R$ is still acyclic. By the universal property satisfied by *S* we conclude that δ is the only possible ring homomorphism extending $\mu = \delta \circ \lambda$.

Now we can relate the result stated in Theorem 6.1 with the notion of generalized universal localization.

Proposition 7.3. Let B be a ring and let T_B be a classical partial n-tilting module with endomorphism ring A. Let P be a projective resolution of T_B in $\mathcal{D}(B)$.

If condition (2) in Theorem 6.1 is satisfied, then $\lambda \colon B \to S$ is a generalized universal localization of B at the set $\{P\}$.

Proof. As usual let $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$. By assumptions $\lambda_*(S) \in \mathcal{Y}$, thus $T \bigotimes_{B}^{\mathbb{L}} S = 0$, so $P \bigotimes_{B} S$ is acyclic. Moreover, $\mathcal{Y} \cap B$ -Mod = \mathcal{E} is bireflective and, by [GL, Proposition 3.8], we have that $\lambda_*(S) = l(B)$, where l: B-Mod $\rightarrow \mathcal{E}$ is the left adjoin of the inclusion of $i: \mathcal{E} \rightarrow B$ -Mod. Let $\mu: B \rightarrow S'$ be a ring homomorphism such that $P \bigotimes_{B} S'$ is acyclic, then also $T \bigotimes_{B}^{\mathbb{L}} S' = 0$, hence $S' \in \mathcal{E}$. Thus, $\operatorname{Hom}_{B}(l(B), S') \cong \operatorname{Hom}_{B}(B, S')$, hence there is a unique morphism $\rho: l(B) \rightarrow S'$ of right B-modules such that $\rho \circ \eta_B = \mu$, where $\eta_B: B \rightarrow l(B)$ is the unit morphism of the adjunction. Using the fact that $S = \operatorname{End}_{B}(l(B))$ and the naturality of the maps induced by the adjunction (l, j), it is not hard to see that ρ induces a unique ring homomorphism $\nu: S \rightarrow S'$ such that $\nu \circ \lambda = \mu$.

Remark 7. Note that the converse of the above statement does not hold in general. In fact, as shown in [AKL, Example 5.4] even in the case of a classical 1-tilting module over an algebra, the universal localization does not give rise to a homological epimorphism.

We now illustrate another property of the "generalized universal localization" .

Proposition 7.4. Let P be a compact complex in $\mathcal{D}(B)$. Assume that $\lambda: B \to S$ is a "generalized universal localization" of B at $\{P\}$. Let $\mathcal{E}_P = \{N \in B\text{-Mod} \mid P \bigotimes_B N \text{ is acyclic }\}$. Then, the following hold:

- (1) $\lambda_*(S-Mod) \subseteq \mathcal{E}_P$.
- (2) $\lambda_*(S-Mod) = \mathcal{E}_P$ if and only if \mathcal{E}_P is a bireflective subcategory of B-Mod.

Proof. (1) Let ${}_{B}M \in \lambda_{*}(S\text{-Mod})$. We have

$$P \underset{B}{\otimes} M \cong P \underset{B}{\otimes} (S \underset{S}{\otimes} M) \cong (P \underset{B}{\otimes} S) \underset{S}{\otimes} M$$

and $(P \bigotimes_B S)$ is a complex in $\mathcal{D}(S)$ whose terms are finitely generated projective right S-modules and by assumption it is acyclic. Thus, $P \bigotimes_B M$ is acyclic too, so $M \in \mathcal{E}_P$.

(2) By Lemma 7.2, λ is a ring epimorphism, hence, if $\lambda_*(S-Mod) = \mathcal{E}_P$, then \mathcal{E}_P is bireflective, by Lemma 2.11.

Conversely, assume that \mathcal{E}_P is bireflective. By Lemma 2.11, there is a ring R and a ring epimorphism $\mu: B \to R$ such that $\mu_*(R\text{-Mod}) = \mathcal{E}_P$. In particular, $\mu_*(R) \in \mathcal{E}_P$, hence $P \otimes R$ is an acyclic complex. Thus, by B the universal property satisfied by S, there is a unique ring homomorphism $\nu: S \to R$ such that $\nu \circ \lambda = \mu$. By Lemma 2.11, $\mu: B \to R$ is an \mathcal{E}_P -reflection of B and $S \in \mathcal{E}_P$ by part (1). We infer that there is a unique morphism $\rho: R \to S$ such that $\rho \circ \mu = \lambda$. By the unicity of the rings homomorphisms ν and ρ it follows that they are inverse to each other.

