CONTRAMODULES OVER PRO-PERFECT TOPOLOGICAL RINGS, THE COVERING PROPERTY IN CATEGORICAL TILTING THEORY, AND HOMOLOGICAL RING EPIMORPHISMS

SILVANA BAZZONI AND LEONID POSITSELSKI

Abstract. For four wide classes of topological rings $\mathfrak{R}$, we show that all flat left $\mathfrak{R}$-contramodules have projective covers if and only if all flat left $\mathfrak{R}$-contramodules are projective if and only if all left $\mathfrak{R}$-contramodules have projective covers if and only if all the discrete quotient rings of $\mathfrak{R}$ are left perfect. The key technique on which the proofs are based is the contramodule Nakayama lemma for topologically T-nilpotent ideals. In the second half of the paper, we present applications of these results to tilting theory, pure module theory, and the Enochs conjecture. In the categorical n-tilting-cotilting correspondence situation, if A is a Grothendieck abelian category and the related abelian category B is equivalent to the category of contramodules over a topological ring $\mathfrak{R}$ belonging to one of our four classes, then the left tilting class is covering in A if and only if it is closed under direct limits in A, and if and only if the topological ring $\mathfrak{R}$ is pro-perfect. We also prove that all the discrete quotient rings of the topological ring of endomorphisms of a Σ-pure-split module are perfect, and discuss the relations of covers with direct limit closedness properties of the class $\mathsf{Add}(M)$ for a Σ-rigid or self-pure-projective module/object M. The example of the tilting object related to an injective ring epimorphism of projective dimension 1 is considered at the end.

Contents

Introduction

0.0. In a classical paper of Bass [5] (based on his Ph. D. dissertation), it was shown that every left module over an associative ring R has a projective cover if and only if every flat left R-module is projective. Such rings were called *left perfect*, and a number of further equivalent characterizations of them were provided in the paper. Many years later, Enochs conjectured [14], and subsequently Bican, El Bashir, and Enochs proved [10] that, over any associative ring, all modules have flat covers. This assertion became known as the "flat cover conjecture/theorem".

In the recent paper [35], coathored by the second-named author of the present paper, it is shown that, over any complete, separated topological associative ring with a countable base of neighborhoods of zero formed by open right ideals, all left contramodules have flat covers. This provides a contramodule analogue of the result of Bican, El Bashir, and Enochs. In fact, there are two proofs of the existence of flat covers in their paper [10], one following the approach of Bican and El Bashir, and the other one based on the results of Eklof and Trlifaj [13]. Both of these lines of argumentation are extended to contramodules over topological rings with a countable base of neighborhoods of zero in the paper [35].

0.1. In the present paper we extend the results of Bass' paper [5] to the contramodule realm. One reason why this is interesting is because perfect rings are relatively rare, while pro-perfect topological rings (for many of which we prove that all contramodules have projective covers and all flat contramodules are projective) are more numerous. In particular, our results apply to the contramodules over all commutative pro-perfect topological rings. This class of topological rings includes complete Noetherian local commutative rings and the S-completions of S-almost perfect commutative rings (as defined in our previous preprint [6]).

Here a *pro-perfect* topological ring is a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals such that all its discrete quotient rings are perfect. In fact, it is only because of possible problems with non-well-behaved uncountable projective limits that we do not claim applicability of our results to *all* pro-perfect topological rings. Such problems do not exist for topological rings with a countable base of neighborhoods of zero, and we find them manageable for commutative topological rings and topological rings having only a finite number of semisimple discrete quotient rings.

Thus the three classes of topological rings for which our main results hold are the complete, separated topological associative rings with a base of neighborhoods of zero formed by open two-sided ideals such that either (a) the ring is commutative, or (b) it has a countable base of neighborhoods of zero, or (c) it has only a finite number of semisimple discrete quotient rings.

The assumption of existence of a base of neighborhoods of zero formed by open two-sided ideals is somewhat restrictive, given that the definition of a left contramodule over a topological ring only requires a base of neighborhoods of zero consisting of open right ideals. Some, though not all, of our main results are applicable to topological rings with a base of neighborhoods of zero formed by open right ideals. In particular, we prove the existence of projective covers and projectivity of flat contramodules over many topological rings $\mathfrak{R}$ having a topologically T-nilpotent closed two-sided ideal $\mathfrak{H}$ such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ is a product of simple Artinian discrete rings (endowed with the product topology).

Here a closed ideal $\mathfrak{H}$ in a topological ring $\mathfrak{R}$ is said to be *topologically left T-nilpotent* if, for any sequence of elements a_1, a_2, a_3, … in $\mathfrak{H}$, the sequence of products a_1, a_1a_2, $a_1a_2a_3$, … converges to zero in $\mathfrak{R}$. Among out main technical tools, we present two versions of the Nakayama lemma for topologically T-nilpotent ideals. Firstly, a closed two-sided ideal $\mathfrak{H} \subset \mathfrak{R}$ is topologically left T-nilpotent if and only if any nonzero discrete right $\mathfrak{R}$-module has a nonzero submodule annihilated by $\mathfrak{H}$. Secondly, if $\mathfrak{H} \subset \mathfrak{R}$ is topologically left T-nilpotent and $\mathfrak{C}$ is a nonzero left $\mathfrak{R}$-contramodule, then the contraaction map $\mathfrak{H}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ is not surjective.

Moreover, there is a wider fourth class of topological rings $\mathfrak{R}$, containing the three classes (a), (b), and (c), for which we show all flat left $\mathfrak{R}$-contramodules have projective covers if and only if all flat left $\mathfrak{R}$-contramodules are projective if and only if all left $\mathfrak{R}$-contramodules have projective covers if and only if all the discrete quotient rings of $\mathfrak{R}$ are left perfect. This class (d) consists of complete, separated topological rings with a base of neighborhoods of zero formed by open right ideals having a topologically left T-nilpotent closed ideal $\mathfrak{H} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ is a topological product of topological rings satisfying (a), (b), or (c). Topological rings satisfying the condition (d) do not need to have a base of neighborhoods of zero formed by open two-sided ideals.

0.2. The present paper consists roughly of three parts, the first of them consisting of Sections 1–10, the second one of Sections 11–15, and the third one of Sections 16–19. The above discussion in this introduction covers the first part.

In the second part, we apply the results of the first part in order to study direct limit closedness and covering properties of classes of modules or, more generally, classes of objects in abelian categories of much more general nature than the class of all projective objects. In the third part, we specialize to some classes of modules related to a homological epimorphism of associative rings.

0.3. Enochs proved that a precovering class of modules closed under direct limits is covering [14, Theorems 2.1 and 3.1], and later asked the question whether any covering class of modules is closed under direct limits (see, e. g., [19, Section 5.4]; cf. [3, Section 5]). A hypothetical general positive answer to this question is sometimes called "the Enochs conjecture". A positive answer in many particular cases was obtained in [3, Theorem 5.2 and Corollary 5.5]. In Section 15 of the present paper, we essentially prove the following result (cf. Corollary 15.5 and its proof).

Let A be an associative ring and M be a left A-module. For any sequence of endomorphisms a_1, a_2, a_3, … of the A-module M, consider the left A-module

$$B = \varinjlim\,(M \xrightarrow{a_1} M \xrightarrow{a_2} M \xrightarrow{a_3} \cdots),$$

and assume that the related short exact sequence of left A-modules

$$0 \longrightarrow \bigoplus\nolimits_{n=1}^{\infty} M \longrightarrow \bigoplus\nolimits_{n=1}^{\infty} M \longrightarrow B \longrightarrow 0$$

remains exact after applying the functor $\mathrm{Hom}_A(M,-)$. Assume further that the topological ring $\mathfrak{R} = \mathrm{Hom}_A(M,M)^{\mathrm{op}}$ opposite to the ring of endomorphisms of the left A-module M satisfies one of the conditions (a), (b), (c), or (d) (e. g., if the ring $\mathfrak{R}$ is commutative, then the condition (a) is satisfied).

Let $\mathsf{Add}(M)$ denote the class of all direct summands of (infinite) direct sums of copies of M. Then the left A-module B has an $\mathsf{Add}(M)$-cover for every sequence of endomorphisms a_1, a_2, a_3, … of the A-module M if and only if the class of left A-modules $\mathsf{Add}(M)$ is closed under direct limits and if and only if all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.

0.4. In Section 11 we prove that, given a hereditary complete cotorsion pair (L,E) in an abelian category A, all the objects of A have L-covers if and only if all the objects of E have L-covers in A. In Section 12 we discuss the direct limit closedness properties of the n-tilting and n-cotilting classes in the n-tilting-cotilting correspondence context, as developed in the paper [36]. We show that, whenever the n-tilting cotilting correspondence connects a Grothendieck abelian category A with a certain abelian category B, the left tilting class L is closed under direct limits in A if and only if the class $\mathsf{Add}(T)$ for the tilting object $T \in \mathsf{A}$ is closed under direct limits and if and only if the class of all projective objects is closed under direct limits in B.

Assuming that B is the category of left contramdules over a topological associative ring $\mathfrak{R}$ (which is always the case, e. g., when A is the category of modules over an associative ring) and that the ring $\mathfrak{R}$ satisfies one of the conditions (a), (b), (c), or (d), we prove in Section 13 that the latter three equivalent conditions hold if and only if the class L is covering in A, if and only if the class $\mathsf{Add}(T)$ is covering in A, if and only if the class of projective objects is covering in B, and if and only if all the discrete quotient rings of $\mathfrak{R}$ are left perfect.

It is well-known that, given an n-tilting left module over an associative ring A, the left tilting class L is closed under direct limits in the category of left A-modules if and only if the left A-module T is Σ-pure-split [19, Proposition 13.55]. In Section 14 we show that, for any Σ-pure-split left A-module M, the class of all projective objects in the abelian category of left contramodules over the topological ring $\mathfrak{R} = \operatorname{Hom}_A(M, M)^{\mathrm{op}}$ is closed under direct limits. It follows that all the discrete quotient rings of $\mathfrak{R}$ are left perfect.

Sections 16–18 provide background material for Section 19. We start our discussion of ring epimorphisms in Section 16 with constructing the Matlis additive category equivalences [25, Theorems 3.4 and 3.8] for an associative ring epimorphism. Notice that Matlis category equivalences for certain noninjective epimorphisms of commutative rings were obtained in [32, Section 5], while for certain injective epimorphisms of noncommutative rings the first of two Matlis category equivalences was constructed in [15, Section 4]. We construct both the first and the second Matlis category equivalences in what we believe is the maximal natural generality of an associative ring epimorphism $u\colon R \longrightarrow U$ satisfying $\operatorname{Tor}_1^R(U, U) = 0$ (for the first equivalence) or $\operatorname{Tor}_1^R(U, U) = 0 = \operatorname{Tor}_2^R(U, U)$ (for the second one).

In the case when the ring R commutative, we also show in Section 16, using some results of Hrbek and Angeleri Hügel–Hrbek, that whenever u is a homological ring epimorphism and U is an R-module of projective dimension 1, it follows that U is actually a flat R-module. In the general context of an associative ring R, assuming that u is a homological ring epimorphism such that U has flat dimension at most 1 as a right R-module (resp., projective dimension at most 1 as a left R-module), we discuss the abelian category of *u-comodule* (resp., *u-contramodule*) left R-modules in Section 17. The former is defined as the full subcategory of all left R-modules annihilated by the derived functor $\operatorname{Tor}_{0,1}^R(U, -)$, while the latter is the Geigle–Lenzing right $\operatorname{Ext}_R^{0,1}$-perpendicular subcategory to U in the category of left R-modules. In the respective assumptions, we show that the u-comodules form a Grothendieck abelian category, while the abelian category of u-contramodules is locally presentable with a projective generator. We also discuss adjoint functors to the identity inclusions of these full subcategories into the category of left R-modules.

In Section 18 we construct a triangulated version of the Matlis category equivalences, as developed in the paper [32, Sections 4 and 6] for multiplicative subsets in commutative rings and in the recent preprint by Chen and Xi [12, Section 4.1] for homological epimorphisms of associative rings. Finally, in the last Section 19 we restrict our attention to injective homological ring epimorphisms $u\colon R \longrightarrow U$ and discuss the related 1-tilting-cotilting correspondence situations and their direct limit closedness and covering properties.

0.5. Specifically, we show that, when U has projective dimension at most 1 as a left R-module and flat dimension at most 1 as a right R-module, the quotient module $K = U/R$ is a 1-tilting object in the abelian category of left u-comodules A, and the related abelian category B is the category of left u-contramodules or, which is the same, the category of left contramodules over the topological ring $\mathfrak{R} = \operatorname{Hom}_R(K, K)^{\mathrm{op}}$ opposite to the endomorphism ring of the left R-module K. Assuming that the topological ring $\mathfrak{R}$ satisfies one of the conditions (a), (b), (c), or (d) (e. g., this holds when the ring R is commutative, as the ring $\mathfrak{R}$ is then commutative, too) and denoting by L the left 1-tilting class in A, we observe that the following conditions are equivalent. Every left R-module has an L-cover if and only if every left R-module has an $\mathsf{Add}(K)$-cover if and only if the class of left R-modules L is closed under direct limits if and only if the class $\mathsf{Add}(K)$ is closed under direct limits, and if and only if all the discrete quotient rings of $\mathfrak{R}$ are left perfect.

Furthermore, it is well-known [2] that the left R-module $U\oplus K$ is 1-tilting whenever U has projective dimension at most 1 as a left R-module. Denoting by N the related left tilting class in the category of left R-modules and assuming that the topological ring $\mathfrak{R}$ satisfies one of the conditions (a), (b), (c), or (d), we show that all left R-modules have N-covers if and only if the class of left R-modules N is closed under direct limits and if and only if *both* all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect *and* the associative ring U is left perfect. In particular, these three conditions are equivalent if the ring R is commutative. The point is that whenever the topological ring $\mathfrak{R} = \mathrm{Hom}_R(K,K)^{\mathrm{op}}$ satisfies (a), (b), (c), or (d), the topological ring $\mathfrak{S} = \mathrm{Hom}_R(U\oplus K,\, U\oplus K)^{\mathrm{op}}$ satisfies (d).

Acknowledgment. The first-named author is partially supported by grants BIRD163492 and DOR1690814 of Padova University. The second-named author's research is supported by the Israel Science Foundation grant # 446/15.

1. Preliminaries on Topological Rings

The material in Sections 1.1–1.4 below is well-known (some relevant references are [42, Chapter VI] and [8]). Sections 1.5–1.10 go back to [27, Remark A.3] and [28, Section 1.2]; for later expositions, see [30, Sections 2.1 and 2.3] and [35, Sections 1.2 and 5]. The material of Sections 1.11–1.12 may be somewhat new (it is implicit in [28, Sections 1.3 and B.4]).

Throughout this paper, by "direct limits" in a category we mean inductive limits indexed by directed posets. Otherwise, these are known as the directed or filtered colimits.

1.1. **Abelian groups with linear topology.** A "topological abelian group" in this paper will always mean a topological abelian group with a base of neighborhoods of zero formed by open subgroups. Any collection of subgroups B in an abelian group A such that for any two subgroups $U'\in B$ and $U''\in B$ there exists a subgroup $U\in B$, $U\subset U'\cap U''$, forms a base of neighborhoods of zero in a (uniquely defined) topology on A compatible with the abelian group structure.

The *completion* $\mathfrak{A} = A^{\frown}_B$ of a topological abelian group A is the abelian group $\mathfrak{A} = \varprojlim_{U\in B} A/U$ endowed with the *projective limit topology*, in which a base of neighborhoods of zero $\mathfrak{B}$ in $\mathfrak{A}$ is formed by the kernels $\mathfrak{U} = U^{\frown}$ of the projection maps $\mathfrak{A}\longrightarrow A/U$. The topological group $\mathfrak{A} = A^{\frown}_B$ does not depend on the choice of a particular base of neigborhoods of zero B in a topological group A.

A topological abelian group A is said to be *separated* if the completion map $A \longrightarrow \mathfrak{A}$ is injective, and *complete* if it is surjective. The projection maps $\mathfrak{A} \longrightarrow A/U$ induce isomorphisms $\mathfrak{A}/\mathfrak{U} \longrightarrow A/U$, so the completion $\mathfrak{A}$ of a topological abelian group A is always separated and complete (in the projective limit topology of $\mathfrak{A}$). The completion map $A \longrightarrow \mathfrak{A}$ is an isomorphism of topological abelian groups (i. e., an additive homeomorphism) whenever A is separated and complete.

1.2. **Subgroup and quotient group topologies.** Let A be a topological abelian group and $A' \subset A$ be a subgroup. Then the group A' can be endowed with the topology induced from A. If B is a collection of subgroups forming a base of neighborhoods of zero in A, then the collection of subgroups $B' = \{A' \cap U \mid U \in B\}$ forms a base of neighborhoods of zero in A'.

Lemma 1.1. (a) *If a topological abelian group A is separated, then any subgroup $A' \subset A$ is separated in the induced topology.*

(b) *Let $\mathfrak{A}$ be a complete, separated topological abelian group. Then a subgroup $\mathfrak{A}' \subset \mathfrak{A}$ is complete in the induced topology if and only if it is closed in $\mathfrak{A}$.*

(c) *More generally, if $\mathfrak{A}$ is separated and complete, then for any subgroup $A' \subset \mathfrak{A}$ the induced map between the completions $A'^{\frown} \longrightarrow \mathfrak{A}^{\frown} = \mathfrak{A}$ provides a topological group isomorphism of the completion $A'^{\frown}$ of the group A' with the closure $\mathfrak{A}' \subset \mathfrak{A}$ of the subgroup $A' \subset \mathfrak{A}$ (where the topology on A' and $\mathfrak{A}'$ is induced from $\mathfrak{A}$).* $\square$

Let A be a topological abelian group, $A' \subset A$ be a subgroup, and $A'' = A/A'$ be the quotient group. Then the group A'' can be endowed with the quotient topology, in which a subset of A'' is open if and only if its preimage in A is. If B is a collection of subgroups forming a base of neighborhoods of zero in A, then the collection of subgroups $B'' = \{(A' + U)/A' \mid U \in B\}$ forms a base of neighborhoods of zero in A''.

Lemma 1.2. (a) *If a topological abelian group A is separated, then the quotient group A/A' is separated in the quotient topology if and only if the subgroup A' is closed in A.*

(b) *Let $\mathfrak{A}$ be a topological abelian group with a countable base of neighborhoods of zero formed by open subgroups, and let $\mathfrak{A}' \subset \mathfrak{A}$ be a closed subgroup. Then the quotient group $\mathfrak{A}'' = \mathfrak{A}/\mathfrak{A}'$ is complete in the quotient topology.*

Proof. We will only prove part (b). Let $\mathfrak{B}$ be a countable base of neighborhoods of zero consisting of open subgroups in $\mathfrak{A}$. Then for any $\mathfrak{U} \in \mathfrak{B}$ we have a short exact sequence of abelian groups

$$0 \longrightarrow \mathfrak{A}'/\mathfrak{U}' \longrightarrow \mathfrak{A}/\mathfrak{U} \longrightarrow \mathfrak{A}''/\mathfrak{U}'' \longrightarrow 0,$$

where $\mathfrak{U}' = \mathfrak{A}' \cap \mathfrak{U}$ and $\mathfrak{U}'' = (\mathfrak{A}' + \mathfrak{U})/\mathfrak{A}'$. Now $(\mathfrak{A}'/\mathfrak{U}')_{\mathfrak{U} \in \mathfrak{B}}$ is a projective system of abelian groups and surjective morphisms between them, indexed by a countable directed set $\mathfrak{B}$. Hence the sequence remains exact after the passage to the projective limits,

$$0 \longrightarrow \mathfrak{A}'^{\frown}_{\mathfrak{B}'} \longrightarrow \mathfrak{A}^{\frown}_{\mathfrak{B}} \longrightarrow \mathfrak{A}''^{\frown}_{\mathfrak{B}''} \longrightarrow 0,$$

where $\mathfrak{B}' = \{\mathfrak{U}' \mid \mathfrak{U} \in \mathfrak{B}\}$ and $\mathfrak{B}'' = \{\mathfrak{U}'' \mid \mathfrak{U} \in \mathfrak{B}\}$. By assumption, we have $\mathfrak{A} = \mathfrak{A}^{\frown}_{\mathfrak{B}}$; by Lemma 1.1(b), $\mathfrak{A}' = \mathfrak{A}'^{\frown}_{\mathfrak{B}'}$. Thus the completion map $\mathfrak{A}'' \longrightarrow \mathfrak{A}''^{\frown}_{\mathfrak{B}''}$ is bijective, that is, the topological abelian group $\mathfrak{A}''$ is (separated and) complete. $\square$

We are not aware of any counterexamples to the assertion of Lemma 1.2(b) in the case of an uncountable base of neighborhoods of zero, but there seems to be no reason for it to be true in full generality. In a different context of topological vector spaces over the field of real numbers (in its real topology), such a counterexample is suggested in [11, Exercise IV.4.10(b)].

1.3. **Topological rings.** In this paper, the word "ring" means "an associative ring with unit" by default. Unless otherwise mentioned, all ring homomorphisms are supposed to preserve units, and all modules are presumed to be unital. When considering rings without unit or subrings without unit, as we will at some point in Section 4, we will always explicitly refer to them as being "without unit".

Given an (associative and unital) ring R, we denote the abelian category of (arbitrary unital) left R-modules by R–mod and the abelian category of right R-modules by mod–R. The ring with the opposite multiplication to a ring R is denoted by R^{op}.

In this paper we are interested in topological associative rings R such that open right ideals $I \subset R$ form a base of neighborhoods of zero in R. A collection of right ideals B in a ring R forms a base of neighborhoods of zero in a topology compatible with the ring structure on R if and only if it satisfies the following conditions (cf. [42, Section VI.4] and [8, Remark 1.1(ii), Claim 1.4, and Lemma 1.4]):

(i) for any two right ideals I' and $I'' \in B$, there exists a right ideal $J \in B$ such that $J \subset I' \cap I''$; and
(ii) for any right ideal $I \in B$ and any element $r \in R$, there exists a right ideal $J \in B$ such that $rJ \subset I$.

The *completion* $\mathfrak{R} = R^{\frown}_B$ of a topological ring R is the abelian group $\mathfrak{R} = \varprojlim_{I \in B} R/I$ endowed with the projective limit topology (in which a base of neighborhoods of zero $\mathfrak{B}$ in $\mathfrak{R}$ is formed by the kernels $\mathfrak{I} = I^{\frown} \subset \mathfrak{R}$ of the projection maps $\mathfrak{R} \longrightarrow R/I$). One readily checks, using the conditions (i) and (ii), that there exists a unique associative ring structure on $\mathfrak{R}$ that is continuous with respect to the projective limit topology and such that the natural map $R \longrightarrow \mathfrak{R}$ is a ring homomorphism. Given two elements $r' = (r'_I)_{I \in B}$ and $r'' = (r''_I)_{I \in B} \in \mathfrak{R}$, in order to compute the I-component r_I of the product $r = (r_I)_{I \in B} \in \mathfrak{R}$, one has to find an open right ideal $J \in B$ such that $J \subset I$ and $r'_I J \subset I$; then one can set $r_I = r'_I r''_J + I$.

The open subsets $\mathfrak{I} = I^{\frown}$ are right ideals in $\mathfrak{R}$, so the topological ring $\mathfrak{R}$ has a base of neighborhoods of zero consisting of open right ideals. When the base of neighborhoods of zero B in R consists of open two-sided ideals, the topological ring $\mathfrak{R}$ can be simply defined as the projective limit of (discrete) rings R/I. Then the open subsets $\mathfrak{I} = I^{\frown} \subset \mathfrak{R}$ are two-sided ideals.

A topological ring R is said to be *separated* (resp., *complete*) if it is separated (resp., complete) as a topological abelian group.

1.4. **Discrete modules.** Let R be a topological ring with a base of neighborhoods of zero formed open right ideals. A right R-module $\mathcal{N}$ is said to be *discrete* if for every element $b \in \mathcal{N}$ the annihilator $\mathrm{Ann}_R(b) = \{r \in R \mid br = 0\}$ is an open right ideal in $\mathfrak{R}$. The annihilator of an element in a right R-module is always a right ideal in R, so topological rings with a base of neighborhoods of zero formed by open right ideals are a natural setting for considering discrete right modules.

The full subcategory of discrete right R-modules $\mathsf{discr}{-}R \subset \mathsf{mod}{-}R$ is a hereditary pretorsion class in the abelian category of right R-modules $\mathsf{mod}{-}R$, and all hereditary pretorsion classes in $\mathsf{mod}{-}R$ appear in this way [42, Lemma VI.4.1 and Proposition VI.4.2]. Viewed as an abstract category, the category $\mathsf{discr}{-}R$ is a Grothendieck abelian category. So, in particular, the abelian category $\mathsf{discr}{-}R$ is complete and cocomplete, has exact direct limits, and an injective cogenerator.

One readily checks that any discrete right R-module has a unique discrete right $\mathfrak{R}$-module structure compatible with its R-module structure (where $\mathfrak{R} = R\widehat{\ }$ denotes the completion of the topological ring R). Thus the abelian categories of discrete right R-modules and discrete right $\mathfrak{R}$-modules are naturally equivalent (in fact, isomorphic), $\mathsf{discr}{-}R \cong \mathsf{discr}{-}\mathfrak{R}$.

1.5. **Convergent formal linear combinations.** Given an abelian group A and a set X, we denote by $A[X] = A^{(X)}$ the direct sum of X copies of the abelian group A, viewed as the group of all finite formal linear combinations $\sum_{x\in X} a_x x$ of elements of X with the coefficients in A. A formal linear combination $\sum_{x\in X} a_x x$ belongs to $A[X]$ if and only if the set of all indices $x \in X$ for which $a_x \neq 0$ is finite.

Given a separated and complete topological abelian group $\mathfrak{A}$ with a base of neighborhoods of zero $\mathfrak{B}$ consisting of open subgroups, and a set X, we denote by $\mathfrak{A}[[X]]$ the abelian group $\varprojlim_{\mathfrak{U}\in\mathfrak{B}} (\mathfrak{A}/\mathfrak{U})[X]$. Clearly, the group $\mathfrak{A}[[X]]$ does not depend on the choise of a particular base of neighborhoods of zero $\mathfrak{B}$ in $\mathfrak{A}$. We interpret $\mathfrak{A}[[X]]$ as the group of all infinite formal linear combinations $\sum_{x\in X} a_x x$ of elements of X with the coefficients in $\mathfrak{A}$ forming an *X-indexed family of elements in $\mathfrak{A}$ converging to zero in the topology of* $\mathfrak{A}$. This means that the subgroup $\mathfrak{A}[[X]] \subset \mathfrak{A}^X$ consists of all the infinite formal linear combinations $\sum_{x\in X} a_x x$ such that, for every open subgroup $\mathfrak{U} \subset \mathfrak{A}$, one has $a_x \in \mathfrak{U}$ for *all but a finite subset of indices $x \in X$*.

The map assigning to a set X the abelian group $\mathfrak{A}[[X]]$ extends naturally to a covariant functor from the category of sets to the category of abelian groups. Given a map of sets $f\colon X \longrightarrow Y$, one defines the induced map $\mathfrak{A}[[f]]\colon \mathfrak{A}[[X]] \longrightarrow \mathfrak{A}[[Y]]$ by the rule $\sum_{x\in X} a_x x \longmapsto \sum_{y\in Y}\bigl(\sum_{f(x)=y} a_x\bigr)y$, where the sum of elements a_x is the parentheses is understood as the limit of finite partial sums in the topology of $\mathfrak{A}$. Such a limit is unique and exists because the topological abelian group $\mathfrak{A}$ is separated and complete, while the family of elements $(a_x)_{x\in X}$, and consequently its subfamily indexed by all $x \in X$ with $f(x) = y$ for a fixed $y \in Y$, converges to zero in $\mathfrak{A}$.

1.6. **The monad structure.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Let us consider the functor $X \longmapsto \mathfrak{R}[[X]]$ as taking values in the category of sets; so it becomes an endofunctor $\mathbb{T}_{\mathfrak{R}} = \mathfrak{R}[[-]]\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$. The key observation is that the functor $\mathbb{T}_{\mathfrak{R}}$ has a natural structure of a *monad* on the category of sets [27, Remark A.3], [28, Section 1.2], [30, Section 2.1], [35, Sections 1.1–1.2 and 5], [33, Section 1].

This means that the functor $\mathbb{T}_{\mathfrak{R}}$ is endowed with natural transformations $\epsilon\colon \mathrm{Id} \longrightarrow \mathbb{T}_{\mathfrak{R}}$ and $\phi\colon \mathbb{T}_{\mathfrak{R}} \circ \mathbb{T}_{\mathfrak{R}} \longrightarrow \mathbb{T}_{\mathfrak{R}}$ satisfying the monad equations (of unitality and associativity). The monad unit $\epsilon_X\colon X \longrightarrow \mathfrak{R}[[X]]$ is the "point measure" map, assigning to an element $x_0 \in X$ the (finite) formal linear combination $\sum_{x\in X} r_x x \in \mathfrak{R}[[X]]$, where $r_{x_0} = 1$ and $r_x = 0$ for all $x \neq x_0$. The monad multiplication $\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ is the "opening of parentheses" map, assigning a formal linear combination to a formal linear combination of formal linear combination.

Given a set X and an element $\mathbf{r} \in \mathfrak{R}[[\mathfrak{R}[[X]]]]$, computing the element $\phi_X(\mathbf{r}) \in \mathfrak{R}[[X]]$ involves opening the parentheses, computing the products of pairs of elements in the ring $\mathfrak{R}$, and then computing the infinite sums. The coefficient t_x of $\phi(\mathbf{r}) = \sum_{x\in X} t_x x$ at an element $x \in X$ is an infinite sum of products of pairs of elements in $\mathfrak{R}$, understood as the limit of finite partial sums in the topology of $\mathfrak{R}$. Thus it is crucial for the definition of ϕ_X that $\mathfrak{R}$ is separated and complete, and that all the infinite sums involved converge. The latter is guaranteed by the assumption that open right ideals form a base of neighborhoods of zero in $\mathfrak{R}$.

