COTILTING MODULES AND HOMOLOGICAL RING EPIMORPHISMS

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Dedicated to Professor Alberto Facchini on the occasion of his 60th birthday

Abstract. We show that every injective homological ring epimorphism $f\colon R \to S$ where S_R has flat dimension at most one gives rise to a 1-cotilting R-module and we give sufficient conditions under which the converse holds true. Specializing to the case of a valuation domain R, we illustrate a bijective correspondence between equivalence classes of injective homological ring epimorphisms originating in R and cotilting classes of certain type and in turn, a bijection with a class of smashing localizing subcategories of the derived category of R. Moreover, we obtain that every cotilting class over a valuation domain is a Tor-orthogonal class, hence it is of cocountable type even though in general cotilting classes are not of cofinite type.

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Introduction

Tilting and cotilting theory has its origin in the context of finitely generated modules over finite dimensional algebras. The theory studied equivalences between subcategories of module categories over two algebras and was in essence a generalization of Morita Theory. A remarkable result by

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Happel ([Hap87]) shows that tilting theory provides a Morita Theorem at the level of derived categories.

The notions of tilting and cotilting modules were further extended to modules of homological dimension $n \geq 1$ (called n-tilting and n-cotilting modules), to arbitrary rings and, what is of more interest to us, to infinitely generated modules which we call *big* modules. In [Baz10] and [BMT11] it was proved that big tilting modules induce equivalences between suitable localizations of the derived categories of two rings. Moreover, big tilting modules induce recollements of derived categories of rings and differential graded algebra which specialise to recollements of derived categories of rings in case the tilting module is of projective dimension one (see [CX12a], [BP13]).

Analogous results about dualities induced by big cotilting modules are not available up to now. Some partial results were obtained in [CX12b, Section 6]. Originally the notion of cotilting module was shadowed, since they were just duals of tilting modules. As soon as big modules enter the picture, the substantial difference between the two concepts became apparent.

In particular, it was proved that every tilting class associated to a big tilting module is of *finite type* ([BH08], [BŠ07]) meaning that it is the Ext-orthogonal of a class of *compact modules*, that is modules with a finite projective resolution consisting of finitely generated projective modules. The corresponding property for a cotilting class, is the *cofinite type* meaning means that it is the Tor-orthogonal of a class of *compact modules*. A cotilting module is of cofinite type if and only if it is the dual of a tilting module (see[AHHT06]). Recently in [AHPŠT14] it has been proved that the cofinite type holds for big 1-cotilting modules over one-sided noetherian rings and it is valid for all n-cotilting modules over commutative noetherian rings. At our knowledge the only available counterexamples to the cofinite type are in the case of valuation domains. In fact, in [Baz07] it is shown that every cotilting class over a valuation domain is of cofinite type if and only if the domain is strongly discrete, that is if and only if it doesn't admit non zero idempotent ideals and, moreover, explicit examples of cotilting classes not of cofinite type are exhibited.

One important question which should be investigated is whether cotilting classes are in any case Tor-orthogonal to some class of modules, not necessarily compact ones. In the case of 1-cotilting modules, we state a necessary and sufficient condition on a cotilting class to be a Tor-orthogonal class (Propostition 7.3) which in particular implies the *cocountable* type. In the case of valuation domains R we are able to prove that every cotilting class is a Tor-orthogonal class (Theorem 7.11).

The relevance of big cotilting modules is also supported by a recent paper [Šťo14] where it is shown that big cotilting modules are in bijective correspondence (up to equivalence) with duals (with respect to an injective cogenerator) of a *classical* tilting object of a Grothendieck category.

In the present paper we carry on an investigation of 1-cotilting modules. Inspired by results in [AHS11] we investigate the relation between 1-cotilting modules and homological ring epimorphisms. In fact, in [AHS11] it is proved they every injective homological ring epimorphism $R \to S$ where S has projective dimension at most one, gives rise to the 1-tilting module $S \oplus S/R$.

We relax the condition on the projective dimension and we prove that every injective homological ring epimorphism $R \to S$ where the flat dimension of S is at most one gives rise to the 1-cotilting module $(S \oplus S/R)^*)$ where * denotes the character module (Theorem 3.4). The converse is proved under some assumptions (Proposition 3.7).

To obtain better results concerning the relation between 1-cotilting modules and homological ring epimorphism we clearly need a good understanding of homological ring epimorphisms. In this respect we take advantage of a recent paper [BŠ14] by Šťovíček and the author where a complete classification of homological ring epimorphisms starting from valuation domains R is achieved. The classification is obtained via a bijective correspondence between equivalence classes of homological ring epimorphisms originating in R, and chains of intervals of prime ideals of R satisfying certain conditions. These conditions amount to order completeness and to a property sometimes referred to as *weakly atomic* meaning that between two distinct intervals there is always a gap.

In [Baz07], the author developed a method to associate to a cotilting module over a valuation domain R a chain of intervals of prime ideals which determine the cotilting class. Here we show that the chain of intervals of prime ideals associated to a cotilting module is order complete and we call a cotilting module *non dense* in case its associated chain of intervals satisfies also the *weakly atomic* property. In Theorem 5.7 we prove a bijective correspondence between equivalence classes of injective homological ring epimorphisms starting in a valuation domain R and equivalence classes of non dense cotilting modules. We also show the existence of *dense* cotilting modules which don't correspond to injective homological ring epimorphism (Proposition 6.4).

The paper is organized as follows. In Section 1 we recall the notions and properties of ring epimorphisms and homological ring epimorphisms. In Section 2 we illustrate the classification of homological ring epimorphisms starting in a valuation domain proved in [BŠ14].

In Section 3 we investigate the relation between 1-cotilting modules over an arbitrary ring R and homological ring epimorphisms describing the cotilting classes associated to homological ring epimorphisms.

In Section 4 we restate some results from [Baz07] about the properties of cotilting modules over valuation domains and examine the properties of the chain of intervals of prime ideals associated to a cotilting module.

In Section 5 we prove that, up to equivalence, there is a bijective correspondence between injective homological ring epimorphisms starting in a valuation domain R and non dense cotilting modules over R. We also mention the related correspondence with a class of smashing localizing subcategories of the derived category of R.

In Section 6 we show examples of *dense* cotilting modules and in the final Section 7 we prove that every cotilting class over a valuation domain is a Tor-orthogonal class.

1. Preliminaries

All rings consider will be associative with identity.

For any class $\mathcal{C}$ of left R-modules we define the following classes:

$$^{\perp}\mathcal{C} = \{X \in R\text{-Mod} \mid \mathrm{Ext}^i_R(X, C) = 0,\ \forall i \geq 1, ,\ \forall C \in \mathcal{C}\},$$

$$^{\intercal}\mathcal{C} = \{X \in \text{Mod-}R \mid \mathrm{Tor}^R_i(X, C) = 0,\ \forall i \geq 1,\ \forall C \in \mathcal{C}\},$$

if $\mathcal{S}$ is a class of right R-modules we define;

$$\mathcal{S}^{\intercal} = \{X \in R\text{-Mod} \mid \mathrm{Tor}^R_i(S, X) = 0,\ \forall i \geq 1,\ \forall S \in \mathcal{S}\},$$

and we say that $\mathcal{C}$ is a *Tor-orthogonal class* if there is a class $\mathcal{S}$ of right R-modules such that $\mathcal{C} = \mathcal{S}^{\intercal}$.

If $\mathcal{C} = \{M\}$ we simply write $^{\perp}M$, $^{\intercal}M$ and $M^{\intercal}$. For every R-module M, i.d.M, p.d.M, w.d.M will denote the injective, projective, weak (flat) dimension of M.

Definition 1.1. Let R a ring. An R-module C is an n-cotilting module if the following conditions hold ([AHC01]):

(C1) i.d.$C \leq n$;
(C2) $\mathrm{Ext}^i_R(C^{\lambda}, C) = 0$ for each $i > 0$ and for every cardinal λ;
(C3) there exists a long exact sequence:

$$0 \to C_r \to \cdots \to C_1 \to C_0 \cdots \to W \to 0,$$

where $C_i \in \mathrm{Prod}C$, for every $0 \leq i \leq r$ and W is an injective cogenerator of R-Mod.

In case $n = 1$ there is an alternative definition of 1-cotilting modules. A module C is 1-cotilting if and only if $\mathrm{Cogen}\, C = {}^{\perp}C$, where $\mathrm{Cogen}\, C$ denotes the class of modules cogenerated by C. Moreover, if C is a 1-cotilting module, then $\mathrm{Cogen}\, C$ is a torsion free class. (For results on torsion and torsion free classes we refer to [Ste75].)

If C is a n-cotilting module the class $^{\perp}C$ is called an n-cotilting class and two cotilting modules are said to be equivalent if the corresponding cotilting classes coincide.

n-cotilting classes have been characterized in [AHC01], [GT06] and [GT12]. In particular $^{\perp}C$ is closed under direct products. Moreover, since every n-cotilting module C is pure injective ([Baz03], [Šťo06]), n-cotilting classes are also closed under direct limits and pure submodules. In other words they are definable classes, that is they are closed under elementary equivalence. Thus, if C is n-cotilting, a module belongs to $^{\perp}C$ if and only if its pure injective envelope belongs to $^{\perp}C$ (see [JL89] or [CB94]).

An R-module M is said to be *compact* if it admits a finite projective resolution consisting of finitely generated projective modules.

Definition 1.2. An n-cotilting class $\mathcal{F} = {}^{\perp}C$ is said to be of *cofinite type* if there is a set $\mathcal{S}$ of compact modules such that $\mathcal{F} = \mathcal{S}^{\intercal}$. In this case we also say that C is of *cofinite type*.

Note that if $\mathcal{F}$ is an n-cotilting class, any compact module in $^{\intercal}\mathcal{F}$ has projective dimension at most n and $\mathcal{F}$ is of cofinite type if and only if $\mathcal{F} = \mathcal{S}^{\intercal}$ where $\mathcal{S}$ is the set of all compact modules in $^{\intercal}\mathcal{F}$. Moreover, by [AHHT06] an n-cotilting module C is of cofinite type if and only if it is the dual of an n-tilting module.

1.1. **Homological ring epimorphisms.** Let us recall some standard facts on ring epimorphisms and on homological ring epimorphisms which we will need in the sequel.

If R, S are associative rings, we denote by Mod-R (R-Mod) and Mod-S (S-Mod) the categories of right (left) R and S modules, respectively. A ring homomorphism $f\colon R \to S$ is a ring epimorphism if it is an epimorphism in the category of rings. Ring epimorphisms have been investigated in [Sil67, Ste75, GlP87, Laz69].

From those papers we infer that a ring homomorphism $f\colon R \to S$ is a ring epimorphism if and only if $S \otimes_R S \cong_S S_S$, if and only if the restriction functor $f_*\colon \text{Mod-}S \to \text{Mod-}R$ is fully faithful (or the same holds for left modules).

Two ring epimorphisms $f\colon R \to S$ and $f'\colon R \to S'$ are said to be *equivalent* if there exists a ring isomorphism $\varphi\colon S \to S'$ such that $f' = \varphi f$. Equivalently, the essential images of f_* and f'_* in Mod-R coincide.

By [Sil67, Corollary 1.2], if R is a commutative ring and $f\colon R \to S$ is a ring epimorphism, then S is a commutative ring.

Definition 1.3. A ring epimorphism $f\colon R \to S$ is a *homological epimorphism* if $\operatorname{Tor}_i^R(S, S) = 0$ for every $i \geq 1$.

A ring epimorphism $f\colon R \to S$ with S a flat R-module is clearly a homological epimorphism. It is called a *flat epimorphism.*

Homological ring epimorphisms have been introduced by Geigle and Lenzing in [GL91] and we refer to [GL91, Theorem 4.4] for their characterization.

We just note that, while a ring epimorphism $R \to S$ implies that the category of S-modules is equivalent to a subcategory of the category of R-modules, homological ring epimorphisms give the analogous result for the derived categories of the rings.

In the sequel the weak global dimension of a ring R will be denoted by w.gl. dim R. If $f\colon R \to S$ is a homological ring epimorphism, then from [GL91, Theorm4.4] it follows that w.gl. dim $S \leq$ w.gl.dim R.