8. Examples

In the notations of Section 6 we give some examples of different behaviors of classical *n*-partial tilting modules with respect to the class \mathcal{E} . In what follows k will indicate an algebraically closed field.

Example 1. We exhibit examples of classical partial tilting modules T_B of projective dimension two over an artin algebra B such that there exists a "generalized universal localization" S of B at the projective resolution of T_B and moreover, there exists a homological ring epimorphism $\lambda \colon B \to S$ such that the class $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -)$ is triangle equivalent to $\mathcal{D}(S)$. In this case the classical partial tilting module T_B doesn't arise from a good tilting module.

Consider a representation-finite type algebra $\Lambda := kQ/I$ of an acyclic connected quiver Q (with n > 1 vertices) with a unique sink j and the category mod- Λ of the finite dimensional right Λ -modules. Note that we use the just-apposition of arrows for the product in Λ).

Let $T_{\Lambda} = \tau^{-1}(S(j)) \oplus (\bigoplus_{i \neq j} P(i))$ be an APR tilting module over Λ (see [APR]). Then proj.dim. $(T_{\Lambda}) = 1$ and its projective resolution is given by

$$0 \longrightarrow S(j) \longrightarrow (\bigoplus_{i \neq j} P(i)) \oplus E \longrightarrow T_{\Lambda} \longrightarrow 0$$

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where

$$0 \longrightarrow S(j) \longrightarrow E \longrightarrow \tau^{-1}(S(j)) \longrightarrow 0$$

is an almost split exact sequence with E a projective Λ -module. Let $S(j)^d := \operatorname{Hom}_k(S(j), k)$ and consider $B := \begin{pmatrix} k & 0 \\ S(j)^d & \Lambda \end{pmatrix}$ the one point coextension of Λ by the non injective simple $S(j)_{\Lambda}$ (see [ASS]). In particular $B \simeq kQ'/J$ where Q' is the quiver Q with the adjoint of a sink * and of an arrow $j \longrightarrow *$. Let I(*) and S(*) be the indecomposable injective and simple right B-modules at the vertex *, respectively. Then $I(*) = \begin{array}{c} j \\ * \\ * \end{array}$ and letting $P(*) = I(*)^d = \operatorname{Hom}_k(I(*), k)$ be the indecomposable projective at the vertex * (regarded as right module on B^{op}), then $P(*) = \begin{array}{c} * \\ * \\ i \end{array}$.

Every Λ -module can be regarded as a *B*-module via the natural embedding $\varphi : \operatorname{mod} \Lambda \hookrightarrow \operatorname{mod} B$.

Proposition 8.1. The following hold:

- (1) T_B has projective dimension 2.
- (2) T_B is self orthogonal.
- (3) $\mathcal{E}_{\Lambda} = \{ M \in \Lambda \text{-Mod} \mid \text{Tor}_{i}^{\Lambda}(T, M) = 0, \forall i \geq 0 \} = 0 \text{ and}$

$$\mathcal{E}_B = \{ M \in B \text{-Mod} \mid \text{Tor}_i^B(T, M) = 0 \ \forall i \ge 0 \} = \text{Add} \ I(*)^d = \text{Add} \ P(*)$$

where for every module M, Add M denotes the class of all direct summands of arbitrary direct sums of copies of M.