Indeed, let Y denote the set $\mathfrak{R}[[X]]$; then we have $\mathbf{r} = \sum_{y\in Y} r_y y$ for some $r_y \in Y$, and $y = \sum_{x\in x} s_{y,x} x$ for all $y \in Y$ and some $s_{y,x} \in \mathfrak{R}$. For any $x \in X$, the coefficient t_x is to be computed as $t_x = \sum_{y\in Y} r_y s_{y,x} \in \mathfrak{R}$. In order to show that this sum converges in the topology of $\mathfrak{R}$, we have to check that, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, the product $r_y s_{y,x}$ belongs to $\mathfrak{I}$ for all but a finite set of indices $y \in Y$. Now, one has $r_y s_{y,x} \in \mathfrak{I}$ whenever $r_y \in \mathfrak{I}$; and there is only a finite set of indices y with $r_y \notin \mathfrak{I}$, because $\mathbf{r} \in \mathfrak{R}[[Y]]$. So the coefficient $t_x \in \mathfrak{R}$ is well-defined for every $x \in X$. In order to check that $\sum_{x\in X} t_x x \in \mathfrak{R}[[X]]$, that is $t_x \in \mathfrak{I}$ for all but a finite set of indices $x \in X$, one has to use the condition (ii) from Section 1.3.

1.7. **Contramodules.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. By the definition, a left $\mathfrak{R}$-contramodule is an algebra or module (depending on the terminology) over the monad $\mathbb{T}_{\mathfrak{R}}\colon X \longmapsto \mathfrak{R}[[X]]$ on the category of sets.

This means that a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is a set endowed with a map of sets $\pi_{\mathfrak{C}}\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$, called the *left contraaction map.* The map $\pi_{\mathfrak{C}}$ must satisfy the equations of *contraunitality*, telling that the composition $\pi_{\mathfrak{C}}\epsilon_{\mathfrak{C}}$ with the monad unit map $\epsilon_{\mathfrak{C}}$ is the identity map $\mathrm{id}_{\mathfrak{C}}$,

$$\mathfrak{C} \longrightarrow \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C},$$

and *contraassociativity*, asserting that the two maps $\mathfrak{R}[[\mathfrak{R}[[\mathfrak{C}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{C}]]$, one of which is the monad multiplication map $\phi_{\mathfrak{C}}$ and the other one is the map $\mathfrak{R}[[\pi_{\mathfrak{C}}]]$ induced by $\pi_{\mathfrak{C}}$, should have equal compositions with the contraaction map $\pi_{\mathfrak{C}}$,

$$\mathfrak{R}[[\mathfrak{R}[[\mathfrak{C}]]]] \rightrightarrows \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}.$$

We denote the category of left $\mathfrak{R}$-contramodules by $\mathfrak{R}\text{–}\mathsf{contra}$.

In particular, any associative ring R can be considered as a topological ring with the discrete topology. In this case, we have $R[[X]] = R[X]$, and a left R-contramodule is the same thing as a left R-module. So the above definition of an $\mathfrak{R}$-contramodule, restricted to the particular case when the topological ring $\mathfrak{R} = R$ is discrete, provides a fancy way to define the familiar notion of a module over an associative ring.

Now, for any complete, separated topological associative ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals, and for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, one can compose the contraaction map $\pi_{\mathfrak{C}}\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ with the identity embedding $\mathfrak{R}[\mathfrak{C}] \longrightarrow \mathfrak{R}[[\mathfrak{C}]]$ of the set of all finite formal linear combinations into the set of all convergent infinite ones. This defines a natural structure of an algebra/module over the monad $X \longmapsto \mathfrak{R}[X]$ on the set $\mathfrak{C}$, which means a left $\mathfrak{R}$-module structure. Thus all left $\mathfrak{R}$-contramodules have underlying structures of left modules over the ring $\mathfrak{R}$, viewed as an abstract (nontopological) ring. We have constructed the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$. In particular, it means that all left $\mathfrak{R}$-contramodules, which were originally defined as only sets endowed with a contraaction map, are actually abelian groups.

The category $\mathfrak{R}\text{–}\mathsf{contra}$ is abelian [33, Lemma 1.1], and the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ is exact. The category $\mathfrak{R}\text{–}\mathsf{contra}$ is also complete and cocomplete, with the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ preserving infinite products (but not coproducts). Consequently, infinite products (but, generally speaking, not coproducts) are exact in $\mathfrak{R}\text{–}\mathsf{contra}$. Given two left $\mathfrak{R}$-contramodules $\mathfrak{C}$ and $\mathfrak{D}$, we denote by $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C},\mathfrak{D})$ the abelian group of morphisms $\mathfrak{C} \longrightarrow \mathfrak{D}$ in $\mathfrak{R}\text{–}\mathsf{contra}$.

For any set X, the map $\pi_{\mathfrak{R}[[X]]} = \phi_X$ endows the set/abelian group $\mathfrak{R}[[X]]$ with the structure of a left $\mathfrak{R}$-contramodule. It is called the *free* left $\mathfrak{R}$-contramodule generated by a set X. For any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, morphisms $\mathfrak{R}[[X]] \longrightarrow \mathfrak{C}$ in the category $\mathfrak{R}\text{–}\mathsf{contra}$ correspond bijectively to maps of sets $X \longrightarrow \mathfrak{C}$,

$$\operatorname{Hom}^{\mathfrak{R}}(R[[X]],\mathfrak{C}) \cong \operatorname{Hom}_{\mathsf{Sets}}(X,\mathfrak{C}).$$

Hence free left $\mathfrak{R}$-contramodules are projective objects of the category $\mathfrak{R}\text{–}\mathsf{contra}$. There are also enough of them: for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, the contraaction map $\pi_{\mathfrak{C}}\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ is an $\mathfrak{R}$-contramodule morphism presenting $\mathfrak{C}$ as a quotient contramodule of the free left $\mathfrak{R}$-contramodule $\mathfrak{R}[[\mathfrak{C}]]$. So the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ has enough projectives, and a left $\mathfrak{R}$-contramodule is projective if and only if it is a direct summand of a free one.

1.8. **Contratensor product.** As above, we denote by $\mathfrak{R}$ a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Some of the simplest examples of non-free left $\mathfrak{R}$-contramodules are obtained by dualizing discrete right $\mathfrak{R}$-modules.

Let A be an associative ring, and let $\mathcal{N}$ be an A-$\mathfrak{R}$-bimodule whose right $\mathfrak{R}$-module structure is that of a discrete right $\mathfrak{R}$-module. Let V be a left A-module. Then the induced left $\mathfrak{R}$-module structure of the abelian group $\operatorname{Hom}_A(\mathcal{N},V)$ extends naturally to a left $\mathfrak{R}$-contramodule structure. Indeed, to construct a left $\mathfrak{R}$-contraaction map for the set $\mathfrak{D} = \operatorname{Hom}_A(\mathcal{N},V)$, consider an element $\mathbf{r} = \sum_{d\in\mathfrak{D}} r_d d \in \mathfrak{R}[[\mathfrak{D}]]$. Set $f = \pi_{\mathfrak{D}}(\mathbf{r})$ to be the left A-module map $f\colon \mathcal{N} \longrightarrow V$ taking any element $b \in \mathcal{N}$ to the element $\sum_{d\in\mathfrak{D}} d(br_d) \in V$,

$$\pi_{\mathfrak{D}}\left(\sum\nolimits_{d\in\mathfrak{D}} r_d d\right)(b) = \sum\nolimits_{d\in\mathfrak{D}} c(br_d),$$

where the sum in the right-hand side is finite, because one has $br_d = 0$ for all but a finite number of elements $d \in \mathfrak{D}$. Indeed, the annihilator ideal $\operatorname{Ann}_{\mathfrak{R}}(b) \subset \mathfrak{R}$ is open by assumption, and consequently, it has to contain the coefficient r_d for all but a finite number of elements $d \in \mathfrak{D}$.

Let $\mathcal{N}$ be a discrete right $\mathfrak{R}$-module and $\mathfrak{C}$ be a left $\mathfrak{R}$-contramodule. The *contratensor product* $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C}$ is an abelian group defined as the cokernel of (the difference of) a natural pair of abelian group homomorphisms

$$\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \rightrightarrows \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}.$$

Here one of the maps $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}$ is just the map $\mathcal{N} \otimes \pi_{\mathfrak{C}}$ induced by the contraaction map $\pi_{\mathfrak{C}} \colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$, while the other map is the composition $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathcal{N}[\mathfrak{C}] \longrightarrow \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}$, where (following our general notation system) $\mathcal{N}[\mathfrak{C}]$ denotes the group of all finite formal linear combinations of elements of $\mathfrak{C}$ with the coefficients in $\mathcal{N}$. The map $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathcal{N}[\mathfrak{C}]$, induced by the right action map $\mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{R} \longrightarrow \mathcal{N}$, is well-defined due to the assumption that $\mathcal{N}$ is a discrete right $\mathfrak{R}$-module. The map $\mathcal{N}[\mathfrak{C}] \longrightarrow \mathcal{N} \otimes_{\mathbb{Z}} \mathfrak{C}$ is just the obvious one, taking a finite formal linear combination $\sum_{c\in\mathfrak{C}} b_c c$, $b_c \in \mathcal{N}$, to the tensor $\sum_{c\in\mathfrak{C}} b_c \otimes c$.

For any A-$\mathfrak{R}$-bimodule $\mathcal{N}$ whose right $\mathfrak{R}$-module structure is that of a discrete right $\mathfrak{R}$-module, any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, and any abelian group V, there is a natural adjunction isomorphism of abelian groups [35, Section 5]

$$\operatorname{Hom}_A(\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C},\ V) \cong \operatorname{Hom}^{\mathfrak{R}}(\mathfrak{C}, \operatorname{Hom}_A(\mathcal{N}, V)).$$

The functor of contratensor product $\odot_{\mathfrak{R}} \colon \mathsf{discr}\text{–}\mathfrak{R} \times \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathbb{Z}\text{–}\mathsf{mod}$ preserves colimits (i. e., is right exact and preserves coproducts) in both its arguments. For any discrete right R-module $\mathcal{N}$ and any set X, there is a natural isomorphism of abelian groups

$$\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{R}[[X]] \cong \mathcal{N}[X].$$

1.9. **Change of scalars.** Let $f \colon \mathfrak{R} \longrightarrow \mathfrak{S}$ be a continuous homomorphism of complete, separated topological rings, each of them having a base of neighborhoods of zero formed by open right ideals. Then for any set X there is the induced map of sets/abelian groups $f[[X]] \colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{S}[[X]]$.

Let $\mathfrak{C}$ be a left $\mathfrak{S}$-contramodule. Composing the map $f[[\mathfrak{C}]] \colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{S}[[\mathfrak{C}]]$ with the contraaction map $\pi_{\mathfrak{C}} \colon \mathfrak{S}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$, we obtain a map $\mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ defining a left $\mathfrak{R}$-contramodule structure on the set $\mathfrak{C}$. We have constructed an exact, faithful functor of *contrarestriction of scalars* $f_\sharp \colon \mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ forming a commutative square diagram with the forgetful functors $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$, $\mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{S}\text{–}\mathsf{mod}$ and the restriction-of-scalars functor $\mathfrak{S}\text{–}\mathsf{mod} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$. The functor $f_\sharp$ also preserves infinite products.

The functor $f_\sharp$ has a left adjoint functor of *contraextension of scalars* $f^\sharp\colon \mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{S}\text{–}\mathsf{contra}$. To construct the functor $f^\sharp$, one can first define it on free left $\mathfrak{R}$-contramodules by the rule $f^\sharp(\mathfrak{R}[[X]]) = \mathfrak{S}[[X]]$ for all sets X, and then extend to a right exact functor on the whole category $\mathfrak{R}\text{–}\mathsf{contra}$. As any left adjoint functor, the functor $f^\sharp$ preserves coproducts.

Similarly, the map f endows any discrete right $\mathfrak{S}$-module with a discrete right $\mathfrak{R}$-module structure. In other words, the conventional functor of restriction of scalars $\mathsf{mod}\text{–}\mathfrak{S} \longrightarrow \mathsf{mod}\text{–}\mathfrak{R}$ takes discrete right S-modules to discrete right $\mathfrak{R}$-modules. So we have an exact, faithful functor of restriction of scalars $f_\diamond\colon \mathsf{discr}\text{–}\mathfrak{S} \longrightarrow \mathsf{discr}\text{–}\mathfrak{R}$. The functor $f_\diamond$ also preserves infinite coproducts.

As any colimit-preserving functor between Grothendieck abelian categories, the functor $f_\diamond$ has a right adjoint functor of *coextension of scalars* $f^\diamond\colon \mathsf{discr}\text{–}\mathfrak{R} \longrightarrow \mathsf{discr}\text{–}\mathfrak{S}$. The functor $f^\diamond$ is left exact and preserves products.

For any discrete right $\mathfrak{S}$-module $\mathcal{M}$ and any left $\mathfrak{R}$-contramodule $\mathfrak{C}$ there is a natural isomorphism of abelian groups

$$f_\diamond(\mathcal{M}) \odot_{\mathfrak{R}} \mathfrak{C} \cong \mathcal{M} \odot_{\mathfrak{S}} f^\sharp(\mathfrak{C}).$$

1.10. **Reductions modulo ideals.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

Let $\mathcal{N}$ be a discrete right $\mathfrak{R}$-module, and $\mathfrak{J} \subset \mathfrak{R}$ be a closed right ideal in $\mathfrak{R}$. Then we denote by $\mathcal{N}_{\mathfrak{J}} \subset \mathcal{N}$ the additive subgroup in $\mathcal{N}$ consisting of all the elements $b \in \mathcal{N}$ such that $br = 0$ for all $r \in \mathfrak{J}$. If $\mathfrak{H}$ is a closed two-sided ideal in $\mathfrak{R}$, then the subgroup $\mathcal{N}_{\mathfrak{H}}$ is an $\mathfrak{R}$-submodule in $\mathcal{N}$.

Let R be an associative ring, C be a left R-module, and $A \subset R$ be an additive subgroup in $\mathfrak{R}$. As usually, we will denote by $AC \subset C$ the subgroup in C spanned by all the elements ac with $a \in A$ and $c \in C$. Clearly, one has $AC = A(RC) = (AR)C$, where $AR \subset R$ is the right ideal generated by A. When $J \subset R$ is a left ideal, the subgroup JC is an R-submodule in C.

Let $\mathfrak{C}$ be a left $\mathfrak{R}$-contramodule, and $\mathfrak{A} \subset \mathfrak{R}$ be a closed additive subgroup in $\mathfrak{R}$. Then $\mathfrak{A}$ is a complete, separated topological abelian group in the topology induced from $\mathfrak{R}$, and for any set X the group $\mathfrak{A}[[X]]$ is a subgroup in $\mathfrak{R}[[X]]$. Following [27, Remark A.3], [28, Section 1.3], [29, Section D.1], [35, Section 5], we will denote by $\mathfrak{A} \leftthreetimes \mathfrak{C} \subset \mathfrak{C}$ the image of the map $\mathfrak{A}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ obtained by restricting the contraaction map $\mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ to the subgroup $\mathfrak{A}[[\mathfrak{C}]] \subset \mathfrak{R}[[\mathfrak{C}]]$. Clearly, one has $\mathfrak{A}\mathfrak{C} \subset \mathfrak{A} \leftthreetimes \mathfrak{C}$.

Let $\mathfrak{J} \subset \mathfrak{R}$ be a closed left ideal. Then the composition of the identity embedding $\mathfrak{R}[[\mathfrak{J}[[X]]]] \longrightarrow \mathfrak{R}[[\mathfrak{R}[[X]]]]$ with the map $\phi_X\colon \mathfrak{R}[[\mathfrak{R}[[X]]]] \longrightarrow \mathfrak{R}[[X]]$ takes values inside the subset $\mathfrak{J}[[X]] \subset \mathfrak{R}[[X]]$. It follows that, for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, the subgroup $\mathfrak{J} \leftthreetimes \mathfrak{C} \subset \mathfrak{C}$ is an $\mathfrak{R}$-subcontramodule in $\mathfrak{C}$.

For any closed right ideal $\mathfrak{J} \subset \mathfrak{R}$ and any set X, one has

$$\mathfrak{J} \leftthreetimes (\mathfrak{R}[[X]]) = \mathfrak{J}[[X]] \subset \mathfrak{R}[[X]].$$

Let $\mathfrak{J} \subset \mathfrak{R}$ be an open right ideal. Then the right $\mathfrak{R}$-module $\mathfrak{R}/\mathfrak{J}$ is discrete, and for any left $\mathfrak{R}$-contramodule there is a natural isomorphism of abelian groups

$$(\mathfrak{R}/\mathfrak{J}) \odot_{\mathfrak{R}} \mathfrak{C} \cong \mathfrak{C}/(\mathfrak{J} \leftthreetimes \mathfrak{C}).$$

In particular, for any set X one has $\mathfrak{R}[[X]]/(\mathfrak{J} \leftthreetimes \mathfrak{R}[[X]]) \cong (\mathfrak{R}/\mathfrak{J})[X]$ and therefore $\mathfrak{R}[[X]] \cong \varprojlim_{\mathfrak{J}} \mathfrak{R}[[X]]/(\mathfrak{J} \leftthreetimes \mathfrak{R}[[X]])$, where the projective limit is taken over all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$. It follows that the natural map

$$\mathfrak{P} \longrightarrow \varprojlim\nolimits_{\mathfrak{J}} \mathfrak{P}/(\mathfrak{J} \leftthreetimes \mathfrak{P})$$

is an isomorphism for every projective left $\mathfrak{R}$-contramodule $\mathfrak{P}$.

1.11. **Strongly closed subgroups.** Let $\mathfrak{A}$ be a complete, separated topological abelian group (with a base of neighborhoods of zero formed by open subgroups). Let $\mathfrak{H} \subset \mathfrak{A}$ be a closed subgroup. Then the quotient group $Q = \mathfrak{A}/\mathfrak{H}$ is separated by Lemma 1.2(a), but it does not seem to follow from anything that it needs to be complete. Let $\mathfrak{Q} = Q\hat{}$ be the completion of the topological group Q. Then he natural morphism $p\colon \mathfrak{A} \longrightarrow \mathfrak{Q}$ is the cokernel of the morphism $\mathfrak{H} \longrightarrow \mathfrak{A}$ in the category of complete, separated topological abelian groups. The group Q is a dense subgroup in $\mathfrak{Q}$, and $\mathfrak{H}$ is the kernel of p; but p needs not be surjective.

Given a set X, we have the induced map of sets/abelian groups $p[[X]]\colon \mathfrak{A}[[X]] \longrightarrow \mathfrak{Q}[[X]]$. The subgroup $\mathfrak{H}[[X]] \subset \mathfrak{A}[[X]]$ is the kernel of $p[[X]]$. But even if the map p is surjective (i. e., the topological group Q is complete), it does not seem to follow from anything that the map $p[[X]]$ is surjective. Essentially, the problem consists in the following: given an X-indexed family of elements in the group $\mathfrak{Q}$ converging to zero in the topology of $\mathfrak{Q}$, how to lift it to an X-indexed family of elements in the group $\mathfrak{A}$ converging to zero in the topology of $\mathfrak{A}$?

We will say that a closed subgroup $\mathfrak{H}$ in a complete, separated topological abelian group $\mathfrak{A}$ is *strongly closed* if the quotient group $\mathfrak{A}/\mathfrak{H}$ is complete and, for every set X, the induced map $\mathfrak{A}[[X]] \longrightarrow (\mathfrak{A}/\mathfrak{H})[[X]]$ is surjective. Clearly, any open subgroup in a complete, separated topological abelian group is strongly closed.

Lemma 1.3. *Let $\mathfrak{A}$ be a complete, separated topological abelian group with a countable base of neighborhoods of zero consisting of open subgroups. Then any closed subgroup in $\mathfrak{A}$ is strongly closed.* □

Lemma 1.4. *Let $\mathfrak{K} \subset \mathfrak{H} \subset \mathfrak{A}$ be two embedded closed subgroups in a complete, separated topological abelian group $\mathfrak{A}$. In this situation,*

(a) *if $\mathfrak{K}$ is strongly closed in $\mathfrak{A}$, then $\mathfrak{K}$ is strongly closed in $\mathfrak{H}$;*

(b) *if $\mathfrak{K}$ is strongly closed in $\mathfrak{A}$ and $\mathfrak{H}/\mathfrak{K}$ is strongly closed in $\mathfrak{A}/\mathfrak{K}$, then $\mathfrak{H}$ is strongly closed in $\mathfrak{A}$;*

(c) *if $\mathfrak{H}$ is strongly closed in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{K}$ is complete, then $\mathfrak{H}/\mathfrak{K}$ is strongly closed in $\mathfrak{A}/\mathfrak{K}$;*

(d) *if $\mathfrak{H}$ is strongly closed in $\mathfrak{A}$ and $\mathfrak{K}$ is strongly closed in $\mathfrak{H}$, then $\mathfrak{K}$ is strongly closed in $\mathfrak{A}$.* □

1.12. **Strongly closed two-sided ideals.** Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Let $\mathfrak{H} \subset \mathfrak{R}$ be a closed two-sided ideal. Then the quotient ring $S = \mathfrak{R}/\mathfrak{H}$ in its quotient topology is a separated topological ring with a base of neighborhoods of zero formed by open right ideals. Hence the completion $\mathfrak{S} = S^{\frown}$ is a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals.

The natural morphism $p\colon \mathfrak{R} \longrightarrow \mathfrak{S}$ is a continuous homomorphism of complete, separated topological rings with the kernel $\mathfrak{H}$, and the universal one with this property; but it needs not be surjective. If $\mathfrak{R}$ has a base of neighborhoods of zero consisting of open two-sided ideals, then so do S and $\mathfrak{S}$.

The abelian category of discrete right S-modules or, which is equivalent, discrete right $\mathfrak{S}$-modules is a full subcategory of the abelian category of discrete right $\mathfrak{R}$-modules. In other words, the exact functor $p_\diamond\colon \mathsf{discr}\text{–}\mathfrak{S} \longrightarrow \mathsf{discr}\text{–}\mathfrak{R}$ is fully faithful. The full subcategory $\mathsf{discr}\text{–}\mathfrak{S} \subset \mathsf{discr}\text{–}\mathfrak{R}$ is closed under arbitrary subobjects, quotient objects, and coproducts.

For any discrete right $\mathfrak{R}$-module $\mathcal{N}$, the discrete right $\mathfrak{R}$-module structure on the submodule $\mathcal{N}_\mathfrak{H}$ comes from a discrete right $\mathfrak{S}$-module structure. In other words, the discrete right $\mathfrak{R}$-module $\mathcal{N}_\mathfrak{H}$ belongs to the essential image of the functor $p_\diamond\colon \mathsf{discr}\text{–}\mathfrak{S} \longrightarrow \mathsf{discr}\text{–}\mathfrak{R}$. This is the maximal $\mathfrak{R}$-submodule in $\mathcal{N}$ with this property. The functor $\mathcal{N} \longmapsto \mathcal{N}_\mathfrak{H}$ is the coextension-of-scalars functor with respect to the morphism $p\colon \mathfrak{R} \longrightarrow \mathfrak{S}$, that is

$$p^\diamond(\mathcal{N}) \cong \mathcal{N}_\mathfrak{H} \qquad \text{for all } \mathcal{N} \in \mathsf{discr}\text{–}\mathfrak{R}.$$

Now let us assume that $\mathfrak{H} \subset \mathfrak{R}$ is a strongly closed two-sided ideal. Then surjectivity of the maps $p[[X]]\colon \mathfrak{R}[[X]] \longrightarrow \mathfrak{S}[[X]]$ for all sets X implies that the exact functor of contraextension of scalars $p_\sharp\colon \mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$ is fully faithful. So the abelian category $\mathfrak{S}\text{–}\mathsf{contra}$ is a full subcategory in the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$. One easily observes that $\mathfrak{S}\text{–}\mathsf{contra}$ is closed under arbitrary subobjects, quotient objects, and products in $\mathfrak{R}\text{–}\mathsf{contra}$.

For any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, the left $\mathfrak{R}$-contramodule structure of the quotient contramodule $\mathfrak{C}/(\mathfrak{H} \leftthreetimes \mathfrak{C})$ comes from a left $\mathfrak{S}$-contramodule structure. In other words, the left $\mathfrak{R}$-contramodule $\mathfrak{C}/(\mathfrak{H} \leftthreetimes \mathfrak{C})$ belongs to the essential image of the functor $p_\sharp\colon \mathfrak{S}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{contra}$. This is the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{C}$ with this property. The functor $\mathfrak{C} \longmapsto \mathfrak{C}/(\mathfrak{H} \leftthreetimes \mathfrak{C})$ is the contraextension-of-scalars functor with respect to the morphism $p\colon \mathfrak{R} \longrightarrow \mathfrak{S}$, that is

$$p^\sharp(\mathfrak{C}) \cong \mathfrak{C}/(\mathfrak{H} \leftthreetimes \mathfrak{C}) \qquad \text{for all } \mathfrak{C} \in \mathfrak{R}\text{–}\mathsf{contra}.$$

When open two-sided ideals form a base of neighborhoods of zero in $\mathfrak{R}$, one can compute the contratensor product $\mathcal{N} \odot_\mathfrak{R} \mathfrak{C}$ as

$$\mathcal{N} \odot_\mathfrak{R} \mathfrak{C} \cong \varinjlim\nolimits_\mathfrak{I} \mathcal{N}_\mathfrak{I} \odot_\mathfrak{R} \mathfrak{C} \cong \varinjlim\nolimits_\mathfrak{I} \mathcal{N}_\mathfrak{I} \odot_{\mathfrak{R}/\mathfrak{I}} (\mathfrak{C}/\mathfrak{I} \leftthreetimes \mathfrak{C})$$

for any discrete right $\mathfrak{R}$-module $\mathcal{N}$ and left $\mathfrak{R}$-contramodule $\mathfrak{C}$, where the inductive limits are taken over all the open two-sided ideals $\mathfrak{I} \subset \mathfrak{R}$ (cf. [29, Section D.2]).

1.13. **Example.** Let A be an associative ring and M be a left A-module. Consider the associative ring $\mathfrak{R} = \operatorname{Hom}_A(M, M)^{\mathrm{op}}$ opposite to the ring of endomorphisms of the A-module M. Then the ring A acts in M on the left and the ring $\mathfrak{R}$ acts in M on the right; so M is an A-$\mathfrak{R}$-bimodule.

For every finitely generated A-submodule $E \subset M$, consider the subgroup $\operatorname{Ann}(E) = \operatorname{Hom}_A(M/E, M) \subset \operatorname{Hom}_A(M, M)$ consisting of all the endomorphisms of the A-module M which annihilate the submodule E. Then $\mathsf{Add}(E)$ is a left ideal in the ring $\operatorname{Hom}_A(M, M)$ and a right ideal in the ring $\mathfrak{R}$. Let $\mathfrak{B}$ denote the set of all right ideals in $\mathfrak{R}$ of the form $\operatorname{Ann}(E)$, where E ranges over all the finitely generated submodules in M. Then $\mathfrak{B}$ is a base of a complete, separated topology compatible with the associative ring structure on $\mathfrak{R}$ [36, Theorem 7.1]. The right action of $\mathfrak{R}$ in M makes M a discrete right $\mathfrak{R}$-module [36, Lemma 7.5].

This example play a key role in the categorical tilting theory [36, 37], and it will be also our intended example of a topological ring in Sections 13–15 and 19. Besides the categories of left modules over an associative rings, there are also other/wider classes of additive categories A such that for any object $M \in \mathsf{A}$ there is a natural structure of a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals on the ring $\mathfrak{R} = \operatorname{Hom}_{\mathsf{A}}(M, M)^{\mathrm{op}}$. A detailed discussion of these can be found in [36, Sections 9.1 and 9.3].

2. Flat Contramodules

The interactions of flatness with adic completion were studied by Yekutieli for ideals in Noetherian commutative rings [45] and in the greater generality of weakly proregular finitely generated ideals in commutative rings [46]. In the work of the second-named author of the present paper, the theory of flat contramodules was developed for ideals in Noetherian commutative rings [28, Sections B.8–B.9], for centrally generated ideals in noncommutative Noetherian rings [29, Section C.5], for topological associative rings with a countable base of neighborhoods of zero formed by open two-sided ideals [29, Section D.1], and for topological rings with a countable base of neighborhoods of zero formed by open right ideals [35, Sections 5–7].

In this section, we obtain some very partial results for topological rings with an uncountable base of neighborhoods of zero.

Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is called *flat* [35, Section 5] if the functor of contratensor product with $\mathfrak{F}$

$$- \odot_{\mathfrak{R}} \mathfrak{F} \colon \mathsf{discr}\text{–}\mathfrak{R} \longrightarrow \mathbb{Z}\text{–}\mathsf{mod}$$

is exact as a functor from the abelian category of discrete right $\mathfrak{R}$-modules to the category of abelian groups. The class of flat left $\mathfrak{R}$-contramodules is closed under coproducts and direct limits in the category $\mathfrak{R}\text{–}\mathsf{contra}$ [35, Lemma 5.6]. All projective left $\mathfrak{R}$-contramodules are flat.

If $\mathfrak{I}\subset\mathfrak{R}$ is an open two-sided ideal, then the left $\mathfrak{R}/\mathfrak{I}$-module $\mathfrak{F}/\mathfrak{I}\leftthreetimes\mathfrak{F}$ is flat for any flat left $\mathfrak{R}$-contramodule $\mathfrak{F}$. Indeed, the functor $-\otimes_{\mathfrak{R}/\mathfrak{I}}(\mathfrak{F}/\mathfrak{I}\leftthreetimes\mathfrak{F})\colon \mathsf{mod}\text{–}\mathfrak{R}/\mathfrak{I}\longrightarrow \mathbb{Z}\text{–}\mathsf{mod}$ is exact, because there is a natural isomorphism

$$N\otimes_{\mathfrak{R}/\mathfrak{I}}(\mathfrak{F}/\mathfrak{I}\leftthreetimes\mathfrak{F})\cong N\odot_{\mathfrak{R}}\mathfrak{F} \qquad \text{for all right } \mathfrak{R}/\mathfrak{I}\text{-modules } N.$$

If open two-sided ideals form a base of neighborhoods of zero in $\mathfrak{R}$, then the converse assertion also holds: a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is flat if and only if the left $\mathfrak{R}/\mathfrak{I}$-module $\mathfrak{F}/\mathfrak{I}\leftthreetimes\mathfrak{F}$ is flat for every open two-sided ideal $\mathfrak{I}\subset\mathfrak{R}$. (Cf. Sections 1.10 and 1.12.)