Moreover, if w.gl.dim $R \leq 1$ and $f\colon R \to S$ is a ring epimorphism, then clearly f is a homological epimorphism if and only if $\operatorname{Tor}_1^R(S, S) = 0$. In this case, given a two-sided ideal I of R, the canonical projection $\pi\colon R \to R/I$ is a homological epimorphism if and only if I is an idempotent two-sided ideal of R, since $\operatorname{Tor}_1^R(R/I, R/I) \cong I/I^2$.

Lemma 1.4. *Let $f\colon R \to S$ be a homological ring epimomorphism such that S and $S/f(R)$ have weak dimension ≤ 1 as right R–modules. Then the following hold:*

(1) *The canonical projection $\pi\colon R \to R/\operatorname{Ker} f$ is a homological epimorphism and $\operatorname{Ker} f$ is an idempotent two-sided ideal of R.*
(2) *The induced homomorphism $\overline{f}\colon R/\operatorname{Ker} f \to S$ is a homological epimorphisms.*

Proof. (1) By the assumptions, the right R-module $f(R)$ has weak dimension ≤ 1 hence the two sided ideal $K := \operatorname{Ker} f$ is a flat right ideal. Thus, $\operatorname{Tor}_i^R(R/K, R/K) = 0$ for every $i \geq 2$. Applying the functors $S \otimes_R -$ and $- \otimes_R R/K$ to the exact sequence

$$0 \longrightarrow R/K \xrightarrow{\overline{f}} S \longrightarrow S/f(R) \longrightarrow 0,$$

we get

$$0 = \mathrm{Tor}_2^R(S, S/f(R)) \to \mathrm{Tor}_1^R(S, R/K) \to \mathrm{Tor}_1^R(S, S) = 0,$$

$$0 = \mathrm{Tor}_2^R(S/f(R), R/K) \to \mathrm{Tor}_1^R(R/K, R/K) \to \mathrm{Tor}_1^R(S, R/K) = 0.$$

Consequently π is a homological epimorphism and from the isomorphism $\mathrm{Tor}_1^R(R/K, R/K) \cong K/K^2$ we conclude that K is an idempotent ideal. (2) By part (1) π is a homological epimorphism, thus $\mathrm{Tor}_n^{R/K}(S, S) \cong \mathrm{Tor}_n^R(S, S) = 0$, since S is an R/K-bimodule. Moreover, $\overline{f}$ is clearly a ring epimorphism, so also homological.

□

2. Homological ring epimorphisms originating in valuation domains

A commutative ring is a valuation ring if the lattice of the ideals is linearly ordered by inclusion. Recall that an idempotent ideal of a valuation domain is a prime ideal and also that if $J \subseteq L$ are prime ideals, then J is canonically an R_L-module, so that $J_L = J$ and $(R/J)_L = R_L/J$.

For other definitions and results on valuation domains we refer to [FS01].

A complete classification of homological ring epimorphisms originating in valuation domains has been obtained by Šťovíček and the author in [BŠ14].

The next Sections 5 and 4 will heavily rely on that classification, thus for the reader convenience, we collect here the relevant definitions and results proved in [BŠ14].

A key observation is given by the following.

Remark 2.1. If R is a valuation domain and $f\colon R \to S$ is a homological ring epimorphism, S is a commutative ring with w.gl.dim $S \leq 1$. Thus, if $\mathfrak{n}$ is any maximal (prime) ideal of S, the localization $S_{\mathfrak{n}}$ of S at $\mathfrak{n}$ is a valuation domain (see [Gla89, Corollary 4.2.6])

Lemma 2.2. *(*[BŠ14, Proposition 6.5] *and its proof) Let R be a valuation domain, $0 \neq f\colon R \to S$ be a homological ring epimorphism, and let $I =$* Ker f*. Then the following hold:*

(1) *There exists a prime ideal $P \in \operatorname{Spec} R$ with $I \subseteq P$ and a surjective homological epimorphism $g\colon S \to R_P/I$ such that the composition $gf\colon R \to R_P/I$ is the canonical morphism. Moreover, there is a unique maximal ideal $\mathfrak{m}$ of S such that $f^{-1}(\mathfrak{m}) = P$ and $g\colon S \to R_P/I$ is equivalent to the localization of S at $\mathfrak{m}$.*
(2) *For every maximal ideal $\mathfrak{n}$ of S, there are two prime ideals J, L of R such that the composition $R \xrightarrow{f} S \xrightarrow{\mathrm{can}} S_{\mathfrak{n}}$ is a homological epimorphism equivalent to $g\colon R \to R_L/J$, where J is idempotent, $J \leq L$ and $L = f^{-1}(\mathfrak{n})$,*
(3) *If $\mathfrak{n}' \neq \mathfrak{n}$ are two distinct maximal ideals of S with corresponding pairs of prime ideals J, L and J', L', then the intervals $[J, L]$ and $[J', L']$ in* $(\operatorname{Spec} R, \subseteq)$ *are disjoint and*

$$\mathrm{Tor}_n^R(R_L/J, R_{L'}/J') = 0$$

for $n \geq 0$.

Consequences of the previous facts are:

Corollary 2.3. *Let R be a valuation domain, $0 \neq f \colon R \to S$ be a homological ring epimorphism and let $I = I^2 \leq R$ be such that the kernel of the composition $R \to S \to S/SI$ is I.*

Then there is a maximal ideal $\mathfrak{m}$ of S, $\mathfrak{m} \supseteq SI$ such that the homological epimorphism $R \xrightarrow{f} S \xrightarrow{\text{can}} S_{\mathfrak{m}}$ is equivalent to $R \to R_P/I'$ for ideals I' and P satisfying $I' \leq I \leq P$ and $f^{-1}(\mathfrak{m}) = P$.

Proof. By Lemma 1.4, the morphism $p \colon S \to \overline{S} = S/SI$ is a homological epimorphism, since SI is idempotent. Then $pf \colon R \to \overline{S}$ is a homological epimorphism with kernel I. Applying Lemma 2.2 (1) to pf we have that there is a maximal ideal $\mathfrak{m}$ of S containing SI such that the homological epimorphism $R \to \overline{S}_{\mathfrak{m}}$ is equivalent to R_P/I. Then the kernel I' of the composition $R \to S \to S_{\mathfrak{m}}$ is contained in I and the homological epimorphism $R \to S \to S_{\mathfrak{m}}$ is equivalent to R_P/I'. Clearly $f^{-1}(\mathfrak{m}) = P$. □

Definition 2.4. ([BŠ14, Section6]) If R is a valuation domain denote by $\operatorname{Inter} R$ the set of intervals $[J, L]$ in $(\operatorname{Spec} R, \subseteq)$ such that $J = J^2 \leq L$ endowed with the partial order $[J, L] < [J', L']$ if $L < J'$ as ideals.

If $f \colon R \to S$ is a homological ring epimorphism, we define $\mathcal{I}(f)$ to be the set of intervals $[J, L] \in \operatorname{Inter} R$ arising as in Lemma 2.2.

The classification of homological epimorphisms originating in valuation domains is obtained by means of three conditions satisfied by subchains $(\mathcal{I}, \leq)$ of $(\operatorname{Inter} R, \leq)$ described as follows.

Conditions 2.5.

(1) If $\mathcal{S} = \{[J_\alpha, L_\alpha] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}$ with no minimal element, then $\mathcal{I}$ contains an element of the form $[J, \bigcap_{\alpha \in \Lambda} L_\alpha]$.

(2) If $\mathcal{S} = \{[J_\alpha, L_\alpha] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}$ with no maximal element, then $\mathcal{I}$ contains an element of the form $[\bigcup_{\alpha \in \Lambda} J_\alpha, L]$.

(3) Given any $[J_0, L_0] < [J_1, L_1]$ in $\mathcal{I}$, there are elements $[J, L], [J', L']$ in $\mathcal{I}$ such that

$$[J_0, L_0] \leq [J, L] < [J', L'] \leq [J_1, L_1]$$

and there are no other intervals of $\mathcal{I}$ between $[J, L]$ and $[J', L']$.

Remark 2.6. Conditions (1) and (2) express the fact that $\mathcal{I}$ is order complete, while condition (3) is typically satisfied by the partially order set of the prime spectrum of a commutative ring (see [Kap74]). Condition (3) is sometimes referred to as *weakly atomic*.

Every subchain $(\mathcal{I}, \leq)$ of $(\operatorname{Inter} R, \leq)$ satisfying Conditions 2.5 gives rise to a ring $R_{(\mathcal{I})}$ and a homological ring epimorphism $f_{(\mathcal{I})} \colon R \to R_{(\mathcal{I})}$. For the reader convenience we sketch the construction of the ring $R_{(\mathcal{I})}$ (for more details see [BŠ14, Construction 4.9]).

Consider a partition of $\mathcal{I}$ into a finite disjoint union $\mathcal{I} = \mathcal{I}_0 \cup \cdots \cup \mathcal{I}_n$ of chains in $\operatorname{Inter} R$ which satisfy again conditions (1)–(3).

Let $R_{\mathcal{I}} = \prod_{[J,L]\in\mathcal{I}} R_L/J$. Define a map

$$g_{(\mathcal{I}_0,\dots,\mathcal{I}_n)}\colon \prod_{i=0}^{n} R_{L_i}/J_i \longrightarrow R_{\mathcal{I}}$$

where $[J_i, K_i]$ and $[J'_i, L_i]$ are the minimal and the maximal elements of $\mathcal{I}_i$, respectively. The images of $g_{(\mathcal{I}_0,\dots,\mathcal{I}_n)}$, where $(\mathcal{I}_0,\dots,\mathcal{I}_n)$ varies over all partitions of $\mathcal{I}$ such that $\mathcal{I}_i$ satisfy conditions (1)–(3), form a direct system of subrings of $R_{\mathcal{I}}$ whose direct union is the ring $R_{(\mathcal{I})}$.

We can now state the classification theorem proved in [BŠ14].

Classification 2.7. ([BŠ14, Theorem 6.23] Let R be a valuation domain. Then there is a bijection between:

(1) Subchains $\mathcal{I}$ of Inter R consisting of disjoint intervals satisfying Conditions 2.5.
(2) Equivalence classes of homological ring epimorphisms $f\colon R \to S$.

The bijection is given by assigning to a non-empty set $\mathcal{I}$ from (1) the ring homomorphism $f_{(\mathcal{I})}\colon R \to R_{(\mathcal{I})}$. We assign $R \to 0$ to $\mathcal{I} = \emptyset$.

The converse is given by sending $f\colon R \to S$ to $\mathcal{I} = \mathcal{I}(f)$ as in Definition 2.4

Remark 2.8. If $[J, L]$ is the minimal element of $\mathcal{I}(f)$, then $J = \operatorname{Ker} f$. Moreover, if $\mathcal{I}(f)$ is finite, then $R_{(\mathcal{I})} = \prod_{[J,L]\in\mathcal{I}} R_L/J$.

3. 1-Cotilting modules versus homological ring epimorphisms

3.1. 1-Cotilting modules arising from homological ring epimorphisms.

In this subsection we show a method to construct 1-cotilting modules from homological ring epimorphisms. The main result of this subsection, Theorem 3.4 can be viewed as a counterpart of the results in [AHS11] where tilting modules arising from injective homological ring epimorphisms $f\colon R \to S$ with p.d. $S \le 1$ are considered. In fact, we prove that every injective homological ring epimorphism $f\colon R \to S$ where w.d.$S_R \le 1$ gives rise to a 1-cotilting left R-module whose corresponding cotilting class consists of the left R-modules cogenerated by S^*. Moreover, we obtain that the bireflective subcategory of R-Mod equivalent to S-Mod via f can be interpreted as a suitable perpendicular category.