Proof. (1) We have that S(j) regarded as B module is non projective and its projective cover is given by

$$I(*) \longrightarrow S(j) \longrightarrow 0.$$

Hence a projective resolution of T_B is

$$0 \longrightarrow S(*) \longrightarrow I(*) \longrightarrow (\bigoplus_{i \neq j} P(i)) \oplus E \longrightarrow \tau^{-1}(S(j)) \oplus (\bigoplus_{i \neq j} P(i)) \longrightarrow 0.$$

(2) To prove the self-orthogonality of T_B we can observe that mod- Λ is equivalent to the class mod- $B \cap \text{Ker}(\text{Hom}_B(-, I(*)))$, then, in particular, it is closed under extensions in mod-B. Hence it is clear that

$$\operatorname{Ext}_{B}^{1}(T_{B}, T_{B}) \simeq \operatorname{Ext}_{\Lambda}^{1}(T_{\Lambda}, T_{\Lambda}) = 0.$$

Moreover

$$\operatorname{Ext}_{B}^{2}(T_{B}, T_{B}) = \operatorname{Ext}_{B}^{1}(S(j)_{B}, T_{B}) = \operatorname{Ext}_{\Lambda}^{1}(S(j), T_{\Lambda}) = 0.$$

(3) $\mathcal{E}_{\Lambda} = 0$ because T_{Λ} is a tilting module. Now, ind- $B \setminus \operatorname{ind} \Lambda = \{I(*), S(*)\}$ and $I(*) = \frac{j}{*}$. We compute the class

$$\mathcal{E}_B = \{ M \in B\text{-Mod} \mid \text{Tor}_i^B(T, M) = 0, \forall i \ge 0 \} =$$
$$= \{ M \in \text{Mod-}B^{op} \mid \text{Ext}_{B^{op}}^i(M, T^d) = 0, \forall i \ge 0 \}$$

where $T_B^d := \operatorname{Hom}_k(T_B, k)$. We can regard *B*-Mod as Mod- B^{op} and B^{op} is the one point extension of Λ^{op} by the simple $S(j)^d = S(j)$. We claim

that $\mathcal{E}_B = \text{Add}(P(*))$. Note that, as in the previous case, $\text{ind}-B^{op}\setminus \text{ind}-\Lambda^{op} = \{P(*), S(*)\}$ and $P(*) = \frac{*}{j}$. From the fact that $\mathcal{E}_{\Lambda} = 0$ and that every Λ -module can be regarded as a *B*-module, only Add $\{P(*), S(*)\}$ could be contained in \mathcal{E}_B .

We prove that $S(*) \notin \operatorname{KerExt}_B^2(-, T_B^d)$. Since S(j) is the first cosyzygy of the injective resolution of $T_{\Lambda}^d = \tau^{-1}(S(j))^d \oplus (\bigoplus_{i \neq j} I(i))$, we show that $S(*) \notin I(i)$

 $\operatorname{KerExt}^{1}_{B}(-, S(j))$. Indeed there is the non split short exact sequence

$$0 \longrightarrow S(j) \longrightarrow \begin{array}{c} *\\ j \end{array} \longrightarrow S(*) \longrightarrow 0.$$

Hence $S(*) \notin \mathcal{E}_B$. To show that $P(*) \in \mathcal{E}_B$ we only have to check that $\operatorname{Hom}_{B^{op}}(P(*), T_B^d) = 0$, since P(*) is projective. This is true from the fact that the top of P(*) = S(*) does not belongs to any composition series of T_B^d . Then $\mathcal{E}_B = \operatorname{Add}(P(*)) = \operatorname{Add}(I(*)^d)$.

Set now $A := \operatorname{End}_B(T_B) = \operatorname{End}_\Lambda(T_\Lambda)$, then $\Lambda = \operatorname{End}_A(_AT)$ because T_Λ is tilting (hence balanced) over Λ . Hence $_AT$ is 1-tilting, but $\operatorname{End}_A(_AT) \neq B$, so $_AT$ is not faithfully balanced.

In the sequel we will simply write \mathcal{E} for the class \mathcal{E}_B .

Lemma 8.2. For each projective left B-module P the unit morphism

$$\overline{\eta}_P: P \longrightarrow \operatorname{Hom}_A(T, T \otimes_B P)$$

of the adjunction $(T \otimes_B -, \operatorname{Hom}_A(_AT, -))$, is surjective and $\operatorname{Ker}\overline{\eta}_P \in \mathcal{E} = \operatorname{Add}(P(*))$.