The left derived functor

$$\mathrm{Ctrtor}_i^{\mathfrak{R}}\colon \mathsf{discr}\text{–}\mathfrak{R}\times\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow\mathbb{Z}\text{–}\mathsf{mod}$$

is constructed using projective resolutions of the second (contramodule) argument. So, if $\cdots\longrightarrow\mathfrak{P}_2\longrightarrow\mathfrak{P}_1\longrightarrow\mathfrak{P}_0\longrightarrow\mathfrak{C}\longrightarrow 0$ is an exact complex in the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{P}_i$ are projective left $\mathfrak{R}$-contramodules for all $i\ge 0$, then

$$\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})=H_i(\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{P}_\bullet) \qquad \text{for all } \mathcal{N}\in\mathsf{discr}\text{–}\mathfrak{R} \text{ and } i\ge 0.$$

As always with derived functors of one argument, for any short exact sequence of left $\mathfrak{R}$-contramodules $0\longrightarrow\mathfrak{A}\longrightarrow\mathfrak{B}\longrightarrow\mathfrak{C}\longrightarrow 0$ and any discrete right $\mathfrak{R}$-module $\mathcal{N}$ there is a natural long exact sequence of abelian groups

$$(1)\qquad \cdots\longrightarrow\mathrm{Ctrtor}_{i+1}^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})\longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N},\mathfrak{A}) \\ \longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N},\mathfrak{B})\longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})\longrightarrow\cdots$$

Since the functor of contratensor product $\mathcal{N}\odot_{\mathfrak{R}}-\colon\mathfrak{R}\text{–}\mathsf{contra}\longrightarrow\mathbb{Z}\text{–}\mathsf{mod}$ is right exact on the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ for every $\mathcal{N}\in\mathsf{discr}\text{–}\mathfrak{R}$, one has

$$\mathrm{Ctrtor}_0^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})=\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{C}.$$

Furthermore, for any short exact sequence of discrete right $\mathfrak{R}$-modules $0\longrightarrow\mathcal{L}\longrightarrow\mathcal{M}\longrightarrow\mathcal{N}\longrightarrow 0$ and any complex of projective left $\mathfrak{R}$-contramodules $\mathfrak{P}_\bullet$, the short sequence of complexes of abelian groups $0\longrightarrow\mathcal{L}\odot_{\mathfrak{R}}\mathfrak{P}_\bullet\longrightarrow\mathcal{M}\odot_{\mathfrak{R}}\mathfrak{P}_\bullet\longrightarrow\mathcal{N}\odot_{\mathfrak{R}}\mathfrak{P}_\bullet\longrightarrow 0$ is exact (because projective left $\mathfrak{R}$-contramodules are flat). Therefore, for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$ there is a long exact sequence of abelian groups

$$(2)\qquad \cdots\longrightarrow\mathrm{Ctrtor}_{i+1}^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})\longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{L},\mathfrak{C}) \\ \longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{M},\mathfrak{C})\longrightarrow\mathrm{Ctrtor}_i^{\mathfrak{R}}(\mathcal{N},\mathfrak{C})\longrightarrow\cdots$$

We will say that a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is *1-strictly flat* if $\mathrm{Ctrtor}_1^{\mathfrak{R}}(\mathcal{N}\,\mathfrak{F})=0$ for all discrete right $\mathfrak{R}$-modules $\mathcal{N}$. More generally, a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is *n-strictly flat* if $\mathrm{Ctrtor}_i^R(\mathcal{N},\mathfrak{F})=0$ for all discrete right $\mathfrak{R}$-modules $\mathcal{N}$ and all $1\le i\le n$. A left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is *∞-strictly flat* if it is n-strictly flat for all $n>0$.

Clearly, all projective left $\mathfrak{R}$-contramodules are ∞-strictly flat. It follows from the exact sequence (1) that the class of all n-strictly flat left $\mathfrak{R}$-contramodules is closed under extensions in $\mathfrak{R}\text{–}\mathsf{contra}$ for every $n\ge 0$, and that the class of all ∞-strictly flat left $\mathfrak{R}$-contramodules is closed under (extensions and) the passage to the kernels of surjective morphisms. Given some $n\ge 1$, the class of all n-strictly flat left

$\mathfrak{R}$-contramodules is closed under the kernels of surjective morphisms if and only if it coincides with the class of all ∞-strictly flat left $\mathfrak{R}$-contramodules.

From the exact sequence (2) one can conclude that every 1-strictly flat left $\mathfrak{R}$-contramodule is flat. According to [35, proof of Lemma 6.10, Remark 6.11 and Corollary 6.15], when the topological ring $\mathfrak{R}$ has a countable base of neighborhoods of zero (formed by open right ideals), the classes of flat, 1-strictly flat, and ∞-strictly flat left $\mathfrak{R}$-contramodules coincide.

Let us say that a short exact sequence of left $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{C} \longrightarrow 0$ is *contratensor pure* if the induced sequence $0 \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{A} \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{B} \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{C} \longrightarrow 0$ is exact (i. e., the map $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{A} \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{B}$ is injective) for every discrete right $\mathfrak{R}$-module $\mathcal{N}$. If the left $\mathfrak{R}$-contramodule $\mathfrak{B}$ is 1-strictly flat, then the sequence $0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{C} \longrightarrow 0$ is contratensor pure if and only if the left $\mathfrak{R}$-contramodule $\mathfrak{C}$ is 1-strictly flat.

Lemma 2.1. (a) *The class of all* 1*-strictly flat left* $\mathfrak{R}$*-contramodules is closed under infinite coproducts in* $\mathfrak{R}\text{–}\mathsf{contra}$*.*

(b) *The class of all* 1*-strictly flat left* $\mathfrak{R}$*-contramodules is closed under countable direct limits in* $\mathfrak{R}\text{–}\mathsf{contra}$*.*

Proof. Part (a): let $(\mathfrak{F}_\alpha)_\alpha$ be a family of 1-strictly flat left $\mathfrak{R}$-contramodules. Choose short exact sequences of left $\mathfrak{R}$-contramodules $0 \longrightarrow \mathfrak{K}_\alpha \longrightarrow \mathfrak{P}_\alpha \longrightarrow \mathfrak{F}_\alpha \longrightarrow 0$, where $\mathfrak{P}_\alpha$ are projective left $\mathfrak{R}$-contramodules. Then the sequence $\coprod_\alpha \mathfrak{K}_\alpha \longrightarrow \coprod_\alpha \mathfrak{P}_\alpha \longrightarrow \coprod_\alpha \mathfrak{F}_\alpha \longrightarrow 0$ (where the coproducts are taken in $\mathfrak{R}\text{–}\mathsf{contra}$) is exact, as the functors of coproduct are right exact in any abelian category. Therefore, there is a natural surjective $\mathfrak{R}$-contramodule morphism from $\coprod_\alpha \mathfrak{K}_\alpha$ onto the kernel $\mathfrak{K}$ of the morphism $\coprod_\alpha \mathfrak{P}_\alpha \longrightarrow \coprod_\alpha \mathfrak{F}_\alpha$. The contratensor product functor $\odot_{\mathfrak{R}}$ preserves colimits, hence for any discrete right $\mathfrak{R}$-module $\mathcal{N}$ the morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_\alpha \mathfrak{K}_\alpha \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \coprod_\alpha \mathfrak{P}_\alpha$ is injective (being isomorphic to the morphism $\coprod_\alpha \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K}_\alpha \longrightarrow \coprod_\alpha \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{P}_\alpha$, where the coproducts are taken in the category of abelian groups). At the same time, the morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_\alpha \mathfrak{K}_\alpha \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K}$ is surjective. It follows that the morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_\alpha \mathfrak{K}_\alpha \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K}$ is an isomorphism and the morphism $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{K} \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \coprod_\alpha \mathfrak{P}_\alpha$ is injective, that is, the short exact sequence $0 \longrightarrow \mathfrak{K} \longrightarrow \coprod_\alpha \mathfrak{P}_\alpha \longrightarrow \coprod_\alpha \mathfrak{F}_\alpha \longrightarrow 0$ is contratensor pure. Since the left $\mathfrak{R}$-contramodule $\coprod_\alpha \mathfrak{P}_\alpha$ is projective, it follows that the left $\mathfrak{R}$-contramodule $\coprod_\alpha \mathfrak{F}_\alpha$ is 1-strictly flat.

Part (b): let $\mathfrak{F}_1 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{F}_3 \longrightarrow \cdots$ be a sequence of left $\mathfrak{R}$-contramodules and $\mathfrak{R}$-contramodule morphisms between them. Then the colimit $\varinjlim_n \mathfrak{F}_n$ is the cokernel of the morphism $\mathrm{id} - \mathit{shift}\colon \coprod_{n=1}^\infty \mathfrak{F}_n \longrightarrow \coprod_{n=1}^\infty \mathfrak{F}_n$. Denote the image of this morphism by $\mathfrak{L}$. Arguing as in part (a), we have a surjective morphism $\coprod_{n=1}^\infty \mathfrak{F}_n \longrightarrow \mathfrak{L}$ and an exact sequence $0 \longrightarrow \mathfrak{L} \longrightarrow \coprod_{n=1}^\infty \mathfrak{F}_n \longrightarrow \varinjlim_n \mathfrak{F}_n \longrightarrow 0$. The morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_n \mathfrak{F}_n \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \coprod_n \mathfrak{F}_n$ is injective (being isomorphic to the morphism $\coprod_n \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{F}_n \longrightarrow \coprod_n \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{F}_n$) for every discrete right $\mathfrak{R}$-module $\mathcal{N}$. At the same time, the morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_n \mathfrak{F}_n \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \varinjlim_n \mathfrak{F}_n$ is surjective. It follows that the morphism $\mathcal{N} \odot_{\mathfrak{R}} \coprod_n \mathfrak{F}_n \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \mathfrak{L}$ is an isomorphism and the morphism $\mathcal{N} \odot_{\mathfrak{R}} \mathfrak{L} \longrightarrow \mathcal{N} \odot_{\mathfrak{R}} \coprod_n \mathfrak{F}_n$ is injective, that is, the short exact sequence $0 \longrightarrow \mathfrak{L} \longrightarrow \coprod_n \mathfrak{F}_n \longrightarrow \varinjlim_n \mathfrak{F}_n \longrightarrow 0$

is contratensor pure. Now, if the left $\mathfrak{R}$-contramodules $\mathfrak{F}_n$ are 1-strictly flat for all $n \geq 1$, then the left $\mathfrak{R}$-contramodule $\coprod_n \mathfrak{F}_n$ is 1-strictly flat by part (a), and it follows that the left $\mathfrak{R}$-contramodule $\varinjlim_n \mathfrak{F}_n$ is also 1-strictly flat. $\square$

Before formulating the next corollary, we notice that, if a left $\mathfrak{R}$-contramodule $\mathfrak{F}$ has projective dimension not exceeding n (as an object of the abelian category $\mathfrak{R}$–contra) and $\mathfrak{F}$ is n-strictly flat, then $\mathfrak{F}$ is also ∞-strictly flat.

Corollary 2.2. *Any countable direct limit of projective left $\mathfrak{R}$-contramodules has projective dimension not exceeding* 1 *in* $\mathfrak{R}$–contra*. In particular, any such $\mathfrak{R}$-contramodule is ∞-strictly flat.*

Proof. The second assertion follows immediately from the first one together with Lemma 2.1(b), while the first assertion is a corollary of the proof of Lemma 2.1(b). In the notation of the latter, let us show that the morphism $\coprod_n \mathfrak{F}_n \longrightarrow \mathfrak{L}$ is an isomorphism (or, in other words, the morphism $\mathrm{id} - \mathit{shift}\colon \coprod_n \mathfrak{F}_n \longrightarrow \coprod_n \mathfrak{F}_n$ is injective) when all the left $\mathfrak{R}$-contramodules $\mathfrak{F}_n$, $n \geq 1$, are projective.

Indeed, we have seen that the functor $\mathcal{N} \odot_{\mathfrak{R}} -$ transforms the morphism $\coprod_n \mathfrak{F}_n \longrightarrow \mathfrak{L}$ into an isomorphism, for every discrete right $\mathfrak{R}$-contramodule. In particular, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, the map $\coprod_n \mathfrak{F}_n/\mathfrak{I} \leftthreetimes \mathfrak{F}_n \longrightarrow \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L}$ is an isomorphism. Passing to the projective limit over all the open right ideals $\mathfrak{I}$ in $\mathfrak{R}$, we get an isomorphism $\coprod_n \mathfrak{F}_n \longrightarrow \varprojlim_{\mathfrak{I}} \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L}$ (because the map $\mathfrak{P} \longrightarrow \varprojlim_{\mathfrak{I}} \mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P}$ is an isomorphism for any projective left $\mathfrak{R}$-contramodule $\mathfrak{P}$). Now commutativity of the triangle diagram (of abelian groups) $\coprod_n \mathfrak{F}_n \longrightarrow \mathfrak{L} \longrightarrow \varprojlim_{\mathfrak{I}} \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L}$ together with surjectivity of the morphism $\coprod_n \mathfrak{F}_n \longrightarrow \mathfrak{L}$ imply that both the maps $\coprod_n \mathfrak{F}_n \longrightarrow \mathfrak{L}$ and $\mathfrak{L} \longrightarrow \varprojlim_{\mathfrak{I}} \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L}$ are isomorphisms. $\square$

In particular, let a_1, a_2, a_3, … be a sequence of elements in the topological ring $\mathfrak{R}$. For every $n \geq 1$, the multiplication by a_n on the right is a left $\mathfrak{R}$-contramodule morphism $\mathfrak{R} \longrightarrow \mathfrak{R}$ (where $\mathfrak{R}$ is viewed as a free left $\mathfrak{R}$-contramodule with one generator). The direct limit

$$\mathfrak{B} = \varinjlim \, (\mathfrak{R} \xrightarrow{a_1} \mathfrak{R} \xrightarrow{a_2} \mathfrak{R} \xrightarrow{a_3} \cdots)$$

is called the *Bass flat left $\mathfrak{R}$-contramodule* associated with the sequence of elements $(a_n \in \mathfrak{R})_{n \geq 1}$. According to Corollary 2.2, the Bass flat left $\mathfrak{R}$-contramodules are ∞-strictly flat and have projective dimension not exceeding 1.

In particular, when the topological ring $\mathfrak{R} = R$ is discrete, the above construction specializes to the classical definition of a *Bass flat left R-module.*

Lemma 2.3. *If all Bass flat left modules over an associative ring R are projective, then all flat left R-modules are projective (i. e., the ring R is left perfect).*

Proof. Clear from the proof of the implication (5) $\Longrightarrow$ (6) in [5, Theorem P], which only uses projectivity of the Bass flat modules. $\square$

Corollary 2.4. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that all Bass flat left*

$\mathfrak{R}$*-contramodules are projective. Then the discrete ring* $R = \mathfrak{R}/\mathfrak{I}$ *is left perfect for every open two-sided ideal* $\mathfrak{I} \subset \mathfrak{R}$.

Proof. In view of Lemma 2.3, it suffices to show that all Bass flat left R-modules are projective. Let $\bar a_1$, $\bar a_2$, $\bar a_3$, … be a sequence of elements in R and $B = \varinjlim(\mathfrak{R}/\mathfrak{I} \xrightarrow{\bar a_1} \mathfrak{R}/\mathfrak{I} \xrightarrow{\bar a_2} \mathfrak{R}/\mathfrak{I} \xrightarrow{a_3} \cdots)$ be the related Bass flat left R-module. Lift the elements $\bar a_n \in R$ to some elements $a_n \in \mathfrak{R}$, and consider the related Bass flat left $\mathfrak{R}$-contramodule $\mathfrak{B}$. Then we have $\mathfrak{B}/\mathfrak{I} \leftthreetimes \mathfrak{B} \cong B$, since the reduction functors preserve colimits (see Sections 1.9–1.10). The left $\mathfrak{R}$-contramodule $\mathfrak{B}$ is projective by assumption, hence the left R-module $\mathfrak{B}/\mathfrak{I} \leftthreetimes \mathfrak{B}$ is projective, too. $\square$

3. Projective Covers of Flat Contramodules

Let B be an abelian category with enough projective objects. An epimorphism $p\colon P \longrightarrow C$ in B is called a *projective cover* (of the object C) if the object P is projective and, for any endomorphism $e\colon P \longrightarrow P$, the equation $pe = p$ implies that e is an automorphism of P (i. e., e is invertible).

A subobject K of an object $Q \in \mathsf{B}$ is said to be *superfluous* if, for any other subobject $G \subset Q$, the equation $K + G = Q$ implies that $G = Q$. If a suboboject $K \subset Q$ is superfluous then, for any subobject $E \subset Q$, the quotient $K/E \cap K$ is a superfluous subobject of the quotient Q/E.

Lemma 3.1. *Let* $P \in \mathsf{B}$ *be a projective object. Then an epimorpism* $p\colon P \longrightarrow C$ *in* B *is a projective cover if and only if its kernel* K *is a superfluous subobject in* P.

Proof. Let $p\colon P \longrightarrow C$ be a projective cover with the kernel K, and let $G \subset P$ be a subobject such that $K + G = P$. Then the restriction of p onto G is an epimorphism $s\colon G \longrightarrow C$. Since P is projective, there exists a morphism $f\colon P \longrightarrow G$ making the triangle diagram $P \longrightarrow G \longrightarrow C$ commutative. Let $e\colon P \longrightarrow P$ be the composition of the morphism f with the embedding $G \longrightarrow P$. Then $pe = p$, and by assumption it follows that e is invertible. Hence $G = P$.

Conversely, let $p\colon P \longrightarrow C$ be an epimorphism with a superfluous kernel $K \subset P$, and let $e\colon P \longrightarrow P$ be an endomorphism satisfying $pe = p$. Let $G \subset P$ be the image of e; then $K + G = P$. By assumption, it follows that $G = P$, so e is an epimorphism. Then, since P is projective, the kernel L of e must be a direct summand of P. Denote by $E \subset P$ a complementary direct summand. The equation $pe = p$ implies that $L \subset K$, hence $K + E = P$. Again by assumption, it follows that $E = P$, so $L = 0$ and e is an automorphism of P. $\square$

Lemma 3.2. *Let* R *be an associative ring and* F *be a flat left* R*-module. Assume that* F *has a projective cover* $p\colon P \longrightarrow F$ *in* $R\text{–}\mathsf{mod}$*, whose kernel* $L = \ker(p)$ *has a projective cover* $q\colon Q \longrightarrow L$*. Then* $Q = 0$*,* $L = 0$*, and the* R*-module* $F = P$ *is projective.*

Proof. Let $H \subset R$ be the Jacobson radical of the ring R. Then for any projective left R-module T the submodule $JT \subset T$ is the intersection of all maximal R-submodules in T. Hence any superfluous submodule $K \subset T$ is contained in JT. If the quotient module T/K is flat, then $K \cap JT = JK$, so we can conclude that $K = JK$.

Returning to the situation at hand, consider the R-module $M = \ker(q)$. Then we have $L = JL$ and $M = JM$, hence $Q = JQ$. According to [5, Proposition 2.7], $T = JT$ implies $T = 0$ for a projective left R-module T. Thus we have $Q = 0$, and the remaining assertions follow. $\square$

The following assertion is a slightly stronger version of the implication (2) $\Longrightarrow$ (3) in [5, Theorem P].

Corollary 3.3. *If all flat left modules over an associative ring R have projective covers, then all flat left R-modules are projective (i. e., the ring R is left perfect).*

Proof. Follows immediately from Lemma 3.2. $\square$

In fact, a stronger assertion holds (cf. [5, proof of the last claim of Theorem 2.1]).

Corollary 3.4. (a) *If a flat left module F of projective dimension not exceeding* 1 *over an associative ring R has a projective cover, then F is projective. In particular, if a Bass flat left R-module B has a projective cover, then B is projective.*

(b) *If all Bass flat left modules over an associative ring R have projective covers, then the ring R is left perfect.*

Proof. Part (a) is a particular case of Lemma 3.2. Part (b) follows from part (a) together with Lemma 2.3. $\square$

Lemma 3.5. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, $\mathfrak{I}$ be an open two-sided ideal in $\mathfrak{R}$, and $R = \mathfrak{R}/\mathfrak{I}$ be the discrete quotient ring. Assume that a left $\mathfrak{R}$-contramodule $\mathfrak{C}$ has a projective cover $p\colon \mathfrak{P} \longrightarrow \mathfrak{C}$ in $\mathfrak{R}$–contra. Then the induced map $\bar{p}\colon \mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P} \longrightarrow \mathfrak{C}/\mathfrak{I} \leftthreetimes \mathfrak{P}$ is a projective cover of the left R-module $\mathfrak{C}/\mathfrak{I} \leftthreetimes \mathfrak{C}$.*

Proof. Set $\mathfrak{K} = \ker(p)$. Then $0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{C} \longrightarrow 0$ is a short exact sequence in $\mathfrak{R}$–contra and $0 \longrightarrow \mathfrak{K}/(\mathfrak{I} \leftthreetimes \mathfrak{P}) \cap \mathfrak{K} \longrightarrow \mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P} \longrightarrow \mathfrak{C}/\mathfrak{I} \leftthreetimes \mathfrak{C} \longrightarrow 0$ is a short exact sequence in R–mod. The left R-module $\mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P}$ is projective, since the left $\mathfrak{R}$-contramodule $\mathfrak{P}$ is; and the R-submodule $\mathfrak{K}/(\mathfrak{I} \leftthreetimes \mathfrak{P}) \cap \mathfrak{K} \subset \mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P}$ is superfluous, since the $\mathfrak{R}$-subcontramodule $\mathfrak{K} \subset \mathfrak{P}$ is. $\square$

The following corollary is a stronger version of Corollary 2.4.

Corollary 3.6. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Assume that all Bass flat left $\mathfrak{R}$-contramodules have projective covers. Then the discrete ring $R = \mathfrak{R}/\mathfrak{I}$ is left perfect for every open two-sided ideal $\mathfrak{I} \subset \mathfrak{R}$.*

Proof. In view of Corollary 3.4(b), it suffices to show that all Bass flat left R-modules have projective covers in R–mod. As in the proof of Corollary 2.4, let $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, …

be a sequence of elements in R and B be the related Bass flat left R-module. Lift the elements $\bar{a}_n \in R$ to some elements $a_n \in \mathfrak{R}$, and consider the related Bass flat left $\mathfrak{R}$-contramodule $\mathfrak{B}$. Then we have $\mathfrak{B}/\mathfrak{I} \leftthreetimes \mathfrak{B} \cong B$. By assumption, we know that the left $\mathfrak{R}$-contramodule $\mathfrak{B}$ has a projective cover in $\mathfrak{R}\text{–}\mathsf{contra}$; and by Lemma 3.5 it follows that the left R-module B is has a projective cover in $R\text{–}\mathsf{mod}$. $\square$

Proposition 3.7. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals, and let $\mathfrak{F}$ be a 1-strictly flat left $\mathfrak{R}$-contramodule. Assume that $\mathfrak{F}$ has a projective cover $p\colon \mathfrak{P} \longrightarrow \mathfrak{F}$ in $\mathfrak{R}\text{–}\mathsf{contra}$, whose kernel $\mathfrak{L} = \ker(p)$ has a projective cover $q\colon \mathfrak{Q} \longrightarrow \mathfrak{L}$. Then $\mathfrak{Q} = 0$, $\mathfrak{L} = 0$, and the $\mathfrak{R}$-contramodule $\mathfrak{F} = \mathfrak{P}$ is projective.*

Proof. For any open two-sided ideal $\mathfrak{I} \subset \mathfrak{R}$, we have a short sequence of left $\mathfrak{R}/\mathfrak{I}$-modules $0 \longrightarrow \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L} \longrightarrow \mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P} \longrightarrow \mathfrak{F}/\mathfrak{I} \leftthreetimes \mathfrak{F} \longrightarrow 0$, which is exact since $\mathrm{Ctrtor}_1^{\mathfrak{R}}(\mathfrak{R}/\mathfrak{I}, \mathfrak{F}) = 0$. The left $\mathfrak{R}/\mathfrak{I}$-module $\mathfrak{F}/\mathfrak{I} \leftthreetimes \mathfrak{F}$ is flat, since the left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is. Furthermore, the morphisms $\mathfrak{P}/\mathfrak{I} \leftthreetimes \mathfrak{P} \longrightarrow \mathfrak{F}/\mathfrak{I} \leftthreetimes \mathfrak{F}$ and $\mathfrak{Q}/\mathfrak{I} \leftthreetimes \mathfrak{Q} \longrightarrow \mathfrak{L}/\mathfrak{I} \leftthreetimes \mathfrak{L}$ are projective covers in the category of left $\mathfrak{R}/\mathfrak{I}$-modules by Lemma 3.5. Applying Lemma 3.2, we conclude that $\mathfrak{Q}/\mathfrak{I} \leftthreetimes \mathfrak{Q} = 0$ for every open two-sided ideal $\mathfrak{I} \subset \mathfrak{R}$. Since $\mathfrak{Q}$ is a projective left $\mathfrak{R}$-contramodule, one has $\mathfrak{Q} = \varprojlim_{\mathfrak{I}} \mathfrak{Q}/\mathfrak{I} \leftthreetimes \mathfrak{Q}$ (see Section 1.10), and therefore $\mathfrak{Q} = 0$. $\square$

Corollary 3.8. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals. Assume that all 1-strictly flat left $\mathfrak{R}$-contramodules have projective covers. Then all 2-strictly flat left $\mathfrak{R}$-contramodules are projective.* $\square$

Proof. Follows from Proposition 3.7 (since the kernel of a surjective morphism from a projective $\mathfrak{R}$-contramodule to a 2-strictly flat one is 1-strictly flat). $\square$

Corollary 3.9. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals, and let $\mathfrak{F}$ be an ∞-strictly flat left $\mathfrak{R}$-contramodule of projective dimension not exceeding 1. Assume that $\mathfrak{F}$ has a projective cover in $\mathfrak{R}\text{–}\mathsf{contra}$. Then $\mathfrak{F}$ is a projective left $\mathfrak{R}$-contramodule.*

Proof. Let $p\colon \mathfrak{P} \longrightarrow \mathfrak{F}$ be a projective cover; set $\mathfrak{L} = \ker(p)$. Then $\mathfrak{L}$ is a projective left $\mathfrak{R}$-contramodule, since the projective dimension of $\mathfrak{F}$ does not exceed 1. Setting $\mathfrak{Q} = \mathfrak{L}$, $q = \mathrm{id}$, and applying Proposition 3.7, we conclude that $\mathfrak{L} = 0$ and $\mathfrak{F} = \mathfrak{P}$. $\square$

4. Topologically T-Nilpotent Ideals

Let H be a separated topological ring without unit. We will say that H is *topologically nil* if for any element $a \in H$ the sequence of elements a, a^2, a^3, … converges to zero in the topology of H. Furthermore, we will say that H is *topologically left T-nilpotent* if for any sequence of elements a_1, a_2, a_3, … in H the sequence of elements a_1, a_1a_2, $a_1a_2a_3$, …, $a_1a_2\cdots a_n$, … converges to zero in the topology of H.

For examples of topologically left T-nilpotent two-sided ideals in topological rings we refer to Section 9 and Examples 10.1–10.2.

The following discrete module version of Nakayama lemma is also a topological version of the condition (7) and the implication (7) $\Longrightarrow$ (1) in [5, Theorem P].

Lemma 4.1. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{H} \subset \mathfrak{R}$ be a closed two-sided ideal. Then $\mathfrak{H}$ is topologically left T-nilpotent (as a topological ring without unit in the topology induced from $\mathfrak{R}$) if and only if for any nonzero discrete right $\mathfrak{R}$-module $\mathcal{N}$ the submodule $\mathcal{N}_{\mathfrak{H}} \subset \mathcal{N}$ of all the elements annihilated by $\mathfrak{H}$ in $\mathcal{N}$ is also nonzero.*

Proof. Assume that $\mathfrak{H}$ is topologically left T-nilpotent, and let $\mathcal{N}$ be a nonzero discrete right $\mathfrak{R}$-module. We have to show that $\mathcal{N}$ contains a nonzero element annihilated by $\mathfrak{H}$. Choose an arbitrary nonzero element $x \in \mathcal{N}$. Suppose x is not annihilated by $\mathfrak{H}$; then there exists an element $a_1 \in \mathfrak{H}$ such that $xa_1 \neq 0$ in $\mathcal{N}$. Suppose xa_1 is not annihilated by $\mathfrak{H}$; then there exists an element $a_2 \in \mathfrak{H}$ such that $xa_1a_2 \neq 0$ in $\mathcal{N}$, etc. Assuming that $\mathcal{N}_{\mathfrak{H}} = 0$, we can proceed indefinitely in this way and construct a sequence of elements $(a_n \in \mathfrak{H})_{n\geq 1}$ such that $xa_1 \cdots a_n \neq 0$ in $\mathcal{N}$ for all $n \geq 1$.

Let $\mathfrak{I} \subset \mathfrak{R}$ be the annihilator of x; then $\mathfrak{I}$ is an open right ideal in $\mathfrak{R}$, so $\mathfrak{I} \cap \mathfrak{H}$ is a neighborhood of zero in $\mathfrak{H}$. Since $\mathfrak{H}$ is topologically left T-nilpotent, there exists $n \geq 1$ such that $a_1a_2 \cdots a_n \in \mathfrak{I} \cap \mathfrak{H}$. Hence $xa_1 \cdots a_n = 0$ in $\mathcal{N}$. The contradiction proves that $\mathcal{N}_{\mathfrak{H}} \neq 0$.