Proposition 3.1. *Let $f\colon R \to S$ be a homological ring epimomorphism such that S and $S/f(R)$ have weak dimension ≤ 1 as right R-modules and let* $\operatorname{Ker} f = K$. *Then, $C = S^* \oplus (S/f(R))^*$ is a 1-cotilting left R/K-module. The corresponding cotilting class in R/K-*Mod *coincides with* $\operatorname{Cogen} S^*$ *and with ${}^{\perp}(S/f(R))^* = (S/f(R))^{\intercal}$. Moreover, an R/K-module M belongs to the cotilting class if and only the map $f \otimes_{R/K} M\colon M \to S \otimes_{R/K} M$ is injective.*

Proof. By Lemma 1.4, $\pi\colon R \to R/K$ is a homological epimorphism, thus S and $S/f(R)$ have weak dimension ≤ 1 also as R/K-modules. Moreover, f induces a homological epimorphism $\overline{f}\colon R/K \to S$. Thus, without loss of generality, we may assume that f is injective and identify R with $f(R)$.

Consider S and S/R as right R-modules and let $C = S^* \oplus (S/R)^*$. Then, i.d.$C \leq 1$. The exact sequence $0 \to (S/R)^* \to S^* \to R^* \to 0$ shows that condition $(C3)$ in Definition 1.1 is satisfied. It remains to show that $\mathrm{Ext}^1_R(C^\lambda, C) = 0$, for every cardinal λ. We have

$$\mathrm{Ext}^1_R(C^\lambda, C) = 0 \Leftrightarrow \mathrm{Tor}^R_1(S \oplus S/R, ((S \oplus S/R)^*)^\lambda) = 0.$$

Since w.dim $(S \oplus S/R) \leq 1$ and $(S/R)^*$ is a submodule of S^*, it is enough to check that $\mathrm{Tor}^R_1(S \oplus S/R, (S^*)^\lambda) = 0$. First note that $\mathrm{Tor}^R_1(S, (S^*)^\lambda) = 0$, since $(S^*)^\lambda$ is an S-module and f is a homological ring epimorphism. Thus, what is left to be proved is that $\mathrm{Tor}^R_1(S/R, (S^*)^\lambda) = 0$.

Consider the exact sequence or right R-modules

$$(1) \qquad 0 \to R \xrightarrow{f} S \to S/R \to 0.$$

Applying the functor $- \otimes_R (S^*)^\lambda$ to (1) we get

$$0 \to \mathrm{Tor}^R_1(S/R, (S^*)^\lambda) \to R \otimes_R (S^*)^\lambda \xrightarrow{f \otimes_R (S^*)^\lambda} S \otimes_R (S^*)^\lambda \to S/R \otimes_R (S^*)^\lambda \to 0$$

so $\mathrm{Tor}^R_1(S/R, (S^*)^\lambda) = 0$, since $S \otimes_R (S^*)^\lambda \cong S \otimes_S (S^*)^\lambda$ implies that $f \otimes_R (S^*)^\lambda$ is injective. The characterization of the cotilting class follows at once from $(S/f(R))^* \leq S^*$.

To prove the last statement we apply the functor $- \otimes_R M$ to (1) to obtain

$$(2) \qquad 0 \to \mathrm{Tor}^R_1(S, M) \to \mathrm{Tor}^R_1(S/R, M) \to M \xrightarrow{f \otimes_R M} S \otimes_R M$$

Thus $\mathrm{Tor}^R_1(S/R, M) = 0$ implies the injectivity of $f \otimes_R M$. Conversely, assume that $f \otimes_R M \colon M \to S \otimes_R M$ is injective. Tensoring by $S \otimes_R -$ the exact sequence $0 \to M \xrightarrow{f \otimes M} S \otimes M \to (S \otimes M)/M \to 0$ we get the exact sequence

$$0 = \mathrm{Tor}^R_2(S, (S \otimes_R M)/M) \to \mathrm{Tor}^R_1(S, M) \to \mathrm{Tor}^R_1(S, S \otimes_R M) \cong \mathrm{Tor}^S_1(S, S \otimes_R M) = 0$$

where the end terms vanish since $\mathrm{Tor}^R_2(S, -) = 0$ and $f \colon R \to S$ is a homological epimorphism. Thus, from (2) also $\mathrm{Tor}^R_1(S/R, M) = 0$. $\square$

We will apply Proposition 3.1 to the case of an injective homological ring epimorphism. For that purpose we first recall some definitions.

Definition 3.2. Let $\mathcal{C}$ be a class of left R modules. The left perpendicular category ${}_\perp\mathcal{C}$ of $\mathcal{C}$ is defined as

$${}_\perp\mathcal{C} = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(M, \mathcal{C}) = 0 = \mathrm{Ext}^1_R(M, \mathcal{C})\}.$$

A full subcategory $\mathcal{C}$ of R-Mod is said to be *bireflective* if the inclusion functor $j \colon \mathcal{C} \to R$-Mod has both a left and a right adjoint.

Remark 3.3. By [GL91] and [GL87] a full subcategory $\mathcal{C}$ of R-Mod is bireflective if and only if there is a ring epimorphism $f : R \longrightarrow S$ such that $\mathcal{C}$ is the essential image of the restriction of scalars functor $f_* : S$-Mod $\to R$-Mod.

Theorem 3.4. *Let $f \colon R \to S$ be an injective homological ring epimomorphism such that S has weak dimension ≤ 1 as a right R-module. The following hold true:*

(1) $C = S^* \oplus (S/f(R))^*$ *is a* 1*-cotilting* R*-module. The corresponding cotilting class in* R-Mod *coincides with* $\operatorname{Cogen} S^*$ *and with* ${}^{\perp}(S/f(R))^* = (S/f(R))^{\intercal}$.
Moreover, an R*-module* M *belongs to the cotilting class if and only the map* $f \otimes_R M \colon M \to S \otimes_R M$ *is injective.*

(2) *The left perpendicular category* ${}_{\perp}(S/f(R)^*$ *is a bireflective subcategory of* R-Mod *equivalent to* S-Mod.

(3) $\operatorname{Hom}_R(S^*, (S/f(R))^*) = 0$ *and the perpendicular category* ${}_{\perp}(S/f(R)^*$ *is closed under direct products.*

Proof. (1) Follows by Proposition 3.1.

(2) Let $f_* \colon S$-Mod $\to R$-Mod be the restriction functor. A left R-module M belongs to the image of f_* if and only if $f \otimes M$ is an isomorphism. By part (1) $f \otimes M$ is injective if and only if $M \in S/f(R))^{\intercal} = {}^{\perp}(S/f(R))^*$ and from the exact sequence $0 \to R \xrightarrow{f} S \to S/f(R) \to 0$ we see that $f \otimes M$ is surjective if and only if $S/f(R) \otimes M = 0$ that is, if and only if $\operatorname{Hom}_R(M, (S/f(R))^*) = 0$. Hence $\operatorname{Im} f_* = {}_{\perp}(S/f(R)^*$ and by Remark 3.3 $\operatorname{Im} f_*$ is a bireflective subcategory of R-Mod equivalent to S-Mod.

(3) $\operatorname{Hom}_R(S^*, (S/f(R))^*) = 0$ if and only if $S/f(R) \otimes_R S^* = 0$. Apply the functor $- \otimes_R S^*$ to the exact sequence $0 \to R \xrightarrow{f} S \to S/f(R) \to 0$ to obtain the exact sequence $S^* \xrightarrow{f \otimes_R S^*} S \otimes_R S^* \to S/f(R) \otimes_R S^* \to 0$, hence $S/f(R) \otimes_R S^* = 0$, since $f \otimes_R S^*$ is an isomorphism.

By part (2), the left perpendicular category ${}_{\perp}(S/f(R)^*$ is bireflective, hence closed under direct products. □

In Proposition 3.7 we will prove a converse of Theorem 3.4 under some extra assumptions on a 1-cotilting module.

If R is a valuation domain, then the weak global dimension of R is at most one, thus the results in this section hold true. Moreover, in Section 5 we will show that for valuation domains there is an even stronger relation between cotilting modules and homological ring epimorphisms.

3.2. 1-Cotilting modules giving rise to homological ring epimorphisms.

Proposition 3.5. *Let* ${}_RC$ *be a* 1*-cotilting module over an arbitrary ring* R*. By condition (C3) in Definition 1.1 choose an exact sequence*

$$(*) \qquad 0 \to C_1 \to C_0 \to R^* \to 0,$$

with $C_0, C_1 \in \operatorname{Prod} C$*. Then, the following hold true:*

(1) C *is equivalent to* $C_0 \oplus C_1$ *and* ${}^{\perp}C = {}^{\perp}C_1 = \operatorname{Cogen} C_0$.

(2) *The left perpendicular category* ${}_{\perp}C_1$ *is closed under extensions, kernels, cokernels and direct sum; moreover, the inclusion* $\iota \colon {}_{\perp}C_1 \to R$-Mod *has a right adjoint* μ.

(3) *If* ${}_{\perp}C_1$ *is also closed under direct products, then the inclusion functor* ι *has a left adjoint* ℓ *and there is a ring epimorphism* $f \colon R \to S$ *such that* ${}_{\perp}C_1$ *is equivalent to* S-Mod.

Proof. (1) Clearly $C' = C_0 \oplus C_1$ is a 1-cotilting module and ${}^{\perp}C \subseteq {}^{\perp}C'$, $\operatorname{Cogen} C' \subseteq \operatorname{Cogen} C$. Thus C and C' are equivalent cotilting modules. The other statement follows immediately by the exact sequence $(*)$.

(2) The closure properties of ${}_{\perp}C_1$ follow from [GL91, Proposition 1.1], since i.d. $C_1 \leq 1$. We show now how to construct a right adjoint of the inclusion functor. Note first that C_1 defines a torsion pair in R-Mod whose torsion free class coincides with $\operatorname{Cogen} C_1$ and a module X is torsion if and only if $\operatorname{Hom}_R(X, C_1) = 0$. Using the fact that $\operatorname{Ext}^1_R(C_1^{\lambda}, C_1) = 0$, for every cardinal λ the dual of Bongartz's Lemma (see [Bon81] and [Trl96, Lemma 6.9]) shows that for every left R-module M there is a short exact sequence (in particular, a special ${}^{\perp}C_1$-precover) of the form

$$0 \to C_1^{\alpha} \to M_0 \to M \to 0$$

with $M_0 \in {}^{\perp}C_1$. Then, the torsion submodule $t_{C_1}(M_0)$ of M_0 with respect to the torsion theory induced by C_1, belongs to the perpendicular class ${}_{\perp}C_1$. It is routine to check that the assignment $M \to t_{C_1}(M_0)$ defines indeed a functor $\mu\colon R\text{-Mod} \to {}_{\perp}C_1$ and that it is a right adjoint of the inclusion $\iota\colon {}_{\perp}C_1 \to R$-Mod (see for instance [CTT07] for a proof of the dual statement).

(3) Assume that ${}_{\perp}C_1$ is also closed under products. Then ${}_{\perp}C_1$ is a bireflective subcategory, hence the inclusion functor $\iota\colon {}_{\perp}C_1 \to R$-Mod admits a left adjoint ℓ and there is an epimorphism $f\colon R \to S$ of rings such that $S = \operatorname{End}_R(\ell(R), \ell(R))$ and the essential image of S-Mod under the fully faithful restriction functor f_* coincides with ${}_{\perp}C_1$ (see [GlP87], [GL91]) $\square$

Remark 3.6. The situation considered in Proposition 3.5 (3) can be illustrated by the following diagram:

$$\begin{tikzcd}
S\text{-Mod} \arrow[r, "\sim", "\rho"'] & {}_{\perp}C_1 \arrow[r, "incl"] & R\text{-Mod} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "\mu"] \arrow[ll, bend right=50, "S\otimes_R -"'] \arrow[ll, bend left=50, "\operatorname{Hom}_R(S,-)"]
\end{tikzcd}$$

Note that, by the unicity of a right adjoint up to natural isomorphisms, we have that $\rho^{-1}\mu \sim \operatorname{Hom}_R(S, -)$.