Proof. We can regard

$$\overline{\eta}_B: B \longrightarrow \operatorname{Hom}_A({}_AT, T_B \otimes_B B) \simeq \Lambda$$

as the projection $\pi : B \longrightarrow \Lambda \simeq B/Be_*B$, hence it is surjective and the kernel is the annihilator of T_B as right *B*-module, that is $\operatorname{Ker}\overline{\eta}_P$ is the projective *B*-module P(*). Now, since \mathcal{E} is closed under direct summand, we can prove the statement just for free modules. Let α be a cardinal, then the map

$$\overline{\eta}_{B(\alpha)}: B^{(\alpha)} \longrightarrow \operatorname{Hom}_A(T, T \otimes_B B^{(\alpha)}) = \Lambda^{(\alpha)}$$

is exactly $\pi^{(\alpha)}$ and the kernel of $\overline{\eta}_{B^{(\alpha)}}$ is $P(*)^{(\alpha)}$.

Proposition 8.3. There is a homological ring epimorphism

 $\lambda: B \longrightarrow S$

with $S = \operatorname{End}(P(*)^{\oplus 2})$.

Proof. We claim that $\mathcal{E} = \text{Add}(P(*))$ is bireflective. A linear representation of $P(*)^{(I)}$ for some cardinal I is of the form

$$k^{(I)} \xrightarrow{\varphi} k^{(I)}$$

at the verteces * and j and zero at the other vertices, where φ is an isomorphism of k-vector spaces. An object is in Add P(*) if and only if it is

a summand of a k-linear representation of this form. Hence, for every morphism $f: P(*)^{(I)} \longrightarrow P(*)^{(J)}$, Kerf, Cokerf, and Im f belong to Add P(*), that is, by [GL], \mathcal{E} is bireflective. Thus, there exists an object $M \in \mathcal{E}$ such that $S := \operatorname{End}_B(M)$ is isomorphic to M as B-module and \mathcal{E} is equivalent to S-Mod. Since $\dim_k P(*) = 2$, then $M = P(*)^{\oplus 2}$, $S \simeq M_2(k)$ and there exists a ring epimorphism $\lambda : B \longrightarrow S$. We now prove that λ is homological. In view of Theorem 6.1 we have just to prove that every object in $\mathcal{Y} = \operatorname{Ker}(T \bigotimes_B^{\mathbb{L}} -)$ is quasi-isomorphic to a complex with terms in \mathcal{E} . Set $H = \mathbb{R}\operatorname{Hom}_A({}_AT_B, -)$ and $G = {}_AT_B \bigotimes_B^{\mathbb{L}} -$ and consider the triangle $B \xrightarrow{\eta_B} HG(B) \longrightarrow Y \longrightarrow B[1].$

We have

 $HG(B) = \mathbb{R}\operatorname{Hom}_A({}_AT_{B,A}T_B \overset{\mathbb{L}}{\otimes}_B B) = \mathbb{R}\operatorname{Hom}_A({}_AT_{B,A}T_B) \cong \operatorname{End}_A({}_AT_B) = \Lambda$ (because $T_B \simeq T_{\Lambda}$ is self orthogonal in A-Mod, hence it is $\operatorname{Hom}_A({}_AT, -)$ -acyclic). Then $\eta_B = \overline{\eta}_B$ and, considering the long exact sequence of the homologies, we can conclude that Y is quasi-isomorphic to the stalk complex $\operatorname{Ker}\eta_B[1]$, that is P(*)[1]. We now follow [CX, Prop. 4.6]. Let M be an object in $\mathcal{D}(B)$, then there is the triangle

$$M \xrightarrow{\eta_M} HG(M) \longrightarrow Y_M \longrightarrow M[1]$$

where

$$HG(M) = \mathbb{R}\operatorname{Hom}_A({}_AT_B, {}_AT_B \overset{\mathbb{L}}{\otimes}_B M) = \operatorname{Hom}_A({}_AT_B, {}_AT_B \otimes_B W)$$