Conversely, assume that $\mathcal{N}_{\mathfrak{H}} \neq 0$ for every nonzero discrete right $\mathfrak{R}$-module $\mathcal{N}$. Assuming that $\mathcal{N}_{\mathfrak{H}} \neq \mathcal{N}$, one then also has $(\mathcal{N}/\mathcal{N}_{\mathfrak{H}})_{\mathfrak{H}} \neq 0$. Proceeding in a transfinite induction, one constructs a filtration $0 = F_0\mathcal{N} \subset F_1\mathcal{N} \subset F_2\mathcal{N} \subset \cdots \subset F_\alpha\mathcal{N} = \mathcal{N}$ of the $\mathfrak{R}$-module $\mathcal{N}$, indexed by some ordinal α, such that $F_{i+1}\mathcal{N}/F_i\mathcal{N} = (\mathcal{N}/F_i\mathcal{N})_{\mathfrak{H}} \neq 0$ for all ordinals $i < \alpha$ and $F_j\mathcal{N} = \bigcup_{i<j} F_i\mathcal{N}$ for all limit ordinals $j \leq \alpha$.

Now let $(a_n \in \mathfrak{H})_{n\geq 1}$ be a sequence of elements. In order to prove that $\mathfrak{H}$ is topologically left T-nilpotent, we have to show that, for every open right ideal $\mathfrak{I} \subset \mathfrak{R}$, there exists $n \geq 1$ such that $a_1 \cdots a_n \in \mathfrak{I} \cap \mathfrak{H}$. Consider the discrete right $\mathfrak{R}$-module $\mathcal{N} = \mathfrak{R}/\mathfrak{I}$ and its filtration $(F_i\mathcal{N})_{i=0}^{\alpha}$, as constructed above. Let $x \in \mathcal{N}$ denote the image of the element $1 \in \mathfrak{R}$.

We follow the argument in the proof of (7) $\Longrightarrow$ (1) in [5, Theorem P]. Let $i_0 \leq \alpha$ be the minimal ordinal such that $x \in F_{i_0}\mathcal{N}$. Then i_0 cannot be a limit ordinal; so either $i_0 = 0$, or $i_0 = i_0' + 1$ for some ordinal i_0'. Since the $\mathfrak{R}$-module $F_{i_0}\mathcal{N}/F_{i_0'}\mathcal{N}$ is annihilated by $\mathfrak{H}$, we have $xa_1 \in F_{i_0'}\mathcal{N}$. Let i_1 be the minimal ordinal such that $xa_1 \in F_{i_1}\mathcal{N}$; then $i_1 < i_0$. Once again, i_1 cannot be a limit ordinal; so either $i_1 = 0$, or $i_1 = i_1' + 1$ for some ordinal i_1', and then $xa_1a_2 \in F_{i_1'}\mathcal{N}$. Proceeding in this way, we construct a decreasing chain of ordinals $i_0 > i_1 > i_2 > \cdots$, which must terminate. Thus there exists $n \geq 1$ such that $xa_1 \cdots a_n \in F_0\mathcal{N} = 0$, hence $a_1 \cdots a_n \in \mathfrak{I}$. $\square$

The following version of contramodule Nakayama lemma is a generalization of [28, Lemma 1.3.1] (which is, in turn, a generalization of [27, Lemma A.2.1]). For other versions of contramodule Nakayama lemma, see [29, Lemma D.1.2] and [35, Lemma 6.14].

Lemma 4.2. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{H} \subset \mathfrak{R}$ be a closed subring without unit. Assume that $\mathfrak{H}$ is topologically left T-nilpotent (in the topology induced from $\mathfrak{R}$). Then for any nonzero left $\mathfrak{R}$-contramodule $\mathfrak{C}$ one has $\mathfrak{H} \leftthreetimes \mathfrak{C} \neq \mathfrak{C}$.*

Proof. The argument follows the proof of [28, Lemma 1.3.1] with an additional consideration based on the König lemma (in the spirit of a paragraph from [5, proof of Theorem 2.1]). We will assume that $\mathfrak{H} \leftthreetimes \mathfrak{C} = \mathfrak{C}$ and prove that $\mathfrak{C} = 0$ in this case. Indeed, let $b \in \mathfrak{C}$ be an element.

By assumption, the contraaction map $\pi\colon \mathfrak{H}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ is surjective. Let $h\colon \mathfrak{C} \longrightarrow \mathfrak{H}[[\mathfrak{C}]]$ be a section of the map π (so $\pi \circ h = \mathrm{id}_{\mathfrak{C}}$). Introduce the notation $h(d) = \sum_{c\in\mathfrak{C}} h_{d,c}c \in \mathfrak{H}[[\mathfrak{C}]]$ for all $d \in \mathfrak{C}$, where $h_{d,c} \in \mathfrak{H}$ and the $\mathfrak{C}$-indexed family of elements $c \longmapsto h_{d,c}$ converges to zero in the topology of $\mathfrak{H}$ for every $d \in \mathfrak{C}$.

For any set X, define inductively $\mathfrak{H}^{(0)}[[[X]] = X$ and $\mathfrak{H}^{(n)}[[X]] = \mathfrak{H}[[\mathfrak{H}^{(n-1)}[[X]]]]$ for $n \geq 1$. Let $\phi_X^{(n)}\colon \mathfrak{H}^{(n)}[[X]] \longrightarrow \mathfrak{H}[[X]]$ denote the iterated monad multiplication ("opening of parentheses") map. Set $b_1 = h(b) \in \mathfrak{H}[[\mathfrak{C}]]$, and define inductively $b_n = \mathfrak{H}^{(n-1)}[[h]](b_{n-1}) \in \mathfrak{H}^{(n)}[[\mathfrak{C}]]$ for each $n \geq 2$, where $\mathfrak{H}^{(n-1)}[[h]]\colon \mathfrak{H}^{(n-1)}[[\mathfrak{C}]] \longrightarrow \mathfrak{H}^{(n)}[[\mathfrak{C}]]$ is the map induced by h. Put $a_n = \phi_{\mathfrak{C}}^{(n)}(b_n) \in \mathfrak{H}[[\mathfrak{C}]]$ for all $n \geq 1$.

Furthermore, set $q_n = \phi^{(n-1)}_{\mathfrak{H}[[\mathfrak{C}]]}(b_n) = \mathfrak{H}[[h]](a_{n-1}) \in \mathfrak{H}[[\mathfrak{H}[[\mathfrak{C}]]]]$ for all $n \geq 2$. Then $\mathfrak{H}[[\pi]](q_n) = a_{n-1}$ and $\phi_{\mathfrak{C}}(q_n) = a_n$.

The abelian group $\mathfrak{H}[[X]]$ is separated and complete in its natural topology with a base of neighborhoods of zero formed by the subgroups $(\mathfrak{I} \cap \mathfrak{H})[[X]] \subset \mathfrak{H}[[X]]$, where $\mathfrak{I} \subset \mathfrak{R}$ are open right ideals. For any map of sets $f\colon X \longrightarrow Y$, the map $\mathfrak{H}[[f]]$ is continuous with respect to such topologies on $\mathfrak{H}[[X]]$ and $\mathfrak{H}[[Y]]$. Besides, the map $\phi_X\colon \mathfrak{H}[[\mathfrak{H}[[X]]]] \longrightarrow \mathfrak{H}[[X]]$ is continuous, too, with respect to the above-described topology on $\mathfrak{H}[[X]]$ and the similar topology of $\mathfrak{H}[[\mathfrak{H}[[X]]]] = \mathfrak{H}[[Y]]$, where $Y = \mathfrak{H}[[X]]$ is viewed as an abstract set.

The key observation is that the sequence of elements a_n converges to zero in the topology of $\mathfrak{H}[[\mathfrak{C}]]$ as $n \to \infty$. In order to prove this convergence, we will represent the sequence of elements $b_n \in \mathfrak{H}^{(n)}[[\mathfrak{C}]]$ by an infinite rooted tree B in the following way. The root vertex (that is, the only vertex of depth 0) is marked by the element $b \in \mathfrak{C}$. Its children (i. e., the vertices of depth 1) are marked by all the elements $c \in \mathfrak{C}$, one such child for every element c. The edge leading from the root vertex b to its child c is marked by the coefficient $h_{b,c} \in \mathfrak{H}$ in the formal linear combination $b_1 = h(b) = \sum_{c\in\mathfrak{C}} h_{b,c}c \in \mathfrak{H}[[\mathfrak{C}]]$.

The element $b_2 = \mathfrak{H}[[h]](b_1) \in \mathfrak{H}[[\mathfrak{H}[[\mathfrak{C}]]]]$ has the form $b_2 = \sum_{c\in\mathfrak{C}} h_{b,c}h(c)$, where $h(c) = \sum_{c_2\in\mathfrak{C}} h_{c,c_2}c_2$ for every $c \in \mathfrak{C}$. The children of a vertex of depth 1 marked by c in the tree B are marked by all the elements $c_2 \in \mathfrak{C}$; and the edge leading from c to c_2 is marked by the element $h_{c,c_2} \in \mathfrak{H}$.

Generally, the children of any vertex in B are marked by all the elements of $\mathfrak{C}$; we will write that the children of any fixed vertex of degree $n-1$ are marked by all the elements $c_n \in \mathfrak{C}$, one such child for every element $c_n \in \mathfrak{C}$. Thus, a vertex v of depth n in B is characterized by its root path, which passes from the root vertex b through vertices marked by $c_1 = c$, c_2, …, and comes to the vertex v marked by $c(v) = c_n$. So the set of all vertices of depth n in B is bijective to $\mathfrak{C}^n = \{(c_1, \dots, c_n) \mid c_i \in \mathfrak{C}\}$. The edge going from a vertex marked by c_{n-1} to its child marked by c_n is marked by the element $h_{c_{n-1},c_n} \in \mathfrak{H}$.

In addition to marking all the vertices and edges of B, let us also mark all the root paths. A root path going from the root vertex b to a vertex marked by c_1, to a vertex marked by c_2, etc., and coming to a vertex v marked by c_n, goes along the edges marked by the elements h_{b,c_1}, h_{c_1,c_2}, …, $h_{c_{n-1},c_n} \in \mathfrak{H}$. We mark such a root path by the product $r(v) = h_{b,c_1} \cdots h_{c_{n-1},c_n} \in \mathfrak{H}$ of the elements marking its edges.

The purpose of this construction is to observe that the element $a_n \in \mathfrak{H}[[\mathfrak{C}]]$ can be expressed as the infinite sum $a_n = \sum_{v \in B_n} r(v)c(v)$ over the set B_n of all vertices of depth n in the tree B. This sum converges in the topology of $\mathfrak{H}[[\mathfrak{C}]]$.

In order to show that the sequence of elements a_n converges to zero in $\mathfrak{H}[[\mathfrak{C}]]$, choose a proper open right ideal $\mathfrak{J} \subset \mathfrak{R}$. Denote by $B^{\mathfrak{J}}$ the subtree of B formed by all the vertices $v \in B$ with $r(v) \notin \mathfrak{J}$. The root vertex belongs to $B^{\mathfrak{J}}$, since $1 \notin \mathfrak{J}$; and whenever $r(v) \in \mathfrak{J}$ for some $v \in B$, one also has $r(w) \in B$ for all the descendants $w \in B$ of the vertex v; so $B^{\mathfrak{J}}$ is indeed a tree.

Furthermore, the tree $B^{\mathfrak{J}}$ is locally finite, because for every vertex $v \in B$ with $c(v) = c_{n-1}$ there exists an open right ideal $\mathfrak{I} \subset \mathfrak{R}$ such that $r(v)\mathfrak{I} \subset \mathfrak{J}$, and the marking element h_{c_{n-1},c_n} of all but a finite subset of the edges going down from v belongs to $\mathfrak{I} \cap \mathfrak{H}$ (as $h(c_{n-1}) = \sum_{c_n \in \mathfrak{C}} h_{c_{n-1},c_n} c_n$ is an element of $\mathfrak{H}[[\mathfrak{C}]]$). So, denoting by $vc_n \in B$ the child of v marked by c_n, we have $r(vc_n) = r(v)h_{c_{n-1},c_n} \in \mathfrak{J}$ for all but a finite subset of $c_n \in \mathfrak{C}$.

Finally, the tree $B^{\mathfrak{J}}$ has no infinite branches, since the ring $\mathfrak{H}$ is topologically left T-nilpotent. Indeed, the sequence of the marking elements $r(v_n)$ of the root paths of the vertices v_n along any infinite branch in B converges to zero in $\mathfrak{H}$, hence $r(v_n) \in \mathfrak{J} \cap \mathfrak{H}$ for $n \gg 0$. By the König lemma, it follows that the tree $B^{\mathfrak{J}}$ is finite, so it has a finite depth m. Thus $a_n \in (\mathfrak{J} \cap \mathfrak{H})[[\mathfrak{C}]]$ for all $n > m$.

Now we can finish the proof of the lemma. Since the sequence a_n converges to zero in $\mathfrak{H}[[\mathfrak{C}]]$ as $n \to \infty$, the sequence $q_n = \mathfrak{H}[[h]](a_{n-1}) \in \mathfrak{H}[[\mathfrak{H}[[\mathfrak{C}]]]]$ converges to zero in the topology of $\mathfrak{H}[[Y]]$, where $Y = \mathfrak{H}[[\mathfrak{C}]]$. So the sum $\sum_{n=2}^{\infty} q_n \in \mathfrak{H}[[\mathfrak{H}[[\mathfrak{C}]]]]$ is well-defined as the limit of finite partial sums. Furthermore, we have $\mathfrak{H}[[\pi]](q_{n+1}) = a_n = \phi_{\mathfrak{C}}(q_n)$ for all $n \geq 2$ and $\mathfrak{H}[[\pi]](q_2) = a_1 = b_1$. Hence

$$\mathfrak{H}[[\pi]]\left(\sum\nolimits_{n=2}^{\infty} q_n\right) - \phi_{\mathfrak{C}}\left(\sum\nolimits_{n=2}^{\infty} q_n\right) = b_1$$

(we recall that the maps $\mathfrak{H}[[\pi]]$ and $\phi_{\mathfrak{C}}$ are continuous, as mentioned in the above discussion). Therefore, $b = \pi(b_1) = 0$ by the contraassociativity equation,

$$\pi \circ (\mathfrak{H}[[\pi]] - \phi_{\mathfrak{C}}) = 0.$$

□

Lemma 4.3. *Let H be a separated topological ring without unit, with a base of neighborhoods of zero formed by open right ideals, and let $K \subset H$ be a closed two-sided ideal. Then H is topologically left T-nilpotent if and only if both K and H/K are.*

Proof. The "only if" assertion is obvious; let us prove the "if". Let a_1, a_2, a_3, ... be a sequence of element in H, and let $I \subset H$ be an open right ideal. Denote by $\bar{a}_i$ the images of the elements a_i in H/K. For any open right ideal $J \subset H$, we will denote by $\overline{J} \subset H/K$ the image of the ideal J. Then $\overline{J}$ is an open right ideal in H/K.

Since H/K is topologically left T-nilpotent, there exists an integer $n_1 \geq 1$ such that the product $\bar{a}_1 \cdots \bar{a}_{n_1}$ belongs to $\overline{I}$. Let $J_1 \subset H$ be an open right ideal such that $a_1 \cdots a_{n_1} J_1 \subset I$. Then there exists an integer $n_2 > n_1$ such that the product $\bar{a}_{n_1+1} \cdots \bar{a}_{n_2}$ belongs to $\overline{J}_1$. Let $J_2 \subset H$ be an open right ideal such that $a_{n_1+1} \cdots a_{n_2} J_2 \subset J_1$, etc. Proceeding in this way, we construct a sequence of integers $0 = n_0 < n_1 < n_1 < n_2 < \cdots$ and open right ideals $I = J_0$, J_1, J_2, $\ldots \subset H$ such that $\bar{a}_{n_{m-1}+1} \cdots \bar{a}_{n_m} \in \overline{J}_{m-1}$ and $a_{n_{m-1}+1} \cdots a_{n_m} J_m \subset J_{m-1}$ for all $m \geq 1$.

For every $m \geq 1$, we have $a_{n_{m-1}+1} \cdots a_{n_m} \in J_{m-1} + K$. Choose $b_m \in J_{m-1}$ and $c_m \in K$ such that $a_{n_{m-1}+1} \cdots a_{n_m} = b_m + c_m$. Since K is topologically left T-nilpotent, there exists $m \geq 1$ such that the product $c_1 c_2 \cdots c_m$ belongs to $I \cap K$. Now we have

$$\begin{gathered} a_1 \cdots a_{n_m} = (b_1 + c_1) \cdots (b_m + c_m) = c_1 c_2 \cdots c_m + b_1 c_2 c_3 \cdots c_m \\ + (b_1 + c_1) b_2 c_3 c_4 \cdots c_m + \cdots + (b_1 + c_1)(b_2 + c_2) \cdots (b_{m-2} + c_{m-2}) b_{m-1} c_m \\ + (b_1 + c_1)(b_2 + c_2) \cdots (b_{m-1} + c_{m-1}) b_m \\ \in I \cap K + J_0 + (b_1 + c_1) J_1 + \cdots + (b_1 + c_1) \cdots (b_{m-1} + c_{m-1}) J_m = I. \end{gathered}$$

□

5. Products of Topological Rings

Let Γ be a set and $(A_\gamma)_{\gamma \in \Gamma}$ be a family of topological abelian groups, each of them with a base of neighborhoods of zero B_γ consisting of open subgroups. The *product topology* on the Cartesian product $A = \prod_{\gamma \in \Gamma} A_\gamma$ has a base of neighborhoods of zero formed by the subgroups $\prod_{\delta \in \Delta} U_\delta \times \prod_{\gamma \in \Gamma \setminus \Delta} A_\gamma$, where $\Delta \subset \Gamma$ are finite subsets and $U_\delta \in B_\delta$. The topological group $A = \prod_{\gamma \in \Gamma} A_\gamma$ does not depend on the choice of bases of neighborhoods of zero B_γ in topological groups A_γ. When the topological groups A_γ are separated, so is the topological group $\prod_{\gamma \in \Gamma} A_\gamma$.

If $A'_\gamma \subset A_\gamma$ are subgroups in topological abelian groups A_γ and A'_γ are viewed as topological abelian groups in the induced topology, then the product topology on $\prod_{\gamma \in \Gamma} A'_\gamma$ coincides with the induced topology on $\prod_{\gamma \in \Gamma} A'_\gamma \subset \prod_{\gamma \in \Gamma} A_\gamma$. When the subgroups A'_γ are closed in A_γ, so is the subgroup $\prod_{\gamma \in \Gamma} A'_\gamma \subset \prod_{\gamma \in \Gamma} A_\gamma$. If the quotient groups $A''_\gamma = A_\gamma / A'_\gamma$ are viewed as topological abelian groups in the quotient topology, then the product topology on $A'' = \prod_{\gamma \in \Gamma} A''_\gamma$ coincides with the quotient topology on $A'' = \prod_{\gamma \in \Gamma} A_\gamma \big/ \prod_{\gamma \in \Gamma} A'_\gamma$.

Let $(\mathfrak{A}_\gamma)_{\gamma \in \Gamma}$ be a family of complete, separated topological abelian groups. Then the group $\mathfrak{A} = \prod_{\gamma \in \Gamma} \mathfrak{A}_\gamma$ is complete and separated in the product topology. Moreover,

for any set X there is a natural isomorphism of abelian groups
$$\mathfrak{A}[[X]] \cong \prod\nolimits_{\gamma\in\Gamma} \mathfrak{A}_\gamma[[X]].$$
If $\mathfrak{H}_\gamma \subset \mathfrak{A}_\gamma$ are strongly closed subgroups then $\mathfrak{H} = \prod_{\gamma\in\Gamma} \mathfrak{H}_\gamma$ is a strongly closed subgroup in $\mathfrak{A} = \prod_{\gamma\in\Gamma} \mathfrak{A}_\gamma$ (in the sense of Section 1.11).

If $(R_\gamma)_{\gamma\in\Gamma}$ is a family of topological rings (with or without unit), then $R = \prod_{\gamma\in\Gamma} R_\gamma$ is a topological ring (with or without unit, respectively) in the product topology. If each of the topological rings R_γ has a base of neighborhoods of zero formed by open right (resp., two-sided) ideals, then the ring R also has a base of neighborhoods of zero formed by open right (resp., two-sided) ideals. If H_γ are topologically nil (resp., topologically left T-nilpotent) separated topological rings without unit, then their product $H = \prod_{\gamma\in\Gamma} H_\gamma$ in its product topology is also a topologically nil (resp., topologically left T-nilpotent) topological ring without unit.

The following lemma is the main result of this section. Part (a) is an easy version of part (b), which is a generalization of [27, Lemma A.2.2] (see also [41, Theorem 4.5]).

Lemma 5.1. *Let $(\mathfrak{R}_\gamma)_{\gamma\in\Gamma}$ be a family of complete, separated topological rings, each of them having a base of neighborhoods of zero formed by open right ideals; and let $\mathfrak{R} = \prod_{\gamma\in\Gamma} \mathfrak{R}_\gamma$ be their product. Then*

(a) *the coproduct functor $(\mathcal{N}_\gamma)_{\gamma\in\Gamma} \longmapsto \bigoplus_{\gamma\in\Gamma} \mathcal{N}_\gamma$ establishes an equivalence between the Cartesian product of the abelian categories of discrete right $\mathfrak{R}_\gamma$-modules over all $\gamma \in \Gamma$ and the abelian category of discrete right $\mathfrak{R}$-modules;*

(b) *the product functor $(\mathfrak{C}_\gamma)_{\gamma\in\Gamma} \longmapsto \prod_{\gamma\in\Gamma} \mathfrak{C}_\gamma$ establishes an equivalence between the Cartesian product of the abelian categories of left $\mathfrak{R}_\gamma$-contramodules over all $\gamma \in \Gamma$ and the abelian category of left $\mathfrak{R}$-contramodules.*

Proof. Part (a): for every $\gamma \in \Gamma$, denote by $e_\gamma = (e_{\gamma,\gamma'})_{\gamma'\in\Gamma} \in \mathfrak{R}$ the central idempotent element whose γ'-component $e_{\gamma,\gamma'}$ is equal to $0 \in \mathfrak{R}_{\gamma'}$ for all $\gamma' \in \Gamma$, $\gamma' \neq \gamma$, and whose γ-component $e_{\gamma,\gamma}$ is equal to $1 \in \mathfrak{R}_\gamma$. For any discrete right $\mathfrak{R}$-module $\mathcal{N}$, the subgroup $\mathcal{N}e_\gamma \subset \mathcal{N}$ is the maximal $\mathfrak{R}$-submodule in $\mathcal{N}$ whose right $\mathfrak{R}$-module structure comes from a (discrete) right $\mathfrak{R}_\gamma$-module structure via the natural continuous ring homomorphism $p_\gamma\colon \mathfrak{R} \longrightarrow \mathfrak{R}_\gamma$. So, in the notation of Sections 1.9 and 1.12, we have $p_\gamma^\diamond(\mathcal{N}) = \mathcal{N}e_\gamma$. We claim that the functor $\mathcal{N} \longmapsto (\mathcal{N}_\gamma = \mathcal{N}e_\gamma)_{\gamma\in\Gamma}$ is quasi-inverse to the functor $(\mathcal{N}_\gamma)_{\gamma\in\Gamma} \longmapsto \mathcal{N} = \bigoplus_{\gamma\in\Gamma} p_{\gamma\diamond}\mathcal{N}_\gamma$, where $\mathcal{N} \in \mathsf{discr}\text{–}\mathfrak{R}$ and $\mathcal{N}_\gamma \in \mathsf{discr}\text{–}\mathfrak{R}_\gamma$. In other words, this simply means that any discrete right $\mathfrak{R}$-module $\mathcal{N}$ is the direct sum of its submodules $\mathcal{N}e_\gamma \subset \mathcal{N}$.

Indeed, the idempotents $e_\gamma \in \mathfrak{R}$, $\gamma \in \Gamma$ are orthogonal to each other, which easily implies injectivity of the map $\bigoplus_{\gamma\in\Gamma} \mathcal{N}e_\gamma \longrightarrow \mathcal{N}$. To prove surjectivity, consider an element $b \in \mathcal{N}$. Since $\mathcal{N}$ is a discrete right $\mathfrak{R}$-module by assumption, there exists a neighborhood of zero $\mathfrak{U} \subset \mathfrak{R}$ such that $b\mathfrak{U} = 0$. By the definition of the product topology, there exists a finite subset $\Delta \subset \Gamma$ such that $\mathfrak{J} = \prod_{\gamma\in\Gamma\setminus\Delta} \mathfrak{R}_\gamma \subset \mathfrak{U} \subset \mathfrak{R}$. Consider the submodule $\mathcal{N}_\mathfrak{J} \subset \mathcal{N}$ of all elements annihilated by the closed two-sided ideal $\mathfrak{J} \subset \mathfrak{R}$; then we have $b \in \mathcal{N}_\mathfrak{J}$. Now we have $1 - \sum_{\delta\in\Delta} e_\delta \in \mathfrak{J}$, hence $b = \sum_{\delta\in\Delta} be_\delta$ is a decomposition of the element b into the sum of elements $be_\delta \in \mathcal{N}e_\delta$.

Part (b): we keep our notation for the central idempotent elements $e_\gamma \in \mathfrak{R}$. For any left $\mathfrak{R}$-contramodule $\mathfrak{C}$, the map $e_\gamma\colon \mathfrak{C} \longrightarrow e_\gamma\mathfrak{C}$ represents $e_\gamma\mathfrak{C}$ as a quotient group of $\mathfrak{C}$. This is the maximal quotient $\mathfrak{R}$-contramodule of $\mathfrak{C}$ whose left $\mathfrak{R}$-contramodule structure comes from a left $\mathfrak{R}_\gamma$-contramodule structure via the homomorphism p_γ. So, in the notation of Sections 1.9 and 1.12, we have $p_\gamma^\sharp(\mathfrak{C}) = e_\gamma\mathfrak{C}$. We claim that the functor $\mathfrak{C} \longmapsto (\mathfrak{C}_\gamma = e_\gamma\mathfrak{C})_{\gamma\in\Gamma}$ is quasi-inverse to the functor $(\mathfrak{C}_\gamma)_{\gamma\in\Gamma} \longmapsto \mathfrak{C} = \prod_{\gamma\in\Gamma} p_{\gamma\sharp}\mathfrak{C}_\gamma$, where $\mathfrak{C} \in \mathfrak{R}\text{–}\mathsf{contra}$ and $\mathfrak{C}_\gamma \in \mathfrak{R}_\gamma\text{–}\mathsf{contra}$. In other words, this simply means that the natural map

$$e = (e_\gamma)_{\gamma\in\Gamma}\colon \mathfrak{C} \longrightarrow \prod\nolimits_{\gamma\in\Gamma} e_\gamma\mathfrak{C}$$

is an isomorphism for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$.

Indeed, let us construct an inverse map to e. Given a family of elements $c_\gamma \in e_\gamma\mathfrak{C}$, we consider them as elements of $\mathfrak{C}$ and assign to them the element

$$f((c_\gamma)_{\gamma\in\Gamma}) = \pi_\mathfrak{C}\left(\sum\nolimits_{\gamma\in\Gamma} e_\gamma c_\gamma\right).$$

Here it is important that the family of central idempotent elements $e_\gamma \in \mathfrak{R}$ converges to zero in the topology of $\mathfrak{R}$, so the expression $\sum_{\gamma\in\Gamma} e_\gamma c_\gamma$ defines an element of the set $\mathfrak{R}[[\mathfrak{C}]]$ of all convergent infinite formal linear combinations of elements of $\mathfrak{C}$ with the coefficients in $\mathfrak{R}$ (to which the contraaction map $\pi_\mathfrak{C}\colon \mathfrak{R}[[\mathfrak{C}]] \longrightarrow \mathfrak{C}$ can be applied). To check that $e\circ f = \mathrm{id}$, it suffices to compute, for any family of elements $(c_\gamma \in \mathfrak{C})_{\gamma\in\Gamma}$ and any fixed element $\gamma' \in \Gamma$,

$$e_{\gamma'}\pi_\mathfrak{C}\left(\sum\nolimits_{\gamma\in\Gamma} e_\gamma c_\gamma\right) = \pi_\mathfrak{C}\left(\sum\nolimits_{\gamma\in\Gamma} e_{\gamma'}e_\gamma c_\gamma\right) = e_{\gamma'}c_{\gamma'}$$

using the contraassociativity equation. To check that $f\circ e = \mathrm{id}$, one computes, for any element $c \in \mathfrak{C}$,

$$\pi_\mathfrak{C}\left(\sum\nolimits_{\gamma\in\Gamma} e_\gamma(e_\gamma c)\right) = \left(\sum\nolimits_{\gamma\in\Gamma} e_\gamma\right)c = c$$

by the contraassociativity equation and because the infinite sum $\sum_{\gamma\in\Gamma} e_\gamma$ converges to 1 in the topology of $\mathfrak{R}$. $\square$

6. Projectivity of Flat Contramodules

In this section and in the next one, we consider the following setting. Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Let $\mathfrak{H} \subset \mathfrak{R}$ be a strongly closed two-sided ideal in $\mathfrak{R}$ (see Sections 1.11–1.12). Assume that the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is isomorphic, as a topological ring, to the product $\prod_{\gamma\in\Gamma} S_\gamma$ of a family of discrete rings S_γ (viewed as a topological ring in the product topology), and that every ring S_γ is a classically simple (i. e., simple Artinian) ring. In other words, S_γ is the matrix ring of some finite order over a division ring (for every γ). Finally, we will also assume that the ideal $\mathfrak{H}$ is topologically left T-nilpotent.

