Abstract. We show that every injective homological ring epimorphism $f: R \to S$ where $S_R$ has flat dimension at most one gives rise to a 1-cotilting $R$-module and we give sufficient conditions under which the converse holds true. Specializing to the case of a valuation domain $R$, we illustrate a bijective correspondence between equivalence classes of injective homological ring epimorphisms originating in $R$ and cotilting classes of certain type and in turn, a bijection with a class of smashing localizing subcategories of the derived category of $R$. Moreover, we obtain that every cotilting class over a valuation domain is a Tor-orthogonal class, hence it is of countable type even though in general cotilting classes are not of cofinite type.

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Introduction

Tilting and cotilting theory has its origin in the context of finitely generated modules over finite dimensional algebras. The theory studied equivalences between subcategories of module categories over two algebras and was in essence a generalization of Morita Theory. A remarkable result by Happel ([Hap87]) shows that tilting theory provides a Morita Theorem at the level of derived categories.

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The notions of tilting and cotilting modules were further extended to modules of homological dimension $n \geq 1$ (called $n$-tilting and $n$-cotilting modules), to arbitrary rings and, what is of more interest to us, to infinitely generated modules which we call big modules. In [Baz10] and [BMT11] it was proved that big tilting modules induce equivalences between suitable localizations of the derived categories of two rings. Moreover, big tilting modules induce recollements of derived categories of rings and differential graded algebra which specialise to recollements of derived categories of rings in case the tilting module is of projective dimension one (see [CX12a], [BP13]).

Analogous results about dualities induced by big cotilting modules are not available up to now. Some partial results were obtained in [CX12b, Section 6]. Originally the notion of cotilting module was shadowed, since they were just duals of tilting modules. As soon as big modules enter the picture, the substantial difference between the two concepts became apparent.

In particular, it was proved that every tilting class associated to a big tilting module is of finite type ([BH08], [BS07]) meaning that it is the Ext orthogonal of a class of compact modules, that is modules with a finite projective resolution consisting of finitely generated projective modules. The corresponding property for a cotilting class, is the cofinite type meaning that it is the Tor orthogonal of a class of compact modules. A cotilting module is of cofinite type if and only if it is the dual of a tilting module (see [AHHT06]). Recently in [AHPST14] it has been proved that the cofinite type holds for big 1-cotilting modules over one-sided noetherian rings and it is valid for all $n$-cotilting modules over commutative noetherian rings. At our knowledge the only available counterexamples to the cofinite type are in the case of valuation domains. In fact, in [Baz07] it is shown that every cotilting class over a valuation domain is of cofinite type if and only if the domain is strongly discrete, that is if and only if it doesn’t admit non zero idempotent ideals and, moreover, explicit examples of cotilting classes not of cofinite type are exhibited.

One important question which should be investigated is whether cotilting classes are in any case Tor orthogonal to some class of modules, not necessarily compact ones. In the case of 1-cotilting modules, we state a necessary and sufficient condition on a cotilting class to be a Tor-orthogonal class (Proposition 7.3) which in particular implies the cocountable type. In the case of valuation domains $R$ we are able to prove that every cotilting class is a Tor orthogonal class (Theorem 7.11).

The relevance of big cotilting modules is also supported by a recent paper [Sto14] where it is shown that big cotilting modules are in bijective correspondence (up to equivalence) with duals (with respect to an injective cogenerator) of a classical tilting object of a Grothendieck category.

In the present paper we carry on an investigation of 1-cotilting modules. Inspired by results in [AHS11] we investigate the relation between 1-cotilting modules and homological ring epimorphisms. In fact, in [AHS11] it is proved they every injective homological ring epimorphism $R \to S$ where $S$ has projective dimension at most one, gives rise to the 1-tilting module $S \oplus S/R$. We relax the condition on the projective dimension and we prove that every injective homological ring epimorphism $R \to S$ where the flat dimension of
$S$ is at most one gives rise to the 1-cotilting module $(S \oplus S/R)^*$ where $^*$ denotes the character module (Theorem 3.3). The converse is proved under some assumptions (Proposition 3.6).

To obtain better results concerning the relation between 1-cotilting modules and homological ring epimorphism we clearly need a good understanding of homological ring epimorphisms. In this respect we take advantage of a recent paper [BS14] by Šťovíček and the author where a complete classification of homological ring epimorphisms starting from valuation domains $R$ is achieved. The classification is obtained via a bijective correspondence between equivalence classes of homological ring epimorphisms originating in $R$, and chains of intervals of prime ideals of $R$ satisfying certain conditions. These conditions amount to order completeness and to a property sometimes referred to as weakly atomic meaning that between two distinct intervals there is always a gap.

In [Baz07], the author developed a method to associate to a cotilting module over a valuation domain $R$ a chain of intervals of prime ideals which determine the cotilting class. Here we show that the chain of intervals of prime ideals associated to a cotilting module is order complete and we call a cotilting module non dense in case its associated chain of intervals satisfies also the weakly atomic property. In Theorem 5.7 we prove a bijective correspondence between equivalence classes of injective homological ring epimorphisms starting in a valuation domain $R$ and equivalence classes of non dense cotilting modules. We also show the existence of dense cotilting modules which don’t correspond to injective homological ring epimorphism (Proposition 6.4).

The paper is organized as follows. In Section 2 we recall the notions and properties of ring epimorphisms and homological ring epimorphisms and in particular, in 2.1 we illustrate the classification of homological ring epimorphisms starting in a valuation domain as proved in [BS14].

In Section 3 we investigate the relation between 1-cotilting modules over an arbitrary ring $R$ and homological ring epimorphisms describing the cotilting classes associated to homological ring epimorphisms.

In Section 4 we restate some results from [Baz07] about the properties of cotilting modules over valuation domains and examine the properties of the chain of intervals of prime ideals associated to a cotilting module.

In Section 5 we prove that, up to equivalence, there is a bijective correspondence between injective homological ring epimorphisms starting in a valuation domain $R$ and non dense cotilting modules over $R$. We also mention the related correspondence with a class of smashing localizing subcategories of the derived category of $R$.

In Section 6 we show examples of dense cotilting modules and in the final Section 7 we prove that every cotilting class over a valuation domain is a Tor-orthogonal class.

1. Preliminaries

All rings consider will be associative with identity.

For any class $C$ of left $R$-modules we define the following classes:

$$^\perp C = \{X \in R\text{-Mod} | \text{Ext}_R^i(X, C) = 0, \forall i \geq 1, \forall C \in C\},$$
\[ ^\dagger C = \{ X \in \text{Mod-}R \mid \text{Tor}^R_i(X, C) = 0, \forall i \geq 1, \forall C \in C \}, \]

If \( C \) is a class of right \( R \)-modules we define;

\[ C^\dagger = \{ X \in \text{Mod-}R \mid \text{Tor}^R_i(C, X) = 0, \forall i \geq 1, \forall C \in C \}, \]

If \( C = \{ M \} \) we simply write \( ^\dagger M \), \( ^\dagger M \) and \( M^\dagger \). For every \( R \)-module \( M \), i.d.\( M \), p.d.\( M \), w.d.\( M \) will denote the injective, projective, weak (flat) dimension of \( M \).

**Definition 1.1.** Let \( R \) a ring. An \( R \)-module \( C \) is an \( n \)-cotilting module if the following conditions hold ([AHC01]):

(C1) i.d.\( C \leq n; \)

(C2) \( \text{Ext}^i_R(C^\lambda, C) = 0 \) for each \( i > 0 \) and for every cardinal \( \lambda; \)

(C3) there exists a long exact sequence:

\[ 0 \to C_r \to \cdots \to C_1 \to C_0 \to \cdots \to W \to 0, \]

where \( C_i \in \text{Prod}C \), for every \( 0 \leq i \leq r \) and \( W \) is an injective cogenerator of \( R \)-Mod.

In case \( n = 1 \) there is an alternative definition of 1-cotilting modules. A module \( C \) is 1-cotilting if and only if \( \text{Cogen} C = ^\dagger C \), where \( \text{Cogen} C \) denotes the class of modules cogenerated by \( C \). Moreover, if \( C \) is a 1-cotilting module, then \( \text{Cogen} C \) is a torsion free class. (For results on torsion and torsion free classes we refer to [Ste75].)

If \( C \) is a \( n \)-cotilting module the class \( ^\dagger C \) is called an \( n \)-cotilting class and two cotilting modules are said to be equivalent if the corresponding cotilting classes coincide.

\( n \)-cotilting classes have been characterized in [AHC01], [GT06] and [GT12]. In particular \( ^\dagger C \) is closed under direct products. Moreover, since every \( n \)-cotilting module \( C \) is pure injective ([Baz03], [Sto06]), \( n \)-cotilting classes are also closed under direct limits and pure submodules. In other words they are definable classes, that is they are closed under elementary equivalence. Thus, if \( C \) is \( n \)-cotilting, a module belongs to \( ^\dagger C \) if and only if its pure injective envelope belongs to \( ^\dagger C \) (see [JL89] or [CB94]).

An \( R \)-module \( M \) is said to be compact if it admits a finite projective resolution consisting of finitely generated projective modules.

**Definition 1.2.** An \( n \)-cotilting class \( \mathcal{F} = ^\dagger C \) is said to be of cofinite type if there is a set \( S \) of compact modules such that \( \mathcal{F} = S^\dagger \). In this case we also say that \( C \) is of cofinite type.

Note that if \( \mathcal{F} \) is an \( n \)-cotilting class, any compact module in \( \mathcal{T} \mathcal{F} \) has projective dimension at most \( n \) and \( \mathcal{F} \) is of cofinite type if and only if \( \mathcal{F} = S^\dagger \) where \( S \) is the set of all compact modules in \( \mathcal{T} \mathcal{F} \). Moreover, by [AHHT06] an \( n \)-cotilting module \( C \) is of cofinite type if and only if it is the dual of an \( n \)-tilting module.

2. **Homological ring epimorphisms**

Let us recall some standard facts on ring epimorphisms and on homological ring epimorphisms which we will need in the sequel.

If \( R, S \) are associative rings, we denote by \( \text{Mod-}R \) (\( R \)-Mod) and \( \text{Mod-}S \) (\( S \)-Mod) the categories of right (left) \( R \) and \( S \) modules, respectively. A ring
A homomorphism $f : R \to S$ is a ring epimorphism if it is an epimorphism in the category of rings. Ring epimorphisms have been investigated in [Sil67, Ste75, GlP87, Laz69].

From those papers we infer that a ring homomorphism $f : R \to S$ is a ring epimorphism if and only if $S \otimes_R S \cong S_S$, if and only if the restriction functor $f_* : \text{Mod}-S \to \text{Mod}-R$ is fully faithful (or the same holds for left modules).

Two ring epimorphisms $f : R \to S$ and $f' : R \to S'$ are said to be equivalent if there exists a ring isomorphism $\varphi : S \to S'$ such that $f' = \varphi f$.

Equivalently, the essential images of $f_*$ and $f'_*$ in $\text{Mod}-R$ coincide.

The following results will be useful in the sequel.

**Proposition 2.1.** Let $R$ a commutative ring and $f : R \to S$ a ring homomorphism. The following hold true:

1. [Sil67, Corollary 1.2] If $f$ is a ring epimorphism, then $S$ is a commutative ring.
2. [Laz69, Lemma 1.1] $f$ is a ring epimorphism if and only if $f_p : R_p \to S \otimes_R R_p$ is a ring epimorphism for every prime ideal $p$ of $R$.

The notion of ring epimorphism is strictly related to the notion of bireflective subcategory.

**Definition 2.2.** Let $E$ be a full subcategory of $R$-Mod. A morphism $f : M \to E$, with $E$ in $E$, is called an $E$-reflection if for every map $g : M \to E'$, with $E'$ in $E$, there is a unique map $h : E \to E'$ such that $hf = g$. A subcategory $E$ of $R$-Mod is said to be reflective if every $R$-module $M$ admits an $E$-reflection. The definition of coreflective subcategory is given dually. A subcategory that is both reflective and coreflective is called bireflective.

**Lemma 2.3.** ([GL91] and [GL87]) Let $E$ be a full subcategory of $R$-Mod. The following assertions are equivalent:

1) $E$ is a bireflective subcategory of $R$-Mod;
2) $E$ is closed under isomorphic images, direct sums, direct products, kernels and cokernels;
3) there is a ring epimorphism $f : R \to S$ such that $E$ is the essential image of the restriction of scalars functor $f_* : S$-Mod $\to$ $R$-Mod.