with W an \mathcal{H} -projective resolution of the complex M. Therefore, the complex $\operatorname{Hom}_A({}_AT_B, {}_AT_B \otimes_B W)$ has terms of the form $\operatorname{Hom}_A(T, T_i)$ with $T_i \in \operatorname{Add}({}_AT)$). Being ${}_AT$ finitely generated, we have that the module $\operatorname{Hom}_A(T, T_i)$ is in Add (Λ_{Λ}) . Regard the triangle in (1) as the triangle

$$W \xrightarrow{\eta_M} \operatorname{Hom}(T, T^{\bullet}) \longrightarrow Y_M \longrightarrow W[1]$$

where T^{\bullet} is the complex ${}_{A}T_{B} \otimes_{B} W$. Therefore the morphism η_{M} can be regarded in $\mathcal{C}(B)$ as the family $(\overline{\eta}^{i})_{i \in \mathbb{Z}}$ with $\eta_{i} : W^{i} \longrightarrow \operatorname{Hom}_{A}(T, T_{i})$. Then for Lemma 8.2, noting that $(\operatorname{Ker} \eta_{M})^{i} = \operatorname{Ker} \overline{\eta}^{i} \in \mathcal{E}$, we can conclude that $Y_{M} \simeq \operatorname{Ker} \eta_{M}[1]$ has terms in \mathcal{E} . Now, for every Y in \mathcal{Y} , there is the triangle

$$Y \xrightarrow{\eta_Y} 0 \longrightarrow Y \longrightarrow Y[1]$$

then Y is Ker η_Y which has terms in \mathcal{E} .

Remark 8. The previous example can be generalized considering a situation similar to [Mi, Corollary 5.5]. Let us point out the key steps used in the previous Example 1. Assume that I is a non-zero projective, idempotent two-sided ideal of an ordinary k-algebra B. Then the projective dimension of Λ , viewed as a right B-module, is one. By [NS, Example in Section 4] the canonical projection $\pi : B \to \Lambda := B/I$ is a homological ring epimorphism and Λ_B is self-orthogonal. Let now T_{Λ} be a classical n-tilting module over Λ and view T as a right B-module via π . Then Iis the annihilator of T_B (and of Λ_B) and T_B is a classical n + 1-partial tilting module, since proj.dim $(T_{\Lambda}) \leq \text{proj.dim}(T_B) \leq \text{proj.dim}(T_{\Lambda}) + 1$. Set $A := \text{End}_{\Lambda}(T_{\Lambda}) = \text{End}_B(T_B)$ (where the last equality holds since π

is a ring epimorphism). The functor ${}_{A}T \overset{\mathbb{L}}{\otimes}_{\Lambda} -: \mathcal{D}(\Lambda) \to \mathcal{D}(A)$ is a triangle equivalence, since T_{Λ} is a classical *n*-tilting module. Moreover the functor ${}_{A}T \overset{\mathbb{L}}{\otimes}_{B} -: \mathcal{D}(B) \to \mathcal{D}(\Lambda)$ is given by the composition of functors $({}_{A}T \overset{\mathbb{L}}{\otimes}_{\Lambda} -) \circ ({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -)$, so the kernel of ${}_{A}T \overset{\mathbb{L}}{\otimes}_{B} -$ is exactly the kernel of $({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -)$. Thus, $\operatorname{Ker}({}_{A}T \overset{\mathbb{L}}{\otimes}_{B} -)$ is equivalent to the derived category of a ring via a homological ring epimorphism if and only so is $\operatorname{Ker}({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -)$. But, Λ_{B} is a classical 1-partial tilting module with $\operatorname{End}_{B}(\Lambda) = \Lambda$, so the class $\mathcal{E} = \operatorname{Ker}({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -) \cap B$ -Mod is bireflective. Now, similarly to the proof of Proposition 8.3, we let $G = ({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -)$ and $H = \mathbb{R}\operatorname{Hom}_{\Lambda}({}_{\Lambda}\Lambda_{B}, -)$. Then, a complex $Y \in \operatorname{Ker}({}_{\Lambda}\Lambda \overset{\mathbb{L}}{\otimes}_{B} -)$ if and only if Y is quasi isomorphic to HG(Y). Computing HG(Y) by means of a \mathcal{H} -projective resolution of Y in $\mathcal{D}(B)$ we obtain that HG(Y) is a direct summand of complex with terms of the form $\Lambda^{(I)}$ for some set I, viewed as left B-modules, hence in the class \mathcal{E} . By Theorem 6.1, we conclude that the kernel of the functor ${}_{A}T \overset{\mathbb{L}}{\otimes}_{B}$ is triangle equivalent to the derived category of a ring via the homological epimorphism π .