Proposition 3.7. *In the notations of Proposition 3.5 assume that the left perpendicular category ${}_{\perp}C_1$ is closed under products and that* $\operatorname{Hom}_R(C_0, C_1) = 0$. *Then the ring epimorphism $f\colon R \to S$ existing by Proposition 3.5 (3) is an injective homological epimorphism, w.dim $S_R \leq 1$ and C is equivalent to $S^* \oplus (S/R)^*$.*

Proof. From the exact sequence $0 \to C_1 \to C_0 \to R^* \to 0$ and from $\operatorname{Hom}_R(C_0, C_1) = 0$ we conclude that the right adjoint of R^* is isomorphic to C_0. By Remark 3.6 $C_0 \cong \operatorname{Hom}_R(S, R^*) \cong (R \otimes_R S)^* \cong S^*$. Hence the injective dimension of ${}_R S^*$ is ≤ 1 and w.d. $S_R \leq 1$. The exact sequence $0 \to C_1 \to C_0 \to R^* \to 0$ and the canonical isomorphisms involved show that there is a surjection $S^* \xrightarrow{f^*} R^* \to 0$, hence also the injection $0 \to R \xrightarrow{f} S$. It remains to show that f is a homological epimorphism. Certainly $\operatorname{Tor}^R_n(S, S) = 0$ for every $n \geq 2$, since w.d. $S_R \leq 1$. To see that also $\mathrm{Tor}_1^R(S,S)=0$ we use the fact that the category ${}_\perp C_1$ is closed under extensions. In fact, we have

$$(\mathrm{Tor}_1^R(S,S))^* \cong \mathrm{Ext}_R^1(S,S^*) \cong \mathrm{Ext}_S^1(S,S^*) = 0.$$

By Theorem 3.4, $S^* \oplus (S/R)^*$ is a 1-cotilting R-module and from the exact sequence

$$0 \to C_1 \to S^* \xrightarrow{f^*} R^* \to 0$$

we conclude that C is equivalent to $S^* \oplus (S/R)^*$. $\square$

Note that the perpendicular category ${}_\perp C_1$ is zero in case C_1 is already a 1-cotilting module. We are not aware of examples of 1-cotilting modules such that for every exact sequence satisfying condition (C3) in Definition 1.1 the module C_1 is cotilting. Thus we pose the following question:

Question 3.8. *Assume that C is a 1-cotilting module. Is it always possible to find an exact sequence $0 \to C_1 \to C_0 \to R^* \to 0$ with $C_0, C_1 \in \mathrm{Prod}C$ such that C_1 is not a cotilting module?*

The analogous question for 1-tilting modules has a negative answer. In fact, if L is the Lukas tilting module ([Luk91, Theorem 3.1], see also [AHKL11, Example 5.1]) then, any exact sequence $0 \to R \to L_0 \to L_1 \to 0$ with $L_1, L_0 \in \mathrm{Add}L$ is such that L_1 is a 1-tilting module.

On the other hand, the dual of Lukas tilting module doesn't provide a counterexample

Moreover, we don't know a characterization of the situation in which the assumption in Proposition 3.5 (3) about the closure under products of the perpendicular category ${}_\perp C_1$ holds true. We will show only a sufficient condition for its validity in the next Lemma 3.11. Thus we pose also this other question:

Question 3.9. *Let C be a 1-cotilting R-module and let $0 \to C_1 \to C_0 \to R^* \to 0$ be an exact sequence satisfying condition (C3) in Definition 1.1. When is ${}_\perp C_1$ closed under direct products?*

In order to prove the promised sufficient condition for a positive answer of the above question we need to recall the notion of relative Mittag-Leffler modules.

Definition 3.10. ([AHH08, Definition 1.1] Let M be a right R-module and let $\mathcal{Q}$ be a class of left R-modules. M is said to be $\mathcal{Q}$-Mittag-Leffler if the canonical map

$$M \otimes_R \prod Q_i \to \prod_i (M \otimes Q_i)$$

is injective for every set $\{Q_i\}_{i\in I}$ of modules in $\mathcal{Q}$.

Lemma 3.11. *Let ${}_RC$ be a 1-cotilting module of cofinite type. There is an exact sequence $0 \to C_1 \to C_0 \to R^* \to 0$ satisfying condition (C3) in Definition 1.1 such that the perpendicular category ${}_\perp C_1$ is closed under direct products.*

Proof. Let $\mathcal{F} = {}^{\perp}C$ and let $\mathcal{S}$ be the set of compact right R-modules in ${}^{\intercal}\mathcal{F}$. By assumption $\mathcal{S}^{\intercal} = \mathcal{F}$ and $\mathcal{S}^{\perp} = T^{\perp}$, where T_R is a 1-tilting module. W.l.o.g. we can assume $T = T_0 \oplus T_1$ where $T_0, T_1 \in \mathrm{Add}T$ are the terms fitting in an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ satisfying condition (T3) for 1-tilting modules. By [AHHT06], C is equivalent to $T^* = T_0^* \oplus T_1^*$, thus, up to equivalence, we can choose the exact sequence satisfying condition (C3) to be $0 \to T_1^* \to T_0^* \to R^* \to 0$. It follows that

$$_{\perp}C_1 = \{X \in R\text{-Mod} \mid T_1 \otimes_R X = 0 = \mathrm{Tor}_1^R(T_1, X)\}.$$

Let $\{X_i\}_{i\in I}$ be a family of modules in $_{\perp}C_1$. Then $\prod X_i \in {}^{\perp}C_1 = T_1^{\intercal}$. We need to show that also $T_1 \otimes \prod X_i = 0$. To this aim we use Mittag-Leffler properties of 1-tilting modules. Consider a projective presentation $0 \to P_1 \to P_0 \to T_1 \to 0$ of T_1 with P_0, P_1 projective right modules, then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & P_1 \otimes \prod X_i \arrow[r] \arrow[d, "\rho_{P_1}"'] & P_0 \otimes \prod X_i \arrow[r] \arrow[d, "\rho_{P_0}"'] & T_1 \otimes \prod X_i \arrow[r] \arrow[d, "\rho_{T_1}"'] & 0 \\
0 \arrow[r] & \prod_i (P_1 \otimes_R X_i) \arrow[r] & \prod_i (P_0 \otimes_R X_i) \arrow[r] & \prod_i (T \otimes_R X_i) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows are the canonical maps and the rows are exact since $\mathrm{Tor}_1^R(T_1, X_i)$ and $\mathrm{Tor}_1^R(T_1, \prod X_i)$ are zero. By [AHH08, Corollary 9.8], T_1 is $\mathcal{F}$-Mittag-Leffler, thus ρ_{T_1} is injective and so $T_1 \otimes \prod X_i = 0$, since by assumption $T_1 \otimes X_i = 0$, for every $i \in I$.

□

4. Cotilting modules over valuation domains

In this section R will be a valuation domain with quotient field Q.

In [Baz07], the author developed a method to associate to a cotilting module over a valuation domain R a chain of intervals of prime ideals of R which determine the cotilting class. The aim of the present and next section is to characterize when this chain of prime ideals satisfies the conditions which allow to apply the classification theorem of Section 2.

We will have to generalize or extend some of the results proved in [Baz07], since now we are interested in the connection with homological ring epimorphisms. For more details the reader is referred to [Baz07].

We just recall one important fact. Using properties of finitely generated modules over valuation domains and the fact that a cotilting module C is pure injective, we have that the cotiling class ${}^{\perp}C$ is determined by the cyclic modules that it contains (see [Baz07, Lemmas 3.1]. Moreover, if M is a uniserial R-module, then $M \in {}^{\perp}C$ if and only if every cyclic (torsion) submodule of M belongs to ${}^{\perp}C$.

For the characterization of cotilting modules over valuation domains an important rôle is played by the following sets of ideals of R:

Notation 4.1. Let C be a cotilting module over a valuation domain R. Let

$$\mathcal{G} = \{I \leq R \mid R/I \in {}^{\perp}C\} = \{I \leq R \mid R/I \in \mathrm{Cogen}\, C\}.$$

$$\mathcal{G}' = \{L \in \mathrm{Spec}\, R \mid L \in \mathcal{G}\}.$$

$\mathcal{G}'$ and $\mathcal{G}$ will be called the sets associated to C.

Note that our definition of $\mathcal{G}'$ slightly differs from the one given in [Baz07], since now we allow $\mathcal{G}'$ to contain also the zero ideal.

We restate here a result from [Baz07] since it will be crucial to relate cotilting classes with chains of intervals of prime ideals. The same proof as in [Baz07] works also with our extended definition of $\mathcal{G}'$.

Lemma 4.2. *(*[Baz07, Lemma 3.5]*) Let C be a cotilting module with associated set $\mathcal{G}'$. For every $L \in \mathcal{G}'$, let $H = \sum\{a^{-1}R \mid 0 \neq a \in R,\ a^{-1}R/L \in {}^{\perp}C\}$. Then there is an idempotent prime ideal $L' \leq L$ such that $H = R_{L'}$ and $L' \in \mathcal{G}'$. Moreover, $R_{L'}/L \in {}^{\perp}C$ and $L' = \inf\{N \in \mathcal{G}' \mid R_N/L \in {}^{\perp}C\}$.*

Remark 4.3. Note that L' in the above lemma might be zero and if $L = 0$, then certainly $L' = 0$. (Note that $Q \in {}^{\perp}C$, because Q is flat and C is pure injective.)

4.1. **Disjoint intervals of primes ideals of $\mathcal{G}'$.**

Definition 4.4. Let C be a cotilting module with associated set $\mathcal{G}'$. For every $L \in \mathcal{G}'$ define

$$\phi(L) = \inf\{N \in \mathcal{G}' \mid R_N/L \in {}^{\perp}C\},$$

$$\psi(L) = \sup\{N \in \mathcal{G}' \mid R_{\phi(L)}/N \in {}^{\perp}C\}.$$

By ([Baz07, Lemma 3.3] and Lemma 4.2, ϕ and ψ are maps from $\mathcal{G}'$ to $\mathcal{G}'$; $\phi(L)$ is an idempotent prime ideal (which might be zero).

Note that $\psi(0)$ is the largest prime ideal N such that $Q/N \in {}^{\perp}C$.

The properties of the two maps ϕ and ψ are illustrated in [Baz07, Lemma 6.1, 6.2]. In particular, distinct intervals of the form $[\phi(L), \psi(L)]$ for $L \in \mathcal{G}'$ are disjoint.

We need to have information about the ideals of $\mathcal{G}$ sitting between $\phi(L)$ and $\psi(L)$ for every $L \in \mathcal{G}'$. To this aim we first recall that for every non zero ideal I of R, $I^{\#} = \{r \in R \mid rI \lneq I\}$ is a prime ideal and it is the union of the proper ideals of R isomorphic to I (see[FS01, p. 70 (g)]. We put $0^{\#} = 0$.

Definition 4.5. Let $L_0 \leq L$ be two prime ideals of a valuation domain R. We let

$$\langle L_0, L\rangle = \{I < R \mid L_0 \leq I \leq I^{\#} \leq L\}.$$

Equivalently, $I \in \langle L_0, L\rangle$ if and only if $L_0 \leq I \leq L$ and I is an ideal R_L.

The next proposition is similar to [Baz07, Proposition 6.5], but now we drop the assumption on R to be a maximal valuation domain, so we present an alternative proof. (Recall that a valuation ring is *maximal* if it is linearly compact in the discrete topology.)

Lemma 4.6. *Let C be a cotilting module over a valuation domain R with associated set $\mathcal{G}'$ and let M be an R-module. If $M \in {}^{\perp}C$, then for every non zero torsion element $x \in M$ there exists $L \in \mathcal{G}'$ such that* $\mathrm{Ann}(x) \in \langle\phi(L), \psi(L)\rangle$.

If M is moreover uniserial, then the converse holds true.

Proof. (1) Let $M \in {}^{\perp}C$ and let $0 \neq x \in M$ be a torsion element. Then $0 \neq \mathrm{Ann}(x) = I \in \mathcal{G}$, so $I^{\#} \in \mathcal{G}'$. Let $L = I^{\#}$; we claim that $I \in \langle \phi(L), \psi(L) \rangle$. It is enough to show that $\phi(L) \leq I$. Assume by contradiction that $I < \phi(L)$ and let $r \in \phi(L) \backslash I$. By[Baz07, Lemma 3.3](3), $rL \in \mathcal{G}$, hence $r^{-1}R/L \in {}^{\perp}C$. But $r^{-1}R > R_{\phi(L)}$, since $rR_{\phi(L)} < \phi(L)$, thus by Lemma 4.2, $r^{-1}R/L \notin {}^{\perp}C$, a contradiction.