In particular there is a bijection between the bireflective subcategories of $R$-Mod and the equivalence classes of ring epimorphisms starting from $R$. Moreover the map $f : R \to S$ as in 3) is an $E$-reflection.

**Definition 2.4.** A ring epimorphism $f : R \to S$ is a homological epimorphism if $\text{Tor}_i^R(S, S) = 0$ for every $i \geq 1$.

A ring epimorphism $f : R \to S$ with $S$ a flat $R$-module is clearly a homological epimorphism. It is called a flat epimorphism.

Homological ring epimorphisms have been introduced and characterized by Geigle and Lenzing in [GL91]. While a ring epimorphism $R \to S$ implies that the category of $S$-modules is equivalent to a subcategory of the category of $R$-modules, homological ring epimorphisms give the analogous result for the derived categories of the rings.
Proposition 2.5. [GL91, 4.4] Let \( R, S \) be rings. A ring homomorphism \( f: R \to S \) is a homological epimorphism if and only if one of the following equivalent conditions holds:

1. \( S \otimes_R S \cong SS \) and \( \text{Tor}_i^R(S, S) = 0 \), for every \( i \geq 1 \) (i.e. the natural map \( S \otimes _R^\mathbb{L} S \to S \) is an isomorphism);
2. for all right \( S \)-modules \( N \) and left \( S \)-modules \( M \), the natural map \( \text{Tor}_i^R(N, M) \to \text{Tor}_i^S(N, M) \) is an isomorphism for every \( i \geq 0 \), (i.e. the natural map \( N \otimes_R^\mathbb{L} M \to N \otimes_S^\mathbb{L} M \) is an isomorphism);
3. for all \( S \)-modules \( M, M' \), the natural map \( \text{Ext}_S^i(M, M') \to \text{Ext}_R^i(M, M') \) is an isomorphism for every \( i \geq 0 \), (i.e. the natural morphism \( \text{RHom}_S(M, M') \to \text{RHom}_R(M, M') \) is an isomorphism);
4. the induced functor \( f_*: \mathcal{D}(S) \to \mathcal{D}(R) \) is a full embedding of triangulated categories.

We collect in the following proposition some easy facts about homological ring epimorphisms. Here the weak global dimension (global dimension) of a ring \( R \) will be denoted by \( \text{w.gl.dim} \ (\text{gl.dim}) \ R \).

Proposition 2.6. The following hold true:

1. The composition of homological ring epimorphisms is a homological ring epimorphism.
2. If \( f: R \to S \) is a homological ring epimorphism, then \( \text{w.gl.dim} \ S \leq \text{w.gl.dim} \ R \) and \( \text{gl.dim} \ S \leq \text{gl.dim} \ R \).

If moreover \( R \) is commutative, then the following hold true:

3. If \( \Sigma \) is a multiplicative subset of \( R \), then \( R \to R\Sigma^{-1} \) is a flat epimorphism.
4. A ring homomorphism \( f: R \to S \) is a homological epimorphism if and only if \( f_p: R_p \to S \otimes_R R_p \) is a homological epimorphism for every prime ideal \( p \in \text{Spec} \, R \).

Proof. (1) Let \( f: R \to S \) and \( g: S \to T \) be two homological ring epimorphisms. Clearly \( gf \) is an epimorphism and, since \( T \) is an \( S \)-bimodule, \( \text{Tor}_i^R(T, T) \cong \text{Tor}_i^S(T, T) = 0 \).

(2) Follows immediately by Proposition 2.5 (2) and (3).

(3) Obvious.

(4) Let \( i \geq 1 \); then \( \text{Tor}^R_i(S, S) = 0 \) if and only if \( \text{Tor}^R_i(S, S) \otimes_R R_p = 0 \), for every prime ideal \( p \) of \( R \) and \( \text{Tor}^R_i(S, S) \otimes_R R_p \cong \text{Tor}^R_{i_p}(S \otimes_R R_p, S \otimes_R R_p) \), for every prime ideal \( p \) of \( R \), since \( - \otimes_R R_p \) is an exact functor (see also [EJ00, Theorem 2.1.11]). Thus the conclusion follows by Proposition 2.1.2. \( \square \)

If \( f: R \to S \) is a ring epimorphism and \( \text{w.gl.dim} \ S \leq 1 \), then clearly \( f \) is a homological epimorphism if and only if \( \text{Tor}^R_1(S, S) = 0 \). Moreover, we have the following useful result.

Lemma 2.7. Let \( f: R \to S \) be a homological ring epimorphism such that \( S \) and \( S/f(R) \) have weak dimension \( \leq 1 \) as right \( R \)-modules. Then the following hold:

1. The canonical projection \( \pi: R \to R/\ker f \) is a homological epimorphism and \( \ker f \) is an idempotent two-sided ideal of \( R \).
(2) The induced homomorphism \( \overline{f} : R/\text{Ker } f \to S \) is a homological epimorphism.

Moreover, if \( w.gl.\dim R \leq 1 \) and \( I \) is a two-sided ideal of \( R \), the canonical projection \( \pi : R \to R/I \) is a homological epimorphism if and only if \( I \) is an idempotent two-sided ideal of \( R \).

Proof. (1) Let \( K = \text{Ker } f \) and apply the functors \( S \otimes_R - \) and \( - \otimes_R R/K \) to the exact sequence

\[
0 \to R/K \xrightarrow{f} S \to S/f(R) \to 0.
\]
For every \( n \geq 1 \) we get

\[
0 = \text{Tor}_{n+1}^R(S, S/f(R)) \to \text{Tor}_n^R(S, R/K) \to \text{Tor}_n^R(S, S) = 0
\]
and

\[
0 = \text{Tor}_{n+1}^R(S/f(R), R/K) \to \text{Tor}_n^R(R/K, R/K) \to \text{Tor}_n^R(S, R/K) = 0.
\]

Consequently \( \text{Tor}_n^R(R/K, R/K) = 0 \), for every \( n \geq 1 \) and \( \pi \) is a homological epimorphism. Consider the exact sequence \( 0 \to K \to R \to R/K \to 0 \) and apply the functor \( R/K \otimes_R - \) to obtain the exact sequence

\[
0 \to R/K \otimes_R K \cong K/K^2 \to R/K \to R/K \otimes_R R/K \to 0,
\]
which yields \( K^2 = K \), since \( R/K \otimes_R R/K \cong R/K \).

(2) By part (1) \( \pi \) is a homological epimorphism, thus \( \text{Tor}_n^R(K, S) \cong \text{Tor}_n^R(S, S) = 0 \), since \( S \) is an \( R/K \)-bimodule. Moreover, \( \overline{f} \) is clearly a ring epimorphism, so also homological.

For the last statement, note that the assumption on \( R \) implies that \( R \to R/I \) is a homological epimorphism if and only if \( \text{Tor}_1^R(R/I, R/I) = 0 \) and this is easily seen to be equivalent to \( I \) being idempotent. \( \square \)

2.1. Homological ring epimorphisms originating in valuation domains. A commutative ring is a valuation ring if the lattice of the ideals is linearly ordered by inclusion. Recall that an idempotent ideal of a valuation domain is a prime ideal and also that if \( J \subseteq L \) are prime ideals, then \( J \) is canonically an \( R_L \)-module, so that \( J_L = J \) and \( (R/J)_L = R_L/J \).

A complete classification of homological ring epimorphisms originating in valuation domains has been obtained by Šťovíček and the author in [BS14]. One of the key facts is given by the following remark.

Remark 2.8. If \( R \) is a valuation domain and \( f : R \to S \) is a homological ring epimorphism, then \( w.gl.\dim S \leq 1 \) (by Proposition 2.6 (2)). Thus, if \( \mathfrak{n} \) is any maximal (prime) ideal of \( S \), the localization \( S_{\mathfrak{n}} \) of \( S \) at \( \mathfrak{n} \) is a valuation domain (see [Gla89, Corollary 4.2.6]).

We recall now the key notations and results from [BS14] which will be strongly used in Section 5.

Proposition 2.9. ([BS14], Proposition 6.5 and its proof) Let \( R \) be a valuation domain, \( 0 \neq f : R \to S \) be a homological ring epimorphism, and let \( I = \text{Ker } f \). Then the following hold:
(1) There exists a prime ideal $P \in \text{Spec} R$ with $I \subseteq P$ and a surjective homological epimorphism $g: S \to R_P/I$ such that the composition $gf: R \to R_P/I$ is the canonical morphism. Moreover, there is a unique maximal ideal $m$ of $S$ such that $f^{-1}(m) = P$ and $g: S \to R_P/I$ is equivalent to the localization of $S$ at $m$.

(2) For every maximal ideal $n$ of $S$, there are two prime ideals $J, L$ of $R$ such that the composition $R \xrightarrow{f} S \xrightarrow{can} S_n$ is a homological epimorphism equivalent to $g: R \to R_L/J$, where $J$ is idempotent, $J \leq L$ and $L = f^{-1}(n)$.

(3) If $n' \neq n$ are two distinct maximal ideals of $S$ with corresponding pairs of prime ideals $J, L$ and $J', L'$, then the intervals $[J, L]$ and $[J', L']$ in $(\text{Spec} R, \subseteq)$ are disjoint and

$$\text{Tor}^R_n(R_L/J, R_L'/J') = 0$$

for $n \geq 0$.

As consequences of the previous proposition we have:

**Corollary 2.10.** Let $R$ be a valuation domain, $0 \neq f: R \to S$ be a homological ring epimorphism and let $I = I^2 \subseteq R$ be such that the kernel of the composition $R \to S/S$ is $I$.

Then there is a maximal ideal $m$ of $S$, $m \supseteq SI$ such that the homological epimorphism $R \xrightarrow{f} S \xrightarrow{can} S_m$ is equivalent to $R \to R_P/I'$ for ideals $I'$ and $P$ satisfying $I' \leq I \leq P$ and $f^{-1}(m) = P$.

**Proof.** By Lemma 2.7, the morphism $p: S \to S/S = S/SI$ is a homological epimorphism, since $SI$ is idempotent. Then $pf: R \to S$ is a homological epimorphism with kernel $I$. Applying Proposition 2.9 (1) to $pf$ we have that there is a maximal ideal $m$ of $S$ containing $SI$ such that the homological epimorphism $R \to S_m$ is equivalent to $R/P/I$. Then the kernel $I'$ of the composition $R \to S \to S_m$ is contained in $I$ and the homological epimorphism $R \to S \to S_m$ is equivalent to $R/P/I'$. Clearly $f^{-1}(m) = P$. \qed

**Corollary 2.11.** [BS14, Lemma 6.8] Let $R$ be a valuation domain and $f: R \to S$ a homological ring epimorphism. Then the canonical map

$$\text{Spec} S \xrightarrow{f} \text{Spec} R, \quad n \mapsto f^{-1}(n)$$

restricts to a poset isomorphism between $(\text{Spec} S, \subseteq)$ and the coproduct (= disjoint union)

$$\coprod_{[J, L] \in \mathcal{I}(f)} [J, L],$$

where $[J, L]$ are viewed as subchains of $(\text{Spec} R, \subseteq)$.

**Definition 2.12.** [BS14, Section 6] If $R$ is a valuation domain denote by $\text{Inter} R$ the set of intervals $[J, L]$ in $(\text{Spec} R, \subseteq)$ such that $J = J^2 \leq L$ endowed with the partial order $[J, L] < [J', L']$ if $L < L'$ as ideals.

If $f: R \to S$ is a homological ring epimorphism, define $\mathcal{I}(f)$ to be the set of intervals $[J, L] \in \text{Inter} R$ arising as in Proposition 2.9.

We will consider non-empty subchains $(\mathcal{I}, \leq)$ of $(\text{Inter} R, \leq)$ consisting of disjoint intervals satisfying the following conditions:
Conditions 2.13.  
(1) If $S = \{ [J_\alpha, L_\alpha] \mid \alpha \in \Lambda \}$ is a non-empty subset of $\mathcal{I}$ with no minimal element, then $\mathcal{I}$ contains an element of the form $[J, \bigcap_{\alpha \in \Lambda} L_\alpha]$.

(2) If $S = \{ [J_\alpha, L_\alpha] \mid \alpha \in \Lambda \}$ is a non-empty subset of $\mathcal{I}$ with no maximal element, then $\mathcal{I}$ contains an element of the form $\bigcup_{\alpha \in \Lambda} J_\alpha, L.$

(3) Given any $[J_0, L_0] < [J_1, L_1]$ in $\mathcal{I}$, there are elements $[J, L], [J', L']$ in $\mathcal{I}$ such that $[J_0, L_0] \leq [J, L] < [J', L'] \leq [J_1, L_1]$ and there are no other intervals of $\mathcal{I}$ between $[J, L]$ and $[J', L'].$

Remark 2.14. Conditions (1) and (2) express the fact that $\mathcal{I}$ is order complete, while condition (3) is typically satisfied by the partially order set of the prime spectrum of a commutative ring (see Kap74). Condition (3) is sometimes referred to as weakly atomic.