Example 2. Now we give a simple example of a finitely generated classical partial tilting module T over a finite dimensional algebra B, such that the class $\mathcal{E} = \bigcap_{i \geq 0} \operatorname{KerTor}_{i}^{B}(T, -)$ is not bireflective (in particular there are no

homological epimorphism of rings $\lambda : B \to S$ such that $\operatorname{Ker}(T \bigotimes_{B}^{\mathbb{L}} -) \simeq \mathcal{D}(S)$).

Consider the quiver $\circ \xrightarrow{a} \circ \xrightarrow{b} \circ \xrightarrow{b} \circ \xrightarrow{a} \circ \xrightarrow{b} \circ \xrightarrow{b} \circ \xrightarrow{a} \circ \xrightarrow{b} \circ$

$$0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0$$

It is easy to see that S_1 is classical partial tilting over B. A calculation shows that the class $\mathcal{E} = \operatorname{Add} \left\{ \begin{array}{c} 2 \\ 1 \end{array}, \begin{array}{c} 3 \\ 2 \end{array} \right\}$ is not bireflective. In fact there is a morphism

$$f: \begin{array}{ccc} 2 \\ 1 \end{array} \longrightarrow \begin{array}{ccc} 3 \\ 2 \end{array}$$

such that the kernel is not in \mathcal{E} .

Example 3. Consider the quiver

$$\overset{b}{\underset{1}{\overset{\circ}{\longrightarrow}}} \overset{\circ}{\underset{2}{\overset{\circ}{\longrightarrow}}} \overset{\circ}{\underset{2}{\overset{\circ}{\longrightarrow}}} \overset{\circ}{\underset{2}{\overset{\circ}{\longrightarrow}}}$$

with relation ab = 0 and consider the classical partial tilting module $\frac{1}{2}$ of projective dimension 2. Here, as shown in [B2, Example 1], $\mathcal{E} = 0$ then it is bireflective but, obviously, the complexes in \mathcal{Y} don't have terms in \mathcal{E} .

Example 4. [CX2, Section 7.1] The following is an example of a good *n*-tilting module ${}_{A}T$ with $B = \operatorname{End}_{A}({}_{A}T)$, such that $\operatorname{Ker}(T_{B} \overset{\mathbb{L}}{\otimes}_{B} -)$ is not triangle equivalent to the derived category of a ring via a homological ring epimorphism.

Let A be a commutative n-Gorestein ring and consider a minimal injective resolution of the regular module $_AA$ of the form:

$$0 \to A \to \bigoplus_{p \in \mathcal{P}_0} E(A/p) \to \ldots \to \bigoplus_{p \in \mathcal{P}_n} E(A/p) \to 0$$

where \mathcal{P}_i is the set of all prime ideals of A of height *i* (see [Bas, Theorem 1, Theorem 6.2]). Then, the module

$$_{A}T := \bigoplus_{0 \le i \le n} \bigoplus_{p \in \mathcal{P}_{i}} E(A/p)$$

is an *n*-tilting module by [GT, Example 5.16]] and it is moreover good. Set, for all $0 \leq i \leq n$, $T_i := \bigoplus_{p \in \mathcal{P}_i} E(A/p)$, then we have $\operatorname{Hom}_A(T_j, T_i) = 0$ for all $0 \leq i \leq j \leq n$. Assume that $n \geq 2$ and that the injective dimension of A is exactly n; then T has projective dimension n (see [B2, Proposition 3.5]). Note that $T_i \neq 0$ for every $2 \leq i \leq n$ so T satisfies the hypotheses of [CX2, Corollary 1.2], hence $\operatorname{Ker}(T_B \bigotimes_B -)$ cannot be realized as the derived category $\mathcal{D}(S)$ of a ring S linked to B via a homological ring epimorphism $B \to S$.

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