For the second statement note that by the remarks at the begining of the section, $M \in {}^{\perp}C$ if and only if every cyclic torsion submodule of M belongs to ${}^{\perp}C$; so it is enough to show that if $0 \neq I \in \langle \phi(L), \psi(L) \rangle$, for some $L \in \mathcal{G}'$, then $R/I \in {}^{\perp}C$. We can assume $I \lneq \psi(L)$, so $I \leq r\psi(L)$ for every $r \in \psi(L) \setminus I$, since $r^{-1}I \leq I^{\#} \leq \psi(L)$. Moreover, $I = \bigcap_{r \in \psi(L) \setminus I} r\psi(L)$. In fact, assume on the contrary that $I \lneq \bigcap_{r \in \psi(L) \setminus I} r\psi(L) = J$ and let $a \in J \setminus I$. Then, $J = a\psi(L) \gneq I$ and choosing any $b \in a\psi(L) \setminus I$ we have $b\psi(L) = a\psi(L)$ and so $b = ac$ for some $c \in \psi(L)$. But $b\psi(L) = a\psi(L)$ implies $c\psi(L) = \psi(L)$ contradicting the assumption $c \in \psi(L)$. Thus R/I is embedded in $\prod_{r \in \psi(L) \setminus I} R/r\psi(L) \cong \prod_{r \in \psi(L) \setminus I} r^{-1}R/\psi(L)$ and $r^{-1}R/\psi(L) \in {}^{\perp}C$, since $r^{-1}R/\psi(L) \leq R_{\phi(L)}/\psi(L) \in {}^{\perp}C$. □

Definition 4.7. Let C be a cotilting module over a valuation domain R with associated set $\mathcal{G}'$. Denote by $\mathcal{I}(C)$ the set of intervals of prime ideals of the form $[\phi(L), \psi(L)]$, for every $L \in \mathcal{G}'$ ordered by

$$[\phi(L), \psi(L)] < [\phi(L'), \psi(L')] \quad \text{if} \quad \psi(\mathrm{L}) < \phi(\mathrm{L}').$$

By Remark 4.3 and [Baz07, Lemmas 6.1, 6.2], $[0, \psi(0)]$ is the unique minimal element of $\mathcal{I}(C)$ and distinct intervals of $\mathcal{I}(C)$ are disjoint.

Our task is now to analyze the properties satisfied by the set $\mathcal{I}(C)$ and in particular, to determine when it fulfils Conditions 2.5 (1)-(3). We first show that it satisfies conditions (1) and (2).

Proposition 4.8. *Let C be a cotilting module over a valuation domain R. The set $\mathcal{I}(C)$ defined in Definition 4.7 has a minimal element $[0, \psi(0)]$ and it satisfies conditions (1) and (2) of Conditions 2.5, namely:*

(1) *If $\mathcal{S} = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}(C)$ with no minimal element, then $\mathcal{I}(C)$ contains an element of the form $[J, \bigcap_{\alpha \in \Lambda} \psi(L_\alpha)]$.*

(2) *If $\mathcal{S} = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}(C)$ with no maximal element, then $\mathcal{I}(C)$ contains an element of the form $[\bigcup_{\alpha \in \Lambda} \phi(L_\alpha), L]$.*

Proof. We will make repeated use of [Baz07, Lemmas 6.1, 6.2].

(1) Assume that $\mathcal{S} = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}(C)$ with no minimal element. Let $L_0 = \bigcap_\alpha \psi(L_\alpha)$. Then, L_0 is a prime ideal and $L_0 \in \mathcal{G}'$ by ([Baz07, Lemma 3.3]. For every $\alpha \in \Lambda$ we have $\psi(L_0) \leq \psi(L_\alpha)$, then $\psi(L_0) \leq L_0$ and thus $\psi(L_0) = L_0$ by [Baz07, Lemmas 6.1, 6.2]. Hence, $[\phi(L_0), \psi(L_0) = \bigcap_\alpha \psi(L_\alpha)]$ is in $\mathcal{I}(C)$.

(2) Assume that $\mathcal{S} = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $\mathcal{I}(C)$ with no maximal element.

Let $L_0 = \bigcup_{\alpha\in\Lambda} \phi(L_\alpha)$. Then L_0 is a prime ideal and by ([Baz07, Lemma 3.3], $L_0 \in \mathcal{G}'$. For every $\alpha \in \Lambda$ we have $\phi(L_\alpha) \leq \phi(L_0)$, Thus, $L_0 = \bigcup_{\alpha\in\Lambda} \phi(L_\alpha) \leq \phi(L_0)$ and so $L_0 = \phi(L_0)$ by [Baz07, Lemmas 6.1, 6.2]. Hence, $[L_0 = \bigcup_{\alpha\in\Lambda} \phi(L_\alpha), \psi(L_0)]$ belongs to $\mathcal{I}(C)$.

□

In Section 6 we will show that there exist cotilting modules C whose associated set of intervals $\mathcal{I}(C)$ does not satisfy condition (3) of Conditions 2.5, that is $\mathcal{I}(f)$ contains a dense subset of intervals.

5. A bijective correspondence

In this section again R will be a valuation domain.

For every homological ring epimorphism $f\colon R \to S$ and every cotilting R-module C, $\mathcal{I}(f)$ and $\mathcal{I}(C)$ will denote the chains of intervals of Inter R as defined in Definition 2.4 and Definition 4.7.

From [BŠ14, Theorem 6.23] we know that the set $\mathcal{I}(f)$ satisfies all the three conditions in Conditions 2.5. By Proposition 4.8, the set $\mathcal{I}(C)$ satisfies the first two conditions of Conditions 2.5 but, as we will see in Section 6 it may not satisfy the third condition.

Thus, we distinguish the two possible situations for a cotilting module and at this aim we introduce the following:

Definition 5.1. We say that a cotilting module C is *non dense* if the set $\mathcal{I}(C)$ does not contain any dense subset, that is if $\mathcal{I}(C)$ satisfies

(3) Given any $[\phi(L_0), \psi(L_0)] < [\phi(L_1), \psi(L_1)]$ in $\mathcal{I}(C)$, there are two intervals $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$ of $\mathcal{I}(C)$ such that

$$[\phi(L_0), \psi(L_0)] \leq [\phi(L), \psi(L)] < [\phi(L'), \psi(L')] \leq [\phi(L_1), \psi(L_1)]$$

and there no other intervals of $\mathcal{I}(C)$ properly between $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$

The corresponding cotilting class will also be called *non dense*.

Combining results from Section 2 and Section 4.1 we are in a position to assign to every non dense cotilting module over a valuation domain R an injective homological ring epimorphisms $f\colon R \to S$.

Proposition 5.2. *Let C be a non dense cotilting module over a valuation domain R with associated set $\mathcal{I}(C)$ of intervals as in Definition 4.7. Then there is an injective homological ring epimorphism $f\colon R \to S$ such that $\mathcal{I}(f) = \mathcal{I}(C)$.*

Proof. By assumption and Proposition 4.8 the ordered set $\mathcal{I}(C)$ satisfies Conditions 2.5. So, by Classification 2.7, there is a homological ring epimorphism $f\colon R \to S$ such that $\mathcal{I}(f) = \mathcal{I}(C)$. Since the minimal element of $\mathcal{I}(C)$ is $[0, \psi(0)]$ we infer that f is injective. □

Remark 5.3. In the notations of Proposition 5.2, if $\mathcal{I}(C)$ is finite, say $\mathcal{I}(C) = [0, \psi(0)] \cup \{[\phi(L_i), \psi(L_i) \mid 1 \leq i \leq n\}$, then $S \cong Q/\psi(0) \oplus \prod_{1\leq i\leq n} \frac{R_{\psi(L_i)}}{\phi(L_i)}$.

From Theorem 3.4 we already know that to every injective homological ring epimorphism $f\colon R \to S$ with w.d.$S \leq 1$ we can associate a cotilting R-module C. A crucial fact is now to prove that, when R is a valuation domain, then C is non dense and the sets $\mathcal{I}(f)$ and $\mathcal{I}(C)$ coincide. This needs some work.

Proposition 5.4. *Let R be a valuation domain and $f\colon R \to S$ be an injective homological epimomorphism with associated set $\mathcal{I}(f)$. Then, $C = S^* \oplus (S/R)^*$ is a non dense 1-cotilting R-module such that $\mathcal{I}(C) = \mathcal{I}(f)$.*

Proof. Identifying R with $f(R)$, Theorem 3.4 tells us that $C = S^* \oplus (S/R)^*$ is a cotilting R-module and that $R/I \in {}^{\perp}C$ if and only if $R/I \to S/SI$ is injective, that is if and only if $R \cap SI = I$. Our goal is to prove that $\mathcal{I}(C) = \mathcal{I}(f)$.

(a) First of all we note that for every interval $[J, L] \in \mathcal{I}(f)$ and every prime ideal $P \in [J, L]$ we have that $R/P \to S/SP$ is an injective ring homomorphism, hence $R/P \in {}^{\perp}C$. In fact, by [BŠ14, Lemma 6.8] $P = f^{-1}(\mathfrak{n}) = R \cap \mathfrak{n}$ for some prime ideal $\mathfrak{n}$ of S, hence $P \leq R \cap SP \leq R \cap \mathfrak{n} = P$.

CLAIM 1 Let $[J, L] \in \mathcal{I}(f)$. We claim that $[J, L]$ is contained in the interval $[\phi(J), \psi(J)]$ of $\mathcal{I}(C)$.

Let $\mathfrak{m}$ be a maximal ideal of S such that $S_{\mathfrak{m}} \cong R_L/J$ (by Lemma 2.2).

(b) We first show that R_J/L is contained in ${}^{\perp}C$. In fact, consider the valuation domain $V = R_L/J$. Its maximal ideal is $\mathfrak{p} = L/J$ and its quotient field $Q(V)$ is isomorphic to R_J/J. Thus $Q(V)/\mathfrak{p} \cong R_J/L$. An injective cogenerator E_V of $(V, \mathfrak{p})$ is the pure injective envelope of $Q(V)/\mathfrak{p}$ as V-module (see [FS01, XII Lemma 4.3].) Thus, in our case we have that R_J/L is an R-submodule of an injective cogenerator of R_L/J. Now the injective cogenerator of R_L/J is an S-module, since so is R_L/J, thus $R_J/L \in \operatorname{Cogen} S^* = {}^{\perp}C$

By (a) $R/J \in {}^{\perp}C$, hence also $R_{\phi(J)}/J \in {}^{\perp}C$, by definition of $\phi(J)$.

(c) We show that $R_{\phi(J)}/L \in {}^{\perp}C$, so that $L \leq \psi(J)$ by the definition of the map ψ and thus $[J, L]$ is contained in $[\phi(J), \psi(J)] \in \mathcal{I}(C)$ and the claim is proved.

If $\phi(J) = J$, then by (b) $R_{\phi(J)}/L \in {}^{\perp}C$.

If $\phi(J) \lneq J$ we consider the exact sequence

$$0 \to R_J/L \to R_{\phi(J)}/L \to R_{\phi(J)}/R_J \to 0.$$

Let $0 \neq x \in R_{\phi(J)}/R_J$, that is $x = a^{-1} + R_J$ with $a \in J \setminus \phi(J)$. Then $\operatorname{Ann}_R x = aR_J \geq \phi(J)$, hence $\operatorname{Ann}_R x \in \langle \phi(J), \psi(J) \rangle$. $R_{\phi(J)}/R_J$ is uniserial, so by Lemma 4.6 we conclude that $R_{\phi(J)}/R_J \in {}^{\perp}C$. Since by (b) $R_J/L \in {}^{\perp}C$, the above exact sequence tells us that also $R_{\phi(J)}/L \in {}^{\perp}C$.

CLAIM 2 Let $[\phi(N), \psi(N)] \in \mathcal{I}(C)$. We claim that $[\phi(N), \psi(N)]$ is contained in an interval of $\mathcal{I}(f)$.

Applying Corollary 2.3 to the idempotent ideal $\phi(N)$ we get an interval $[J, L] \in \mathcal{I}(f)$ such that $J \leq \phi(N) \leq L$.