Denote the natural continuous ring homomorphisms by $p\colon \mathfrak{R} \longrightarrow \mathfrak{S}$, $q_\gamma\colon \mathfrak{S} \longrightarrow S_\gamma$, and $p_\gamma = pq_\gamma\colon \mathfrak{R} \longrightarrow S_\gamma$. Set $\mathfrak{J}_\gamma = \ker(p_\gamma) \subset \mathfrak{R}$. Recall that, according to the discussion in Sections 1.9 and 1.12, the fully faithful functor of contrarestriction of scalars $p_\sharp\colon \mathfrak{S}\text{--}\mathsf{contra} \longrightarrow \mathfrak{R}\text{--}\mathsf{contra}$ has a left adjoint functor of contraextension of scalars $p^\sharp\colon \mathfrak{R}\text{--}\mathsf{contra} \longrightarrow \mathfrak{S}\text{--}\mathsf{contra}$ computable as $p^\sharp(\mathfrak{C}) = \mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C}$. Similarly, the fully faithful functor $p_{\gamma\sharp}\colon S_\gamma\text{--}\mathsf{mod} = S_\gamma\text{--}\mathsf{contra} \longrightarrow \mathfrak{R}\text{--}\mathsf{contra}$ has a left adjoint functor $p_\gamma^\sharp\colon \mathfrak{R}\text{--}\mathsf{contra} \longrightarrow S_\gamma\text{--}\mathsf{mod}$ computable as $p_\gamma^\sharp(\mathfrak{C}) = \mathfrak{C}/\mathfrak{J}_\gamma \leftthreetimes \mathfrak{C}$. The fully faithful functor $q_{\gamma\sharp}\colon S_\gamma\text{--}\mathsf{mod} \longrightarrow \mathfrak{S}\text{--}\mathsf{contra}$ has a left adjoint functor $q_\gamma^\sharp\colon \mathfrak{S}\text{--}\mathsf{contra} \longrightarrow S_\gamma\text{--}\mathsf{mod}$, which can be computed in the same fashion.

Finally, according to Lemma 5.1(b), for any left $\mathfrak{S}$-contramodule $\mathfrak{D}$ we have a natural direct product decomposition $\mathfrak{D} \cong \prod_{\gamma\in\Gamma} q_{\gamma\sharp}q_\gamma^\sharp\mathfrak{D}$. So, in particular, for any left $\mathfrak{R}$-contramodule $\mathfrak{C}$ one has $\mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C} \cong \prod_{\gamma\in\Gamma} \mathfrak{C}/\mathfrak{J}_\gamma \leftthreetimes \mathfrak{C}$.

The analogous assertions hold for discrete right modules. The fully faithful functor of corestriction of scalars $p_\diamond\colon \mathsf{discr}\text{--}\mathfrak{S} \longrightarrow \mathsf{discr}\text{--}\mathfrak{R}$ has a right adjoint functor of coextension of scalars $p^\diamond\colon \mathsf{discr}\text{--}\mathfrak{R} \longrightarrow \mathsf{discr}\text{--}\mathfrak{S}$ computable as $p^\diamond(\mathcal{N}) = \mathcal{N}_\mathfrak{H}$. The fully faithful functor $p_{\gamma\diamond}\colon \mathsf{mod}\text{--}S_\gamma = \mathsf{discr}\text{--}S_\gamma \longrightarrow \mathsf{discr}\text{--}\mathfrak{R}$ has a right adjoint functor $p_\gamma^\diamond\colon \mathsf{discr}\text{--}\mathfrak{R} \longrightarrow \mathsf{mod}\text{--}S_\gamma$ computable as $p_\gamma^\diamond(\mathcal{N}) = \mathcal{N}_{\mathfrak{J}_\gamma}$. The fully faithful functor $q_{\gamma\diamond}\colon \mathsf{mod}\text{--}S_\gamma \longrightarrow \mathsf{discr}\text{--}\mathfrak{S}$ is a right adjoint functor $q_\gamma^\diamond\colon \mathsf{discr}\text{--}\mathfrak{S} \longrightarrow \mathsf{mod}\text{--}S_\gamma$, which can be computed similarly. Finally, by Lemma 5.1(a), for any discrete right $\mathfrak{S}$-module $\mathcal{M}$ we have a natural direct sum decomposition $\mathcal{M} \cong \bigoplus_{\gamma\in\Gamma} q_{\gamma\diamond}q_\gamma^\diamond\mathcal{M}$; so, in particular, for any discrete right $\mathfrak{R}$-module $\mathcal{N}$ one has $\mathcal{N}_\mathfrak{H} \cong \bigoplus_{\gamma\in\Gamma} \mathcal{N}_{\mathfrak{J}_\gamma}$.

Lemma 6.1. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{H} \subset \mathfrak{R}$ be a topologically left T-nilpotent strongly closed two-sided ideal. Let $f\colon \mathfrak{F}' \longrightarrow \mathfrak{F}''$ be a morphism of flat left $\mathfrak{R}$-contramodules such that the induced morphism of left $\mathfrak{S}$-contramodules $\mathfrak{F}'/\mathfrak{H} \leftthreetimes \mathfrak{F}' \longrightarrow \mathfrak{F}''/\mathfrak{H} \leftthreetimes \mathfrak{F}''$ is an isomorphism. Then the morphism f is surjective and its kernel is contained in $\bigcap_{\mathfrak{I}\subset\mathfrak{R}} \mathfrak{I} \leftthreetimes \mathfrak{F}' \subset \mathfrak{F}'$, where the intersection is taken over all the open right ideals $\mathfrak{I} \subset \mathfrak{R}$.*

Proof. The conclusion that f is surjective does not depend on the flatness assumption on $\mathfrak{F}'$ and $\mathfrak{F}''$, and only requires surjectivity of the map $\mathfrak{F}'/\mathfrak{H} \leftthreetimes \mathfrak{F}' \longrightarrow \mathfrak{F}''/\mathfrak{H} \leftthreetimes \mathfrak{F}''$. It suffices to set $\mathfrak{C} = \operatorname{coker}(f)$, observe that $\mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C} = 0$, and apply the contramodule Nakayama Lemma 4.2 in order to conclude that $\mathfrak{C} = 0$.

In order to prove the assertion about $\ker(f)$, we will show that the map of abelian groups $\mathcal{N} \odot_\mathfrak{R} f\colon \mathcal{N} \odot_\mathfrak{R} \mathfrak{F}' \longrightarrow \mathcal{N} \odot_\mathfrak{R} \mathfrak{F}''$ is an isomorphism for any discrete right $\mathfrak{R}$-module $\mathcal{N}$. In particular, it will follow that the map $\mathfrak{F}'/\mathfrak{I} \leftthreetimes \mathfrak{F}' \longrightarrow \mathfrak{F}''/\mathfrak{I} \leftthreetimes \mathfrak{F}''$ is an isomorphism for any open right ideal $\mathfrak{I} \subset \mathfrak{R}$, hence $\ker(f) \subset \mathfrak{I} \leftthreetimes \mathfrak{F}' \subset \mathfrak{F}'$.

Indeed, for any discrete right $\mathfrak{S}$-module $\mathcal{M}$, one has $p_\diamond\mathcal{M} \odot_\mathfrak{R} \mathfrak{F}' = \mathcal{M} \odot_\mathfrak{S} p^\sharp\mathfrak{F}' = \mathcal{M} \odot_\mathfrak{S} p^\sharp\mathfrak{F}'' = p_\diamond\mathcal{M} \odot_\mathfrak{R} \mathfrak{F}''$ (see Section 1.9), so the map $\mathcal{M} \odot_\mathfrak{R} f$ is an isomorphism for any discrete right $\mathfrak{R}$-module $\mathcal{M}$ annihilated by $\mathfrak{H}$. Now, according to the discrete module Nakayama Lemma 4.1, any discrete right $\mathfrak{R}$-module $\mathcal{N}$ has an increasing filtration $0 = F_0\mathcal{N} \subset F_1\mathcal{N} \subset F_2\mathcal{N} \subset \cdots \subset F_\alpha\mathcal{N} = \mathcal{N}$, indexed by some ordinal α, such that the quotient module $F_{i+1}\mathcal{N}/F_i\mathcal{N}$ is annihilated by $\mathfrak{H}$ for all ordinals $i < \alpha$

and $F_j\mathcal{N} = \bigcup_{i<j} F_i\mathcal{N}$ for all limit ordinals $j \le \alpha$. Since the functors of contratensor product with $\mathfrak{F}'$ and $\mathfrak{F}''$ are exact on the abelian category $\mathsf{discr}\text{–}\mathfrak{R}$ by assumption, and since they also preserve colimits, it follows by induction on i that $F_i\mathcal{N} \odot_{\mathfrak{R}} f$ is an isomorphism for all $0 \le i \le \alpha$. $\square$

Theorem 6.2. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, let $\mathfrak{H} \subset \mathfrak{R}$ be a topologically left T-nilpotent strongly closed two-sided ideal, and let $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ be the quotient ring. Let $\mathfrak{F}$ be a flat left $\mathfrak{R}$-contramodule. Then the left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective if and only if the left $\mathfrak{S}$-contramodule $\mathfrak{F}/\mathfrak{H} \leftthreetimes \mathfrak{F}$ is projective.*

Proof. The functor of contraextension of scalars $f^\sharp$ with respect to a continuous homomorphism of topological rings f always takes projective contramodules to projective contramodules, since it is left adjoint to an exact functor of contrarestriction of scalars $f_\sharp$ (cf. Sections 1.9 and 1.12). So the "only if" assertion is obvious.

To prove the "if", choose a set X_0 such that the projective left $\mathfrak{S}$-contramodule $\mathfrak{Q} = \mathfrak{F}/\mathfrak{H} \leftthreetimes \mathfrak{F}$ is a direct summand of the free left $\mathfrak{S}$-contramodule $\mathfrak{S}[[X_0]]$. Setting $X = \mathbb{Z}_{\ge 0} \times X_0$, so that $\mathfrak{S}[[X]]$ is the coproduct of a countable family of copies of $\mathfrak{S}[[X_0]]$ in $\mathfrak{S}\text{–}\mathsf{contra}$, and using the cancellation trick, one can see that the left $\mathfrak{S}$-contramodule $\mathfrak{Q} \oplus \mathfrak{S}[[X]]$ is isomorphic to $\mathfrak{S}[[X]]$.

Consider the left $\mathfrak{R}$-contramodule $\mathfrak{F}'' = \mathfrak{F} \oplus \mathfrak{R}[[X]]$ and put $\mathfrak{Q}'' = \mathfrak{F}''/\mathfrak{H} \leftthreetimes \mathfrak{F}'' \cong \mathfrak{Q} \oplus \mathfrak{S}[[X]]$. Then $\mathfrak{Q}''$ is a free left $\mathfrak{S}$-contramodule. Let us write $\mathfrak{Q}'' = \mathfrak{S}[[Y]]$ (where Y is a subset in $\mathfrak{Q}''$ bijective to X). Set $\mathfrak{F}' = \mathfrak{R}[[Y]]$ to be the free left $\mathfrak{R}$-contramodule with Y generators. Then we have natural surjective left $\mathfrak{R}$-contramodule morphisms $\mathfrak{F}'' \longrightarrow p_\sharp\mathfrak{Q}'' = \mathfrak{S}[[Y]]$ and $\mathfrak{F}' \longrightarrow \mathfrak{S}[[Y]]$. Since $\mathfrak{F}'$ is a projective left $\mathfrak{R}$-contramodule, the latter morphism lifts to a left $\mathfrak{R}$-contramodule morphism $f\colon \mathfrak{F}' \longrightarrow \mathfrak{F}''$ satisfying the assumption of Lemma 6.1. (Notice that both the left $\mathfrak{R}$-contramodules $\mathfrak{F}'$ and $\mathfrak{F}''$ are flat.) Thus the morphism f is surjective with $\ker(f) \subset \bigcap_{\mathfrak{I}\subset\mathfrak{R}} \mathfrak{I} \leftthreetimes \mathfrak{F}'$.

Since the left $\mathfrak{R}$-contramodule $\mathfrak{F}'$ is projective (and even free) by construction, the natural map $\mathfrak{F}' \longrightarrow \varprojlim_{\mathfrak{I}\subset\mathfrak{R}} \mathfrak{F}'/\mathfrak{I} \leftthreetimes \mathfrak{F}'$ is an isomorphism (see Section 1.10). So one has $\bigcap_{\mathfrak{I}\subset\mathfrak{R}} \mathfrak{I} \leftthreetimes \mathfrak{F}' = 0$. Hence the morphism f is an isomorphism. We have shown that $\mathfrak{F}''$ is a free left $\mathfrak{R}$-contramodule. Finally, we can conclude the left $\mathfrak{R}$-contramodule $\mathfrak{F}$ is projective as a direct summand of $\mathfrak{F}''$. $\square$

The following corollary is a generalization of [27, Lemma A.3].

Corollary 6.3. *In the assumptions formulated in the beginning of this section, all flat left $\mathfrak{R}$-contramodules are projective.*

Proof. The abelian category $\mathfrak{S}\text{–}\mathsf{contra} \cong \prod_{\gamma\in\Gamma} S_\gamma\text{–}\mathsf{mod}$ (see Lemma 5.1(b)) is semisimple in these assumptions. So all left $\mathfrak{S}$-contramodules are projective, and the assertion of the corollary follows from Theorem 6.2. $\square$

7. Existence of Projective Covers

This section contains two proofs of its main result, which is Theorem 7.4. The first one is very short, consisting only of two references: one of them to the main result of the previous section, and the other one to a general theorem from category theory. The second proof is longer and more explicit.

Theorem 7.1. *Let B be a locally presentable abelian category with enough projective objects. Assume that the class of all projective objects is closed under direct limits in B. Then every object of B has a projective cover.*

Proof. This is a particular case of [35, Theorem 2.7, or Corollary 3.7, or Corollary 4.17]. (Cf. Sections 11 and 13 below for some background.) $\square$

Corollary 7.2. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero consisting of open right ideals. Assume that all flat left $\mathfrak{R}$-contramodules are projective. Then every left $\mathfrak{R}$-contramodule has a projective cover.*

Proof. The abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ is locally presentable [35, Section 5]. All the projective left $\mathfrak{R}$-contramodules are flat, and the class of all flat left $\mathfrak{R}$-contramodules is closed under direct limits (see Section 2). Thus the assertion of the corollary follows from Theorem 7.1. $\square$

This is essentially all we need for our first proof of Theorem 7.4. To prepare ground for the second one, we have to address the question of lifting of idempotents.

It is a classical fact in the associative ring theory that idempotents can be lifted modulo any nil ideal. The following lemma provides a topological generalization. We refer to Section 4 for the definition of a *topologically nil* topological ring without unit.

Lemma 7.3. *Let $\mathfrak{R}$ be complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Let $\mathfrak{H} \subset \mathfrak{R}$ be a topologically nil closed two-sided ideal, and let $S = \mathfrak{R}/\mathfrak{H}$ be the quotient ring. Then any idempotent element in S can be lifted to an idempotent element in $\mathfrak{R}$.*

Proof. We adopt the argument from [22, Tag 00J9] to the situation at hand. Let $\bar{e} \in S$ be an idempotent element. Choose any preimage $f \in \mathfrak{R}$ of the element $\bar{e} \in S$. Proceeding by induction, we construct a sequence of elements $e_k \in \mathfrak{R}$, $k \geq 0$, starting from $e_0 = f$ and passing from e_k to e_{k+1} by the rule

$$e_{k+1} = e_k - (2e_k - 1)(e_k^2 - e_k) = 3e_k^2 - 2e_k^3, \qquad k \geq 0.$$

A straightforward computation yields

$$e_{k+1}^2 - e_{k+1} = (4e_k^2 - 4e_k - 3)(e_k^2 - e_k)^2.$$

Now let us show that the sequence of elements $e_k \in \mathfrak{R}$ converges in the topology of $\mathfrak{R}$ as $k \to \infty$, and that its limit e is an idempotent element in $\mathfrak{R}$ whose image in S is equal to $\bar{e}$. Indeed, set $h = f^2 - f$; then we have $h \in \mathfrak{H}$, since $\bar{e}^2 - \bar{e} = 0$ in S. Notice that all the elements f, h, and e_k belong to the subring generated by f in $\mathfrak{R}$

over $\mathbb{Z}$; so they commute with each other. It follows from the above formulas by a simple induction on k that $e_k^2 - e_k \in h^{2^k}\mathfrak{R}$ for all $k \geq 0$.

Let $\mathfrak{J} \subset \mathfrak{R}$ be an open right ideal. Since the ideal $\mathfrak{H} \subset \mathfrak{R}$ is topologically nil, there exists $n \geq 1$ such that $h^n \in \mathfrak{J} \cap \mathfrak{H}$. Choosing m such that $2^m \geq n$, we find that $e_{k+1} - e_k = (2e_k - 1)(e_k^2 - e_k) \in \mathfrak{J}$ for all $k \geq m$. Thus the sequence of elements e_k converges in $\mathfrak{R}$ as $k \to \infty$, and we can consider its limit $e \in \mathfrak{R}$. We also have $e_k^2 - e_k \in \mathfrak{J}$ for all $k \geq m$, hence $e^2 - e \in \mathfrak{J}$, and, as this holds for all the open right ideals $\mathfrak{J} \subset \mathfrak{R}$, it follows that $e^2 - e = 0$ in $\mathfrak{R}$. Finally, $e_{k+1} - e_k \in h\mathfrak{R} \subset \mathfrak{H}$ for all $k \geq 0$, hence $e - f \in \mathfrak{H}$, and therefore the image of e in S is equal to $\bar{e}$. $\square$

Theorem 7.4. *In the assumptions formulated in the beginning of Section 6, every left $\mathfrak{R}$-contramodule has a projective cover.*

First proof. The assertion follows from Corollaries 6.3 and 7.2. $\square$

Second proof. Let us first show that for every left $\mathfrak{S}$-contramodule $\mathfrak{D}$ there exists a projective left $\mathfrak{R}$-contramodule $\mathfrak{P}$ such that the left $\mathfrak{S}$-contramodule $\mathfrak{P}/\mathfrak{H} \leftthreetimes \mathfrak{P}$ is isomorphic to $\mathfrak{D}$. Indeed, Lemma 5.1(b) applied to the ring $\mathfrak{S} = \prod_{\gamma\in\Gamma} S_\gamma$ implies that any left $\mathfrak{S}$-contramodule, viewed as an object of $\mathfrak{S}\text{–}\mathsf{contra}$, is a coproduct of irreducible left $\mathfrak{S}$-contramodules. The irreducible left $\mathfrak{S}$-contramodules are indexed by the elements $\gamma \in \Gamma$ and have the form $q_{\gamma\sharp}I_\gamma$, where I_γ is the (unique) irreducible left S_γ-module.

One easily finds a (noncentral) idempotent element $i_\gamma \in \mathfrak{S}$ such that $q_{\gamma\sharp}I_\gamma \cong \mathfrak{S}i_\gamma$. Lifting i_γ to an idempotent element $\tilde{i}_\gamma \in \mathfrak{R}$ using Lemma 7.3, one can produce a projective left $\mathfrak{R}$-contramodule $\mathfrak{P}_\gamma = \mathfrak{R}\tilde{i}_\gamma$ such that $\mathfrak{P}_\gamma/\mathfrak{H} \leftthreetimes \mathfrak{P}_\gamma \cong \mathfrak{S}i_\gamma$. Finally, coproducts of projective objects are projective, and the reduction functor $\mathfrak{C} \longmapsto p^\sharp(\mathfrak{C}) = \mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C}$ preserves coproducts, which allows to construct a projective left $\mathfrak{R}$-contramodule $\mathfrak{P}$ such that $\mathfrak{P}/\mathfrak{H} \leftthreetimes \mathfrak{P} \cong \mathfrak{D}$.

Now let $\mathfrak{C}$ be a left $\mathfrak{R}$-contramodule. Consider the left $\mathfrak{S}$-contramodule $\mathfrak{D} = \mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C}$ and find a projective left $\mathfrak{R}$-contramodule $\mathfrak{P}$ such that $\mathfrak{P}/\mathfrak{H} \leftthreetimes \mathfrak{P} \cong \mathfrak{D}$. Then we have two surjective left $\mathfrak{R}$-contramodule morphisms $\mathfrak{P} \longrightarrow p_\sharp\mathfrak{D}$ and $\mathfrak{C} \longrightarrow p_\sharp\mathfrak{D}$. Since $\mathfrak{P} \in \mathfrak{R}\text{–}\mathsf{contra}$ is a projective object, we can lift the former morphism to a left $\mathfrak{R}$-contramodule morphism $f\colon \mathfrak{P} \longrightarrow \mathfrak{C}$ such that the induced morphism $\mathfrak{P}/\mathfrak{H} \leftthreetimes \mathfrak{P} \longrightarrow \mathfrak{C}/\mathfrak{H} \leftthreetimes \mathfrak{C}$ is an isomorphism.

Arguing as in Lemma 6.1 and using the contramodule Nakayama Lemma 4.2, one shows that the map f is surjective. We claim that the morphism f is a projective cover of a left $\mathfrak{R}$-contramodule $\mathfrak{C}$. Indeed, in view of Lemma 3.1, it suffices to check that $\mathfrak{K} = \ker(f)$ is a superfluous $\mathfrak{R}$-subcontramodule in $\mathfrak{P}$.

Let $\mathfrak{G} \subset \mathfrak{P}$ be an $\mathfrak{R}$-subcontramodule such that $\mathfrak{K} + \mathfrak{G} = \mathfrak{P}$. The morphism of left $\mathfrak{S}$-contramodules $p^\sharp(\mathfrak{K}) \longrightarrow p^\sharp(\mathfrak{P}/\mathfrak{G})$ is surjective, because the morphism of left $\mathfrak{R}$-contramodules $\mathfrak{K} \longrightarrow \mathfrak{P}/\mathfrak{G}$ is. On the other hand, the morphism $p^\sharp(\mathfrak{K}) \longrightarrow p^\sharp(\mathfrak{P})$ is zero, since the composition $\mathfrak{K} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{C}$ vanishes and the morphism $p^\sharp(\mathfrak{P}) \longrightarrow p^\sharp(\mathfrak{C})$ is an isomorphism. Therefore, the composition $p^\sharp(\mathfrak{K}) \longrightarrow p^\sharp(\mathfrak{P}) \longrightarrow p^\sharp(\mathfrak{P}/\mathfrak{G})$ also vanishes. It follows that $p^\sharp(\mathfrak{P}/\mathfrak{G}) = 0$, that is $\mathfrak{P}/\mathfrak{G} = \mathfrak{H} \leftthreetimes (\mathfrak{P}/\mathfrak{G})$. Applying Lemma 4.2 again, we conclude that $\mathfrak{P}/\mathfrak{G} = 0$, as desired. $\square$

8. Proof of Main Theorem

Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open two-sided ideals. We will need to assume that one of the following three conditions holds:

(a) the ring $\mathfrak{R}$ is commutative; or
(b) $\mathfrak{R}$ has a countable base of neighborhoods of zero; or
(c) $\mathfrak{R}$ has only a finite number of classically semisimple (semisimple Artinian) discrete quotient rings.

The following theorem is the main result of this paper.

Theorem 8.1. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open two-sided ideals. Assume that one of the conditions (a), (b), or (c) is satisfied. Then the following conditions are equivalent:*

(i) *all flat left $\mathfrak{R}$-contramodules have projective covers;*
($\mathrm{i}^\flat$) *all Bass flat left $\mathfrak{R}$-contramodules have projective covers;*
(ii) *all left $\mathfrak{R}$-contramodules have projective covers;*
(iii) *all flat left $\mathfrak{R}$-contramodules are projective;*
($\mathrm{iii}^\flat$) *all Bass flat left $\mathfrak{R}$-contramodules are projective;*
(iv) *all discrete quotient rings of $\mathfrak{R}$ are left perfect;*
(v) *$\mathfrak{R}$ has a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{H}$ such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology.*

Proof. The implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ ($\mathrm{i}^\flat$) and (iii) $\Longrightarrow$ ($\mathrm{iii}^\flat$) are obvious. So are the implications (iii) $\Longrightarrow$ (i) and ($\mathrm{iii}^\flat$) $\Longrightarrow$ ($\mathrm{i}^\flat$).

For any complete, separated topological ring $\mathfrak{R}$ with a countable base of neighborhoods of zero formed by open right ideals, any left $\mathfrak{R}$-contramodule has a flat cover [35, Corollary 7.9]. Hence the condition (iii) implies (ii) under the assumption of (b). Moreover, Corollary 7.2 provides the implication (iii) $\Longrightarrow$ (ii) for any complete, separated topological ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals. (But we do not need to use either of these arguments.)

The condition (v) was already formulated in the beginning of Section 6. The implications (v) $\Longrightarrow$ (iii) and (v) $\Longrightarrow$ (ii) are provided by Corollary 6.3 and Theorem 7.4, respectively, and hold for any complete, separated topological ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals.

The implication ($\mathrm{iii}^\flat$) $\Longrightarrow$ (iv) is provided by Corollary 2.4, and the implication ($\mathrm{i}^\flat$) $\Longrightarrow$ (iv) by Corollary 3.6. Using the assumption of open two-sided ideals forming a base of neighborhoods of zero in $\mathfrak{R}$, one can obtain the implication ($\mathrm{i}^\flat$) $\Longrightarrow$ ($\mathrm{iii}^\flat$) from Corollaries 2.2 and 3.9.

It is the implication (iv) $\Longrightarrow$ (v) that needs both the assumption that $\mathfrak{R}$ has a base of neighborhoods of zero formed by open two-sided ideals and one of the conditions (a), (b), or (c). Assuming (iv) and denoting by $H(R)$ the Jacobson radical

(= nilradical) of a left perfect discrete ring R, we set

$$\mathfrak{H} = \varprojlim\nolimits_{\mathfrak{I}\subset\mathfrak{R}} H(\mathfrak{R}/\mathfrak{I}) \subset \mathfrak{R},$$

where the projective limit is taken over all the open two-sided ideals $\mathfrak{I}$ in $\mathfrak{R}$. Here the projective limit is well-defined, because for any surjective morphism of left perfect rings $f\colon R' \longrightarrow R''$ one has $f(H(R')) = H(R'')$. In order to finish the proof of the theorem, it remains to apply the next proposition. □

Proposition 8.2. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open two-sided ideals such that all the discrete quotient rings of $\mathfrak{R}$ are left perfect. Assume that one of the conditions (a), (b), or (c) is satisfied. Then $\mathfrak{H} = \varprojlim_{\mathfrak{I}\subset\mathfrak{R}} H(\mathfrak{R}/\mathfrak{I})$ is a topologically left T-nilpotent strongly closed two-sided ideal in $\mathfrak{R}$, and the quotient ring $\mathfrak{S} = \mathfrak{R}/\mathfrak{H}$ is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology.*

Proof. The two-sided ideal $\mathfrak{H} \subset \mathfrak{R}$ is closed by construction and, viewed as a topological ring without unit, it is topologically left T-nilpotent as the projective limit of T-nilpotent discrete rings without unit. In order to prove the remaining assertions, let us consider the three cases separately.

(b) First of all, any closed subgroup in a topological abelian group with a countable base of neighborhoods of zero is strongly closed (see Lemma 1.3).

Furthermore, for any discrete quotient ring $R = \mathfrak{R}/\mathfrak{I}$ of the topological ring $\mathfrak{R}$, we have a short exact sequence

$$0 \longrightarrow H(R) \longrightarrow R \longrightarrow R/H(R) \longrightarrow 0.$$

The transition maps in the projective system $(H(\mathfrak{R}/\mathfrak{I}))_{\mathfrak{I}\subset\mathfrak{R}}$ are surjective, so passing to the (countable filtered) projective limit we get a short exact sequence

$$0 \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{S} = \varprojlim\nolimits_{\mathfrak{I}\subset\mathfrak{R}} R/H(R) \longrightarrow 0.$$

This proves that the topological ring $\mathfrak{S}$ is the topological projective limit of the countable filtered projective system of semisimple Artinian discrete rings $R/H(R)$ and surjective morphisms between them. All such ring homomorphisms are projections onto direct factors, and it follows that $\mathfrak{S}$ is a topological product of simple Artinian discrete rings.

(c) Let $\mathfrak{J}_1$ and $\mathfrak{J}_2 \subset \mathfrak{R}$ be two open two-sided ideals such that the quotient rings $\mathfrak{R}/\mathfrak{J}_1$ and $\mathfrak{R}/\mathfrak{J}_2$ are semisimple Artinian. Since $R = \mathfrak{R}/(\mathfrak{J}_1 \cap \mathfrak{J}_2)$ is a left perfect discrete ring by assumption, we have $H(R) \subset \mathfrak{J}_1/(\mathfrak{J}_1\cap\mathfrak{J}_2)$ and $H(R) \subset \mathfrak{J}_2/(\mathfrak{J}_1\cap\mathfrak{J}_2)$, so $H(R) = 0$ and R is a semisimple Artinian ring, too. Since $\mathfrak{R}$ only has a finite number of semisimple Artinian discrete quotient rings, it follows that there exists a unique minimal open two-sided ideal $\mathfrak{J} \subset \mathfrak{R}$ such that $\mathfrak{R}/\mathfrak{J}$ is semisimple Artinian.

Now if $\mathfrak{I} \subset \mathfrak{J} \subset \mathfrak{R}$ is an open two-sided ideal, then $H(\mathfrak{R}/\mathfrak{I}) = \mathfrak{J}/\mathfrak{I}$. Thus we have $\mathfrak{H} = \mathfrak{J}$, so $\mathfrak{H}$ is an open (hence strongly closed) two-sided ideal in $\mathfrak{R}$ and the quotient ring $\mathfrak{R}/\mathfrak{H}$ is a finite product of simple Artinian rings.

(a) Any perfect commutative ring uniquely decomposes as a finite product of perfect commutative local rings, while any semisimple commutative ring uniquely decomposes as a finite product of fields.

Let Γ be set of all open ideals $\mathfrak{G} \subset \mathfrak{R}$ such that the discrete quotient ring $\mathfrak{R}/\mathfrak{G}$ is a field. Then for any open ideal $\mathfrak{I} \subset \mathfrak{R}$ the subset $\Delta_\mathfrak{I} \subset \Gamma$ of all $\mathfrak{G} \in \Gamma$ such that $\mathfrak{I} \subset \mathfrak{G}$ is finite (and bijective to the spectrum of $\mathfrak{R}/\mathfrak{I}$). Furthermore, there exists a unique collection of open ideals $\mathfrak{I}_\mathfrak{G} \subset \mathfrak{R}$, $\mathfrak{G} \in \Delta_\mathfrak{I}$ such that $\mathfrak{I} \subset \mathfrak{I}_\mathfrak{G} \subset \mathfrak{G}$, the quotient rings $\mathfrak{R}/\mathfrak{I}_\mathfrak{G}$ are local, and the natural ring homomorphism

$$\mathfrak{R}/\mathfrak{I} \longrightarrow \prod\nolimits_{\mathfrak{G}\in\Delta_\mathfrak{I}} \mathfrak{R}/\mathfrak{I}_\mathfrak{G}$$

is an isomorphism. Conversely, for any finite subset $\Delta \subset \Gamma$ and any collection of open ideals $\mathfrak{I}_\mathfrak{G} \subset \mathfrak{G}$, $\mathfrak{G} \in \Delta$ with local quotient rings $\mathfrak{R}/\mathfrak{I}_\mathfrak{G}$, the intersection $\mathfrak{I} = \bigcap_{\mathfrak{G}\in\Delta} \mathfrak{I}_\mathfrak{G}$ is an open ideal in $\mathfrak{R}$ and the map $\mathfrak{R}/\mathfrak{I} \longrightarrow \prod_{\mathfrak{G}\in\Delta} \mathfrak{R}/\mathfrak{I}_\mathfrak{G}$ is an isomorphism.

