The heart of a t-structure induced by an n-tilting module

Silvana Bazzoni

Università di Padova
bazzoni@math.unipd.it

1 Introduction

The notion of a t-structure in a triangulated category was introduced by Beilinson, Bernstein and Deligne in [**1**]. A t-structure is a pair of full subcategories satisfying suitable axioms which guarantee that their intersection is an abelian category $\mathcal{H}$, called the heart of the t-structure.

From the memoire [**2**] it is known that every torsion pair in a Grothendieck category $\mathcal{A}$ induces a t-structure in the unbounded derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$.

In particular, every (infinitely generated) 1-tilting module T over a ring R induces torsion pair, hence a t-structure in $\mathcal{D}(R)$ such that $\mathcal{D}(\mathcal{H})$ is triangle equivalent to $\mathcal{D}(R)$. Moreover, if T is finitely generated, then the heart $\mathcal{H}$ is equivalent to the module category over the endomorphism ring of T.

We describe the t-structure induced by an infinitely generated n-tilting module and we ask when its heart is a Grothendieck category.

In a recent paper [**7**] Parra and Saorín proved that, if T is a 1-tilting module, then the heart of the induced t-structure is a Grothendieck category if and only if the torsion free class associated to T is closed under direct limits.

We prove that the latter condition holds if and only if T is a pure projective 1-tilting module. Moreover, we show that if R is a commutative ring, then T is projective, so equivalent to a finitely generated tilting module. We also exhibit a class of rings for which there exist pure projective 1-tilting modules which are not equivalent to finitely generated tilting modules.

2 Definitions and notations

Definition 1 (Beilinson, Bernstein, Deligne '82) A t-structure in a triangulated category $(\mathcal{D}, [-])$ is a pair $(\mathcal{U}, \mathcal{V})$ of subcategories such that:

(1) $\mathcal{U}[1] \subseteq \mathcal{U}$;
(2) $\mathcal{V} = \mathcal{U}^{\perp}[1]$, where $\mathcal{U}^{\perp} = \{Y \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, Y) = 0\}$;
(3) for every object $D \in \mathcal{D}$ there is a triangle $U \to D \to Y \to U[1]$ with $U \in \mathcal{U}$ and $Y \in \mathcal{U}^{\perp}$.

A t-structure plays the rôle of a torsion pair in a triangulated category.

Theorem 2 ([**1**]) *The heart $\mathcal{H} = \mathcal{U} \cap \mathcal{V}$ of a t-structure $(\mathcal{U}, V)$ is an abelian category.*

We are interested in the t-structures in the unbounded derived category of a ring R induced by n-tilting modules. To this aim we recall the notion of tilting modules.

Definition 3 An R-module T is n-tilting if:

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(T1) p.dim.$T \leq n$;
(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$, for all $i \geq 1$, for all sets I;
(T3) there exists an exact sequence $0 \to R \to T_0 \to \cdots \to T_r \to 0$
where $T_i \in \mathrm{Add}(T)$, i.e. they are direct summands of direct sums of copies of T.

If T is finitely generated then it is called *classical*.

The class $T^\perp = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i(T, M) = 0, \forall i \geq 1\}$ is called the tilting class.

If T is a 1-tilting module, then $T^\perp$ coincides with the class of modules generated by T and it is a torsion class.

3 The t-structure induced by an n-tilting module

Every n-tilting R-module T gives rise to a t-structure $(\mathcal{U}, \mathcal{U}^\perp[1])$ in $\mathcal{D}(R)$, which can be better described using the tool of model structures (see e.g. [**5**]).

The starting point is to note that if T is an n-tilting module, then $({}^\perp\mathcal{T}, \mathcal{T})$ is a complete cotorsion pair, that is a pair of mutually orthogonal classes with respect to the Ext-functor providing for approximations.

By Hoevey's Theorems [**6**] there is a model structure on $\mathcal{C}h(R)$ corresponding to the tilting cotorsion pair.

The fibrant objects of this model structure can be explicetly described. They are exactly the complexes quasi isomorphic to complexes with terms in $\mathcal{T}$, and if

$$\mathrm{Hom}_{\mathcal{D}(R)}(T[i], X) = 0, \text{for all } i < 0,$$

then X is quasi isomorphic to $\cdots \to X^{-n} \to \cdots \to X^{-1} \to X^0 \to 0$, with $X^i \in \mathcal{T}$, for all i. Thus, if we let:

- $\mathcal{U} = \{X \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(T[i], X) = 0, \text{for all } i < 0\}$.
- $\mathcal{U}^\perp[1] = \{Y \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(T[i], Y) = 0, \text{for all } i > 0\}$.
- The pair $(\mathcal{U}, \mathcal{U}^\perp[1])$ in $\mathcal{D}(R)$ is a t-structure whose heart is $\mathcal{H} = \{Z \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(T[i], Y) = 0, \text{for all } i \neq 0\}$.

Remark 4 If T is a 1-tilting module, the above t-structure coincides with the t-structure induced by the torsion pair associated to T as in definition by Happel, Reiten, Smalø in [**2**]. In this case the heart of the t-structure can be easily described: It consists of the complexes Z such that $\mathrm{H}^i(Z) = 0$ for $i \neq 0, -1$, $\mathrm{H}^0(Z)$ is a torsion object and $\mathrm{H}^{-1}(Z)$ is torsion free.