We can now state the classification theorem proved in [BS14].

Theorem 2.15. ([BS14, Theorem 6.23] Let $R$ be a valuation domain. Then there is a bijection between:

(1) Subchains $\mathcal{I}$ of $\text{Inter} R$ consisting of disjoint intervals satisfying Conditions 2.13.

(2) Equivalence classes of homological ring epimorphisms $f : R \to S$.

The bijection is given by assigning to a non-empty set $I$ from (1) the ring homomorphism $f(I) : R \to R(I)$ where $R(I)$ is directed union of subrings of $\prod_{[J, L] \in I} (R/J)_L$ (see [BS14, Construction 4.9]). We assign $R \to 0$ to $I = \emptyset$.

The converse is given by sending $f : R \to S$ to $I = I(f)$ as in Definition 2.12.

Remark 2.16. If $[J, L]$ is the minimal element of $I(f)$, then $J = \text{Ker} f$. Moreover, if $I(f)$ is finite, then $R(I) = \prod_{[J, L] \in I} R_L/J$.

3. 1-Cotilting modules versus homological ring epimorphisms

3.1. 1-Cotilting modules arising from homological ring epimorphisms.

In this subsection we show a method to construct 1-cotilting modules from homological ring epimorphisms. The main result of this subsection, Theorem 3.3 can be viewed as a counterpart of the results in [AHS11] where tilting modules arising from injective homological ring epimorphisms $f : R \to S$ with p.d. $S \leq 1$ are considered. In fact, we prove that every injective homological ring epimorphism $f : R \to S$ where w.d.$S_R \leq 1$ gives rise to a 1-cotilting left $R$-module whose corresponding cotilting class consists of the left $R$-modules cogenerated by $S^*$. Moreover, we obtain that the bireflective subcategory of $R$-Mod equivalent to $S$-Mod via $f$ can be interpreted as a suitable perpendicular category.

Proposition 3.1. Let $f : R \to S$ be a homological ring epimorphism such that $S$ and $S/f(R)$ have weak dimension $\leq 1$ as right $R$-modules and let $\text{Ker} f = K$. Then, $C = S^* \oplus (S/f(R))^*$ is a 1-cotilting left $R/K$-module. The corresponding cotilting class in $R/K$-Mod coincides with $\text{Cogen} S^*$ and
with $(S/f(R))^\ast = (S/f(R))^\Gamma$. Moreover, an $R/K$-module $M$ belongs to the cotilting class if and only the map $f \otimes_{R/K} M : M \to S \otimes_{R/K} M$ is injective.

**Proof.** By Lemma 2.7, $\pi : R \to R/K$ is a homological epimorphism, thus $S$ and $S/f(R)$ have weak dimension $\leq 1$ also as $R/K$-modules. Moreover, $f$ induces a homological epimorphism $\overline{f} : R/K \to S$. Thus, without loss of generality, we may assume that $f$ is injective and identify $R$ with $f(R)$.

Consider $S$ and $S/R$ as right $R$-modules and let $C = S^\ast \oplus (S/R)^\ast$. Then, $\text{i.d.} C \leq 1$. The exact sequence $0 \to (S/R)^\ast \to S^\ast \to R^\ast \to 0$ shows that condition $(C3)$ in Definition 1.1 is satisfied. It remains to show that $\text{Ext}^1_R(C, C) = 0$, for every cardinal $\lambda$. We have

$$\text{Ext}^1_R(C, C) = 0 \iff \text{Tor}^R_1(S \oplus S/R, ((S \oplus S/R)^\ast)^\lambda) = 0.$$  

Since w.dim $(S \oplus S/R) \leq 1$ and $(S/R)^\ast$ is a submodule of $S^\ast$, it is enough to check that $\text{Tor}^R_1(S \oplus S/R, (S^\ast)^\lambda) = 0$. First note that $\text{Tor}^R_1(S, (S^\ast)^\lambda) = 0$, since $(S^\ast)^\lambda$ is an $S$-module and $f$ is a homological ring epimorphism. Thus, what is left to be proved is that $\text{Tor}^R_1(S/R, (S^\ast)^\lambda) = 0$.

Consider the exact sequence or right $R$-modules

$$(1) \quad 0 \to R \xrightarrow{f} S \to S/R \to 0.$$  

Applying the functor $- \otimes_R (S^\ast)^\lambda$ to (1) we get

$$0 \to \text{Tor}^R_1(S/R, (S^\ast)^\lambda) \to R \otimes_R (S^\ast)^\lambda \xrightarrow{f \otimes_R (S^\ast)^\lambda} S \otimes_R (S^\ast)^\lambda \to S/R \otimes_R (S^\ast)^\lambda \to 0$$  

so $\text{Tor}^R_1(S/R, (S^\ast)^\lambda) = 0$, since $S \otimes_R (S^\ast)^\lambda \cong S \otimes_S (S^\ast)^\lambda$ implies that $f \otimes_R (S^\ast)^\lambda$ is injective. The characterization of the cotilting class follows at once from $(S/f(R))^\ast \leq S^\ast$.

To prove the last statement we apply the functor $- \otimes_R M$ to (1) to obtain

$$(2) \quad 0 \to \text{Tor}^R_1(S, M) \to \text{Tor}^R_1(S/R, M) \to M \xrightarrow{f \otimes_R M} S \otimes_R M$$  

Thus $\text{Tor}^R_1(S/R, M) = 0$ implies the injectivity of $f \otimes_R M$. Conversely, assume that $f \otimes_R M : M \to S \otimes_R M$ is injective. Tensoring by $S \otimes_R -$ the exact sequence $0 \to M \xrightarrow{f \otimes_M} S \otimes M \to (S \otimes M)/M \to 0$ we get the exact sequence

$$0 = \text{Tor}^R_2(S, (S \otimes_R M)/M) \to \text{Tor}^R_1(S, M) \to \text{Tor}^R_1(S, S \otimes_R M) \cong \text{Tor}^S_1(S, S \otimes_R M) = 0$$  

where the end terms vanish since $\text{Tor}^R_2(S, -) = 0$ and $f : R \to S$ is a homological epimorphism. Thus, from (2) also $\text{Tor}^R_1(S/R, M) = 0$.  

We now apply Proposition 3.1 to the case of an injective homological ring epimorphism.

**Definition 3.2.** Let $C$ be a class of left $R$ modules. The left perpendicular category $\perp C$ of $C$ is defined as

$$\perp C = \{ M \in R\text{-Mod} \mid \text{Hom}_R(M, C) = 0 = \text{Ext}^1_R(M, C) \}.$$  

**Theorem 3.3.** Let $f : R \to S$ be an injective homological ring epimorphism such that $S$ has weak dimension $\leq 1$ as a right $R$-module. The following hold true:

- $\text{Hom}_R(M, C) = 0$ for all $M \in \perp C$.
- $\text{Ext}^1_R(M, C) = 0$ for all $M \in \perp C$.
- $\text{Tor}_n^R(M, C) = 0$ for all $M \in \perp C$ and $n \geq 2$.
Proof. (1) Follows by Proposition 3.1.

(2) By Lemma 2.3, the essential image of the restriction functor \( f_* : S\text{-Mod} \to R\text{-Mod} \) is a bireflective category equivalent to \( S\text{-Mod} \) and a left \( R \)-module \( f \) belongs to the cotilting class if and only if \( f \otimes M \) is injective. By part (1) \( f \otimes M \) is injective if and only if \( M \in (S/f(R))^* \) and from the exact sequence \( 0 \to R \xrightarrow{i} S \to S/f(R) \to 0 \) we see that \( f \otimes M \) is surjective if and only if \( S/f(R) \otimes M = 0 \) that is, if and only if \( \text{Hom}_R(M, (S/f(R))^*) = 0 \). Hence \( \text{Im} f_* = (S/f(R))^* \).

(3) \( \text{Hom}_R(S^*, (S/f(R))^*) = 0 \) if and only if \( S/f(R) \otimes_R S^* = 0 \). Apply the functor \( - \otimes_R S^* \) to the exact sequence \( 0 \to R \xrightarrow{i} S \to S/f(R) \to 0 \) to obtain the exact sequence \( S^* \otimes_R S \to S/f(R) \otimes_R S^* \to 0 \), hence \( S/f(R) \otimes_R S^* = 0 \), since \( f \otimes R S^* \) is an isomorphism.

By part (2), the left perpendicular category \( \perp (S/f(R))^* \) is bireflective, hence closed under direct products. \( \square \)

In Proposition 3.6 we will prove a converse of Theorem 3.3 under some extra assumptions on a 1-cotilting module.

If \( R \) is a valuation domain, then the weak global dimension of \( R \) is at most one, thus the results in this section hold true. Moreover, in Section 5 we will show that for valuation domains there is an even stronger relation between cotilting modules and homological ring epimorphisms.

3.2. 1-Cotilting modules giving rise to homological ring epimorphisms.

Proposition 3.4. Let \( R \) be a 1-cotilting module over an arbitrary ring \( R \). By condition (C3) in Definition 1.1 choose an exact sequence

\[ \begin{array}{c}
0 \to C_0 \to C_1 \to R^* \to 0,
\end{array} \]

with \( C_0, C_1 \in \text{ProdC} \). Then, the following hold true:

(1) \( C \) is equivalent to \( C_0 \oplus C_1 \) and \( \perp C = \perp C_1 \cong \text{Cogen} C_0 \).

(2) The left perpendicular category \( \perp C_1 \) is closed under extensions, kernels, cokernels and direct sum; moreover, the inclusion \( \iota : \perp C_1 \to R\text{-Mod} \) has a right adjoint \( \mu \).

(3) If \( \perp C_1 \) is also closed under direct products, then the inclusion functor \( \iota \) has a left adjoint \( \ell \) and there is a ring epimorphism \( f : R \to S \) such that \( \perp C_1 \) is equivalent to \( S\text{-Mod} \).

Proof. (1) Clearly \( C' = C_0 \oplus C_1 \) is a 1-cotilting module and \( \perp C \subseteq \perp C' \), \( \text{Cogen} C' \subseteq \text{Cogen} C \). Thus \( C \) and \( C' \) are equivalent cotilting modules. The other statement follows immediately by the exact sequence (\( \ast \)).
(2) The closure properties of $\perp C_1$ follow from [GL91, Proposition 1.1], since i.d. $C_1 \leq 1$. We show now how to construct a right adjoint of the inclusion functor. Note first that $C_1$ defines a torsion pair in $R$-$\text{Mod}$ whose torsion free class coincides with Cogen $C_1$ and a module $X$ is torsion if and only if $\text{Hom}_R(X, C_1) = 0$. Using the fact that $\text{Ext}_R^1(C_1, C_1) = 0$, for every cardinal $\lambda$ the dual of Bongartz’s Lemma (see [Bon81] and [Trl96, Lemma 6.9]) shows that for every left $R$-module $M$ there is a short exact sequence (in particular, a special $\perp C_1$-precover) of the form

$$0 \rightarrow C_1^\perp \rightarrow M_0 \rightarrow M \rightarrow 0$$

with $M_0 \in \perp C_1$. Then, the torsion submodule $t_{C_1}(M_0)$ of $M_0$ with respect to the torsion theory induced by $C_1$, belongs to the perpendicular class $\perp C_1$. It is routine to check that the assignment $M \mapsto t_{C_1}(M_0)$ defines a functor $\mu : R$-$\text{Mod} \rightarrow \perp C_1$ and that it is a right adjoint of the inclusion $\iota : \perp C_1 \rightarrow R$-$\text{Mod}$ (see for instance [CTT07] for a proof of the dual statement).

(3) Assume that $\perp C_1$ is also closed under products. Then $\perp C_1$ is a bifree subcategory, hence the inclusion functor $\iota : \perp C_1 \rightarrow R$-$\text{Mod}$ admits a left adjoint $\ell$ and there is an epimorphism $f : R \rightarrow S$ of rings such that $S = \text{End}_R(\ell(R), \ell(R))$ and the essential image of $S$-$\text{Mod}$ under the fully faithful restriction functor $f_*$ coincides with $\perp C_1$ (see [Glp87], [GL91]).