For any two open ideals $\mathfrak{I}'_\mathfrak{G}$, $\mathfrak{I}''_\mathfrak{G} \subset \mathfrak{G}$, $\mathfrak{G} \in \Gamma$ such that the quotient rings $\mathfrak{R}/\mathfrak{I}'_\mathfrak{G}$ and $\mathfrak{R}/\mathfrak{I}''_\mathfrak{G}$ are local, the quotient ring $\mathfrak{R}/(\mathfrak{I}'_\mathfrak{G} \cap \mathfrak{I}''_\mathfrak{G})$ is local, too. Hence those of the quotient rings $\mathfrak{R}/\mathfrak{I}_\mathfrak{G}$ by open ideals $\mathfrak{I}_\mathfrak{G} \subset \mathfrak{G}$ that are local rings form a directed projective system, and we can form their projective limit

$$\mathfrak{R}_\mathfrak{G} = \varprojlim\nolimits_{\mathfrak{I}_\mathfrak{G}\subset\mathfrak{G}} \mathfrak{R}/\mathfrak{I}_\mathfrak{G},$$

endowing it with the projective limit topology. There is a natural ring homomorphism $\mathfrak{R} \longrightarrow \mathfrak{R}_\mathfrak{G}$ whose compositions with the projections $\mathfrak{R}_\mathfrak{G} \longrightarrow \mathfrak{R}/\mathfrak{I}_\mathfrak{G}$ are surjective, so these projections are surjective, too. In particular, there is a natural surjective ring homomorphism $\mathfrak{R}_\mathfrak{G} \longrightarrow \mathfrak{R}/\mathfrak{G}$, whose open kernel we denote by $\mathfrak{H}_\mathfrak{G} \subset \mathfrak{R}_\mathfrak{G}$.

It follows from these considerations that the topological ring $\mathfrak{R}$ decomposes as the product of topological rings $\mathfrak{R}_\mathfrak{G}$,

$$\mathfrak{R} \cong \prod\nolimits_{\mathfrak{G}\in\Gamma} \mathfrak{R}_\mathfrak{G},$$

and the topology on $\mathfrak{R}$ coincides with the product topology. Furthermore, under this isomorphism one has

$$\mathfrak{H} = \prod\nolimits_{\mathfrak{G}\in\Gamma} \mathfrak{H}_\mathfrak{G}.$$

Now the ideal $\mathfrak{H} \subset \mathfrak{R}$ is strongly closed as a product of open ideals $\mathfrak{H}_\mathfrak{G} \subset \mathfrak{R}_\mathfrak{G}$ (cf. the discussion in the beginning of Section 5), and the quotient ring

$$\mathfrak{S} = \mathfrak{R}/\mathfrak{H} \cong \prod\nolimits_{\mathfrak{G}\in\Gamma} \mathfrak{R}_\mathfrak{G}/\mathfrak{H}_\mathfrak{G} = \prod\nolimits_{\mathfrak{G}\in\Gamma} \mathfrak{R}/\mathfrak{G}$$

is the topological product of discrete fields. □

We will say that a topological ring $\mathfrak{R}$ is *left pro-perfect* if it is separated and complete, has a base of neighborhoods of zero consisting of open two-sided ideals, and all the discrete quotient rings of $\mathfrak{R}$ are left perfect. According to Theorem 8.1, over a left pro-perfect topological ring satisfying one of the conditions (a), (b), or (c) all left contramodules have projective covers and all flat left contramodules are projective. Conversely, any complete, separated topological ring with a base of neighborhoods of zero consisting of open two-sided ideals over which all Bass flat left contramodules have projective covers is pro-perfect.

9. Examples

Two (classes of) examples of pro-perfect topological rings are discussed below. Both of them are commutative topological rings.

Example 9.1. Let $\mathfrak{R}$ be a complete Noetherian commutative local ring with the maximal ideal $\mathfrak{m} \subset \mathfrak{R}$. We view $\mathfrak{R}$ as a topological ring in the $\mathfrak{m}$-adic topology. Then $\mathfrak{R} = \varprojlim_{n\ge1} \mathfrak{R}/\mathfrak{m}^n$ is a separated and complete topological ring with a base of neighborhoods of zero formed by the ideals $\mathfrak{m}^n \subset \mathfrak{R}$. Furthermore, $\mathfrak{R}$ is pro-perfect, as all of its discrete quotient rings are Artinian and consequently perfect. The maximal ideal $\mathfrak{m} \subset \mathfrak{R}$ is strongly closed and topologically T-nilpotent.

By [28, Theorem B.1.1] or [33, Example 2.2(4)], the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow \mathfrak{R}\text{–}\mathsf{mod}$ is fully faithful, so the abelian category of $\mathfrak{R}$-contramodules is a full subcategory in the category of arbitrary $\mathfrak{R}$-modules. This full subcategory consists of all the so-called $\mathfrak{m}$-*contramodule* $\mathfrak{R}$-*modules*, which means the $\mathfrak{R}$-modules C such that $\mathrm{Ext}^i_{\mathfrak{R}}(\mathfrak{R}[s^{-1}], C) = 0$ for all $i = 0$, 1 and all $s \in \mathfrak{m}$. It suffices to check this condition for any chosen set of generators $s_1, \dots, s_m \in \mathfrak{m}$ of the ideal $\mathfrak{m}$ (or of any ideal in $\mathfrak{R}$ whose radical is equal to $\mathfrak{m}$) [31, Theorem 5.1].

According to Theorem 8.1, all $\mathfrak{R}$-contramodules have projective covers and all flat left $\mathfrak{R}$-contramodules are projective. Let us explain how to obtain these results from the previously existing literature. An $\mathfrak{R}$-contramodule is flat if and only if it is flat as an $\mathfrak{R}$-module [28, Lemma B.9.2], [31, Corollary 10.3(a)]. All flat $\mathfrak{R}$-contramodules are projective by [28, Corollary B.8.2] or [31, Theorem 10.5]. Moreover, the projective objects of the category $\mathfrak{R}\text{–}\mathsf{contra}$ are precisely the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]] = \varprojlim_{n\ge1}(\mathfrak{R}/\mathfrak{m}^n)[[X]]$ (see [28, Lemma 1.3.2] or [31, Corollary 10.7]).

Concerning the projective covers, one observes that all $\mathfrak{R}$-contramodules are Enochs cotorsion $\mathfrak{R}$-modules [28, Proposition B.10.1], [31, Theorem 9.3]. Let $\mathfrak{C}$ be an $\mathfrak{R}$-contramodule, and let $f\colon F \longrightarrow \mathfrak{C}$ be a flat cover of the $\mathfrak{R}$-module $\mathfrak{C}$. Let $p\colon \mathfrak{P} \longrightarrow \mathfrak{C}$ be a surjective morphism onto $\mathfrak{C}$ from a projective $\mathfrak{R}$-contramodule $\mathfrak{P}$ with the kernel $\mathfrak{K}$. Then $\mathfrak{P}$ is also a flat $\mathfrak{R}$-module, while $\mathfrak{K}$ is a cotorsion $\mathfrak{R}$-module; so p is a special flat precover of the $\mathfrak{R}$-module $\mathfrak{C}$. It follows that the $\mathfrak{R}$-module F is a direct summand of $\mathfrak{P}$; hence F is also an $\mathfrak{R}$-contramodule. Thus the morphism f is a projective cover of $\mathfrak{C}$ in the category $\mathfrak{R}\text{–}\mathsf{contra}$.

Example 9.2. Let R be a commutative ring and $S \subset R$ be a multiplicative subset. The S-*topology* on an R-module M has a base of neighborhoods of zero formed by the R-submodules $sM \subset M$, where $s \in S$. In particular, the ring R itself is a topological ring in the S-topology. Let $\mathfrak{R} = \varprojlim_{s\in S} R/sR$ be its completion, endowed with the projective limit topology [32, Section 2]. Then $\mathfrak{R}$ is a complete, separated topological commutative ring with a base of neighborhoods of zero formed by open ideals.

Assume that the quotient ring R/sR is perfect for all $s \in S$. Then $\mathfrak{R}$ is a pro-perfect commutative topological ring, so Theorem 8.1 tells that all $\mathfrak{R}$-contramodules have projective covers and all flat $\mathfrak{R}$-contramodules are projective. These results do not seem to follow easily from the previously existing literature.

Let us discuss the category of $\mathfrak{R}$-contramodules $\mathfrak{R}\text{–}\mathsf{contra}$ in some more detail. Following the proof of Proposition 8.2(a), the topological ring $\mathfrak{R}$ decomposes as the topological product $\mathfrak{R} \cong \prod_{\mathfrak{G}\in\mathfrak{R}} \mathfrak{R}_{\mathfrak{G}}$ over the open ideals $\mathfrak{G} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{G}$ is a field. (The same argument allows to obtain such a decomposition in the slightly more general case of an *S-h-nil* ring R [6, Section 6].) Such open ideals $\mathfrak{G} \subset \mathfrak{R}$ correspond bijectively to the maximal ideals $\mathfrak{m} \subset \mathfrak{R}$ for which the intersection $\mathfrak{m} \cap S$ is nonempty, and the topological ring $\mathfrak{R}_{\mathfrak{G}}$ can be described as the S-completion of the localization $R_{\mathfrak{m}}$ of the ring R. By Lemma 5.1(b), the abelian category $\mathfrak{R}\text{–}\mathsf{contra}$ decomposes as the Cartesian product of the abelian categories $\mathfrak{R}_{\mathfrak{G}}\text{–}\mathsf{contra}$.

By [6, Corollary 6.13], the localization R_S of the ring R at the multiplicative subset S, viewed as an R-module, has projective dimension at most 1. Thus the full subcategory of *S-contramodule R-modules* $R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}} \subset R\text{–}\mathsf{mod}$, consisting of all the R-modules C such that $\mathrm{Ext}^i_R(R_S, C) = 0$ for $i = 0$ and 1, is an abelian category, and the identity embedding $R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$ is an exact functor [32, Theorem 3.4(a)]. The forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ factorizes as $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$ [33, Example 2.4(2)]. We studied the category $R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ in [6, Sections 4 and 6].

Still, the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is *not* an equivalence of categories, generally speaking [33, Example 1.3(6)]. A sufficient condition for it to be an equivalence is that the S-torsion of R be bounded [33, Example 2.4(3)]. More generally, using the decomposition of the category $\mathfrak{R}\text{–}\mathsf{contra}$ into the Cartesian product over the maximal ideals $\mathfrak{m}$ of the ring R with $\mathfrak{m} \cap S \neq \varnothing$ and the similar decomposition of the category $R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ [6, Corollary 6.15], one shows that the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is an equivalence whenever the S-torsion in $R_{\mathfrak{m}}$ is bounded for every $\mathfrak{m}$. When S is countable, the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is an equivalence of categories if and only if the S-torsion in $\mathfrak{R} = \varprojlim_{s\in S} R/sR$ is bounded [33, Example 5.4(2)].

Furthermore, [33, Example 3.7(1)] lists two conditions which, taken together, are sufficient for the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ to be fully faithful. By [6, Proposition 2.1(2)], every S-divisible R-module is S-h-divisible, so one of the two conditions always holds in our case. Hence the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is fully faithful whenever the other condition holds, that is, whenever for every set X the free $\mathfrak{R}$-contramodule $\mathfrak{R}[[X]] = \varprojlim_{s\in S}(R/sR)[X]$ is complete in its S-topology (or in other words, its S-topology coincides with its projective limit topology [32, Theorem 2.3]).

In particular, the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is fully faithful whenever S is countable [33, Example 3.7(2)]. In this case, every object of $R\text{–}\mathsf{mod}_{S\text{-}\mathsf{ctra}}$ is an extension of two objects from $\mathfrak{R}\text{–}\mathsf{contra}$ [33, Example 5.4(2)].

10. Generalization of Main Theorem

The aim of this section is to generalize the result of Theorem 8.1 so that the class of topological rings covered by its equivalent conditions includes all the rings satisfying the assumptions formulated in the beginning of Section 6.

Let $\mathfrak{R}$ be a complete, separated topological ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals. We are interested in the following condition on the topological ring $\mathfrak{R}$, generalizing the conditions (a-c) of Section 8:

(d) there is a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{K} \subset \mathfrak{R}$ such that the quotient ring $\mathfrak{R}/\mathfrak{K}$ is isomorphic, as a topological ring, to the product $\prod_{\delta\in\Delta} \mathfrak{T}_\delta$ of a family of topological rings $\mathfrak{T}_\delta$, each of which has a base of neighborhoods of zero consisting of open two-sided ideals and satisfies one of the conditions (a), (b), or (c) of Section 8.

Here the quotient ring $\mathfrak{R}/\mathfrak{K}$ is endowed with the quotient topology and the product of topological rings $\prod_\delta \mathfrak{T}_\delta$ is endowed with the product topology. The following example shows that a topological ring satisfying (d) does not need to have a base of neighborhoods of zero consisting of open two-sided ideals.

Example 10.1. Let $\mathfrak{R} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}, \mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z})^{\mathrm{op}}$ be the opposite ring to the ring of endomorphisms of the abelian group $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$, endowed with the topology defined in Section 1.13. This topological ring, occuring in tilting theory, was described in [36, Example 7.12] as the matrix ring

$$\mathfrak{R} = \begin{pmatrix} \mathbb{Q} & \mathbb{A}^f_{\mathbb{Q}} \\ 0 & \widehat{\mathbb{Z}} \end{pmatrix},$$

where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the product over the prime numbers p of the topological rings of p-adic integers $\mathbb{Z}_p$ endowed with the p-adic topology, $\mathbb{A}^f_{\mathbb{Q}} = \mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ is the ring of finite adeles of the field of rational numbers $\mathbb{Q}$ endowed with the adelic topology, and the field $\mathbb{Q}$ itself is endowed with the discrete topology.

Consider the two-sided ideal $\mathfrak{K} = \mathbb{A}^f_{\mathbb{Q}} \subset \mathfrak{R}$. Then one has $\mathfrak{K}^2 = 0$, so this ideal is even (finitely) nilpotent. It is also clearly strongly closed in $\mathfrak{R}$. The quotient ring $\mathfrak{R}/\mathfrak{K}$ is commutative, so it satisfies the condition (a). The topological ring $\mathfrak{R}$ has a base of neighborhoods of zero formed by the open right ideals

$$\begin{pmatrix} 0 & r\widehat{\mathbb{Z}} \\ 0 & n\widehat{\mathbb{Z}} \end{pmatrix} \subset \mathfrak{R},$$

where $r \in \mathbb{Q}_{>0}$ and $n \in \mathbb{Z}_{>0}$ are an arbitrary positive rational number and a positive integer. But every open two-sided (or even left) ideal in $\mathfrak{R}$ contains $\mathfrak{K}$, so such ideals do not form a base of neighborhoods of zero.

Example 10.2. Let α be an ordinal, and let (M_i) be an α-indexed sequence of left modules over an associative ring R. Assume that all the morphisms between the R-modules M_i go backwards, that is, $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $0 \le i < j < \alpha$. Let $\mathfrak{T}_i = \mathrm{Hom}_R(M_i, M_i)^{\mathrm{op}}$ be topological rings opposite to the endomorphism rings of the R-modules M_i, and let $\mathfrak{S} = \operatorname{Hom}_R(M, M)^{\mathrm{op}}$ be the topological ring opposite to the ring of endomorphisms of the R-module $M = \bigoplus_{i<\alpha} M_i$.

Then there is a natural surjective morphism of topological rings $p\colon \mathfrak{S} \longrightarrow \prod_{i\in\alpha} \mathfrak{T}_i$. Set $\mathfrak{K} = \ker(p) \subset \mathfrak{S}$; then $\mathfrak{K}$ is a strongly closed two-sided ideal in $\mathfrak{S}$ and the topological quotient ring $\mathfrak{S}/\mathfrak{K}$ is isomorphic to the topological product $\prod_{i\in\alpha} \mathfrak{T}_i$. Moreover, the ideal $\mathfrak{K}$ is topologically left T-nilpotent, because for every element $b \in M$ and any sequence of endomorphisms $a_1, a_2, a_3, \ldots \in \mathfrak{K}$ one has $ba_1a_2\cdots a_n = 0$ in M for n large enough (as one easily shows using König's lemma).

Thus if for every $i < \alpha$ the topological ring $\mathfrak{T}_i$ has a base of neighborhoods of zero consisting of open two-sided ideals and satisfies one of the conditions (a), (b), or (c) of Section 8, then the topological ring $\mathfrak{S}$ satisfies the condition (d). Moreover, if for every index i the topological ring $\mathfrak{T}_i$ satisfies the condition (d), then so does the topological ring $\mathfrak{S}$ (as we will see below in Lemma 10.6(b)).

Lemma 10.3. (a) *Let $(R_\gamma)_{\gamma\in\Gamma}$ be a family of topological rings. Then all the discrete quotient rings of the topological ring $R = \prod_{\gamma\in\Gamma} R_\gamma$ are left perfect if and only if all the discrete quotient rings of the topological rings R_γ, $\gamma \in \Gamma$, are left perfect.*

(b) *Let R be a separated topological ring, and let $K \subset R$ be a topologically left T-nilpotent closed two-sided ideal. Then all the discrete quotient rings of the ring R are left perfect if and only if all the discrete quotient rings of the topological ring R/K are left perfect.*

Proof. Part (a) holds, because the set of all discrete quotient rings of R coincides with the union of the sets of all discrete quotient rings of R_γ. Part (b) follows from its discrete version: if $\overline{K}$ is a left T-nilpotent two-sided ideal in an associative ring $\overline{R}$ and the quotient ring $\overline{R}/\overline{K}$ is left perfect, then the ring R is left perfect. The latter is obtainable from the characterization of left perfect rings in [5, Theorem P (1)] and the discrete version of Lemma 4.3. $\square$

The following theorem is our generalization of Theorem 8.1.

Theorem 10.4. *Let $\mathfrak{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. Assume that the condition (d) is satisfied. Then the following conditions are equivalent:*

- (i) *all flat left $\mathfrak{R}$-contramodules have projective covers;*
- (i$^\flat$) *all Bass flat left $\mathfrak{R}$-contramodules have projective covers;*
- (ii) *all left $\mathfrak{R}$-contramodules have projective covers;*
- (iii) *all flat left $\mathfrak{R}$-contramodules are projective;*
- (iii$^\flat$) *all Bass flat left $\mathfrak{R}$-contramodules are projective;*
- (iv) *all discrete quotient rings of $\mathfrak{R}$ are left perfect;*
- (iv$'$) *all discrete quotient rings of $\mathfrak{R}/\mathfrak{K}$ are left perfect;*
- (v) *$\mathfrak{R}$ has a topologically left T-nilpotent strongly closed two-sided ideal $\mathfrak{H}$ such that the quotient ring $\mathfrak{R}/\mathfrak{H}$ is topologically isomorphic to a product of simple Artinian discrete rings endowed with the product topology.*

Conversely, if a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals satisfies (v), then it also satisfies (d).

Proof. The first three paragraphs of the proof of Theorem 8.1 apply in our present context as well. Furthermore, as in Theorem 8.1, the implication (iii$^\flat$) $\Longrightarrow$ (iv) is provided by Corollary 2.4, and the implication (i$^\flat$) $\Longrightarrow$ (iv) by Corollary 3.6. The implication (iv) $\Longrightarrow$ (iv$'$) is obvious, and the inverse implication (iv$'$) $\Longrightarrow$ (iv) is provided by Lemma 10.3(b).

The final implication (iv$'$) $\Longrightarrow$ (v) holds in the assumption of the condition (d). This one, as well as the converse implication (v) $\Longrightarrow$ (d), are provided by the following proposition. In other words, the proposition below shows that (v) is equivalent to the combination of (iv$'$) and (d). $\square$

Proposition 10.5. *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals. Suppose that there exists an ideal $\mathfrak{K} \subset \mathfrak{R}$ such that the condition (d) is satisfied and all the discrete quotient rings of the ring $\mathfrak{R}/\mathfrak{K}$ are left perfect. Then there exists an ideal $\mathfrak{H} \subset \mathfrak{R}$ satisfying (v). Conversely, if an ideal $\mathfrak{H} \subset \mathfrak{R}$ satisfies (v), then the same ideal $\mathfrak{K} = \mathfrak{H}$ also satisfies (d).*

Proof. The converse assertion is obvious: any simple Artinian ring endowed with the discrete topology satisfies both (b) and (c), so a product of such rings is a product of topological rings satisfying (b) and (c). To prove the direct implication, suppose that $\mathfrak{K} \subset \mathfrak{R}$ is an ideal satisfying (d) such that all the discrete quotient rings of $\mathfrak{R}/\mathfrak{K}$ are left perfect. Then we have $\mathfrak{R}/\mathfrak{K} \cong \prod_{\delta\in\Delta} \mathfrak{T}_\delta$, so any discrete quotient ring of $\mathfrak{T}_\delta$ is at the same time a discrete quotient ring of $\mathfrak{R}/\mathfrak{K}$.

Applying Proposition 8.2 to the topological ring $\mathfrak{T}_\delta$, we conclude that there exists a left T-nilpotent strongly closed two-sided ideal $\mathfrak{J}_\delta \subset \mathfrak{T}_\delta$ such that the quotient ring $\mathfrak{S}_\delta = \mathfrak{T}_\delta/\mathfrak{J}_\delta$ is topologically isomorphic to a product of discrete simple Artinian rings, $\mathfrak{S}_\delta \cong \prod_{\gamma\in\Gamma_\delta} S_\gamma$. According to the discussion in the beginning of Section 5, it follows that $\mathfrak{J} = \prod_\delta \mathfrak{J}_\delta$ is a left T-nilpotent strongly closed two-sded ideal in $\mathfrak{T} = \prod_\delta \mathfrak{T}_\delta$. Furthermore, the topological quotient ring $\mathfrak{T}/\mathfrak{J} \cong \prod_{\delta\in\Delta} \mathfrak{S}_\delta$ is isomorphic to the topological product $\prod_{\gamma\in\Gamma} S_\gamma$ of the discrete simple Artinian rings S_γ over the disjoint union $\Gamma = \coprod_{\delta\in\Delta} \Gamma_\delta$ of the sets of indices Γ_δ.

Now we have a surjective continuous ring homomorphism $\mathfrak{R} \longrightarrow \mathfrak{R}/\mathfrak{K} \cong \mathfrak{T}$. Let $\mathfrak{H} \subset \mathfrak{R}$ be the full preimage of the closed ideal $\mathfrak{J} \subset \mathfrak{T}$ under this homomorphism. Then the ideal $\mathfrak{H}$ is strongly closed in $\mathfrak{R}$ by Lemma 1.4(b), $\mathfrak{H}$ is left T-nilpotent by Lemma 4.3, and the topological ring $\mathfrak{R}/\mathfrak{H} \cong \mathfrak{T}/\mathfrak{J}$ is the topological product of discrete simple Artinian rings S_γ. $\square$

The following lemma shows the class of all topological rings satisfying (d) is closed under the operations that were used to define it.

Lemma 10.6. (a) *Let $(\mathfrak{R}_\gamma)_{\gamma\in\Gamma}$ be a family of topological rings satisfying the condition (d). Then the topological ring $\mathfrak{R} = \prod_{\gamma\in\Gamma} \mathfrak{R}_\gamma$ also satisfies (d).*

(b) *Let $\mathfrak{R}$ be a complete, separated topological ring with a base of neighborhoods of zero formed by open right ideals, and let $\mathfrak{J} \subset \mathfrak{R}$ be a left T-nilpotent strongly closed two-sided ideal. Assume that the topological quotient ring $\mathfrak{R}/\mathfrak{J}$ satisfies (d). Then the topological ring $\mathfrak{R}$ satisfies (d).*

Proof. Similar to the proof of Proposition 10.5. The proof of part (a) is based on the discussion in the beginning of Section 5, while the proof of part (b) uses Lemmas 1.4(b) and 4.3. $\square$

11. Covers in Hereditary Cotorsion Pairs

In Section 3, we defined and discussed *projective covers* in abelian categories. Now we will define covers by (objects from) arbitrary classes of objects.

Let A be a category and $\mathsf{L} \subset \mathsf{A}$ be a class of objects. A morphism $l\colon L \longrightarrow C$ in A is called an L-*precover* (of the object C) if $L \in \mathsf{L}$ and all the morphisms from objects of L to the object C factorize through the morphism l in the category A, that is, for every morphism $l'\colon L' \longrightarrow C$ with $L' \in \mathsf{L}$ there exists a morphism $f\colon L' \longrightarrow L$ such that $l' = lf$. For example, if A is an abelian category with enough projective objects and $\mathsf{L} \subset \mathsf{A}$ is the class of all projective objects, then a morphism $L \longrightarrow C$ with $L \in \mathsf{L}$ is an L-precover if and only if it is an epimorphism.

A morphism $l\colon L \longrightarrow C$ in A is called an L-*cover* if it is an L-precover and, for any endomorphism $e\colon L \longrightarrow L$, the equation $le = l$ implies that e is an automorphism of L. Given another class of objects $\mathsf{E} \subset \mathsf{A}$, the definitions of an E-*preeenvepe* and an E-*envelope* of an object of an object $C \in \mathsf{A}$ are dual to the above definitions of an L-precover and an L-cover. These notions are due to Enochs [14]; a detailed discussion of their properties in a relevant context can be found in the book [44].

Furthermore, suppose that A is an abelian category, and let L and $\mathsf{E} \subset \mathsf{A}$ be two classes of objects. Let $\mathsf{L}^{\perp_1} \subset \mathsf{A}$ denote the class of all objects $X \in \mathsf{A}$ such that $\operatorname{Ext}^1_{\mathsf{A}}(L, X) = 0$ for all $L \in \mathsf{L}$, and let ${}^{\perp_1}\mathsf{E} \subset \mathsf{A}$ be the class of all objects $Y \in \mathsf{A}$ such that $\operatorname{Ext}^1_{\mathsf{A}}(Y, E) = 0$ for all $E \in \mathsf{E}$. The pair of classes of objects (L, E) in A is called a *cotorsion pair* (or a *cotorsion theory*) if $\mathsf{E} = \mathsf{L}^{\perp_1}$ and $\mathsf{L} = {}^{\perp_1}\mathsf{E}$. A cotorsion pair (L, E) is called *hereditary* if $\operatorname{Ext}^n_{\mathsf{A}}(L, E) = 0$ for all $L \in \mathsf{L}$, $E \in \mathsf{E}$, and $n \geq 1$. These definitions go back to Salce [38].

An epimorphism $l\colon L \longrightarrow C$ in A is called a *special* L-*precover* if $L \in \mathsf{L}$ and $\ker(l) \in \mathsf{L}^{\perp_1}$. A monomorphism $b\colon B \longrightarrow E$ in A is called a *special* E-*preenvelope* if $E \in \mathsf{E}$ and $\operatorname{coker}(b) \in {}^{\perp_1}\mathsf{E}$. The following lemma summarizes the properties of precovers, special precovers, and covers.

Lemma 11.1. (a) *Any special* L-*precover is an* L-*precover.*

(b) *If the class* L *is closed under extensions in* A, *then the kernel of any* L-*cover belongs to* $\mathsf{L}^{\perp_1}$. *In particular, any epic* L-*cover is special in this case.*

(c) *Let* $l\colon L \longrightarrow C$ *be an* L-*cover, and let* $l'\colon L' \longrightarrow C$ *be an* L-*precover. Then there exists a split epimorphism* $f\colon L' \longrightarrow L$ *forming a commutative triangle diagram with the morphisms* l *and* l'. *The kernel* K *of the morphism* f *is a direct summand of* L' *contained in* $\ker(l') \subset L'$.

(d) *Assume that an object* $C \in \mathsf{A}$ *has an* L-*cover, and let* $l'\colon L' \longrightarrow C$ *be an* L-*precover. Then the morphism* l' *is an* L-*cover if and only if the object* L' *has no nonzero direct summands contained in* $\ker(l')$.

Proof. Part (a) is [44, Proposition 2.1.3 or 2.1.4]. Part (b) is known as Wakamatsu lemma; this is [44, Lemma 2.1.1 or 2.1.2]. Part (c) is [44, Proposition 1.2.2 or Theorem 1.2.7], and part (d) is [44, Corollary 1.2.3 or 1.2.8]. $\square$

Let (L, E) be a cotorsion pair in A. If $c\colon L \longrightarrow C$ is an epimorphism in A with $L \in \mathsf{L}$ and the object $\ker(c) \in \mathsf{A}$ has a special E-preenvelope, then the object C has a special L-precover. If $b\colon B \longrightarrow E$ is a monomorphism in A with $E \in \mathsf{E}$ and the object $\operatorname{coker}(b) \in \mathsf{A}$ has a special L-precover, then the object B has a special E-preenvelope. In particular, if there are enough injective and projective objects in A, then, given a cotorsion pair (L, E) in A, every object of A has a special L-precover if and only if every object of A has a special E-preenvelope. These results are known as Salce lemmas [38]. A cotorsion pair (L, E) in A is called *complete* if every object of A has a special L-precover and a special E-preenvelope.