(d) We show that $\psi(N) \leq L$ so that $[J, L]$ contains $[\phi(N), \psi(N)]$ and the claim is proved.

Assume by way of contradiction that $L \lneq \psi(N)$.

We have $S\psi(N) \cap R = \psi(N)$, because $R/\psi(N) \in {}^{\perp}C$. Consider the set $\{\mathfrak{n}_\alpha \mid \alpha \in \Lambda\}$ of maximal ideals of S containing $S\psi(N)$ and let

$$\mathcal{S} = \{[J_\alpha, L_\alpha] \mid \alpha \in \Lambda\}$$

be the set of intervals of $\mathcal{I}(f)$ corresponding to $\mathfrak{n}_\alpha$, for every $\alpha \in \Lambda$. Note that $\psi(N) \leq L_\alpha$, for every α, since $f^{-1}(\mathfrak{n}_\alpha) = L_\alpha$. The assumption $L \lneq \psi(N)$ implies that $L \lneq J_\alpha$ for every $\alpha \in \Lambda$. In fact, if $J_\alpha \leq L$ for some α, then $L \in [J_\alpha, L_\alpha]$, hence $[J, L] = [J_\alpha, L_\alpha]$ giving $\psi(N) \leq L$, contradicting the assumption. We show now that there is an interval $[J', L'] \in \mathcal{I}(f)$ which is minimal among the intervals of $\mathcal{I}(f)$ satisfying $\psi(N) \leq L'$. Indeed, if $\mathcal{S}$ has a minimal element the claim is immediate. Otherwise, by Classification 2.7, Conditions 2.5 (1) ensures that $\mathcal{I}(f)$ contains an interval $[J', L' = \bigcap_{\alpha\in\Lambda} L_\alpha]$ and $\psi(N) \leq \bigcap_{\alpha\in\Lambda} L_\alpha$. Hence $[J', L']$ satisfies our claim.

We must have $[J, L] \lneq [J', L']$ hence, by Conditions 2.5 (3), there are two intervals $[J_0, L_0] \lneq [J_1, L_1]$ with no other intervals of $\mathcal{I}(f)$ between them and such that

$$[J, L] \leq [J_0, L_0] \lneq [J_1, L_1] \leq [J', L'].$$

Thus $L_0 \lneq \psi(N)$. Choose $a \in (\psi(N) \cap J_1) \setminus L_0$. Then the canonical localization map $S \to S[\frac{1}{a}]$ is surjective. Indeed, this can be proved locally by using the properties of $\mathcal{I}(f)$ and Lemma 2.2. In fact, for every maximal ideal $\mathfrak{m}$ of S with corresponding interval $[J'', L'']$, if $[J'', L''] \leq [J_0, L_0]$ the morphism $S_\mathfrak{m} \to (S[\frac{1}{a}])_\mathfrak{m}$ is an isomorphism; if $[J'', L''] \geq [J_1, L_1]$ the morphism $S_\mathfrak{m} \to (S[\frac{1}{a}])_\mathfrak{m}$ is zero. The surjectivity of $S \to S[\frac{1}{a}]$ implies the existence of $n \geq 1$ and of an element $t \in S$ such that $a^n = ta^{n+1}$. Let $I = a^n\psi(N)$; then $a^n \in SI$, since $a \in \psi(N)$, hence $a^n \in R \cap SI$, but clearly $a^n \notin I$. This implies $R/I \notin {}^{\perp}C$. But, the annihilator of every $0 \neq r + I \in R/I$ is $r^{-1}I$ and $r^{-1}I \in \langle \phi(N), \psi(N) \rangle$. In fact, $\phi(N) \leq L_0 \lneq a^n\psi(N) = I \leq r^{-1}I$ and $(r^{-1}I)^\# = \psi(N)$, hence by Lemma 4.6 $R/I \in {}^{\perp}C$, a contradiction.

By claims (a) and (b) and by the disjointness of the intervals in $\mathcal{I}(f)$ and $\mathcal{I}(C)$, we conclude that the two sets of intervals coincide. □

At this point, we have all the ingredients to formulate the existence of a bijection between non dense cotilting modules and injective homological ring epimorphisms (up to equivalence). But before doing that, we give some results in order to characterize the injective homological epimorphisms $f\colon R \to S$ among the homological epimorphisms.

Lemma 5.5. *Let R be a valuation domain let $\epsilon\colon R \to Q$ be the canonical inclusion into the quotient field Q of R. Assume that $f\colon R \to S$ is a homological ring epimorphism with* $\operatorname{Ker} f \neq 0$. *The following hold true:*

(1) *The morphism $g = (\epsilon, f)\colon R \to Q \oplus S$ is an injective homological epimorphism and $\mathcal{I}(g) = [0] \cup \mathcal{I}(f)$.*

(2) $\left(\dfrac{Q \oplus S}{g(R)}\right)^* \cong \left(\dfrac{Q}{\operatorname{Ker} f}\right)^* \oplus \left(\dfrac{S}{f(R)}\right)^*.$

Proof. (1) g is an injective ring homomorphism and it is an epimorphism since $Q \otimes_R S = 0 = S \otimes_R Q$, due to the fact that Q is divisible and S is a torsion R-module annihilated by K. It is obvious that g is homological. Let $S' = Q \oplus S$, and let $\mathfrak{n}'$ be a maximal ideal of S'. Then $\mathfrak{n}' = Q \oplus \mathfrak{n}$, with $\mathfrak{n}$ a maximal ideal of S or $\mathfrak{n}' = S$ viewed as an ideal of S'. It is easy to check that $S'_{\mathfrak{n}'} \cong S_{\mathfrak{n}}$ whenever $\mathfrak{n}' = Q \oplus \mathfrak{n}$ and $S'_S \cong Q$. Thus the interval in $\mathcal{I}(g)$ corresponding to the maximal ideal S of S' is $[0]$ and the intervals of $\mathcal{I}(g)$ corresponding to the other maximal ideals of S' are the same as the intervals of $\mathcal{I}(f)$.

(2) $\dfrac{Q \oplus S}{g(R)}$ is a pushout of f and $-\epsilon$, so we get the exact sequence

$$0 \to \frac{Q}{\operatorname{Ker} f} \to \frac{Q \oplus S}{g(R)} \to \frac{S}{f(R)} \to 0$$

whose dual sequence splits since $\left(\dfrac{Q}{\operatorname{Ker} f}\right)^*$ is torsion-free (hence flat) and $\left(\dfrac{S}{f(R)}\right)^*$ is pure injective. □

Lemma 5.6. *Let R be a valuation domain and let $f\colon R \to S$ be a homological ring epimorphism. Then f is injective if and only if there are a prime ideal L of R and a homological epimorphism $g\colon R \to S'$ such that $S \cong R_L \oplus S'$ and f is equivalent to (ψ_L, g) where ψ_L is the canonical localization of R at the prime ideal L.*

Proof. If f is an injective homological epimorphism, then the minimal element of $\mathcal{I}(f)$ is an interval $[0, L]$, for some prime ideal L of R. By [BŠ14, Section 4], $\mathcal{I}(f)$ is the disjoint union of $[0, L]$ with a set $\mathcal{I}'$, where $\mathcal{I}'$ satisfies Conditions 2.5, hence there is a ring S' and a homological epimorphism $g\colon R \to S'$ such that $\mathcal{I}(g) = \mathcal{I}'$. Then $(\psi(L), g)\colon R \to R_L \oplus S'$ is a homogical epimorphism, by Lemma 2.2 (3) and $\mathcal{I}(\psi(L), g) = \mathcal{I}$. By Classification 2.7 we conclude that f and $(\psi(L), g)$ are equivalent homological epimorphisms.

The converse is clear from the fact that $\psi_L\colon R \to R_L$ is an injective homological epimorphism. □

With the aid of the classification theorem (Classification 2.7, the results proved in this section can be summarized by the following theorem.

Theorem 5.7. *Let R be a valuation domain. Then there is a bijection between:*

1. *Equivalence classes of non dense cotilting modules.*
2. *Equivalence classes of injective homological ring epimorphisms*

$$f\colon R \to S.$$

The bijection is given by assigning to a non dense cotilting module C the homological ring epimorphism $f_{\mathcal{I}(C)}\colon R \to R_{\mathcal{I}(C)}$ constructed in [BŠ14, Construction 4.12].

The converse is given by sending an injective homological ring epimorphism $f\colon R \to S$ to the cotilting module $C = S^* \oplus S/f(R)^*$.

Moreover, for every homological ring epimorphism $f\colon R \to S$ with $\operatorname{Ker} f \neq 0$ *there is an injective homological ring epimorphism $g\colon R \to S' \cong Q \oplus S$ such that $\mathcal{I}(g) = [0] \cup \mathcal{I}(f)$ and associated cotilting module $S'^* \oplus (Q/\operatorname{Ker} f)^* \oplus (S/f(R))^*$.*

By [BŠ14, Theorem 3.13] there is a bijective correspondence between equivalence classes of homological ring epimorphism originating in valuation domains R and smashing localizing subcategories of the derived category $\mathbf{D}(R)$ of R. In the next Proposition 5.9, we will restrict the correspondence between non dense cotilting classes and some particular smashing localizing subcategories of $\mathbf{D}(R)$.

First recall that a triangulated subcategory $\mathcal{X}$ of the derived category $\mathbf{D}(R)$ of a ring R is *smashing localizing* if it is closed under coproducts and its orthogonal class $\mathcal{Y} = \{Y \in \mathbf{D}(R) \mid \mathrm{Hom}_{\mathbf{D}(R)}(\mathcal{X}, Y) = 0\}$ is closed under coproducts as well.

We recall also the following notion.

Definition 5.8. Let R be a commutative ring. The *cohomological support* of $X \in \mathbf{D}(R)$ is:

$$\mathrm{Supp}\, X = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid R_{\mathfrak{p}} \otimes_R X \neq 0\}.$$

For a class of complexes $\mathcal{X}$, we define $\mathrm{Supp}\,\mathcal{X} = \bigcup_{X\in\mathcal{X}} \mathrm{Supp}\, X$

Combining the previous results with [BŠ14, Theorem 4.10] we obtain:

Proposition 5.9. *Let R be a valuation domain. Then there is a bijection between:*

(1) *Equivalence classes of non dense cotilting modules.*
(2) *Smashing localizing subcategories $\mathcal{X}$ of $\mathbf{D}(R)$ for which there exists a prime ideal L of R such that $L \notin \mathrm{Supp}\,\mathcal{X}$, or equivalently such that $H^n(X)$ is a torsion R/L-module for every $n \in \mathbb{Z}$ and every $X \in \mathcal{X}$.*

Proof. By Theorem 5.7 and Lemma 5.6 a non dense cotilting class corresponds bijectively to the equivalence classes of a homological epimorphism $f\colon R \to R_L \oplus S$ for some prime ideal L of R. By [BŠ14, Theorem 4.10] the smashing localizing subcategory $\mathcal{X}$ corresponding to f satisfies the condition as in (2).

Conversely, let $\mathcal{X}$ be a smashing localizing subcategory of $\mathbf{D}(R)$ as in (2) and let $f\colon R \to S$ be a homological ring epimorphism corresponding to $\mathcal{X}$ under [BŠ14, Theorem 4.10].

We claim that f is injective. Assume on the contrary that $0 \neq J = \mathrm{Ker}\, f$. We have that J, as a complex concentrated in degree zero, belongs to $\mathcal{X}$. In fact, by [BŠ14, Theorem 4.10] $J \in \mathcal{X}$ if and only if $S \otimes_R J = 0$. The latter is zero since S is annihilated by J and every element of J is of the form ab for a and b in J, because J is idempotent. By assumption, there is a prime ideal $L \in \mathrm{Spec}\, R$ such that $L \notin \mathrm{Supp}\, J$, that is $R_L \otimes_R J = J_L = 0$, a contradiction.