Remark 3.5. The situation considered in Proposition 3.4 (3) can be illustrated by the following diagram:

\[
\begin{array}{ccc}
S \otimes_R - & \xrightarrow{\rho} & \perp C_1 \\
\downarrow & & \downarrow \\
S \text{-Mod} & \xrightarrow{\sim} & R \text{-Mod} \\
\downarrow & & \downarrow \\
\text{Hom}_R(S, -) & \sim & \mu \\
\end{array}
\]

Note that, by the unicity of a right adjoint up to natural isomorphisms, we have that $\rho^{-1} \mu \sim \text{Hom}_R(S, -)$.

**Proposition 3.6.** In the notations of Proposition 3.4 assume that the left perpendicular category $\perp C_1$ is closed under products and that $\text{Hom}_R(C_0, C_1) = 0$. Then the ring epimorphism $f : R \rightarrow S$ existing by Proposition 3.4 (3) is an injective homological epimorphism, w.dim $S_R \leq 1$ and $C$ is equivalent to $S^* \oplus (S/R)^*$. 

**Proof.** From the exact sequence $0 \rightarrow C_1 \rightarrow C_0 \rightarrow R^* \rightarrow 0$ and from $\text{Hom}_R(C_0, C_1) = 0$ we conclude that the right adjoint of $R^*$ is isomorphic to $C_0$. By Remark 3.5 $C_0 \cong \text{Hom}_R(S, R^*) \cong (R \otimes_R S)^* \cong S^*$. Hence the injective dimension of $R^*$ is $\leq 1$ and w.d. $S_R \leq 1$. The exact sequence $0 \rightarrow C_1 \rightarrow C_0 \rightarrow R^* \rightarrow 0$ and the canonical isomorphisms involved show that there is a surjection $S^* \rightarrow R^* \rightarrow 0$, hence also the injection $0 \rightarrow R \rightarrow S$. It remains to show that $f$ is a homological epimorphism. Certainly $\text{Tor}_n^R(S, S) = 0$ for every $n \geq 2$, since w.d. $S_R \leq 1$. To see that
also Tor\textsubscript{R}^1(S, S) = 0 we use the fact that the category \(\perp C_1\) is closed under extensions. In fact, we have
\[
(Tor^R_S(S, S))^* \cong Ext^1_R(S, S^*) \cong Ext^1_S(S, S^*) = 0.
\]
By Theorem 3.3, \(S^* \oplus (S/R)^*\) is a 1-tilting \(R\)-module and from the exact sequence
\[
0 \rightarrow C_1 \rightarrow S^* \xrightarrow{L^*} R^* \rightarrow 0
\]
we conclude that \(C\) is equivalent to \(S^* \oplus (S/R)^*\). 

Note that the perpendicular category \(\perp C_1\) is zero in case \(C_1\) is already a 1-tilting module. We are not aware of examples of 1-tilting modules such that for every exact sequence satisfying condition (C3) in Definition 1.1 the module \(C_1\) is cotilting. Thus we pose the following question:

**Question 3.7.** Assume that \(C\) is a 1-tilting module. Is it always possible to find an exact sequence \(0 \rightarrow C_1 \rightarrow C_0 \rightarrow R^* \rightarrow 0\) with \(C_0, C_1 \in \text{Prod} C\) such that \(C_1\) is not a cotilting module?

The analogous question for 1-tilting modules has a negative answer. In fact, if \(L\) is the Lukas tilting module ([Luk91, Theorem 3.1], see also [AHKL11, Example 5.1]) then, any exact sequence \(0 \rightarrow R \rightarrow L_0 \rightarrow L_1 \rightarrow 0\) with \(L_1, L_0 \in \text{Add} L\) is such that \(L_1\) is a 1-tilting module.

On the other hand, the dual of Lukas tilting module doesn’t provide a counterexample.

Moreover, we don’t know a characterization of the situation in which the assumption in Proposition 3.4 (3) about the closure under products of the perpendicular category \(\perp C_1\) holds true. We will show only a sufficient condition for its validity in the next Lemma 3.10. Thus we pose also this other question:

**Question 3.8.** Let \(C\) be a 1-tilting \(R\)-module and let \(0 \rightarrow C_1 \rightarrow C_0 \rightarrow R^* \rightarrow 0\) be an exact sequence satisfying condition (C3) in Definition 1.1. When is \(\perp C_1\) closed under direct products?

In order to prove the promised sufficient condition for a positive answer of the above question we need to recall the notion of relative Mittag-Leffler modules.

**Definition 3.9.** ([AHH08, Definition 1.1] Let \(M\) be a right \(R\)-module and let \(\mathcal{Q}\) be a class of left \(R\)-modules. \(M\) is said to be \(\mathcal{Q}\)-Mittag-Leffler if the canonical map
\[
M \otimes_R \prod_i Q_i \rightarrow \prod_i (M \otimes Q_i)
\]
is injective for every set \(\{Q_i\}_{i \in I}\) of modules in \(\mathcal{Q}\).

**Lemma 3.10.** Let \(rC\) be a 1-tilting module of cofinite type. There is an exact sequence \(0 \rightarrow C_1 \rightarrow C_0 \rightarrow R^* \rightarrow 0\) satisfying condition (C3) in Definition 1.1 such that the perpendicular category \(\perp C_1\) is closed under direct products.
Proof. Let $\mathcal{F} = \perp C$ and let $\mathcal{S}$ be the set of compact right $R$-modules in $\uparrow \mathcal{F}$. By assumption $S^\perp = \mathcal{F}$ and $S^\perp = T^\perp$, where $T_R$ is a 1-tilting module. W.l.o.g. we can assume $T = T_0 \oplus T_1$ where $T_0, T_1 \in \text{Add} T$ are the terms fitting in an exact sequence $0 \to R \to T_0 \to T_1 \to 0$ satisfying condition (T3) for 1-tilting modules. By $[\text{AHHT06}]$, $C$ is equivalent to $T^* = T_0^* \oplus T_1^*$, thus, up to equivalence, we can choose the exact sequence satisfying condition (C3) to be $0 \to T_1^* \to T_0^* \to R^* \to 0$. It follows that

$$\perp C_1 = \{ X \in R\text{-Mod} \mid T_1 \otimes_R X = 0 = \text{Tor}_1^R(T_1, X) \}.$$ 

Let $\{ X_i \}_{i \in I}$ be a family of modules in $\perp C_1$. Then $\prod X_i \in \perp C_1 = T_1^\perp$. We need to show that also $T_1 \otimes \prod X_i = 0$. To this aim we use Mittag-Leffler properties of 1-tilting modules. Consider a projective presentation $0 \to P_1 \to P_0 \to T_1 \to 0$ of $T_1$ with $P_0, P_1$ projective right modules, then we have a commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & P_1 \otimes \prod X_i & \longrightarrow & P_0 \otimes \prod X_i & \longrightarrow & T_1 \otimes \prod X_i & \longrightarrow & 0 \\
& & \rho_{P_1} & & \rho_{P_0} & & \rho_{T_1} & & \\
0 & \longrightarrow & \prod_i(P_1 \otimes_R X_i) & \longrightarrow & \prod_i(P_0 \otimes_R X_i) & \longrightarrow & \prod_i(T \otimes_R X_i) & \longrightarrow & 0
\end{array}
$$

where the vertical arrows are the canonical maps and the rows are exact since $\text{Tor}_1^R(T_1, X_i)$ and $\text{Tor}_1^1(T_1, \prod X_i)$ are zero. By $[\text{AHH08}]$ Corollary 9.8, $T_1$ is $\mathcal{F}$-Mittag-Leffler, thus $\rho_{T_1}$ is injective and so $T_1 \otimes \prod X_i = 0$, since by assumption $T_1 \otimes X_i = 0$, for every $i \in I$.

\end{proof}

4. Cotilting modules over valuation domains

In this section $R$ will be a valuation domain with quotient field $Q$. For terminology and definitions we refer to $[\text{FS01}]$. We will illustrate properties of cotilting modules over valuation domains mostly taken from $[\text{Baz07}]$. We will have to generalize or extend some of the proofs given there, since now we are interested in the connection with homological ring epimorphisms.

Recall that flat modules over valuation domains are exactly the torsion free module. A valuation ring is maximal if it is linearly compact in the discrete topology and it is almost maximal if $R/I$ is maximal for every non zero ideal $I < R$. Every valuation domain $R$ can be embedded in a maximal immediate extension $S$ which is a maximal valuation domain and moreover, a pure injective envelope of $R$.

We also recall that, by $[\text{FS01}]$ I, 7.8], a finitely generated torsion module $F$ over a valuation domain $R$ admits a finite chain of pure submodules with cyclic successive factors. If $R$ is moreover almost maximal, then $F$ is a direct sum of cyclic modules (see $[\text{FS01}]$ V, 10.4).

Combining these observations with a famous Auslander’s result ($[\text{Aus78}]$) stating that for every pure injective $R$-module the functor $\text{Ext}_R^1(-, C)$ sends direct limits into inverse limits, we obtain the following

Proposition 4.1. ($[\text{Baz07}]$ Lemmas 3.1, 5.3) Let $R$ be valuation domain $R$. The following hold true.
(1) If $C$ is a pure injective $R$-module, then $\perp C$ is determined by the cyclic modules that it contains. Moreover if $R$ is an almost maximal valuation domain a module $M \in \perp C$ if and only if every cyclic (torsion) submodule of $M$ belongs to $\perp C$.

(2) If $M$ is a uniserial $R$-module, then $M \in \perp C$ if and only if every cyclic (torsion) submodule of $M$ belongs to $\perp C$.

For the characterization of cotilting modules over valuation domains an important role is played by the following sets of ideals of $R$:

Notation 4.2. Let $C$ be a cotilting module over a valuation domain $R$. Let

$$G = \{ I \leq R \mid R/I \in \perp C \} = \{ I \leq R \mid R/I \in \text{Cogen } C \}.$$  

$G'$ will be called the sets associated to $C$.

Note that our definition of $G'$ slightly differs from the one given in [Baz07], since now we allow $G'$ to contain also the zero ideal.

For every non zero ideal $I$ of a valuation domain $R$, $I^#$ denotes the prime ideal associated to $I$, that is $I^# = \{ r \in R \mid rI \leq I \}$. $I^#$ is the union of the proper ideals of $R$ isomorphic to $I$ (see [FS01, p. 70 (g)]). We put $0^# = 0$.

For further use we recall the following result proved in [Baz07].

Lemma 4.3. ([Baz07, Lemma 3.3]) Let $C$ be a cotilting $R$-module. The following hold:

1. The sets $G$ and $G'$ are closed under arbitrary unions and arbitrary intersections.
2. If $0 \neq I \in G$, then for every $r \in R \setminus I$, $r^{-1}I \in G$. Moreover, $I^# \in G'$ and $R_{I^#}/I^# \in \perp C$.
3. If $0 \neq I \in G$ and $I < I^#$, then for every $r \in I^# \setminus I$, $rR_{I^#}$ and $rI^#$ belong to $G$.

The next result will be crucial to relate cotilting classes with chains of intervals of prime ideals. The same proof as in [Baz07] works also with our extended definition of $G'$.

Lemma 4.4. ([Baz07, Lemma 3.5]) Let $C$ be a cotilting module with associated set $G'$. For every $L \in G'$, let $H = \sum \{ a^{-1}R \mid a^{-1}R/L \in \perp C \}$. Then there is an idempotent prime ideal $L' \leq L$ such that $H = RL'$ and $L' \in G'$. Moreover, $R_{L'}/L \in \perp C$ and $L' = \inf \{ N \in G' \mid R_N/L \in \perp C \}$.

Remark 4.5. Note that $L'$ in the above lemma might be zero and if $L = 0$, then certainly $L' = 0$. (Note that $Q \in \perp C$, because $Q$ is flat and $C$ is pure injective.)

4.1. Disjoint intervals of primes ideals of $G'$.

Definition 4.6. Let $C$ be a cotilting module with associated set $G'$. For every $L \in G'$ define

$$\phi(L) = \inf \{ N \in G' \mid R_N/L \in \perp C \},$$

$$\psi(L) = \sup \{ N \in G' \mid R_{\phi(L)}/N \in \perp C \}.$$
By Lemma 4.3 and 4.4, \( \phi \) and \( \psi \) are maps from \( \mathcal{G}' \) to \( \mathcal{G}' \); \( \phi(L) \) is an idempotent prime ideal (which might be zero).

Note that \( \psi(0) \) is the largest prime ideal \( N \) such that \( Q/N \in \downarrow C \).

The properties of the two maps \( \phi \) and \( \psi \) are illustrated in \cite[Lemma 6.1]{Baz07}. We collect in the next result the relevant ones.