Facts

- T is a projective generator of the abelian category $\mathcal{H}$.
- If T is a classical n-tilting module, then the t-structure is compactly generated (T is a compact object in $\mathcal{D}(R)$) and the heart is a Grothendieck category, even a module category equivalent to Mod-$\mathrm{End}_{\mathcal{H}}(T)$.

Our main concern is to answer the following:

Question 5 When is the heart of the t-structure induced by an n-tilting module a Grothendieck category?

For the 1-tilting case we have a complete answer.

4 The 1-tilting case

Theorem 6 (Parra-Saorín [7]) *The heart $\mathcal{H}$ of the t-structure induced by a 1-tilting module T is a Grothendieck category if and only if the torsion free class $\mathcal{F}$ in the torsion pair corresponding to T is closed under direct limits.*

Question 7 (Saorín's Question) If T is a 1-tilting module such that the heart of the t-structure induced by T is Grothendieck, is T equivalent to a classical tilting module?

(Two tilting modules T, T' are equivalent if they define the same tilting class.)

A first key result to help answering the above question is given by:

Proposition 8 *Let T a 1-tilting module and let $\mathcal{F}$ be the associated torsion free class. Then $\mathcal{F}$ is closed under direct limits if and only if T is a pure projective module, i.e. T is a direct summand of a direct sum of finitely presented modules.*

Proposition 9 *Let T be a pure projective 1-tilting module. Up to equivalence $T \underset{\oplus}{\leq} \bigoplus_{n\in\mathbb{N}} A_n$ with A_n finitely presented modules of p.d≤ 1 and such that $T^{\perp} = \{A_n\}^{\perp}$.*

An important result about direct sums decomposition which is very useful in our context is the following famous theorem.

Theorem 10 (Azumaya, Crawley-Jónsson, Warfield) *If $M = \oplus_{i\in I} M_i$, M_i indecomposable countably generated with local endomorphism ring, then any other decomposition of M refines to a decomposition isomorphic to this. In particular, any direct summand of M is a direct sum of modules, each isomorphic to one of the summands M_i.*

Applying the above theorem we can solve our problem in the case of a commutative bring R.

Proposition 11 *If R is a commutative ring and T is a pure projective 1-tilting module, then T is projective, thus equivalent to R.*

In general the answer to Saorin's question is negative. In fact,

Proposition 12 *There exist pure projective 1-tilting modules not equivalent to classical tilting modules.*

Sketch of the construction.

- Let R be a nearly simple uniserial domain, that is a uniserial domain if only one non trivial two-sided ideal, which is $J(R)$.
- The ring for which there is a pure projective tilting module not equivalent to a classical one will be

$$S = \mathrm{End}(R/aR),\ 0 \neq a \in J(R).$$

If R is a nearly simple uniserial domain, then:

- For $0 \neq a, b \in J(R)$, $R/aR \cong R/bR$.
- For every $X = R/aR$ and every finitely generated $A < X$, $A \cong X \cong X/A$.
- Fix X and exact sequence $0 \to X \xrightarrow{f} X \xrightarrow{g} X \to 0$.
- Let $S = \mathrm{End}_R(X)$

Then:

(1) S has only 3 non trivial two-sided ideals which are all idempotent, namely:

(2) I the ideal of non-monomorphisms;
(3) K the ideal of non-epimorphisms;
(4) the Jacobson radical J which coincides with IK.

Facts

- $I_S = gS$ is uniserial; (${}_SK = Sf$ is uniserial.)
- ${}_SI$ is not finitely generated; (K_S is not finitely generated.)
- S/I (S/K) is a division ring;
- the simple left S-module S/I is flat, but not injective (${}_SS/K$ is injective, but not flat).

Thus:

(1) ${}_SI$ is pure in S and a Mittag-Leffler module; $J = IK = If$.
(2) the finitely presented S-modules are (finite) direct sums of ${}_SS$, ${}_SK = Sf \cong S/Sg$, S/K.
(3) p.d. $K = 1$, $Sg \cong S$.

Theorem 13 *the class*

$$\mathcal{T}_I = \{{}_SM \mid IM = M\} = \{{}_SM \mid gM = M\} = (S/Sg)^{\perp}$$

is a tilting torsion class whose corresponding torsion free class is

$$\mathcal{F}_I = \{{}_SY \mid IY = 0\} = (S/I)\text{-Mod}$$

which is also a torsion class.

$\mathcal{F}_I$ is closed under direct limits (even epimorphic images)

T_I is not the tilting class of a finitely generated tilting module since the finitely presented left S-modules S, K are not in $\mathcal{T}_I$ and p.d.$S/K = 2$

(I is idempotent, hence $(\mathcal{T}_I, \mathcal{F}_I, \mathcal{Z}_I)$ is a TTF, that is a torsion torsion-free triple.

If $J(S)$ is countably generated (for instance if R is countable) then

$$I \oplus J = T \text{ is a tilting module such that } T^{\perp} = \mathcal{T}_I.$$

- I is countably generated projective and J is pure projective.
- I has no finitely generated summands.
- The sequence $(*)$ is a special $\mathcal{T}_I$-preenvelope of S.

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