□

6. Cotilting modules with a dense set of intervals

Example 6.1. Let $\Theta = [0, 1]$ be the interval of the real numbers between 0 and 1 and consider the totally ordered set $(\mathcal{T}, \leq)$ where $\mathcal{T} = \Theta \times \{0, 1\}$ and $\leq$ is the lexicographic order:

$$(x, a) < (y, b) \Leftrightarrow x < y \text{ or } (x = y \text{ and } a < b).$$

For every $x \in \Theta$ let $p_x = (x, 0)$, $q_x = (x, 1)$. Fix two elements $(x, a) < (y, b)$ of $\mathcal{T}$. If $x = y$, then there are no elements of $\mathcal{T}$ properly between (x, a) and

(y,b). If $x<y$ and $x<z<y$, then $(x,a)<p_z<q_z<(y,b)$ and there are no elements of $\mathcal{T}$ between p_z and q_z. Moreover,

$$(i)\quad \forall\ 0\neq x\in\Theta,\quad p_x=\sup\{q_z\mid z<x\}=\sup\{p_z\mid z<x\},$$

$$(ii)\quad \forall\ 1\neq x\in\Theta,\quad q_x=\inf\{p_z\mid z>x\}=\inf\{q_z\mid z>x\}.$$

If $t\leq t'\in\mathcal{T}$, let $[t,t']$ be the interval of the elements of $\mathcal{T}$ between t and t'.

For every, $x\in\Theta$ the interval $[p_x,q_x]$ consists just of the two elements p_x,q_x and we have $\mathcal{T}=\bigcup\limits_{x\in I}[p_x,q_x]$.

By [FS01, Theorem 2.5 and Proposition 4.7], $\mathcal{T}$ is order isomorphic to the prime spectrum of a valuation domain R.

For each p_x, q_x, let J_x, L_x be the prime ideals of R corresponding to p_x and q_x, respectively. Then, for every $x\in\Theta$, J_x is idempotent by (i). The set of intervals $\{[p_x,q_x]\mid x\in\Theta\}$ corresponds to the set $\mathcal{I}=\{[J_x,L_x]\mid x\in\Theta\}$ of intervals of prime ideals of R. Define on $\mathcal{I}$ the total order given by $[J_x,L_x]<[J_y,L_y]$ if and only if $x<y$ in Θ.

Remark 6.2. It is easy to see that the totally ordered set $\mathcal{I}$ defined above satisfies properties (1) and (2) of Construction 2.5, but $\mathcal{I}$ does not satisfy (3). This means that there are no homological ring epimorphisms $f\colon R\to S$ such that $\mathcal{I}=\mathcal{I}(f)$.

Indeed, from [BŠ14, Lemm 6.5] one obtains that such an S should be a subring of $\prod\limits_{x\in\Theta}R_{L_x}/J_x$ whose elements satisfy the conditions of [BŠ14, Proposition 6.16], while in our case the only elements of $\prod\limits_{x\in\Theta}R_{L_x}/J_x$ satisfying those conditions are the elements of R.

We show that there is a valuation domain and a cotilting module C whose associated set $\mathcal{I}(C)$ of intervals is the set $\mathcal{I}$ defined above. We first note the following.

Lemma 6.3. *Let R be a valuation domain with prime spectrum order isomorphic to the totally order set $\mathcal{T}$ of Example 6.1. Then, for every ideal I of R there is $x\in\Theta$ such that $J_x\leq I\leq L_x$.*

Proof. Let $S_I=\{y\in\Theta\mid I\lneq J_y\}$. If $S_I=\emptyset$, then $I\geq J_1$ so $J_1\leq I\leq L_1$, since L_1 is the maximal ideal of R. If $S_I\neq\emptyset$ let x_0 be the infimum of S_I. Then $I\geq\bigcup\limits_{z<x_0}J_z=\bigcup\limits_{z<x_0}L_z=J_{x_0}$. Thus $x_0\notin S_I$ and $I\leq\bigcap\limits_{x_0<z}J_z=L_{x_0}$, hence the conclusion. □

Proposition 6.4. *Let R be a maximal valuation domain whose prime spectrum is isomorphic to the totally ordered set $\mathcal{T}$ defined in Example 6.1. Then the module:*

$$C=Q\oplus\prod_{x\in\Theta}\frac{R_{J_x}}{L_x}$$

is a cotilting module such that $\mathcal{I}(C)=\mathcal{I}=\{[J_x,L_x]\mid x\in\Theta\}$.

Proof. First note that C is a pure injective module, since R is a maximal valuation domain (see[FS01, Xiii Theorem 5.2]).

CLAIM (A) $\operatorname{Cogen}C\subseteq{}^{\perp}C$.

By [Baz07, Lemmas 3.1] it is enough to show that every cyclic submodule of C^λ belongs to ${}^\perp C$, for every cardinal λ. Let $(c_\alpha)_{\alpha\in\lambda} \in C^\lambda$ and $I_\alpha = \mathrm{Ann}_R(c_\alpha)$ for every $\alpha \in \lambda$. We have to show that, if $I = \cap_{\alpha\in\lambda} I_\alpha$, then R/I belongs to ${}^\perp C$. By [Baz07, Lemma 6.6] $R/I \in {}^\perp(R_{J_y}/L_y)$ if and only if either $I \geq J_y$ or in case $I \lneq J_y$ it must be $I^\# \leq J_y$ and $I \not\cong R_{J_y}$. Every I_α is of the form $\bigcap_{z\in \mathrm{Supp}\, c_\alpha} a_x L_x$ where $a_x \notin J_x$, thus I is also an intersection of ideals of the form $a_x L_x$ where x vary in a subset of Θ. Fix $y \in \Theta$ and assume $I \lneq J_y$. Let $A_I = \{x \in \Theta \mid I \leq a_x L_x \lneq J_y\}$, then $I = \bigcap_{x\in A_I} a_x L_x$ and for every $x \in A_I$ we have $x \lneq y$. In fact, if $y \leq x$, then $J_y \leq J_x$ so $a_x \notin J_y$ and we would get the contradiction $J_y = a_x J_y \lneq a_x L_x$. Let $b \notin J_y$, then $b \notin L_x$, for every $x \in A_I$, so $bI = \bigcap_{x\in A_I} a_x b L_x = I$, hence we conclude that $I^\# \leq J_y$. It remains to show that I cannot be a principal ideal of R_{J_y}. Order $\{a_x L_x \mid x \in A_I\}$ as a descending chain of ideals. Note that if $x' < x \in A_I$, then we have the inclusions

$$(a) \quad J_{x'} \lneq a_{x'} L_{x'} \leq L_{x'} \lneq J_x \lneq a_x L_x.$$

If A_I has a minimum x_0, then $I = a_{x_0} L_{x_0}$, so $I \not\cong R_{J_y}$. If $x_0 = \inf A_I$, then by (a), $I = \bigcap_{x_0<x\in A_I} J_x = \bigcap_{x_0<x\in A_I} L_x = J_{x_0}$, so again $I \not\cong R_{J_y}$.

CLAIM (B) ${}^\perp C \subseteq \mathrm{Cogen}\, C$.

By (A) $\mathrm{Cogen}\, C$ is a torsion free class and ${}^\perp C$ is closed under submodules. Thus to prove the claim it is enough to show that if $M \in {}^\perp C$ and M is torsion in the torsion theory associated to C, that is $\mathrm{Hom}_R(M, C) = 0$, then $M = 0$.

Moreover, since Q is a summand of C, $\mathrm{Cogen}\, C$ contains the class of torsion free modules in the torsion theory of the commutative domain R. Thus we may assume that $M \in {}^\perp C$ is torsion in the classical sense.

Let $0 \neq R/I$ be isomorphic to a non zero cyclic submodule of M.

(i) We show that there is $x_0 \in \Theta$ such that $I \in \langle J_{x_0}, L_{x_0}\rangle$ (see Definition 4.5).
We have $R/I \in {}^\perp R_{J_x}/L_x$, for every $x \in \Theta$. By [Baz07, Lemma 6.6], we infer that if $I \lneq J_y$ for some $y \in \Theta$, then $I^\# \leq J_y$ and $I \not\cong R_{J_y}$. As in the proof of Lemma 6.3, let x_0 be the infimum of the set $S_I = \{y \in \Theta \mid I \lneq J_y\}$. Then $J_{x_0} = \bigcup_{x<x_0} J_x \leq I$ and $I^\# \leq \bigcap_{x_0<y} J_y = L_{x_0}$. Then, $I \in \langle J_{x_0}, L_{x_0}\rangle$.
Let x_0 be as in (i) and let

$$M[J_{x_0}] = \{m \in M \mid mJ_{x_0} = 0\}.$$

(ii) We show that $\frac{M}{M[J_{x_0}]} \in {}^\perp C$, so that $\mathrm{Hom}_R(M[J_{x_0}], C) = 0$.
Let $0 \neq \overline{m} = m + M[J_{x_0}] \in \frac{M}{M[J_{x_0}]}$, that is $A = \mathrm{Ann}_R\, m \lneq J_{x_0}$. By (i) there is $x_1 \in [0, 1]$ such that $A \in \langle J_{x_1}, L_{x_1}\rangle$ and certainly $x_1 < x_0$. Consider $B = \mathrm{Ann}_R\, \overline{m}$, then $B = A\colon J_{x_0}$ and $A \leq B \lneq J_{x_0}$ (since J_{x_0} is idempotent). We get that $B = A$. Indeed, if $b \in B \setminus A$ then $J_{x_0} \leq b^{-1}A \leq A^\# \leq L_{x_1}$ contradicting $x_1 < x_0$. Thus, every cyclic submodule of $\frac{M}{M[J_{x_0}]}$ belongs to ${}^\perp C$, hence, by [Baz07, Lemmas 3.1],

also $\frac{M}{M[J_{x_0}]} \in {}^{\perp}C$ and from the exact sequence

$$0 \to M[J_{x_0}] \to M \to \frac{M}{M[J_{x_0}]} \to 0$$

we conclude that $\mathrm{Hom}_R(M[J_{x_0}], C) = 0$.

(iii) Consider the canonical localization $\psi\colon M[J_{x_0}] \to M[J_{x_0}] \otimes_R R_{L_{x_0}}$. Then $\operatorname{Ker} f = \{x \in M[J_{x_0}] \mid sx = 0, \exists s \notin L_{x_0}\}$.
Now $M[J_{x_0}] \otimes_R R_{L_{x_0}}$ is an $(R_{L_{x_0}}/J_{x_0})$-module, hence it is cogenerated by $R_{J_{x_0}}/R_{L_{x_0}}$ (see part (b) in the proof of Proposition 5.4) which is a direct summand of C. Thus the module $\dfrac{M[J_{x_0}]}{\operatorname{Ker} f}$ is also cogenerated by C and the condition $\mathrm{Hom}_R(M[J_{x_0}], C) = 0$ yields $\mathrm{Hom}_R\left(\dfrac{M[J_{x_0}]}{\operatorname{Ker} f}, C\right) = 0$. The generator ξ of the non zero cyclic submodule R/I of M we started with in (i), belongs to $M[J_{x_0}]$, since $J_{x_0} \leq I$ and its annihilator I is contained in L_{x_0}, thus ξ doesn't belong to $\operatorname{Ker} f$, a contradiction.

□

7. 1-cotilting modules and Tor-orthogonal classes

In this section we consider the problem about cotilting classes being Tor-orthogonal classes (see Section 1 for the definition of Tor-orthogonal classes).

We start by recalling a result relating Tor-orthoganal classes with the Mittag-Leffler condition (see Definition 3.10).

Lemma 7.1. *(*[Her13, Theorem 3.13]*) Let $\mathcal{C}$ be a class of right R-modules. The class $\mathcal{C}^{\intercal}$ is closed under products if and only if the syzygy of every module in $\mathcal{C}$ is $\mathcal{C}^{\intercal}$-Mittag-Leffler. Moreover, if $\mathcal{C}^{\intercal}$ is closed under products, there is a set $\mathcal{S}$ of countably presented right modules such that $\mathcal{S}^{\intercal} = \mathcal{C}^{\intercal}$.*

Remark 7.2. From the proof of [Her13, Theorem 3.13] and from the properties of Mittag-Leffler modules, one can see that the set $\mathcal{S}$ can be chosen to consist of strongly countably presented modules, that is modules whose first syzygy is again countably presented.