**Lemma 4.7.** \cite[Lemma 6.1, 6.2]{Baz07} Let \( \phi, \psi \) be defined as above. Then the following hold:

1. \( \phi, \psi \) are increasing maps; \( \phi(L) \leq L \) and \( L \leq \psi(L) \);
2. \( \phi(\psi(L)) = \phi(L) \) and \( \phi(\phi(L)) = \phi(L) \);
3. \( \psi(\phi(L)) = \psi(L) \) and \( \psi(\psi(L)) = \psi(L) \);
4. For every \( L \in \mathcal{G}' \), the pre-image of \( \phi(L) \) under \( \phi \) is the interval \( \left[ \phi(L), \psi(L) \right] = \{ N \in \mathcal{G}' \mid \phi(L) \leq N \leq \psi(L) \} \);
5. For every \( L \in \mathcal{G}' \), \( \phi(L) \) is an idempotent ideal and distinct intervals of the form \( \left[ \phi(L), \psi(L) \right] \) are disjoint.

We need to have information about the ideals of \( \mathcal{G} \) sitting between \( \phi(L) \) and \( \psi(L) \) for every \( L \in \mathcal{G}' \). To this aim we use the following definition.

**Definition 4.8.** Let \( L_0 \leq L \) be two prime ideals of a valuation domain \( R \). We let

\[
\langle L_0, L \rangle = \{ I \in R \mid L_0 \leq I \leq I^\# \leq L \}.
\]

Equivalently, \( I \in \langle L_0, L \rangle \) if and only if \( I \geq L_0 \) and \( I \) is an ideal of \( R_L \).

**Proposition 4.9.** Let \( C \) be a cotilting module over a valuation domain \( R \) with associated set \( \mathcal{G}' \) and let \( M \) be an \( R \)-module.

1. If \( M \in \downarrow C \), then for every non zero torsion element \( x \in M \) there exists \( L \in \mathcal{G}' \) such that \( \text{Ann}(x) \in \langle \phi(L), \psi(L) \rangle \).

2. The converse of (1) holds if \( R \) is almost maximal or if \( M \) is uniserial.

**Proof.** (1) Let \( M \in \downarrow C \) and let \( 0 \neq x \in M \) be a torsion element. Then \( 0 \neq \text{Ann}(x) = I \in \mathcal{G} \), so \( I^\# \in \mathcal{G}' \). Let \( L = I^\# \); we claim that \( I \in \langle \phi(L), \psi(L) \rangle \).

It is enough to show that \( \phi(L) \leq I \). Assume \( I < \phi(L) \) and let \( r \in \phi(L) \setminus I \).

By Lemma 4.3 (3), \( rL \in \mathcal{G} \), hence \( r^{-1}R/L \in \downarrow C \). But \( r^{-1}R > R_{\phi(L)} \), since \( rR_{\phi(L)} < \phi(L) \), thus by Lemma 4.4, \( r^{-1}R/L \notin \downarrow C \), a contradiction.

(2) By Proposition 4.1, our assumptions imply that \( M \in \downarrow C \) if and only if every cyclic torsion submodule of \( M \) belongs to \( \downarrow C \); so it is enough to show that if \( 0 \neq I \in \langle \phi(L), \psi(L) \rangle \), for some \( L \in \mathcal{G}' \), then \( R/I \in \downarrow C \). We can assume \( I \leq \psi(L) \), so \( I \leq r\psi(L) \) for every \( r \in \psi(L) \setminus I \), since \( r^{-1}I \leq I^\# \leq \psi(L) \). Moreover, \( I = \bigcap_{r \in \psi(L) \setminus I} r\psi(L) \).

In fact, assume on the contrary that \( I \leq \bigcap_{r \in \psi(L) \setminus I} r\psi(L) = J \) and let \( a \in J \setminus I \). Then, \( J = a\psi(L) \geq I \) and choosing any \( b \in a\psi(L) \setminus I \) we have \( b\psi(L) = a\psi(L) \) and so \( b = ac \) for some \( c \in \psi(L) \). But \( b\psi(L) = a\psi(L) \) implies \( c\psi(L) = \psi(L) \) contradicting the assumption \( c \in \psi(L) \). Thus \( R/I \) is embedded in \( \prod_{r \in \psi(L) \setminus I} R/r\psi(L) \cong \prod_{r \in \psi(L) \setminus I} r^{-1}R/\psi(L) \) and \( r^{-1}R/\psi(L) \in \downarrow C \), since \( r^{-1}R/\psi(L) \leq R_{\phi(L)}/\psi(L) \in \downarrow C \).
Definition 4.10. Let $C$ be a cotilting module over a valuation domain $R$ with associated set $G'$. Denote by $I(C)$ the set of intervals of prime ideals of the form $[\phi(L), \psi(L)]$, for every $L \in G'$ ordered by $[\phi(L), \psi(L)] < [\phi(L'), \psi(L')]$ if $\psi(L) < \phi(L')$.

By Remark 4.5 and Lemma 4.7, $[0, \psi(0)]$ is the unique minimal element of $I(C)$ and distinct intervals of $I(C)$ are disjoint.

We now show some properties satisfied by the set $I(C)$.

Proposition 4.11. Let $C$ be a cotilting module over a valuation domain $R$. The set $I(C)$ defined in Definition 4.10 has a minimal element $[0, \psi(0)]$ and satisfies conditions (1) and (2) of Conditions 2.13, namely:

1. If $S = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $I(C)$ with no minimal element, then $I(C)$ contains an element of the form $[J, \bigcap_{\alpha \in \Lambda} \psi(L_\alpha)]$.

2. If $S = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $I(C)$ with no maximal element, then $I(C)$ contains an element of the form $[\bigcup_{\alpha \in \Lambda} \phi(L_\alpha), L]$.

Proof. We will make repeated use of Lemma 4.7.

(1) Assume that $S = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $I(C)$ with no minimal element. Let $L_0 = \bigcap_\alpha \psi(L_\alpha)$. Then, $L_0$ is a prime ideal and $L_0 \in G'$ by Lemma 4.3. For every $\alpha \in \Lambda$ we have $\psi(L_0) \leq \psi(L_\alpha)$, then $\psi(L_0) \leq L_0$ and thus $\psi(L_0) = L_0$ by Lemma 4.7. Hence, $[\phi(L_0), \psi(L_0) = \bigcap_\alpha \psi(L_\alpha)]$ is in $I(C)$.

(2) Assume that $S = \{[\phi(L_\alpha), \psi(L_\alpha)] \mid \alpha \in \Lambda\}$ is a non-empty subset of $I(C)$ with no maximal element.

Let $L_0 = \bigcup_{\alpha \in \Lambda} \phi(L_\alpha)$. Then $L_0$ is a prime ideal and by Lemma 4.3, $L_0 \in G'$. For every $\alpha \in \Lambda$ we have $\phi(L_\alpha) \leq \phi(L_0)$, Thus, $L_0 = \bigcup_{\alpha \in \Lambda} \phi(L_\alpha) \leq \phi(L_0)$ and so $L_0 = \phi(L_0)$ by Lemma 4.7. Hence, $[L_0 = L_0 = L_0, L]$ belongs to $I(C)$.

□

In Section 6 we will show that there exist cotilting modules $C$ whose associated set of intervals $I(C)$ does not satisfy condition (3) of Conditions 2.13, that is $I(f)$ contains a dense subset of intervals.

5. A BIJECTIVE CORRESPONDENCE

In this section again $R$ will be a valuation domain.

For every homological ring epimorphism $f: R \to S$ and every cotilting $R$-module $C$, $I(f)$ and $I(C)$ will denote the chains of intervals of Inter $R$ as defined in Definition 2.12 and Definition 4.10.

From [BS14, Theorem 6.23] we know that the set $I(f)$ satisfies all the three conditions in Conditions 2.13. By Proposition 4.11, the set $I(C)$ satisfies the first two conditions of Conditions 2.13 but, as we will see in Section 6, it may not satisfy the third condition.

Thus, we distinguish the two possible situations for a cotilting module and at this aim we introduce the following:
Definition 5.1. We say that a cotilting module $C$ is non dense if the set $\mathcal{I}(C)$ does not contain any dense subset, that is if $\mathcal{I}(C)$ satisfies

(3) Given any $[\phi(L_0), \psi(L_0)] < [\phi(L_1), \psi(L_1)]$ in $\mathcal{I}(C)$, there are two intervals $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$ of $\mathcal{I}(C)$ such that

$[\phi(L_0), \psi(L_0)] \leq [\phi(L), \psi(L)] < [\phi(L'), \psi(L')] \leq [\phi(L_1), \psi(L_1)]$

and there no other intervals of $\mathcal{I}(C)$ properly between $[\phi(L), \psi(L)]$ and $[\phi(L'), \psi(L')]$

The corresponding cotilting class will also be called non dense.

Combining results from Sections 2.1 and 4.1 we are in a position to assign to every non dense cotilting module over a valuation domain $R$ an injective homological ring epimorphisms $f: R \to S$.

Proposition 5.2. Let $C$ be a non dense cotilting module over a valuation domain $R$ with associated set $\mathcal{I}(C)$ of intervals as in Definition 4.10. Then there is an injective homological ring epimorphism $f: R \to S$ such that $\mathcal{I}(f) = \mathcal{I}(C)$.

Proof. By assumption and Proposition 4.11 the ordered set $\mathcal{I}(C)$ satisfies Conditions 2.13. So, by Theorem 2.15, there is a homological ring epimorphism $f: R \to S$ such that $\mathcal{I}(f) = \mathcal{I}(C)$. Since the minimal element of $\mathcal{I}(C)$ is $[0, \psi(0)]$ we infer that $f$ is injective.

Remark 5.3. In the notations of Proposition 5.2, if $\mathcal{I}(C)$ is finite, say $\mathcal{I}(C) = [0, \psi(0)] \cup \{[\phi(L_i), \psi(L_i) \mid 1 \leq i \leq n\}$, then $S \cong Q/\psi(0) \oplus \prod_{1 \leq i \leq n} R_{\psi(L_i)}[\phi(L_i)]$.

From Theorem 3.3 we already know that to every injective homological ring epimorphism $f: R \to S$ with w.d. $S \leq 1$ we can associate a cotilting $R$-module $C$. Our next task will be to prove that, when $R$ is a valuation domain, then $C$ is non dense and the sets $\mathcal{I}(f)$ and $\mathcal{I}(C)$ coincide.

Proposition 5.4. Let $R$ be a valuation domain and $f: R \to S$ be an injective homological epimorphism with associated set $\mathcal{I}(f)$. Then, $C = S^* \oplus (S/R)^*$ is a non dense 1-cotilting $R$-module such that $\mathcal{I}(C) = \mathcal{I}(f)$.

Proof. Identifying $R$ with $f(R)$, Theorem 3.3 tells us that $C = S^* \oplus (S/R)^*$ is a cotilting $R$-module and that $R/I \in \mathcal{I}(f)$ if and only if $R/I \to S/SI$ is injective, that is if and only if $R \cap SI = I$. Our goal is to prove that $\mathcal{I}(C) = \mathcal{I}(f)$.

(a) First of all we note that for every interval $[J, L] \in \mathcal{I}(f)$ and every prime ideal $P \in [J, L]$ we have that $R/P \to S/SP$ is an injective ring homomorphism, hence $R/P \in \mathcal{I}(C)$. In fact, by Corollary 2.11, $P = f^{-1}(n) = R \cap n$ for some prime ideal $n$ of $S$, hence $P \leq R \cap SP \leq R \cap n = P$.

CLAIM 1 Let $[J, L] \in \mathcal{I}(f)$. We claim that $[J, L]$ is contained in the interval $[\phi(J), \psi(J)]$ of $\mathcal{I}(C)$.

Let $m$ be a maximal ideal of $S$ such that $S_m \cong R_L/J$ (by Proposition 2.9).

(b) We first show that $R_L/J$ is contained in $\mathcal{I}(f)$. In fact, consider the valuation domain $V = R_L/J$. Its maximal ideal is $p = L/J$ and its quotient field $Q(V)$ is isomorphic to $R_L/J$. Thus $Q(V)/p \cong R_L/L$. An injective cogenerator $E_V$ of $(V, p)$ is the pure injective envelope of $Q(V)/p$ as $V$-module.
(see [FS01, XII Lemma 4.3].) Thus, in our case we have that $R_J/L$ is an $R$-submodule of an injective cogenerator of $R_L/J$. Now the injective cogenerator of $R_L/J$ is an $S$-module, since so is $R_L/J$, thus $R_J/L \in \text{Cogen } S^*$.