Lemma 11.2. *Let (L, E) be a hereditary complete cotorsion pair in an abelian category A. Assume that every object of E has an L-cover in A. Then every object of A has an L-cover.*

Proof. Let A be an object in A. By assumption, A has a special E-preenvelope $a\colon A \longrightarrow E$. Set $L = \operatorname{coker}(a)$; then we have a short exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow L \longrightarrow 0$ in A with $E \in \mathsf{E}$ and $L \in \mathsf{L}$. By assumption, the object E has an L-cover $m\colon M \longrightarrow E$ in A. Set $F = \ker(m)$; by Lemma 11.1(b), we have $F \in \mathsf{E}$. Let K denote the kernel of the composition of epimorphisms $M \longrightarrow E \longrightarrow L$; then we have $K \in \mathsf{L}$, since M, $L \in \mathsf{L}$ and the cotorsion pair (L, E) is assumed to be hereditary. We have constructed a commutative diagram of four short exact sequences

$$\begin{tikzcd}
& 0 & 0 & & \\
0 \arrow[r] & A \arrow[u] \arrow[r, "a"] & E \arrow[u] \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "k"] \arrow[r] & M \arrow[u, "m"] \arrow[r] & L \arrow[u, equal] \arrow[r] & 0 \\
& F \arrow[u] \arrow[r, equal] & F \arrow[u] & & \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The morphism $k\colon K \longrightarrow A$ is an epimorphism with the kernel $F \in \mathsf{E}$, so it is a special L-precover. By Lemma 11.1(a), k is an L-precover. Let us show that it is an L-cover. Let $h\colon K \longrightarrow K$ be an endomorpism such that $kh = k$. Consider a pushout of the short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ by the morphism h and denote it by $0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0$. We have $N \in \mathsf{L}$, since K, $L \in \mathsf{L}$ and the class L is closed under extensions in A. In view of the universal property of the pushout, we have a commutative diagram of two morphisms of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{a} & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle k}\big\uparrow & & {\scriptstyle n}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle h}\big\uparrow & & {\scriptstyle s}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

with $kh = k$ and $ns = m$. Since the morphism $m\colon M \longrightarrow E$ is an L-cover and $N \in \mathsf{L}$, there exists a morphism $r'\colon N \longrightarrow M$ such that $mr' = n$. Moreover, one has $mr's = ns = m$, hence $r's\colon M \longrightarrow M$ is automorphism. Setting $r = (r's)^{-1}r'\colon N \longrightarrow M$, we have $rs = \mathrm{id}_M$ and $mr = m(r's)^{-1}r' = mr' = n$.

It follows that the morphism $r\colon N \longrightarrow M$ forms a commutative triangle diagram with the morphisms $N \longrightarrow L$ and $M \longrightarrow L$. Passing to the kernels of the latter two morphisms, we obtain a morphism $g\colon K \longrightarrow K$ such that $gh = \mathrm{id}_K$. We have constructed a commutative diagram of two morphisms of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle g}\big\uparrow & & {\scriptstyle r}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle h}\big\uparrow & & {\scriptstyle s}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

whose composition is the identity endomorphism of the short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$.

Thus we have shown that any endomorphism $h\colon K \longrightarrow K$ such that $kh = k$ is a (split) monomorphism. Furthermore, there is a commutative diagram of two morphisms of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{a} & E & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle k}\big\uparrow & & {\scriptstyle m}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\
 & & {\scriptstyle g}\big\uparrow & & {\scriptstyle r}\big\uparrow & & \big\| & & \\
0 & \longrightarrow & K & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

where $kg = k$, because $mr = n$ (indeed, since a is a monomorphism, it suffices to show that $akg = ak$, which follows from the equation $mr = n$ and the commutativity of the left squares of our diagrams).

Therefore, the morphism $g\colon K \longrightarrow K$ is a (split) monomorphism, too, and we can conclude that both g and h are isomorphisms. $\square$

12. Tilting-Cotilting Correspondence and Direct Limits

Given an exact category E (in Quillen's sense), we denote by $\mathsf{E}_{\mathsf{inj}}$ and $\mathsf{E}_{\mathsf{proj}} \subset \mathsf{E}$ the classes of all injective and projective objects in A, respectively. In particular, this notation applies to abelian categories.

Let A be an additive category with set-indexed coproducts, and let B be an additive category with set-indexed products. For any object $T \in \mathsf{A}$ and any set X, we denote by $T^{(X)} \in \mathsf{A}$ the coproduct of X copies of T in A. For any object $W \in \mathsf{B}$ and any set X, we denote by $W^X \in \mathsf{B}$ the product of X copies of W in B.

Furthermore, we denote by $\mathsf{Add}(T) = \mathsf{Add}_{\mathsf{A}}(T) \subset \mathsf{A}$ the class of all direct summands of the coproducts $T^{(X)}$ of copies of the object T in the category A. Similarly, we denote by $\mathsf{Prod}(W) = \mathsf{Prod}_{\mathsf{B}}(W) \subset \mathsf{B}$ the class of all direct summands of the products W^X of copies of the object W in B.

Let A be a complete, cocomplete abelian category (or, in other words, an abelian category with set-indexed products and coproducts) with a fixed injective cogenerator $J \in \mathsf{A}$. So there are enough injective objects in the category A, and the class of all injective objects is $\mathsf{A}_{\mathsf{inj}} = \mathsf{Prod}(J) \subset \mathsf{A}$.

Let $n \geq 0$ be an integer, and let $T \in \mathsf{A}$ be an object satisfying the following two conditions:

(i) the projective dimension of T (as an object of A) does not exceed n, that is $\operatorname{Ext}^i_{\mathsf{A}}(T, A) = 0$ for all $A \in \mathsf{A}$ and $i > n$; and
(ii) for any set X, one has $\operatorname{Ext}^i_{\mathsf{A}}(T, T^{(X)}) = 0$ for all $i > 0$.

Denote by $\mathsf{E} \subset \mathsf{A}$ the class of all objects $E \in \mathsf{A}$ such that $\operatorname{Ext}^i_{\mathsf{A}}(T, E) = 0$ for all $i > 0$. Notice that, by the definition, one has $\mathsf{A}_{\mathsf{inj}} = \mathsf{Prod}_{\mathsf{A}}(J) \subset \mathsf{E}$ and, by the condition (ii), $\mathsf{Add}_{\mathsf{A}}(T) \subset \mathsf{E}$.

Furthermore, for each integer $m \geq 0$, denote by $\mathsf{L}_m \subset \mathsf{A}$ the class of all objects $L \in \mathsf{A}$ for which there exists an exact sequence of the form

$$0 \longrightarrow L \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^m \longrightarrow 0$$

in the category A with the objects $T^m \in \mathsf{Add}(T)$. By the definition, $\mathsf{Add}(T) = \mathsf{L}_0 \subset \mathsf{L}_1 \subset \mathsf{L}_2 \subset \cdots \subset \mathsf{A}$. According to [36, Lemma 2.2], one has $\mathsf{L}_n = \mathsf{L}_{n+1} = \mathsf{L}_{n+2} = \cdots$ (so we set $\mathsf{L} = \mathsf{L}_n$) and $\mathsf{L} \cap \mathsf{E} = \mathsf{Add}(T) \subset \mathsf{A}$.

According to [36, Theorem 2.4], every object of E is a quotient of an object from $\mathsf{Add}(T)$ in A if and only if every object of A is a quotient of an object from L. If this is the case, we say that the object $T \in \mathsf{A}$ is *n-tilting*. For an n-tilting object T, the pair of classes of objects (L, E) in A is a hereditary complete cotorsion pair, called the *n-tilting cotorsion pair* associated with T.

Let B be a complete, cocomplete abelian category with a fixed projective generator $P \in \mathsf{B}$. So there are enough projective objects in B, and one has $\mathsf{B}_{\mathsf{proj}} = \mathsf{Add}(P) \subset \mathsf{B}$.

The definition of an *n-cotilting object* $W \in \mathsf{B}$ is dual to the above definition of an n-tilting object. In other words, an object $W \in \mathsf{B}$ is said to be *n-cotilting* if the object W^{op} is n-tilting in the abelian category B^{op} opposite to B.

Specifically, this means, first of all, that the two conditions dual to (i) and (ii) have to be satisfied:

(i*) the injective dimension of W (as an object of B) does not exceed n, that is $\operatorname{Ext}^i_{\mathsf{B}}(B, W) = 0$ for all $B \in \mathsf{B}$ and $i > n$; and

(ii*) for any set X, one has $\operatorname{Ext}^i_{\mathsf{B}}(W^X, W) = 0$ for all $i > 0$.

On top of that, denoting by $\mathsf{F} \subset \mathsf{B}$ the class of all objects $F \in \mathsf{B}$ such that $\operatorname{Ext}^i_{\mathsf{B}}(F, W) = 0$ for all $i > 0$, it is required that every object of F should be a subobject of an object from $\mathsf{Prod}(W)$ in B.

The following theorem from [36] describes the phenomenon of *n-tilting-cotilting correspondence.*

Theorem 12.1. *There is a bijective correspondence between (the equivalence classes of) complete, cocomplete abelian categories A with an injective cogenerator J and an n-tilting object $T \in \mathsf{A}$, and (the equivalence classes of) complete, cocomplete abelian categories B with a projective generator P and an n-cotilting object $W \in \mathsf{B}$. The abelian categories A and B corresponding to each other under this correspondence are connected by the following structures:*

(a) *there is a pair of adjoint functors between A and B, with a left adjoint functor $\Phi\colon \mathsf{B} \longrightarrow \mathsf{A}$ and a right adjoint functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$;*

(b) *one has $\Phi(\mathsf{F}) \subset \mathsf{E}$ and $\Psi(\mathsf{E}) \subset \mathsf{F}$; the restrictions of the functors Φ and Ψ are mutually inverse equivalences between the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$;*

(c) *the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under extensions and the cokernels of monomorphisms, while the full subcategory $\mathsf{F} \subset \mathsf{B}$ is closed under extensions and the kernels of epimorphisms; hence they inherit exact category structures from their ambient abelian categories; the equivalence of categories $\mathsf{E} \cong \mathsf{F}$ provided by the functors Φ and Ψ is an equivalence of exact categories (in Quillen's sense); in other words, the functor Φ preserves exactness of short exact sequences of objects from F, and the functor Ψ preserves exactness of short exact sequences of objects from E;*

(d) *both the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under both the products and coproducts in their ambient abelian categories; the functor $\Phi\colon \mathsf{B} \longrightarrow \mathsf{A}$ preserves the products (and coproducts) of objects from F, while the functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ preserves the (products and) coproducts of objects from E;*

(e) *under the equivalence of exact categories $\mathsf{E} \cong \mathsf{F}$, the injective cogenerator $J \in \mathsf{E} \subset \mathsf{A}$ corresponds to the n-cotilting object $W \in \mathsf{F} \subset \mathsf{B}$, and the n-tilting object $T \in \mathsf{E} \subset \mathsf{A}$ corresponds to the projective generator $P \in \mathsf{F} \subset \mathsf{B}$;*

(f) *there are enough projective and injective objects in the exact category $\mathsf{E} \cong \mathsf{F}$; the full subcategories of projectives and injectives in E are $\mathsf{E}_{\mathsf{proj}} = \mathsf{Add}(T)$ and $\mathsf{E}_{\mathsf{inj}} = \mathsf{A}_{\mathsf{inj}} = \mathsf{Prod}(J)$, while the full subcategories of projectives and injectives in F are $\mathsf{F}_{\mathsf{proj}} = \mathsf{B}_{\mathsf{proj}} = \mathsf{Add}(P)$ and $\mathsf{F}_{\mathsf{inj}} = \mathsf{Prod}(W)$.*

Proof. The bijective correspondence is constructed in [36, Corollary 3.12] (based on [36, Theorems 3.10 and 3.11]), and the assertions (e-f) are a part of that construction (cf. [36, Proposition 1.6 and Theorem 1.4]). The adjoint functors Φ and Ψ

are described in [36, beginning of Section 4], and parts (b-c) are also explained there. Part (d) is [36, Lemma 4.3 and Remark 4.4]. ☐

Example 12.2. Suppose that there is an associative ring A such that the abelian category A can be embedded into A–mod as a full subcategory closed under coproducts. So, in particular, the n-tilting object $T \in \mathsf{A}$ can be viewed as a left A-module. Then the abelian category B can be described as the category of left contramodules $\mathfrak{R}$–contra over the topological ring $\mathfrak{R} = \operatorname{Hom}_A(T,T)^{\mathrm{op}}$ from Section 1.13 (see [36, Corollaries 7.2, 7.4, and 7.6]. Further examples of classes of abelian categories A for which the category B admits such a description are discussed in [36, Section 9].

In the rest of this section we discuss the properties of direct limits in the n-tilting-cotilting correspondence context. To make the exposition more accessible, we start with the case of the direct limits indexed by the poset of natural numbers.

Lemma 12.3. *In the context of the n-tilting-cotilting correspondence, assume that countable direct limits are exact in the abelian category A. Then both the full subcategories $\mathsf{E} \subset \mathsf{A}$ and $\mathsf{F} \subset \mathsf{B}$ are closed under countable direct limits in their ambient abelian categories, and the functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ preserves countable direct limits of objects from E. Furthermore, for any sequence of objects and morphisms $F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots$ with $F_i \in \mathsf{F}$, the short sequence $0 \longrightarrow \coprod_{i=1}^\infty F_i \longrightarrow \coprod_{i=1}^\infty F_i \longrightarrow \varinjlim_{i\ge1} F_i \longrightarrow 0$ with the map $\mathrm{id} - shift\colon \coprod_i F_i \longrightarrow \coprod_i F_i$ is exact in B. The functors of countable direct limit are exact in the exact category F.*

Proof. For any sequence of objects and morphisms $B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow \cdots$ in an an abelian category B with countable coproducts, the short sequence $\coprod_{i=1}^\infty B_i \longrightarrow \coprod_{i=1}^\infty B_i \longrightarrow \varinjlim_{i\ge1} B_i \longrightarrow 0$ is right exact in B. Moreover, for any sequence of objects and morphisms $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$ in an abelian category A with exact countable direct limits, the short sequence $0 \longrightarrow \coprod_{i=1}^\infty A_i \longrightarrow \coprod_{i=1}^\infty A_i \longrightarrow \varinjlim_{i\ge1} A_i \longrightarrow 0$ is exact in A, because it is the countable direct limit of split short exact sequences $0 \longrightarrow \coprod_{i=1}^j A_i \longrightarrow \coprod_{i=1}^{j+1} A_j \longrightarrow A_{j+1} \longrightarrow 0$ over $j \ge 1$. In particular, for any sequence of objects and morphisms $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ with $E_i \in \mathsf{E}$, the short sequence $0 \longrightarrow \coprod_i E_i \longrightarrow \coprod_i E_i \longrightarrow \varinjlim_i E_i \longrightarrow 0$ is exact in A. Hence it follows that $\varinjlim_i E_i \in \mathsf{E}$, because the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under coproducts and the cokernels of monomorphisms.

The functor Φ, being a left adjoint, preserves all colimits. Thus, for any sequence of objects and morphisms $F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow \cdots$ in F, the short sequence $0 \longrightarrow \Phi(\coprod_i F_i) \longrightarrow \Phi(\coprod_i F_i) \longrightarrow \Phi(\varinjlim_i F_i) \longrightarrow 0$, being isomorphic to the short sequence $0 \longrightarrow \coprod_i \Phi(F_i) \longrightarrow \coprod_i \Phi(F_i) \longrightarrow \varinjlim_i \Phi(F_i) \longrightarrow 0$, is exact in A. This is a short exact sequence in A with all the three terms belonging to E, so the functor Ψ transforms it into a short exact sequence in B with all the three terms belonging to F. We have a natural (adjunction) morphism from the right exact sequence $\coprod_i F_i \longrightarrow \coprod_i F_i \longrightarrow \varinjlim_i F_i \longrightarrow 0$ to the exact sequence $0 \longrightarrow \Psi\Phi(\coprod_i F_i) \longrightarrow \Psi\Phi(\coprod_i F_i) \longrightarrow \Psi\Phi(\varinjlim_i F_i) \longrightarrow 0$, which is an isomorphism on the first two terms, and therefore on the third term, too. Hence the object $\varinjlim_i F_i \cong \Psi(\varinjlim_i \Phi(F_i))$ belongs to F and the short sequence $0 \longrightarrow \coprod_i F_i \longrightarrow \coprod_i F_i \longrightarrow \varinjlim_i F_i \longrightarrow 0$ is exact. Since the coproduct functors are exact in F (because they are exact in E) and the cokernel of an admissible monomorphism is an exact functor, it follows that the functors of countable direct limit are exact in F. The functor $\Psi|_{\mathsf{E}}\colon \mathsf{E} \longrightarrow \mathsf{B}$ preserves countable direct limits, because both the equivalence of categories $\mathsf{E} \cong \mathsf{F}$ and the inclusion functor $\mathsf{F} \longrightarrow \mathsf{B}$ do. This proves all the assertions of the lemma. □

Corollary 12.4. *In the context of the n-tilting-cotilting correspondence, assume that countable direct limits are exact in the abelian category A. Then the following three conditions are equivalent:*

(i) *the full subcategory L is closed under countable direct limits in A;*
(ii) *the class of objects $\mathsf{Add}(T)$ is closed under countable direct limits in A;*
(iii) *the class of all projective objects $\mathsf{B}_{\mathsf{proj}}$ is closed under under countable direct limits in B.*

Proof. (i) $\Longrightarrow$ (ii) According to Lemma 12.3, the class E is closed under countable direct limits in A. Hence, if the class L is closed under countable direct limits, too, then so is the class $\mathsf{L} \cap \mathsf{E} = \mathsf{Add}(T)$.

(ii) $\Longleftrightarrow$ (iii) By the same lemma, the equivalence of categories $\mathsf{E} \cong \mathsf{F}$ transforms countable direct limits of objects from E computed in A to countable direct limits of objects from F computed in B. Thus the class $\mathsf{B}_{\mathsf{proj}} = \Psi(\mathsf{Add}(T)) \subset \mathsf{F}$ is closed under countable direct limits in B if and only if the class $\mathsf{Add}(T) \subset \mathsf{E}$ is closed under countable direct limits in A.

(ii) $\Longrightarrow$ (i) Given an object $L \in \mathsf{L}$, an exact sequence $0 \longrightarrow L \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$ with $T^j \in \mathsf{Add}(T)$ can be constructed in the following way. Let $L \longrightarrow E$ be a special E-preenvelope of L; then we have a short exact sequence $0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0$ with $E \in \mathsf{E}$ and $M \in \mathsf{L}$. Since the class L is closed under extensions in A, we have $E \in \mathsf{L} \cap \mathsf{E} = \mathsf{Add}(T)$. Set $T^0 = E$ and $M^1 = M$, and let $M^1 \longrightarrow T^1$ be a special E-preenvelope of M^1, etc. Proceeding in this way, one obtains an exact sequence $0 \longrightarrow L \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^{n-1} \longrightarrow M^n \longrightarrow 0$ with $M^n \in \mathsf{L}$; and one also has $M^n \in \mathsf{E}$ by cohomological dimension shifting, since the projective dimension of T does not exceed n. It remains to set $T^n = M^n$. Conversely, in any exact sequence $0 \longrightarrow L \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$ with $L \in \mathsf{L}$ and $T^j \in \mathsf{Add}(T)$, the objects of cocycles belong to L, since the class L, being a left class in a hereditary cotorsion theory, is closed under the kernels of epimorphisms.

Now, for any two objects A' and $A'' \in \mathsf{A}$, their special E-preenvelopes $A' \longrightarrow E'$ and $A'' \longrightarrow E''$, and a morphism $A' \longrightarrow A''$, there is a morphism $E' \longrightarrow E''$ forming a commutative triangle diagram with the composition $A' \longrightarrow A'' \longrightarrow E''$. Using this observation, for any sequence of objects and morphisms $L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow \cdots$ in L and any exact sequences $0 \longrightarrow L_i \longrightarrow T_i^0 \longrightarrow \cdots \longrightarrow T_i^n \longrightarrow 0$ with $T_i^j \in \mathsf{Add}(T)$, one can extend the sequence $L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow \cdots$ to a sequence of morphisms of exact sequences $(0 \to L_1 \to T_1^0 \to \cdots \to T_1^n \to 0) \longrightarrow (L_2 \to T_2^0 \to \cdots \to T_2^n \to$

$0) \longrightarrow \cdots$. Passing to the direct limit, we obtain an exact sequence

$$0 \longrightarrow \varinjlim_{i\ge1} L_i \longrightarrow \varinjlim_{i\ge1} T_i^0 \longrightarrow \cdots \longrightarrow \varinjlim_{i\ge1} T_i^n \longrightarrow 0$$

in the abelian category A. Since $\varinjlim_i T_i^j \in \mathsf{Add}(T)$ for all $j = 0, \ldots, m$, it follows that $\varinjlim_i L_i \in \mathsf{L}$ by the definition. $\square$

The following proposition provides a generalization to noncountable direct limits.

Proposition 12.5. *In the context of the n-tilting-cotilting correspondence, assume that direct limits are exact in the abelian category A. Then both the full subcategories E and F are closed under direct limits in their ambient abelian categories A and B, and the functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ preserves direct limits of objects from E. The functors of direct limit are exact in the exact category F.*

Proof. For any cocomplete abelian category A, a small category Γ, and a functor $A\colon \Gamma \longrightarrow \mathsf{A}$, the colimit of $A(\gamma)$ over $\gamma \in \Gamma$ can be computed as the cokernel of the natural morphism $\coprod_{\phi\colon \gamma'\to\gamma''} A(\gamma') \longrightarrow \coprod_\gamma A(\gamma)$, where the coproduct in the left-hand side ranges over all the morphisms in Γ and the coproduct in the right-hand side ranges over all the objects in Γ. Given an abelian category A with exact direct limits, a directed poset Γ, and a Γ-indexed diagram A in A, the whole bar-complex $(*)$

$$\cdots \longrightarrow \coprod_{\gamma_0\le\gamma_1\le\gamma_2} A(\gamma_0) \longrightarrow \coprod_{\gamma_0\le\gamma_1} A(\gamma_0) \longrightarrow \coprod_{\gamma_0} A(\gamma_0) \longrightarrow \varinjlim_{\gamma\in\Gamma} A(\gamma) \longrightarrow 0$$

is exact in A. Indeed, the complex $(*)$ is the direct limit (over $\beta \in \Gamma$) of the similar bar-complexes related to the subposets $\Gamma_\beta = \{\gamma \in \Gamma\colon \gamma \le \beta\} \subset \Gamma$ and the subdiagrams $A|_{\Gamma_\beta}$ of A. The bar-complex of any diagram indexed by a poset with a maximum element is easily seen to be contractible (by the explicit contracting homotopy given by the morphisms taking the summand $A(\gamma_0)$ indexed by $\gamma_0 \le \cdots \le \gamma_n$ to the summand $A(\gamma_0)$ indexed by $\gamma_0 \le \cdots \le \gamma_n \le \beta$).

Now let $E\colon \Gamma \longrightarrow \mathsf{E}$ be a diagram in the exact category E indexed by a directed poset Γ. Then the complex $(*)$ is an unbounded resolution of an object of A by objects of E (since the full subcategory $\mathsf{E} \subset \mathsf{A}$ is closed under coproducts). Since the full subcategory $\mathsf{E} \subset \mathsf{A}$ is defined as the class of all objects $E \in \mathsf{A}$ such that $\mathrm{Ext}^i_{\mathsf{A}}(T, E) = 0$ for all $i > 0$, and the tilting object $T \in \mathsf{A}$ has finite projective dimension, a simple cohomological dimension shifting argument shows that $\varinjlim_{\gamma\in\Gamma} E(\gamma) \in \mathsf{E}$. Moreover, all the objects of cycles of the exact complex $(*)$ for the diagram E also belong to E. So this complex is exact in the exact category E.

Applying the functor Ψ to the bar-complex for the diagram E, we get an exact complex in the category F, which coincides, except possibly at his rightmost term, with the bar-complex for the diagram $\Psi \circ E$ in B (because both the equivalence of categories $\mathsf{E} \cong \mathsf{F}$ and the inclusion functor $\mathsf{F} \longrightarrow \mathsf{B}$ preserve coproducts). Since the bar-complex of any diagram in a cocomplete abelian category is exact at its rightmost term, it follows that the natural morphism $\varinjlim_\gamma \Psi(E(\gamma)) \longrightarrow \Psi(\varinjlim_\gamma E(\gamma))$ is an isomorphism and $\varinjlim_\gamma \Psi(E(\gamma)) \in \mathsf{F}$. As any diagram in F can be obtained by applying the functor Ψ to a diagram in E, we can conclude that the full subcategory $\mathsf{F} \subset \mathsf{B}$ is also closed under direct limits, and the bar-complexes $(*)$ computing such direct limits in F are exact. Exactness of the direct limit functors in F easily follows. $\square$

Corollary 12.6. *In the context of the n-tilting-cotilting correspondence, assume that direct limits are exact in the abelian category A. Consider the following three properties:*

(i) *the full subcategory L is closed under direct limits in A;*
(ii) *the class of objects $\mathsf{Add}(T)$ is closed under direct limits in A;*
(iii) *the class of all projective objects $\mathsf{B}_{\mathsf{proj}}$ is closed under under direct limits in B.*

Then the implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) *hold.*

If there is a functor from A to the category of morphisms in A assigning to every object $A \in \mathsf{A}$ one of its special E-preenvelopes $A \longrightarrow E$, then all the three conditions (i), (ii), *and* (iii) *are equivalent.*

Proof. Provable in the same way as Corollary 12.4, using Proposition 12.5 in place of Lemma 12.3. $\square$

Lemma 12.7. *Let A be a locally presentable abelian category, (L, E) be a hereditary complete cotorsion pair in A, and T be a set of objects in A such that $\mathsf{E} = \mathsf{T}^{\perp_{\geq 1}}$ is the class of all objects $E \in \mathsf{A}$ such that $\mathrm{Ext}^i_{\mathsf{A}}(T, E) = 0$ for all $T \in \mathsf{T}$ and $i > 0$. Then there exists a functor from A to the category of morphisms in A assigning to every object $A \in \mathsf{A}$ one of its special E-preeenvelopes.*

Proof. One says that a cotorsion pair (L, E) in A is *generated by a set* if there exists a set of objects S in A such that $\mathsf{E} = \mathsf{S}^{\perp_1}$. In a locally presentable abelian category A, if a cotorsion pair (L, E) is generated by a set and every object of A is a subobject of an object of E, then every object of A has a special E-preenvelope and such a special preenvelope can be produced by the small object argument [35, Proposition 3.5 or Theorem 4.8(b)]. The construction of the small object argument in a locally presentable category can be performed functorially [9, Proposition 1.3]. It remains to show that there exists a set of objects $\mathsf{S} \subset \mathsf{A}$ such that $\mathsf{S}^{\perp_1} = \mathsf{E} = \mathsf{T}^{\perp_{\geq 1}}$.

Clearly, one has $\mathsf{T} \subset \mathsf{L}$. Arguing by induction, it suffices to show that for every object $S \in \mathsf{L}$ and an integer $i \geq 2$ there exists a set of objects $\mathsf{S}' \subset \mathsf{L}$ such that for any given $A \in \mathsf{A}$ one has $\mathrm{Ext}^i_{\mathsf{A}}(S, A) = 0$ whenever $\mathrm{Ext}^{i-1}_{\mathsf{A}}(S', A) = 0$ for all $S' \in \mathsf{S}'$. Let λ be a regular cardinal such that the category A is locally λ-presentable and the object S is λ-presentable. For every λ-presentable object $B \in \mathsf{A}$ endowed with an epimorphism $B \longrightarrow S$, choose an epimorphism $L \longrightarrow B$ onto B from an object $L \in \mathsf{L}$, and set S' to be the kernel of the composition $L \longrightarrow B \longrightarrow S$. Then one has $S' \in \mathsf{L}$, since the class L is closed under the kernels of epimorphisms.

Let S' be the set of all objects S' obtained in this way. For any Ext class $\xi \in \mathrm{Ext}^i_{\mathsf{A}}(S, A)$, there exists an object $X \in \mathsf{A}$ and two Ext classes $\eta \in \mathrm{Ext}^1_{\mathsf{A}}(S, X)$ and $\zeta \in \mathrm{Ext}^{i-1}_{\mathsf{A}}(X, A)$ such that $\xi = \zeta\eta$. By [35, Lemma 3.4], any short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$ is A is a pushout of a short exact sequence $0 \longrightarrow X' \longrightarrow B \longrightarrow S \longrightarrow 0$ in which the object B is λ-presentable. The latter short exact sequence is, in turn, a pushout of the short exact sequence $0 \longrightarrow S' \longrightarrow L \longrightarrow S \longrightarrow 0$. It follows easily that $\mathrm{Ext}^{i-1}_{\mathsf{A}}(S', A) = 0$ for all $S' \in \mathsf{S}'$ implies $\mathrm{Ext}^i_{\mathsf{A}}(S, A) = 0$. □

Corollary 12.8. *In the context of the n-tilting-cotilting correspondence, assume that* A *is a Grothendieck abelian category. Then the following three conditions are equivalent:*

(i) *the full subcategory* L *is closed under direct limits in* A*;*
(ii) *the class of objects* $\mathsf{Add}(T)$ *is closed under direct limits in* A*;*
(iii) *the class of all projective objects* $\mathsf{B}_{\mathsf{proj}}$ *is closed under under direct limits in* B.

Proof. Follows from Corollary 12.6 and Lemma 12.7. □

13. When is the Left Tilting Class Covering?

The results of this section are our version of [3, Theorem 3.6, Theorem 5.2, and Corollary 5.5]. Our techniques, which are completely different from those of Šaroch in [39] and Angeleri Hügel, Šaroch, and Trlifaj in [3], only allow us to reproduce a small subset of (the tilting particular case of) their results, and only under the somewhat restrictive assumptions of Theorem 10.4 on the topological ring $\mathfrak{R}$ of endomorphisms of the tilting module. On the other hand, our approach provides some information about the topological ring $\mathfrak{R}$, and is applicable to abelian categories A much more general then the categories of modules.

We will say that a class of object L in a category A is *precovering* if every object of A has an L-precover. Similarly, the class L is said to be *covering* if every object of A has an L-cover.