From the above results, we obtain:

Proposition 7.3. *Let ${}_RC$ be a 1-cotilting module over a ring R.*

1. *The cotilting class $\mathcal{F} = {}^{\perp}C$ is a* Tor *orthogonal class if and only if there is a set $\mathcal{S}$ of countably presented modules in ${}^{\intercal}\mathcal{F}$ such that $\mathcal{F} = \mathcal{S}^{\intercal}$.*
2. *Let M be a right R-module. The class $M^{\intercal}$ is a 1-cotilting class if and only if w.d. $M \leq 1$ and the first syzygy of M is a (countably presented flat) $M^{\intercal}$-Mittag-Leffler module.*

Proof. (1) follows immediately by Lemma 7.1

(2) We have $M^{\intercal} = {}^{\perp}M^*$ and $\text{w.d.}M = \text{i.d.}M^*$. Thus, by [GT12, Theorem 15.9], $M^{\intercal}$ is a 1-cotilting class iff it is closed under direct products. Then, apply Lemma 7.1 to conclude. □

Remark 7.4. By [Trl07, Lemma 4.9], if p.d.$M \leq 1$, then $M^{\intercal}$ is a 1-cotilting class of cofinite type.

Our next task will be to prove that over valuation domains every cotilting class is a Tor-orthogonal class. First we note the following property.

Lemma 7.5. *Let R be a valuation domain. A Tor-orthogonal class is determined by the cyclic modules that it contains.*

Proof. Let $\mathcal{C}$ be a class of modules. Since w.gl.dim $R = 1$ and Tor commutes with direct limits, we have that $\mathrm{Tor}_1^R(\mathcal{C}, M) = 0$ if and only if $\mathrm{Tor}_1^R(\mathcal{C}, F) = 0$ for every finitely generated torsion submodule of M (recall that torsion free modules are flat). By [FS01, I, 7.8], a finitely generated torsion module F over a valuation domain admits a finite chain of pure submodules with cyclic successive factors. It is immediate to conclude that $F \in \mathcal{C}^{\intercal}$ if and only if all the cyclic factors are in $\mathcal{C}^{\intercal}$. $\square$

Now we consider the easy case.

Lemma 7.6. *Let R be a valuation domain. Every non dense cotilting class is a* Tor-*orthogonal class.*

Proof. Let C be a non dense cotilting R-module over a valuation domain R. By Theorem 5.7 there is an injective homological ring epimorphism $f\colon R \to S$ such that C is equivalent to $S^* \oplus (S/f(R))^*$. Thus ${}^{\perp}C$ coincides with $(S/f(R))^{\intercal}$. $\square$

To deal with the case of a cotilting module C with associated set $\mathcal{I}(C)$ containing dense intervals, we introduce an equivalence relation on $\mathcal{I}(C)$ in the following way.

Notation 7.7.

(†) Let $(X, \leq)$ be a totally ordered set. A suborder $(Y, \leq)$ is said to be dense if given any two elements $a < b$ in Y there is $c \in Y$ such that $a < c < b$. Given $x, y \in X$, define $x \sim y$ if either $x = y$ or if the suborder of X consisting of all elements of X between x and y is dense. It is easy to see that $\sim$ is an equivalence relation.

(††) Let C be a cotilting module over a valuation domain and let $\mathcal{I}(C)$ be the totally ordered set as in Definition 4.7. Consider on $\mathcal{I}(C)$ the equivalence relation $\sim$ defined above and for every prime ideal $L \in \mathcal{G}'$ denote by $\mathfrak{i}_L$ the equivalence class determined by the interval $[\phi(L), \psi(L)]$ under the equivalence $\sim$. Applying Proposition 4.8, we see that every $\mathfrak{i}_L$ has a minimal element $[\phi(L_0), \psi(L_0)]$ and a maximal element $[\phi(L_1), \psi(L_1)]$. For every equivalence class $\mathfrak{i}_L$ consider the interval $[\phi(L_0), \psi(L_1)]$ of prime ideals and let $\mathcal{J}$ be totally ordered set consisting of all these intervals, for L varying in $\mathcal{G}'$.

Fact 7.8. The totally ordered set $\mathcal{J}$ defined in Notations 7.7 (††), satisfies all the properties in Conditions 2.5 and has a minimal element of the form $[0, \psi(N)]$ for some $N \in \mathcal{G}'$. So, by Classification 2.7 there is an injective homological ring epimorphism $f\colon R \to S$ such that $\mathcal{I}(f) = \mathcal{J}$.

By Proposition 5.4, $D = S^* \oplus (S/R)^*$ is a cotilting module such that $\mathcal{I}(D) = \mathcal{J}$, hence by Theorem 3.4 ${}^{\perp}D = (S/R)^{\intercal}$.

Remark 7.9. Note that if $\mathcal{I}$ is the totally ordered set of intervals defined in Example 6.1, then the quotient set $\mathcal{I}/_{\sim}$ of $\mathcal{I}$ modulo the equivalence relation in Notation 7.7 (†) consists just of the interval $[0, L_1]$, where L_1 corresponds to the maximal ideal of R.

Compare also with Remark 6.2.

Lemma 7.10. *Let C be a cotilting module over a valuation domain R. In the above notations, let $[\phi(L_0), \psi(L_1)]$ be the interval of $\mathcal{J}$ corresponding to the equivalence class $\mathfrak{i}_L$ determined by a prime ideal $L \in \mathcal{G}'$ and assume $L_0 < L_1$ (that is the equivalence class $\mathfrak{i}_L$ doesn't consist of a single interval). Consider the set $\mathcal{H}$ of prime ideals $N \in \mathcal{G}'$ such that $\phi(L_0) < \phi(N)$ and $\psi(N) < \psi(L_1)$. Then, for every $N \in \mathcal{H}$ it holds:*

$$\phi(N) = \bigcup_{P \in \mathcal{H}, \psi(P) < \phi(N)} \psi(P) \quad \textit{and} \quad \psi(N) = \bigcap_{Q \in \mathcal{H}, \psi(N) < \phi(Q)} \phi(N')$$

Proof. By assumption $\mathcal{H} \neq \emptyset$. Note that, if $N \in \mathcal{H}$ then also $\phi(N)$ and $\psi(N)$ are in $\mathcal{H}$. Fix $N \in \mathcal{H}$ and let $\mathcal{H}_N = \{P \in \mathcal{H} \mid \psi(P) \lneq \phi(N)\}$. By density $\mathcal{H}_N \neq \emptyset$ and if $P_0 = \bigcup_{P \in \mathcal{H}_N} \psi(P)$ then we also have $P_0 = \bigcup_{P \in \mathcal{H}_N} \phi(P)$. Thus, by [Baz07, Lemmas 6.1, 6.2] $\phi(P_0) = P_0$ and it must be $P_0 = \phi(N)$, otherwise $\psi(P_0) < \phi(N)$ contradicting density. Analogously, let $\mathcal{H}^N = \{Q \in \mathcal{H} \mid \psi(N) \lneq \phi(Q)\}$. Again by density, we have $\psi(N) = \bigcup_{Q \in \mathcal{H}^N} \phi(Q) = \bigcup_{Q \in \mathcal{H}^N} \psi(Q)$.

□

In the previous notations we can now prove the main result of this section.

Theorem 7.11. *Let C be a cotilting module over a valuation domain R. Then ${}^{\perp}C$ is a Tor-orthogonal class.*

Proof. If C is non dense, the conclusion follows by Lemma 7.6.

Otherwise let $\mathcal{J}$ be the totally ordered set of intervals of prime ideals obtained from $\mathcal{I}(C)$ as constructed in Notation 7.7 (††). By Fact 7.8 there is an injective homological ring epimorphism $f\colon R \to S$ such that $\mathcal{I}(f) = \mathcal{J}$ and the corresponding cotilting module $D = S^* \oplus (S/R)^*$ satisfies $\mathcal{I}(D) = \mathcal{J}$ and ${}^{\perp}D = (S/R)^{\intercal}$.

Our aim is to prove:

$${}^{\perp}C = \left(\bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \quad \oplus \quad S/R \right)^{\intercal}.$$

The claim will be proved in several steps.

First of all note that by [Baz07, Lemma 3.1] and Lemma 7.5 it is enough to show that the two classes contain the same cyclic modules.

(CLAIM (i) Let $0 \neq I < R$ and $L \in \mathcal{G}'$. Then, $\operatorname{Tor}_1^R\left(\frac{R_{\psi(L)}}{\phi(L)}, R/I\right) = 0$ if and only if either $I \geq \phi(L)$ or if $I \lneq \phi(L)$, then $I^{\#} \leq \phi(L)$ and $I \ncong R_{\phi(L)}$.

In fact,

$$\operatorname{Tor}_1^R\left(\frac{R_{\psi(L)}}{\phi(L)}, R/I\right) \cong \operatorname{Tor}_1^{R_{\psi(L)}}\left(\frac{R_{\psi(L)}}{\phi(L)}, R_{\psi(L)} \otimes R/I\right),$$

since $\dfrac{R_{\psi(L)}}{\phi(L)}$ is an $R_{\psi(L)}$-module and $\operatorname{Tor}_1^V(V/J, V/K) \cong (J \cap K)/KI$, for every valuation domain V and ideals K, J of V. (Recall that, for every ideal K of a valuation domain V, $K^{\#}K < K$ if and only if $K \cong V_{K^{\#}}$.)

(CLAIM (ii)

$${}^{\perp}C \subseteq \left(\bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \quad \oplus \quad S/R\right)^{\intercal}.$$

Let $R/I \in {}^{\perp}C$. By Lemma 4.6 there is $L \in \mathcal{G}'$ such that I and every $r^{-1}I$, for $r \notin I$ belong to $\in \langle \phi(L), \psi(L) \rangle$. Thus I and $r^{-1}I$, $r \notin I$ belong to $\langle \phi(L_0), \psi(L_1) \rangle$ where $[\phi(L_0), \psi(L_1)]$ is the interval of $\mathcal{J}$ corresponding to the equivalence class $\mathfrak{i}_L$. By Lemma 4.6 again $R/I \in {}^{\perp}D = (S/R)^{\intercal}$. Moreover, if $I \lneq \phi(N)$ for some $N \in \mathcal{G}'$, then $\psi(L) \lneq \phi(N)$, hence $R/I \in \left(\bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)}\right)^{\intercal}$ by (i).

(CLAIM (iii)

$$\left(\bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \oplus S/R\right)^{\intercal} \subseteq {}^{\perp}C.$$

Let R/I belong to the left hand side of the above inclusion. Since $R/I \in (S/R)^{\intercal} = {}^{\perp}D$, there is an interval $[\phi(L_0), \psi(L_1)]$ of $\mathcal{J}$ such that $\phi(L_0) \le I \le I^{\#} \le \psi(L_1)$. Now R/I belongs also to $\left(\bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)}\right)^{\intercal}$. As in Lemma 7.10, consider the set $\mathcal{H}$ of prime ideals $N \in \mathcal{G}'$ such that $\phi(L_0) < \phi(N)$ and $\psi(N) < \psi(L_1)$. Let $N_0 = \sup\{\phi(N) \in \mathcal{H} \mid \phi(N) \le I\}$. Then, by [Baz07, Lemmas 6.1, 6.2], $\phi(N_0) = N_0 \le I$. If $\phi(L_1) \le I$, then we conclude that $I \in \langle \phi(L_1), \psi(L_1) \rangle$, hence $R/I \in {}^{\perp}C$. Otherwise, the set $\mathcal{T} = \{Q \in \mathcal{H} \mid I < \phi(Q)\}$ is non empty and by (i) $I^{\#} \le \phi(Q)$ for every $Q \in \mathcal{T}$. Then $I^{\#} \le \bigcap_{Q \in \mathcal{T}} \phi(Q)$ and by Lemma 7.10, $\bigcap_{Q \in \mathcal{T}} \phi(Q) = \psi(N_0)$ since $\mathcal{T}$ coincides with the set $\{Q \in \mathcal{H} \mid \psi(N_0) \lneq \phi(Q)\}$. Hence $I \in \langle \phi(N_0), \psi(N_0) \rangle$ and again $R/I \in {}^{\perp}C$.

□

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