By (a) $R/J \in \perp C$, hence also $R_{\phi(J)}/J \in \perp C$, by definition of $\phi(J)$.

(c) We show that $R_{\phi(J)}/L \in \perp C$, so that $L \leq \psi(J)$ by the definition of the map $\psi$ and thus $[J, L]$ is contained in $[\phi(J), \psi(J)] \in \mathcal{I}(C)$ and the claim is proved.

If $\phi(J) = J$, then by (b) $R_{\phi(J)}/L \in \perp C$.

If $\phi(J) \leq J$ we consider the exact sequence

$$0 \to R/J \to R_{\phi(J)}/L \to R_{\phi(J)}/R_J \to 0.$$ 

Let $0 \neq x \in R_{\phi(J)}/R_J$, that is $x = a^{-1} + R_J$ with $a \in J \setminus \phi(J)$. Then $\text{Ann}_{R/J} = aR_J \geq \phi(J)$, hence $\text{Ann}_{R/J} x \in (\phi(J), \psi(J))$. $R_{\phi(J)}/R_J$ is uniserial, so by Proposition 4.9, we conclude that $R_{\phi(J)}/R_J \in \perp C$. Since by (b) $R_J/L \in \perp C$, the above exact sequence tells us that also $R_{\phi(J)}/L \in \perp C$.

CLAIM 2 Let $[\phi(N), \psi(N)] \in \mathcal{I}(C)$. We claim that $[\phi(N), \psi(N)]$ is contained in an interval of $\mathcal{I}(f)$.

Applying Corollary 2.10 to the idempotent ideal $\phi(N)$ we get an interval $[J, L] \in \mathcal{I}(f)$ such that $J \leq \psi(N) \leq L$.

(d) We show that $\psi(N) \leq L$ so that $[J, L]$ contains $[\phi(N), \psi(N)]$ and the claim is proved.

Assume by way of contradiction that $L \leq \psi(N)$.

We have $S\psi(N) \cap R = \psi(N)$, because $R/\psi(N) \in \perp C$. Consider the set $\{n_\alpha \mid \alpha \in \Lambda\}$ of maximal ideals of $S$ containing $S\psi(N)$ and let

$$S = \{[J_\alpha, L_\alpha] \mid \alpha \in \Lambda\}$$

be the set of intervals of $\mathcal{I}(f)$ corresponding to $n_\alpha$, for every $\alpha \in \Lambda$. Note that $\psi(N) \leq L_\alpha$, for every $\alpha$, since $f^{-1}(n_\alpha) = L_\alpha$. The assumption $L \leq \psi(N)$ implies that $L \leq j_\alpha$ for every $\alpha \in \Lambda$. In fact, if $J_\alpha \leq L$ for some $\alpha$, then $L \in [J_\alpha, L_\alpha]$, hence $[J, L] = [J_\alpha, L_\alpha]$ giving $\psi(N) \leq L$, contradicting the assumption. We show now that there is an interval $[J', L'] \in \mathcal{I}(f)$ which is minimal among the intervals of $\mathcal{I}(f)$ satisfying $\psi(N) \leq L'$. Indeed, if $S$ has a minimal element the claim is immediate. Otherwise, by Theorem 2.13, Conditions 2.13(1) ensures that $\mathcal{I}(f)$ contains an interval $[J', L'] = \bigcap_{\alpha \in \Lambda} L_\alpha$ and $\psi(N) \leq \bigcap_{\alpha \in \Lambda} L_\alpha$. Hence $[J', L']$ satisfies our claim.

We must have $[J, L] \leq [J', L']$ hence, by Conditions 2.13(3), there are two intervals $[J_0, L_0] \leq [J_1, L_1]$ with no other intervals of $\mathcal{I}(f)$ between them and such that

$$[J, L] \leq [J_0, L_0] \leq [J_1, L_1] \leq [J', L'].$$

Thus $L_0 \leq \psi(N)$. Choose $a \in (\psi(N) \cap J_1) \setminus L_0$. Then the canonical localization map $S \to S_{[1]}^\alpha$ is surjective. Indeed, this can be proved locally by using the properties of $\mathcal{I}(f)$ and Proposition 2.9. In fact, for every maximal ideal $m$ of $S$ with corresponding interval $[J'', L'']$, if $[J'', L''] \leq [J_0, L_0]$ the morphism $S_m \to (S_{[1]}^\alpha)_m$ is an isomorphism; if $[J'', L''] \geq [J_1, L_1]$ the morphism $S_m \to (S_{[1]}^\alpha)_m$ is zero. The surjectivity of $S \to S_{[1]}^\alpha$ implies the existence of $n \geq 1$ and of an element $t \in S$ such that $a^n = ta^{n+1}$. Let $I = a^n \psi(N)$; then $a^n \in SI$, since $a \in \psi(N)$, hence $a^n \in R \cap SI$, but clearly $a^n \notin I$. This implies $R/I \notin \perp C$. But, the annihilator of every
0 \neq r + I \in R/I is \( r^{-1}I \) and \( r^{-1}I \in (\phi(N), \psi(N)) \). In fact, \( \phi(N) \leq L_0 \leq a^n\psi(N) = I \leq r^{-1}I \) and \( (r^{-1}I)^* = \psi(N) \), hence by Proposition 4.9 \( R/I \in \mathcal{I}(C) \), a contradiction.

By claims (a) and (b) and by the disjointness of the intervals in \( \mathcal{I}(f) \) and \( \mathcal{I}(C) \), we conclude that the two sets of intervals coincide. \( \square \)

We illustrate now some results in order to characterize the injective homological ring epimorphisms \( f: R \to S \) among the homological epimorphisms.

**Lemma 5.5.** Let \( R \) be a valuation domain let \( \epsilon: R \to Q \) be the canonical inclusion into the quotient field \( Q \) of \( R \). Assume that \( f: R \to S \) is a homological ring epimorphism with \( \ker f \neq 0 \). The following hold true:

1. The morphism \( g = (\epsilon, f): R \to Q \oplus S \) is an injective homological epimorphism and \( \mathcal{I}(g) = [0] \cup \mathcal{I}(f) \).
2. \( \left( \frac{Q \oplus S}{g(R)} \right)^* = \left( \frac{Q}{\ker f} \right)^* \oplus \left( \frac{S}{f(R)} \right)^* \).

**Proof.** (1) \( g \) is an injective ring homomorphism and it is an epimorphism since \( Q \otimes_R S = 0 = S \otimes_R Q \), due to the fact that \( Q \) is divisible and \( S \) is a torsion \( R \)-module annihilated by \( K \). It is obvious that \( g \) is homological. Let \( S' = Q \oplus S \), and let \( n' \) be a maximal ideal of \( S' \). Then \( n' = Q \oplus n \), with \( n \) a maximal ideal of \( S \) or \( n' = S \) viewed as an ideal of \( S' \). It is easy to check that \( S'_n \cong S_n \) whenever \( n' = Q \oplus n \) and \( S'_n \cong Q \). Thus the interval in \( \mathcal{I}(g) \) corresponding to the maximal ideal \( S \) of \( S' \) is \([0]\) and the intervals of \( \mathcal{I}(g) \) corresponding to the other maximal ideals of \( S' \) are the same as the intervals of \( \mathcal{I}(f) \).

(2) \( \frac{Q \oplus S}{g(R)} \) is a pushout of \( f \) and \( -\epsilon \), so we get the exact sequence

\[
0 \to \frac{Q}{\ker f} \to \frac{Q \oplus S}{g(R)} \to \frac{S}{f(R)} \to 0
\]

whose dual sequence splits since \( \left( \frac{Q}{\ker f} \right)^* \) is torsion-free (hence flat) and \( \left( \frac{S}{f(R)} \right)^* \) is pure injective. \( \square \)

**Lemma 5.6.** Let \( R \) be a valuation domain and let \( f: R \to S \) be a homological ring epimorphism. Then \( f \) is injective if and only if there are a prime ideal \( L \) of \( R \) and a homological epimorphism \( g: R \to S' \) such that \( S \cong R_L \oplus S' \) and \( f \) is equivalent to \((\psi_L, g)\) where \( \psi_L \) is the canonical localization of \( R \) at the prime ideal \( L \).

**Proof.** If \( f \) is an injective homological epimorphism, then the minimal element of \( \mathcal{I}(f) \) is an interval \([0, L]\), for some prime ideal \( L \) of \( R \). By [BS14, Section 4], \( \mathcal{I}(f) \) is the disjoint union of \([0, L]\) with a set \( T' \), where \( T' \) satisfies Conditions 2.13 hence there is a ring \( S' \) and a homological epimorphism \( g: R \to S' \) such that \( \mathcal{I}(g) = T' \). Then \( (\psi(L), g): R \to R_L \oplus S' \) is a homological epimorphism, by Proposition 2.9 (3) and \( \mathcal{I}(\psi(L), g) = \mathcal{I} \). By Theorem 2.15 we conclude that \( f \) and \((\psi(L), g)\) are equivalent homological epimorphisms.

The converse is clear from the fact that \( \psi_L: R \to R_L \) is an injective homological epimorphism. \( \square \)
The results proved in this section can be summarized by the following theorem.

**Theorem 5.7.** Let $R$ be a valuation domain. Then there is a bijection between:

1. Equivalence classes of non dense cotilting modules.
2. Equivalence classes of injective homological ring epimorphisms $f : R \to S$.

The bijection is given by assigning to a non dense cotilting module $C$ the homological ring epimorphism $f_{\mathcal{I}(C)} : R \to R_{\mathcal{I}(C)}$ constructed in [BS14, Construction 4.12].

The converse is given by sending an injective homological ring epimorphism $f : R \to S$ to the cotilting module $C = S^* \oplus S/f(R)^*$. Moreover, for every homological ring epimorphism $f : R \to S$ with $\text{Ker} f \neq 0$ there is an injective homological ring epimorphism $g : R \to S'$ such that $\mathcal{I}(g) = [0] \cup \mathcal{I}(f)$ and associated cotilting module $S^* \oplus (Q/\text{Ker} f)^* \oplus (S/f(R))^*$.

By [BS14] there is a bijective correspondence between equivalence classes of homological ring epimorphism originating in valuation domains $R$ and smashing localizing subcategories of the derived category $\mathcal{D}(R)$ of $R$. We restrict now the correspondence between non dense cotilting classes and some particular smashing localizing subcategories of $\mathcal{D}(R)$ in the way that we are going to describe.

First recall that a triangulated subcategory $\mathcal{X}$ of the derived category $\mathcal{D}(R)$ of a ring $R$ is *smashing localizing* if it is closed under coproducts and its orthogonal class $\mathcal{Y} = \{ Y \in \mathcal{D}(R) \mid \text{Hom}_{\mathcal{D}(R)}(\mathcal{X}, Y) = 0 \}$ is closed under coproducts as well.

A complete description of smashing localizing subcategories of $\mathcal{D}(R)$, for $R$ with w.gl. dim $R \leq 1$ is given by the following theorem.

**Theorem 5.8.** ([BS14, Theorem 3.13]) Let $R$ be a (possibly non-commutative) ring of weak global dimension at most one. Then the assignment $f \mapsto \{ X \in \mathcal{D}(R) \mid S \otimes_R^L X = 0 \}$ is a bijection between

1. equivalence classes of homological ring epimorphisms $f : R \to S$ originating at $R$, and
2. smashing localizing subcategories $\mathcal{X} \subseteq \mathcal{D}(R)$.

Moreover, the class $\mathcal{X}$ corresponding to a given $f$ consists precisely of the complexes $X \in \mathcal{D}(R)$ such that $S \otimes_R H^n(X) = 0 = \text{Tor}_1^R(S, H^n(X))$ for all $n \in \mathbb{Z}$.

**Corollary 5.9.** Let $R$ be a valuation domain and let $f : R \to S$ be a homological ring epimorphism. Let $g : R \to Q \oplus S$ be the injective homological epimorphism defined in Lemma 5.5 and let $\mathcal{X}$ and $\mathcal{X}'$ be the smashing localizing subcategories of $\mathcal{D}(R)$ corresponding to $f$ and $g$, respectively. Then

$$\mathcal{X}' = \{ X \in \mathcal{X} \mid H^n(X) \text{ is a torsion } R\text{-module for all } n \in \mathbb{Z} \}.$$

**Proof.** Follows immediately by Theorem 5.8 and by the fact that $Q \otimes_R M = 0$ if and only if $M$ is a torsion $R$-module. □
We recall also the following notion.

**Definition 5.10.** Let $R$ be a commutative ring. The *cohomological support* of $X \in D(R)$ is:

$$\text{Supp } X = \{ p \in \text{Spec } R \mid R_p \otimes_R X \neq 0 \}.$$ 

For a class of complexes $\mathcal{X}$, we define $\text{Supp } \mathcal{X} = \bigcup_{X \in \mathcal{X}} \text{Supp } X$.

Combining the previous results we can relate non dense cotilting classes over valuation domains $R$ with particular smashing localizing subcategories of $D(R)$ in the following way:

**Proposition 5.11.** Let $R$ be a valuation domain. Then there is a bijection between:

1. Equivalence classes of non dense cotilting modules.
2. Smashing localizing subcategories $\mathcal{X}$ of $D(R)$ for which there exists a prime ideal $L$ of $R$ such that $L \notin \text{Supp } \mathcal{X}$, or equivalently such that $H^n(X)$ is a torsion $R/L$-module for every $n \in \mathbb{Z}$ and every $X \in \mathcal{X}$.

**Proof.** By Theorem 5.7 and Lemma 5.6 a non dense cotilting class corresponds bijectively to the equivalence classes of a homological epimorphism $f : R \to R_L \otimes_S S$ for some prime ideal $L$ of $R$. By Theorem 5.8 the smashing localizing subcategory $\mathcal{X}$ corresponding to $f$ satisfies the condition as in (2).

Conversely, let $\mathcal{X}$ be a smashing localizing subcategory of $D(R)$ as in (2) and let $f : R \to S$ be a homological ring epimorphism corresponding to $\mathcal{X}$ under Theorem 5.8.

We claim that $f$ is injective. Assume on the contrary that $0 \neq J = \text{Ker } f$, then $J$ as a complex concentrated in degree zero belongs to $\mathcal{X}$. In fact, $S \otimes_R J \cong S \otimes_R S \otimes_R J$ and $S \otimes_R J = 0$, since $S$ is annihilated by $J$ and every element of $J$ is of the form $ab$ for $a$ and $b$ in $J$, because $J$ is idempotent. By assumption, there is a prime ideal $L \in \text{Spec } R$ such that $L \notin \text{Supp } J$, that is $R_L \otimes_R J = J_L = 0$, a contradiction.

\[ \square \]

6. **Cotilting modules with a dense set of intervals**

**Example 6.1.** Let $\Theta = [0, 1]$ be the interval of the real numbers between 0 and 1 and consider the totally ordered set $(T, \leq)$ where $T = \Theta \times \{0, 1\}$ and $\leq$ is the lexicographic order:

$$(x, a) < (y, b) \iff x < y \text{ or } (x = y \text{ and } a < b).$$

For every $x \in \Theta$ let $p_x = (x, 0), q_x = (x, 1)$. Fix two elements $(x, a) < (y, b)$ of $T$. If $x = y$, then there are no elements of $T$ properly between $(x, a)$ and $(y, b)$. If $x < y$ and $x < z < y$, then $(x, a) < p_z < q_z < (y, b)$ and there are no elements of $T$ between $p_z$ and $q_z$. Moreover,

(i) $\forall 0 \neq x \in \Theta, \quad p_x = \sup \{q_z \mid z < x\} = \sup \{p_z \mid z < x\}$,

(ii) $\forall 1 \neq x \in \Theta, \quad q_x = \inf \{p_z \mid z > x\} = \inf \{q_z \mid z > x\}$.

If $t \leq t' \in T$, let $[t, t']$ be the interval of the elements of $T$ between $t$ and $t'$.

For every, $x \in \Theta$ the interval $[p_x, q_x]$ consists just of the two elements $p_x, q_x$ and we have $T = \bigcup_{x \in t} [p_x, q_x]$.
By [FS01] Theorem 2.5 and Proposition 4.7, $\mathcal{T}$ is order isomorphic to the prime spectrum of a valuation domain $R$.

For each $p_x$, $q_x$, let $J_x$, $L_x$ be the prime ideals of $R$ corresponding to $p_x$ and $q_x$, respectively. Then, for every $x \in \Theta$, $J_x$ is idempotent by (i). The set of intervals $\{[p_x,q_x] \mid x \in \Theta\}$ corresponds to the set $\mathcal{I} = \{[J_x,L_x] \mid x \in \Theta\}$ of intervals of prime ideals of $R$. Define on $\mathcal{I}$ the total order given by $[J_x,L_x] < [J_y,L_y]$ if and only if $x < y$ in $\Theta$.

**Remark 6.2.** It is easy to see that the totally ordered set $\mathcal{I}$ defined above satisfies properties (1) and (2) of Construction 2.13, but $\mathcal{I}$ does not satisfy (3). This means that there are no homological ring epimorphisms $f : R \to S$ such that $\mathcal{I} = \mathcal{I}(f)$.

Indeed, from [BS14] Lemm 6.5 one obtains that such an $S$ should be a subring of $\prod_{x \in \Theta} (R/J_x)_{L_x}$ whose elements satisfy the conditions of Proposition 6.16, while in our case the only elements of $\prod_{x \in \Theta} (R/J_x)_{L_x}$ satisfying those conditions are the elements of $R$.

We show that there is a valuation domain and a cotilting module $C$ whose associated set $\mathcal{I}(C)$ of intervals is the set $\mathcal{I}$ defined above. We first note the following.

**Lemma 6.3.** Let $R$ be a valuation domain with prime spectrum order isomorphic to the totally ordered set $\mathcal{T}$ of Example 6.7. Then, for every ideal $I$ of $R$ there is $x \in \Theta$ such that $J_x \leq I \leq L_x$.

**Proof.** Let $S_I = \{y \in \Theta \mid I \leq J_y\}$. If $S_I = \emptyset$, then $I \geq J_1$ so $J_1 \leq I \leq L_1$, since $L_1$ is the maximal ideal of $R$. If $S_I \neq \emptyset$ let $x_0$ be the infimum of $S_I$. Then $I \geq \bigcup_{z < x_0} J_z = \bigcup_{z < x_0} L_z = J_{x_0}$. Thus $x_0 \notin S_I$ and $I \leq \bigcap_{x_0 < z} J_z = L_{x_0}$, hence the conclusion. \qed

**Proposition 6.4.** Let $R$ be a maximal valuation domain whose prime spectrum is isomorphic to the totally ordered set $\mathcal{T}$ defined in Example 6.7. Then the module:

$$C = Q \oplus \prod_{x \in \Theta} \frac{R_{J_x}}{L_x}$$

is a cotilting module such that $\mathcal{I}(C) = \mathcal{I} = \{[J_x,L_x] \mid x \in \Theta\}$.

**Proof.** First note that $C$ is a pure injective module, since $R$ is a maximal valuation domain (see [FS01] Xiii Theorem 5.2).

CLAIM (A) Cogen $C \subseteq C^\perp$.

By Proposition 4.1, it is enough to show that every cyclic submodule of $C^\perp$ belongs to $C^\perp$, for every cardinal $\lambda$. Let $(c_\alpha)_{\alpha \in \lambda} \in C^\perp$ and $I_\alpha = \text{Ann}_R(c_\alpha)$ for every $\alpha \in \lambda$. We have to show that, if $I = \bigcap_{\alpha \in \lambda} I_\alpha$, then $R/I$ belongs to $C^\perp$ or in case $I \leq J_y$ it must be $I^\# \leq J_y$ and $I \nmid R_{J_y}$. Every $I_\alpha$ is of the form $\bigcap_{z \in \text{Supp } c_\alpha} a_x L_x$ where $a_x \notin J_x$, thus $I$ is also an intersection of ideals of the form $a_x L_x$ where $x$ vary in a subset of $\Theta$. Fix $y \in \Theta$ and assume $I \leq J_y$. Let $A_I = \{x \in \Theta \mid I \leq a_x L_x \leq J_y\}$, then $I = \bigcap_{x \in A_I} a_x L_x$ and for every $x \in A_I$ we
Thus we may assume that $M$ is torsion free modules in the torsion theory of the commutative domain $C$. Thus to prove the claim it is enough to show that if $I = a_x L_x$, then $I \not\subseteq R_{J_y}$. If $x_0 = \inf A_I$, then

$$\bigcap_{x < x_0 \in A_I} J_x = \bigcup_{x < x_0 \in A_I} L_x = J_{x_0},$$

so

$$x \leq x_0 \leq L_{x_0} \lessgtr J_{x_0} \lessgtr a_x L_x.$$

If $A_I$ has a minimum $x_0$, then $I = a_{x_0} L_{x_0}$, so $I \not\subseteq R_{J_y}$. If $x_0 = \inf A_I$, then by (a), $I = \bigcap_{x < x_0 \in A_I} J_x = \bigcup_{x < x_0 \in A_I} L_x = J_{x_0}$, so again $I \not\subseteq R_{J_y}$.

CLAIM (B) $\perp C \subseteq \text{Cogen } C$.

By (A) Cogen $C$ is a torsion free class and $\perp C$ is closed under submodules. Thus to prove the claim it is enough to show that if $M \in \perp C$ and $M$ is torsion in the torsion theory associated to $C$, that is $\text{Hom}_R(M, C) = 0$, then $M = 0$.

Moreover, since $Q$ is a summand of $C$, Cogen $C$ contains the class of torsion free modules in the torsion theory of the commutative domain $R$. Thus we may assume that $M \in \perp C$ is torsion in the classical sense.

Let $0 \neq R/I$ be isomorphic to a non zero cyclic submodule of $M$.

(i) We show that there is $x_0 \in \Theta$ such that $I \in \langle J_{x_0}, L_{x_0} \rangle$ (see Definition 4.8).

We have $R/I \in \perp R_{J_y}/L_x$, for every $x \in \Theta$. By Lemma 6.6, we infer that if $I \leq J_y$ for some $y \in \Theta$, then $I \not\subseteq J_y$ and $I \not\subseteq R_{J_y}$. As in the proof of Lemma 6.3 let $x_0$ be the infimum of the set $S_I = \{y \in \Theta \mid I \leq J_y \}$. Then $J_{x_0} = \bigcup_{x < x_0 \in A_I} J_x \leq I$ and $I \not\subseteq J_{x_0} \lessgtr L_{x_0}$. Then, $I \in \langle J_{x_0}, L_{x_0} \rangle$.

Let $x_0$ be as in (i) and let

$$M[J_{x_0}] = \{m \in M \mid m J_{x_0} = 0 \}.$$

(ii) We show that $\frac{M}{M[J_{x_0}]} \in \perp C$, so that $\text{Hom}_R(M[J_{x_0}], C) = 0$.

Let $0 \neq \overline{m} = m + M[J_{x_0}] \in \frac{M}{M[J_{x_0}]}$, that is $A = \text{Ann}_R m \leq J_{x_0}$. By (i) there is $x_1 \in [0, 1]$ such that $A \in \langle J_{x_1}, L_{x_1} \rangle$ and certainly $x_1 < x_0$. Consider $B = \text{Ann}_R \overline{m}$, then $B = A : J_{x_0}$. Also $A \leq B \leq J_{x_0}$ (since $J_{x_0}$ is idempotent). We get that $B = A$. Indeed, if $b \in B \setminus A$ then $J_{x_0} \leq b^{-1} A \leq A \not\subseteq L_{x_1}$ contradicting $x_1 < x_0$. Thus, every cyclic submodule of $\frac{M}{M[J_{x_0}]}$ belongs to $\perp C$, hence, by Proposition 4.1 also $\frac{M}{M[J_{x_0}]} \in \perp C$ and from the exact sequence

$$0 \to M[J_{x_0}] \to M \to \frac{M}{M[J_{x_0}]} \to 0$$

we conclude that $\text{Hom}_R(M[J_{x_0}], C) = 0$.

(iii) Consider the canonical localization $\psi : M[J_{x_0}] \to M[J_{x_0}] \otimes_R R_{L_{x_0}}$.

Then $\text{Ker } f = \{x \in M[J_{x_0}] \mid sx = 0, \exists s \not\in L_{x_0} \}$. Now $M[J_{x_0}] \otimes_R R_{L_{x_0}}$ is an $(R_{L_{x_0}}/J_{x_0})$-module, hence it is cogenerated by $R_{J_{x_0}}/R_{L_{x_0}}$ (see part (b) in the proof of Proposition 5.4).
which is a direct summand of $C$. Thus the module $\frac{M[J_{x_0}]}{\text{Ker } f}$ is also cogenerated by $C$ and the condition $\text{Hom}_R(M[J_{x_0}], C) = 0$ yields $\text{Hom}_R \left( \frac{M[J_{x_0}]}{\text{Ker } f}, C \right) = 0$. The generator $\xi$ of the non zero cyclic submodule $R/I$ of $M$ we started with in (i), belongs to $M[J_{x_0}]$, since $J_{x_0} \leq I$ and its annihilator $I$ is contained in $L_{x_0}$, thus $\xi$ doesn’t belong to $\text{Ker } f$, a contradiction.

\[\square\]

### 7. 1-cotilting modules and Tor-orthogonal classes

In this section we consider the problem about cotilting classes being Tor-orthogonal classes.

We start by recalling a result relating Tor-orthogonal classes with the Mittag-Leffler condition (see Definition 3.9).

**Lemma 7.1.** ([Her13, Theorem 3.13]) Let $C$ be a class of right $R$-modules. The class $C^\uparrow$ is closed under products if and only if the syzygy of every module in $C$ is $C^\uparrow$-Mittag-Leffler. Moreover, if $C^\uparrow$ is closed under products, there is a set $S$ of countably presented right modules such that $S^\uparrow = C^\uparrow$.

**Remark 7.2.** From the proof of [Her13, Theorem 3.13] and from the properties of Mittag-Leffler modules, one can see that the set $S$ can be chosen to consist of strongly countably presented modules, that is modules whose first syzygy is again countably presented.

From the above results, we obtain:

**Proposition 7.3.** Let $R$ be a 1-cotilting module over a ring $R$.

1. The cotilting class $\mathcal{F} = \perp C$ is a Tor orthogonal class if and only if there is a set $S$ of countably presented modules in $\mathcal{F}$ such that $\mathcal{F} = S^\uparrow$.

2. Let $M$ be a right $R$-module. The class $M^\uparrow$ is a 1-cotilting class if and only if $\text{w.d. } M \leq 1$ and the first syzygy of $M$ is a (countably presented flat) $M^\uparrow$-Mittag-Leffler module.

**Proof.** (1) follows immediately by Lemma 7.1

(2) We have $M^\uparrow = \perp M^\ast$ and $\text{w.d. } M = \text{i.d. } M^\ast$. Thus, by [GT12, Theorem 15.9], $M^\uparrow$ is a 1-cotilting class if it is closed under direct products. Then, apply Lemma 7.1 to conclude. \[\square\]

**Remark 7.4.** By [Tri07, Lemma 4.9], if $\text{p.d. } M \leq 1$, then $M^\uparrow$ is a 1-cotilting class of cofinite type.

Our next task will be to prove that over valuation domains every cotilting class is a Tor-orthogonal class. First we note the following property.

**Lemma 7.5.** Let $R$ be a valuation domain. A Tor-orthogonal class is determined by the cyclic modules that it contains.

**Proof.** Let $C$ be a class of modules. Since $\text{w.gl.dim } R = 1$ and Tor commutes with direct limits, we have that $\text{Tor}_1^R(C, M) = 0$ if and only if $\text{Tor}_1^R(C, F) = 0$ for every finitely generated torsion submodule of $M$ (recall that torsion free
modules are flat). By \cite[I, 7.8]{FS01}, a finitely generated torsion module $F$ over a valuation domain admits a finite chain of pure submodules with cyclic successive factors. It is immediate to conclude that $F \in C$ if and only if all the cyclic factors are in $C$.

Now we consider the easy case.

**Lemma 7.6.** Let $R$ be a valuation domain. Every non dense cotilting class is a Tor-orthogonal class.

**Proof.** Let $C$ be a non dense cotilting $R$-module over a valuation domain $R$. By Theorem 5.7 there is an injective homological ring epimorphism $f: R \to S$ such that $C$ is equivalent to $S^* \oplus (S/f(R))^\perp$. Thus $\perp C$ coincides with $(S/f(R))^\perp$. □

To deal with the case of a cotilting module $C$ with associated set $I(C)$ containing dense intervals, we introduce an equivalence relation on $I(C)$ in the following way.

**Notation 7.7.** ($\dagger$) Let $(X, \leq)$ be a totally ordered set. A suborder $(Y, \leq)$ is said to be dense if given any two elements $a < b$ in $Y$ there is $c \in Y$ such that $a < c < b$. Given $x, y \in X$, define $x \sim y$ if either $x = y$ or if the suborder of $X$ consisting of all elements of $X$ between $x$ and $y$ is dense. It is easy to see that $\sim$ is an equivalence relation.

($\dagger\dagger$) Let $C$ be a cotilting module over a valuation domain and let $I(C)$ be the totally ordered set as in Definition 4.10. Consider on $I(C)$ the equivalence relation $\sim$ defined above and for every prime ideal $L \in G'$ denote by $i_L$ the equivalence class determined by the interval $[\phi(L), \psi(L)]$ under the equivalence $\sim$. Applying Proposition 4.11, we see that every $i_L$ has a minimal element $[\phi(L_0), \psi(L_0)]$ and a maximal element $[\phi(L_1), \psi(L_1)]$. For every equivalence class $i_L$ consider the interval $[\phi(L_0), \psi(L_1)]$ of prime ideals and let $J$ be totally ordered set consisting of all these intervals, for $L$ varying in $G'$.

**Fact 7.8.** The totally ordered set $J$ defined in Notations 7.7 ($\dagger\dagger$), satisfies all the properties in Conditions 2.13 and has a minimal element of the form $[0, \psi(N)]$ for some $N \in G'$. So, by Theorem 2.15 there is an injective homological ring epimorphism $f: R \to S$ such that $I(J) = J$.

By Proposition 5.4, $D = S^* \oplus (S/R)^*$ is a cotilting module such that $I(D) = J$, hence by Theorem 3.3 $\perp D = (S/R)^\perp$.

**Remark 7.9.** Note that if $I$ is the totally ordered set of intervals defined in Example 6.1, then the quotient set $I/\sim$ of $I$ modulo the equivalence relation in Notation 7.7 ($\dagger$) consists just of the interval $[0, L_1]$, where $L_1$ corresponds to the maximal ideal of $R$.

Compare also with Remark 6.2.

**Lemma 7.10.** Let $C$ be a cotilting module over a valuation domain $R$. In the above notations, let $[\phi(L_0), \psi(L_1)]$ be the interval of $J$ corresponding to the equivalence class $i_L$ determined by a prime ideal $L \in G'$ and assume $L_0 < L_1$ (that is the equivalence class $i_L$ doesn’t consist of a single interval). Consider the set $H$ of prime ideals $N \in G'$ such that $\phi(L_0) < \phi(N)$ and
ψ(N) < ψ(L_1). Then, for every N ∈ ℋ it holds:

\[ \phi(N) = \bigcup_{P \in ℋ, \psi(P) < \phi(N)} \psi(P) \quad \text{and} \quad \psi(N) = \bigcap_{Q \in ℋ, \psi(N) < \phi(Q)} \phi(N') \]

Proof. By assumption ℋ ≠ ∅. Note that, if N ∈ ℋ then also φ(N) and ψ(N) are in ℋ. Fix N ∈ ℋ and let ℋ_N = \{ P ∈ ℋ | ψ(P) ≤ φ(N) \}. By density ℋ_N ≠ ∅ and if \( P_0 = \bigcup_{P \in ℋ_N} \psi(P) \) then we also have \( P_0 = \bigcup_{P \in ℋ_N} \phi(P) \). Thus, by Lemma 4.7 \( \phi(P_0) = P_0 \) and it must be \( P_0 = \phi(N) \), otherwise \( \psi(P_0) < \phi(N) \) contradicting density. Analogously, let \( ℋ^N = \{ Q \in ℋ | \psi(N) ≤ \phi(Q) \} \). Again by density, we have \( \psi(N) = \bigcup_{Q \in ℋ^N} \phi(Q) = \bigcup_{Q \in ℋ^N} \psi(Q) \).

In the previous notations we can now prove the main result of this section.

**Theorem 7.11.** Let C be a cotilting module over a valuation domain R. Then \( ^⊥C \) is a Tor-orthogonal class.

Proof. If C is non dense, the conclusion follows by Lemma 7.6.

Let J be the totally ordered set of intervals of prime ideals obtained from \( ℤ(C) \) as constructed in Notation 7.7 (††). By Fact 7.8 there is an injective homological ring epimorphism \( f: R \to S \) such that \( ℤ(f) = J \) and the corresponding cotilting module \( D = S^∗ \oplus (S/R)^∗ \) satisfies \( ℤ(D) = J \) and \( ^⊥D = (S/R)^∗ \).

Our aim is to prove:

\[ ^⊥C = \left( \bigoplus_{L \in G'} \frac{R_{ψ(L)}}{φ(L)} \oplus S/R \right)^∗. \]

The claim will be proved by several steps.

First of all note that by Proposition 4.1 and Lemma 7.5 it is enough to show that the two classes contain the same cyclic modules.

CLAIM (i) Let \( 0 \neq I < R \) and \( L \in G' \). Then, \( \text{Tor}_1^R \left( \frac{R_{ψ(L)}}{φ(L)}, R/I \right) = 0 \) if and only if either \( I ≥ φ(L) \) or if \( I ≤ φ(L) \), then \( I^♯ ≤ φ(L) \) and \( I \not∈ R_{φ(L)}^∗ \).

In fact,

\[ \text{Tor}_1^R \left( \frac{R_{ψ(L)}}{φ(L)}, R/I \right) \cong \text{Tor}_1^{R_{ψ(L)}} \left( \frac{R_{ψ(L)}}{φ(L)}, R_{ψ(L)} \otimes R/I \right), \]

since \( \frac{R_{ψ(L)}}{φ(L)} \) is an \( R_{ψ(L)}^∗ \)-module and \( \text{Tor}_1^V (V/J, V/K) \cong (J ∩ K)/KI \), for every valuation domain V and ideals K, J of V. (Recall that, for every ideal K of a valuation domain V, \( K^♯ K < K \) if and only if \( K \cong V_{K^♯} \).

CLAIM (ii)

\[ ^⊥C \subseteq \left( \bigoplus_{L \in G'} \frac{R_{ψ(L)}}{φ(L)} \oplus S/R \right)^∗. \]
Let \( R/I \in \perp C \). By Proposition 4.9 there is \( L \in \mathcal{G}' \) such that \( I \) and every \( r^{-1}I, r \notin I \) belong to \( (\phi(L), \psi(L)) \). Thus \( I \) and \( r^{-1}I, r \notin I \) belong to \( (\phi(L_0), \psi(L_1)) \) where \( [\phi(L_0), \psi(L_1)] \) is the interval of \( \mathcal{J} \) corresponding to the equivalence class \( L \). By Proposition 4.9 again \( R/I \in \perp D = (S/R)^\top \). Moreover, if \( I \leq \phi(N) \) for some \( N \in \mathcal{G}' \), then \( \psi(L) \leq \phi(N) \), hence \( R/I \in \left( \bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \right)^\top \) by (i).

(claim (iii))

\[
\left( \bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \oplus S/R \right)^\top \subseteq \perp C.
\]

Let \( R/I \) belong to the left hand side of the above inclusion. Since \( R/I \in (S/R)^\top = \perp D \), there is an interval \([\phi(L_0), \psi(L_1)]\) of \( \mathcal{J} \) such that \( \phi(L_0) \leq I \leq I^\# \leq \psi(L_1) \). Now \( R/I \) belongs also to \( \left( \bigoplus_{L \in \mathcal{G}'} \frac{R_{\psi(L)}}{\phi(L)} \right)^\top \).

As in Lemma 7.10 consider the set \( \mathcal{H} \) of prime ideals \( N \in \mathcal{G}' \) such that \( \phi(L_0) < \phi(N) \) and \( \psi(N) < \psi(L_1) \). Let \( N_0 = \sup \{ \phi(N) \in \mathcal{H} \mid \phi(N) \leq I \} \). Then, by Lemma 4.7 \( \phi(N_0) = N_0 \leq I \). If \( \phi(L_1) \leq I \), then we conclude that \( I \in (\phi(L_1), \psi(L_1)) \), hence \( R/I \in \perp C \). Otherwise, the set \( T = \{ Q \in \mathcal{H} \mid I < \phi(Q) \} \) is non empty and by (i) \( I^\# \leq \phi(Q) \) for every \( Q \in T \). Then \( I^\# \leq \bigcap_{Q \in T} \phi(Q) \) and by Lemma 7.10 \( \bigcap_{Q \in T} \phi(Q) = \psi(N_0) \) since \( T \) coincides with the set \( \{ Q \in \mathcal{H} \mid \psi(N_0) \leq \phi(Q) \} \). Hence \( I \in (\phi(N_0), \psi(N_0) \) and again \( R/I \in \perp C \).


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