For any class of objects M in a cocomplete category A, we denote by $\varinjlim \mathsf{M} = \varinjlim^{\mathsf{A}} \mathsf{M} \subset \mathsf{A}$ the class of all direct limits of objects from M in A. This means the direct limits of diagrams $A\colon \Gamma \longrightarrow \mathsf{A}$ indexed by directed posets Γ and such that $A(\gamma) \in \mathsf{M}$ for all $\gamma \in \Gamma$. The class of all countable direct limits of objects from M in A will be denoted by $\varinjlim_\omega \mathsf{M} = \varinjlim_\omega^{\mathsf{A}} \mathsf{M} \subset \varinjlim^{\mathsf{A}} \mathsf{M}$.

Proposition 13.1. *In the context of the n-tilting-cotilting correspondence, suppose that* A *is a Grothendieck abelian category and* B *is the abelian category of left contramodules over a topological ring* $\mathfrak{R}$ *satisfying the assumptions formulated in the beginning of Section 6. Then the class* $\mathsf{B}_{\mathsf{proj}}$ *is closed under direct limits in* B*, every object of* B *has a projective cover, the classes* L *and* $\mathsf{Add}(T)$ *are closed under direct limits in* A*, and every object of* A *has an* L*-cover and an* $\mathsf{Add}(T)$*-cover.*

Proof. Every left $\mathfrak{R}$-contramodule has a projective cover in our assumptions by Theorem 7.4. Furthermore, by [35, Lemma 5.6] (see the discussion in the beginning of Section 2), the class of flat left $\mathfrak{R}$-contramodules is closed under direct limits in $\mathsf{B} = \mathfrak{R}\text{–}\mathsf{contra}$. By Corollary 6.3, all flat left $\mathfrak{R}$-contramodules are projective in our assumptions. Hence the class of all projective objects $\mathsf{B}_{\mathsf{proj}}$ is closed under direct limits in B. According to Corollary 12.8, we can conclude that the classes L and $\mathsf{Add}(T)$ are closed under direct limits in A.

Notice that the class L is precovering in A by Lemma 11.1(a), since (L, E) is a complete cotorsion pair in A by [36, Theorem 2.4] (see the discussion in the beginning of Section 12). A general argument going back to Enochs [14, Theorems 2.1 and 3.1] tells that a precovering class closed under direct limits is covering. For Grothendieck abelian categories, a proof of this assertion can be found in [4, Theorem 1.2] (and for locally presentable categories, in [35, Theorem 2.7 or Corollary 4.17]).

Finally, for any object T in an additive category A, the class $\mathsf{Add}(T)$ is precovering in A, with the natural morphism $T^{(\mathrm{Hom}_{\mathsf{A}}(T,A))} \longrightarrow A$ being an $\mathsf{Add}(T)$-precover of an object $A \in \mathsf{A}$. If the class $\mathsf{Add}(T) \subset \mathsf{A}$ is closed under direct limits, it follows that it is a covering class. □

Proposition 13.2. *In the context of the n-tilting-cotilting correspondence, the following four conditions are equivalent:*

(i) *the class L is covering in A;*
(ii) *every object of E has an L-cover in A;*
(iii) *the class $\mathsf{Add}(T)$ is covering in E;*
(iv) *the class $\mathsf{B}_{\mathsf{proj}}$ is covering in F.*

Furthermore, assume that countable direct limits are exact in the abelian category A. Then the following three conditions (v-vii) *are equivalent:*

(v) *all the objects from $\varinjlim_\omega \mathsf{Add}(T)$ have L-covers in A;*
(vi) *all the objects from $\varinjlim_\omega \mathsf{Add}(T)$ have $\mathsf{Add}(T)$-covers in A;*
(vii) *all the objects from $\varinjlim_\omega \mathsf{B}_{\mathsf{proj}}$ have projective covers in B.*

. Finally, assume that countable direct limits are exact in A and that B is the abelian category of left contramodules over a complete, separated topological associative ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals, satisfying the condition (d) of Section 10. Then all the conditions (i-vii) are equivalent to each other *and to the following six conditions* (viii-xiii)*:*

(viii) *all the discrete quotient rings of $\mathfrak{R}$ are left perfect;*
(ix) *the class $\mathsf{B}_{\mathsf{proj}}$ is covering in B;*
(x) *the class $\mathsf{B}_{\mathsf{proj}}$ is closed under direct limits in B;*
(xi) *the class $\mathsf{B}_{\mathsf{proj}}$ is closed under countable direct limits in B;*
(xii) *the class $\mathsf{Add}(T)$ is closed under countable direct limits in A;*
(xiii) *the class L is closed under countable direct limits in A.*

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious, and the inverse implication (ii) $\Longrightarrow$ (i) is provided by Lemma 11.2.

(ii) $\Longrightarrow$ (iii) Let $E \in \mathsf{E}$ be an object, and let $L \longrightarrow E$ be its L-cover in A. By Lemma 11.1(b), the morphism $L \longrightarrow E$ is a special L-precover; so its kernel belongs to E. Since the class E is closed under extensions in A, it follows that $L \in \mathsf{L} \cap \mathsf{E} = \mathsf{Add}(T)$. Hence, in particular, $L \longrightarrow E$ is an $\mathsf{Add}(T)$-cover of the object E.

(iii) $\Longrightarrow$ (ii) Given an object $E \in \mathsf{E}$, let $l \colon L \longrightarrow E$ be its $\mathsf{Add}(T)$-cover, and let $l' \colon L' \longrightarrow E$ be one of its special L-precovers in A. By the above argument, we have

$L' \in \mathsf{Add}(T)$; so l' is also an $\mathsf{Add}(T)$-precover of E. According to Lemma 11.1(c) (which does not actually need the category to be abelian, but only idempotent-complete additive; so it can be applied to $\mathsf{Add}(T)$-precovers and $\mathsf{Add}(T)$-covers in E), the object $\ker(l)$ is a direct summand of $\ker(l')$. So l is a special L-precover of E in A. In particular, by Lemma 11.1(a), l is an L-precover of E in A. Since l is an $\mathsf{Add}(T)$-cover of E, it follows that l is an L-cover of E in A.

(iii) $\Longleftrightarrow$ (iv) Holds in view of the equivalence of categories $\mathsf{E} \cong \mathsf{F}$ taking the class $\mathsf{Add}(T) \subset \mathsf{E}$ to the class $\mathsf{B}_{\mathsf{proj}} = \mathsf{F}_{\mathsf{proj}} \subset \mathsf{F}$ (see Theorem 12.1(b,f)).

(v) $\Longleftrightarrow$ (vi) By Lemma 12.3, we have $\varinjlim_\omega \mathsf{Add}(T) \subset \varinjlim_\omega \mathsf{E} = \mathsf{E}$, to the arguments from the above proof of (ii) $\Longleftrightarrow$ (iii) apply.

(vi) $\Longleftrightarrow$ (vii) The equivalence of categories $\mathsf{E} \cong \mathsf{F}$ identifies the class of objects $\mathsf{Add}(T) \subset \mathsf{E}$ with the class $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{F}$. By Lemma 12.3, it also identifies the class $\varinjlim_\omega \mathsf{Add}(T) \subset \mathsf{E}$ with the class $\varinjlim_\omega \mathsf{B}_{\mathsf{proj}} \subset \mathsf{F}$.

The implications (ii) $\Longrightarrow$ (v), (iii) $\Longrightarrow$ (vi), and (iv) $\Longrightarrow$ (vii) are obvious (as $\varinjlim_\omega \mathsf{Add}(T) \subset \mathsf{E}$ and $\varinjlim_\omega \mathsf{B}_{\mathsf{proj}} \subset \mathsf{F}$).

(vii) $\Longrightarrow$ (viii) Holds by Corollary 3.6 or Theorem 10.4 ($\mathrm{i}^\flat$) $\Longrightarrow$ (iv).

(viii) $\Longrightarrow$ (ix) If all the discrete quotient rings of $\mathfrak{R}$ are left perfect and (d) is satisfied, then all left $\mathfrak{R}$-contramodules have projective covers by Theorem 10.4 (iv) $\Longrightarrow$ (ii).

The implications (ix) $\Longrightarrow$ (iv) and (x) $\Longrightarrow$ (xi) are obvious.

(viii) $\Longrightarrow$ (x) Follows from Theorem 10.4 (iv) $\Longrightarrow$ (iii), since the direct limits of projective contramodules are always flat.

(xi) $\Longrightarrow$ (viii) Holds by Corollary 2.4 or Theorem 10.4 ($\mathrm{iii}^\flat$) $\Longrightarrow$ (iv).

The equivalences (xi) $\Longleftrightarrow$ (xii) $\Longleftrightarrow$ (xiii) hold by Corollary 12.4. □

Corollary 13.3. *In the context of the n-tilting-cotilting correspondence, assume that A is a Grothendieck abelian category and B is the abelian category of left contramodules over a complete, separated topological associative ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals, satisfying the condition (d) of Section 10. Then the following conditions are equivalent:*

- (i) *the class L is covering in A;*
- (ii) *the class L is closed under direct limits in A;*
- (iii) *the class $\mathsf{Add}(T)$ is covering in A;*
- (iv) *the class $\mathsf{Add}(T)$ is closed under direct limits in A;*
- (v) *the class $\mathsf{B}_{\mathsf{proj}}$ is covering in B;*
- (vi) *the class $\mathsf{B}_{\mathsf{proj}}$ is closed under direct limits in B;*
- (vii) *all the discrete quotient rings of $\mathfrak{R}$ are left perfect.*

Proof. The equivalences (ii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (vi) hold by Corollary 12.8. The implications (vii) $\Longrightarrow$ (i-vi) are provided by Proposition 13.1 (because, by Proposition 10.5 or Theorem 10.4 (iv) $\Longrightarrow$ (v), the assumptions formulated in the beginning of Section 6 hold for any topological ring $\mathfrak{R}$ satisfying the condition (d) of Section 10 whose discrete quotient rings are left perfect). The equivalences (i) $\Longleftrightarrow$ (v) $\Longleftrightarrow$

(vi) $\Longleftrightarrow$ (vii) are provided by Proposition 13.2 (i) $\Longleftrightarrow$ (viii) $\Longleftrightarrow$ (ix) $\Longleftrightarrow$ (x), and the implication (iii) $\Longrightarrow$ (i) by Proposition 13.2 (iii) $\Longrightarrow$ (i). $\square$

14. Σ-Pure-Split Objects and their Endomorphism Rings

The following setting is more general then the n-tilting-cotilting correspondence context described in Section 12. Let A be a cocomplete abelian category and $M \in \mathsf{A}$ be an object. Then there exists a unique abelian category B with enough projective objects such that the full subcategory $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$ is equivalent to the full subcategory $\mathsf{Add}(M) \subset \mathsf{A}$.

The object $M \in \mathsf{Add}(M)$ corresponds to a projective generator $P \in \mathsf{B}_{\mathsf{proj}}$ of the category B. The abelian category B can be described as the category of modules over the additive monad $\mathbb{T}_M$ on the category of sets assigning the set $\mathbb{T}_M(X) = \mathrm{Hom}_{\mathsf{A}}(M, M^{(X)})$ to an arbitrary set X. The embedding functor $\mathsf{Add}(M) \cong \mathsf{B}_{\mathsf{proj}} \longrightarrow \mathsf{B}$ extends naturally to a right exact functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ assigning to every object $N \in \mathsf{A}$ the set $\mathrm{Hom}_{\mathsf{A}}(M, N)$ endowed with a natural $\mathbb{T}_M$-module structure. The embedding functor $\mathsf{B}_{\mathsf{proj}} \cong \mathsf{Add}(M) \longrightarrow \mathsf{A}$ can be extended uniquely to a right exact functor $\Phi\colon \mathsf{A} \longrightarrow \mathsf{B}$. The functor Φ is left adjoint to the functor Ψ. We refer to [36, Sections 6.1–6.2] and [37, Section 1] for further details.

Let A be a cocomplete abelian category. We will say that a monomorphism $f\colon K \longrightarrow M$ in A is *pure* if for every cocomplete abelian category V with exact direct limit functors, and any additive functor $F\colon \mathsf{A} \longrightarrow \mathsf{V}$ preserving all colimits (that is, a right exact covariant functor preserving coproducts), the morphism $F(f)\colon F(K) \longrightarrow F(M)$ is a monomorphism in V. If this is the case, the object K is said to be a *pure subobject* of the object $M \in \mathsf{A}$.

A short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ in A is called *pure* if the monomorphism $K \longrightarrow M$ is pure, or equivalently, if the short sequence $0 \longrightarrow F(K) \longrightarrow F(M) \longrightarrow F(L) \longrightarrow 0$ is exact in V for every functor $F\colon \mathsf{A} \longrightarrow \mathsf{V}$ as above. A long exact sequence $K^\bullet$ in A is said to be *pure* if it is obtained by splicing pure short exact sequences in A, or equivalently, if the complex $F(K^\bullet)$ is exact in V for every abelian category V with exact direct limits and any colimit-preserving functor $F\colon \mathsf{A} \longrightarrow \mathsf{V}$.

For example, the bar-complex $(*)$ from the proof of Proposition 12.5 is pure exact in any abelian category A with exact direct limits (because the image of $(*)$ under any colimit-preserving functor F is a similar bar-complex in V, which is also exact, since the direct limits are exact in V). As usually, this assertion has a separate (simpler) version for countable direct limits: for any sequence of objects and morphisms $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$ in an abelian category A with exact countable direct limits, the short exact sequence $0 \longrightarrow \coprod_{i=1}^\infty A_i \longrightarrow \coprod_{i=1}^\infty A_i \longrightarrow \varinjlim_{i\ge1} A_i \longrightarrow 0$ is pure exact in A (because its image under F is the similar short exact sequence in V for the sequence of objects and morphisms $F(A_1) \longrightarrow F(A_2) \longrightarrow F(A_3) \longrightarrow \cdots$).

Lemma 14.1. *Let* $\mathsf{A} = R\text{–}\mathsf{mod}$ *be the abelian category of left modules over an associative ring* R*. Then a monomorphism (or a short exact sequence, or a long exact sequence) in* $R\text{–}\mathsf{mod}$ *is pure in the sense of the above definition if and only if it is pure in the conventional sense of the word (as in* [19]*).*

Proof. A functor $R\text{–}\mathsf{mod} \longrightarrow \mathsf{Ab}$ from the category of left R-modules to the category of abelian groups Ab preserves colimits if and only if it is isomorphic to the functor of tensor product $A \longmapsto N \otimes_R A$ with a certain right R-module N [43, Theorem 1]. (Colimit-preserving functors $R\text{–}\mathsf{mod} \longrightarrow \mathsf{V}$ can be similarly described as the functors of tensor product with an object in V endowed with a right action of the ring R.) So any pure exact sequence in $R\text{–}\mathsf{mod}$ in the sense of the above definition has remain exact after taking the tensor product with any right R-module N, i. e., it is pure exact in the conventional sense of the word.

Conversely, any pure short exact sequence of left R-modules the conventional sense is a direct limit of split short exact sequences. Hence its image under any colimit-preserving functor (and more generally, under any direct limit-preserving additive functor) $F\colon R\text{–}\mathsf{mod} \longrightarrow \mathsf{V}$, taking values in an abelian category V with exact direct limits, is exact. $\square$

Furthermore, if $\mathsf{A} = \mathfrak{R}\text{–}\mathsf{contra}$ is the category of contramodules over a topological ring $\mathfrak{R}$, then the functors of contratensor product $\mathcal{N} \odot_{\mathfrak{R}} -$ with discrete right $\mathfrak{R}$-modules $\mathcal{N}$ preserve all colimits. Hence any pure short exact sequence in $\mathfrak{R}\text{–}\mathsf{contra}$ in the sense of our present definition is contratensor pure in the sense of Section 2.

An object $M \in \mathsf{A}$ is said to be *pure-split* if every pure monomorphism $K \longrightarrow M$ is split in A. One says that an object $T \in \mathsf{A}$ is Σ*-pure-split* all the objects M from the class $\mathsf{Add}(T) \subset \mathsf{A}$ are pure-split in A.

Proposition 14.2. *Let* A *be a cocomplete abelian category with exact countable direct limits and* $M \in \mathsf{A}$ *be a* Σ*-pure-split object. Then the class of object* $\mathsf{Add}(M) \subset \mathsf{A}$ *is closed under countable direct limits. In the related abelian category* $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$*, the class of all projective objects* $\mathsf{B}_{\mathsf{proj}}$ *is closed under countable direct limits.*

In particular, if the monad $\mathbb{T}_M\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ *is isomorphic to the monad* $\mathbb{T}_{\mathfrak{R}}$ *for a complete, separated topological associative ring* $\mathfrak{R}$ *with a base of neighborhoods of zero formed by open right ideals, then all the discrete quotient rings of the topological ring* $\mathfrak{R}$ *are left perfect.*

Proof. Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \cdots$ be a short sequence of objects and morphisms in A with $M_i \in \mathsf{Add}(M)$. Then the short sequence $0 \longrightarrow \coprod_{i=1}^\infty M_i \longrightarrow \coprod_{i=1}^\infty M_i \longrightarrow \varinjlim_{i\ge1} M_i \longrightarrow 0$ is pure exact in A. The object $\coprod_{i=1}^\infty M_i$ belongs to $\mathsf{Add}(M)$, so it follows that this short exact sequence splits. Hence the object $\varinjlim_{i\ge1} M_i$ is a direct summand of $\coprod_{i=1}^\infty M_i$, so it also belongs to $\mathsf{Add}(M)$.

Let $P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow \cdots$ be a sequence of projective objects in B and morphisms between them. Then the short sequence

$$\coprod\nolimits_{i=1}^\infty P_i \longrightarrow \coprod\nolimits_{i=1}^\infty P_i \longrightarrow \varinjlim\nolimits_{i\ge1} P_i \longrightarrow 0 \qquad (3)$$

is, at least, right exact in B. Now we use the same kind of going-back-and-forth argument as the one in the proof of Lemma 12.3 in order to show that this sequence is split exact under our new assumptions.

The functor Φ, being a left adjoint, preserves all colimits. Hence the image of (3) under Φ is the similar short exact sequence for the sequence of objects and morphisms $\Phi(P_0) \longrightarrow \Phi(P_1) \longrightarrow \Phi(P_2) \longrightarrow \cdots$ in the category A. According to the above argument, it follows that the morphism $\mathrm{id} - \mathit{shift}\colon \coprod_{i=1}^{\infty} \Phi(P_i) \longrightarrow \coprod_{i=1}^{\infty} \Phi(P_i)$ is a split monomorphism in the full subcategory $\mathsf{Add}(M) \subset \mathsf{A}$. Applying the functor Ψ in order to get back to the category B, we conclude that the morphism $\mathrm{id} - \mathit{shift}\colon \coprod_i P_i \longrightarrow \coprod_i P_i$ is a split monomorphism in $\mathsf{B}_{\mathsf{proj}}$. Thus the object $\varinjlim_{i \ge 1} P_i$ is a direct summand of $\coprod_{i=1}^{\infty} P_i$ in B, so it belongs to $\mathsf{B}_{\mathsf{proj}}$.

In particular, if $\mathsf{B} \cong \mathfrak{R}\text{--}\mathsf{contra}$, then the class of projective left $\mathfrak{R}$-contramodules is closed under countable direct limits in $\mathfrak{R}\text{--}\mathsf{contra}$. By Corollary 2.4, it follows that all the discrete quotient rings of $\mathfrak{R}$ are left perfect. $\square$

Example 14.3. If $\mathsf{A} = A\text{--}\mathsf{mod}$ is the category of left modules over an associative ring A and $M \in \mathsf{A}$ is a left A-module, then the monad $\mathbb{T}_M$ corresponds to the topological ring $\mathfrak{R} = \mathrm{Hom}_A(M, M)^{\mathrm{op}}$ opposite to the endomorphism ring of the module M. The ring A acts in M on the left and the topological ring $\mathfrak{R}$ acts in M on the right, making M a discrete right $\mathfrak{R}$-module [36, Theorem 7.1, Lemma 7.3, and Lemma 7.5] (cf. Section 1.13). According to Proposition 14.2, if the A-module M is Σ-pure-split, then all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.

Notice that all left perfect rings can be obtained in this way. If A is a left perfect associative ring and $M = A$, then the A-module M is Σ-pure split (because all flat left A-modules are projective) and the topology of the ring $\mathfrak{R} = \mathrm{Hom}_A(A, A)^{\mathrm{op}} = A$ is discrete (because the A-module M is finitely generated).

The next proposition is an uncountable version of Proposition 14.2.

Proposition 14.4. *Let A be a cocomplete abelian category with exact direct limits and $M \in \mathsf{A}$ be a Σ-pure-split object. Then the class of object $\mathsf{Add}(M) \subset \mathsf{A}$ is closed under direct limits. In the related abelian category $\mathsf{B} = \mathbb{T}_M\text{--}\mathsf{mod}$, the class of all projective objects $\mathsf{B}_{\mathsf{proj}}$ is closed under direct limits.*

Proof. For any directed poset Γ and a Γ-indexed diagram $A\colon \Gamma \longrightarrow \mathsf{A}$, the bar-complex $(*)$ from the proof of Proposition 12.5 is pure exact in A, according to the above discussion. Now if $A(\gamma) \in \mathsf{Add}(M) \subset \mathsf{A}$ for all $\gamma \in \Gamma$, then all the terms of the complex $(*)$, except perhaps the rightmost one, belong to $\mathsf{Add}(M)$. Since the object M is Σ-pure-split, it follows that the bar-complex $(*)$ is plit exact. In particular, $\varinjlim_{\gamma \in \Gamma} A(\gamma)$ is a direct summand of $\bigoplus_{\gamma_0 \in \Gamma} A(\gamma_0)$, hence $\varinjlim_{\gamma} A(\gamma) \in \mathsf{Add}(M)$.

For any Γ-indexed diagram $B\colon \Gamma \longrightarrow \mathsf{B}$, the related bar-complex

(4)
$$\cdots \longrightarrow \coprod_{\gamma_0 \le \gamma_1 \le \gamma_2} B(\gamma_0) \longrightarrow \coprod_{\gamma_0 \le \gamma_1} B(\gamma_0) \longrightarrow \coprod_{\gamma_0} B(\gamma_0) \longrightarrow \varinjlim_{\gamma \in \Gamma} B(\gamma) \longrightarrow 0$$

is, at least, exact at its rightmost term in the category B. The image of (4) under Φ is the similar bar-complex for the diagram $\Phi\circ\Gamma$ in the category A.

Now, if $B(\gamma)\in\mathsf{B}_{\mathsf{proj}}$ for all $\gamma\in\Gamma$, then $\Phi(B(\gamma))\in\mathsf{Add}(M)$ and, according to the above argument, the image of (4) under Φ is a split exact complex in $\mathsf{Add}(M)$. Applying the functor Ψ to get back to the category B, we conclude that the complex (4) is split exact in $\mathsf{B}_{\mathsf{proj}}$ except, perhaps, at its rightmost term. Since we also know that it is exact at the rightmost term, it follows that the whole complex (4) is split exact in B and $\varinjlim_\gamma B(\gamma)\in\mathsf{B}_{\mathsf{proj}}$. □

We will say that a morphism of left $\mathfrak{R}$-contramodules $f\colon\mathfrak{K}\longrightarrow\mathfrak{M}$ is *a contratensor pure monomorphism* if the short exact sequence $0\longrightarrow\mathfrak{K}\longrightarrow\mathfrak{M}\longrightarrow\mathfrak{M}/\mathfrak{K}\longrightarrow 0$ is contratensor pure in the sense of the definition in Section 2.

Proposition 14.5. *Let $\mathfrak{R}$ be a topological ring satisfying the assumptions formulated in the beginning of Section 6 (e. g., $\mathfrak{R}$ is left pro-perfect and one of the conditions (a), (b), or (c) from Section 8 holds, or all the discrete quotient rings of $\mathfrak{R}$ are left perfect and the condition (d) of Section 10 is satisfied). Let $\mathfrak{M}$ be a projective left $\mathfrak{R}$-contramodule. Then any contratensor pure monomorphism $\mathfrak{K}\longrightarrow\mathfrak{M}$ in $\mathfrak{R}\text{–}\mathsf{contra}$ is split.*

Proof. For any complete, separated topological ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals, all the projective left $\mathfrak{R}$-contramodules are 1-strictly flat. Hence, following the discussion in Section 2, if $\mathfrak{K}\longrightarrow\mathfrak{M}$ is a contratensor pure monomorphism and $\mathfrak{M}\in\mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$, then the quotient contramodule $\mathfrak{M}/\mathfrak{K}$ is 1-strictly flat, too. Furthermore, all 1-strictly flat left $\mathfrak{R}$-contramodules are flat.

Now, if $\mathfrak{R}$ satisfies the assumptions from the beginning of Section 6, then all flat left $\mathfrak{R}$-contramodules are projective by Corollary 6.3. Thus $\mathfrak{M}/\mathfrak{K}\in\mathfrak{R}\text{–}\mathsf{contra}_{\mathsf{proj}}$ and the short exact sequence $0\longrightarrow\mathfrak{K}\longrightarrow\mathfrak{M}\longrightarrow\mathfrak{M}/\mathfrak{K}\longrightarrow 0$ splits. □

Lemma 14.6. *In the context of the n-tilting-cotilting correspondence, suppose that B is the abelian category of left contramodules over a topological ring $\mathfrak{R}$ satisfying the assumptions formulated in the beginning of Section 6. Let $f\colon E\longrightarrow M$ be a morphism in A such that $E\in\mathsf{E}$, $M\in\mathsf{Add}(T)$, and the morphism $\Psi(f)\colon\Psi(E)\longrightarrow\Psi(M)$ is a contratensor pure monomorphism in $\mathsf{B}=\mathfrak{R}\text{–}\mathsf{contra}$. Then f is a split monomorphism.*

Proof. Follows from Proposition 14.5. □

15. Self-Pure-Projective and Σ-Rigid Objects, Direct Limits, and Covers

In this section we reproduce and partly extend some of the results of Šaroch and Angeleri Hügel–Šaroch–Trlifaj [39, 3] in a more general context than that of Section 13. As usually in this paper, we also establish a connection with the left perfectness properties of discrete quotient rings of the topological ring of endomorphisms.

Let A be a cocomplete abelian category. An object $Q \in \mathsf{A}$ is said to be *pure-projective* if, for any pure short exact sequence $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ (in the sense of the definition in Section 14) in the category A, the short sequence of abelian groups $0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Q, K) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Q, M) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(Q, L) \longrightarrow 0$ is exact.

We will say that an object $M \in \mathsf{A}$ is *self-pure-projective* if, for any pure short exact sequence $0 \longrightarrow K \longrightarrow M' \longrightarrow L \longrightarrow 0$ in A with $M' \in \mathsf{Add}(M)$, the short sequence of abelian groups $0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(M, K) \longrightarrow \mathrm{Hom}_{\mathsf{A}}(M, M') \longrightarrow \mathrm{Hom}_{\mathsf{A}}(M, L) \longrightarrow 0$ is exact. The following lemma lists several classes of objects that are known to be self-pure-projective, showing that self-pure-projective objects and, in particular, self-pure-projective modules, are not uncommon.

Lemma 15.1. *The following objects in a cocomplete abelian category* A *are self-pure-projective:*

(a) *all pure-projective objects;*

(b) *all* Σ*-pure-split objects;*

(c) *all the objects belonging to* $\mathsf{Add}(M)$*, if* $M \in \mathsf{A}$ *is a self-pure-projective object;*

(d) *all the objects in the kernel* $\mathsf{L} \cap \mathsf{E}$ *of any cotorsion pair* (L, E) *in* A *such that the class* $\mathsf{E} \subset \mathsf{A}$ *is closed under coproducts and pure subobjects;*

(e) *in particular, when* $\mathsf{A} = R\text{–}\mathsf{mod}$ *is the category of modules over an associative ring* R*—any* n*-tilting left* R*-module is self-pure-projective.*

Proof. The assertions (a-c) follow immediately from the definitions. To prove (d), observe that $M \in \mathsf{L} \cap \mathsf{E}$ and $M' \in \mathsf{Add}(M)$ implies $M' \in \mathsf{E}$; and if a (pure) subobject K of M' also belongs to E, then $\mathrm{Ext}^1_{\mathsf{A}}(M, K) = 0$, and consequently $\mathrm{Hom}_{\mathsf{A}}(M, M') \longrightarrow \mathrm{Hom}_{\mathsf{A}}(M, M'/K)$ is a surjective map. Part (e) holds, because any n-tilting class E in $R\text{–}\mathsf{mod}$ is definable, which implies, in particular, that it is closed under direct sums and pure submodules [19, Definition 6.8 and Corollary 13.42]. $\square$

Notice that if A is a complete, cocomplete abelian category with exact direct limits and (L, E) is a cotorsion pair in A such that the class $\mathsf{E} \subset \mathsf{A}$ is closed under pure subobjects, then the class E is also closed under coproducts in A. Indeed, the right class E in a cotorsion pair (L, E) is always closed under products in A. For any family of objects $A_\alpha \in \mathsf{A}$, the natural morphism $\coprod_\alpha A_\alpha \longrightarrow \prod_\alpha A_\alpha$ is a direct limit of split monomorphisms, hence $\coprod_\alpha A_\alpha$ is a pure subobject of $\prod_\alpha A_\alpha$.

Remark 15.2. Let (L, E) be a cotorsion pair in the category of left modules over an associative ring R. In this context, if the class E is closed under direct limits in $R\text{–}\mathsf{mod}$, then it is definable [39, Theorem 6.1]. If the cotorsion pair (L, E) is hereditary and the class E is closed under unions of well-ordered chains in $R\text{–}\mathsf{mod}$, then the class E is definable as well [40, Theorem 3.5]. In both cases, the class $\mathsf{E} \subset R\text{–}\mathsf{mod}$ is closed under (direct sums and) pure submodules, and it follows that all the R-modules in the class $\mathsf{L} \cap \mathsf{E}$ are self-pure-projective.

We will say that an object M in a cocomplete abelian category A is Σ*-rigid* if $\mathrm{Ext}^1_{\mathsf{A}}(M, M^{(X)}) = 0$ for all sets X. If an object M is Σ-rigid, then all the objects $M' \in \mathsf{Add}(M)$ are Σ-rigid as well. If (L, E) is a cotorsion pair in A and the class E is closed under coproducts, then all the objects $M \in \mathsf{L} \cap \mathsf{E}$ are Σ-rigid.

For the rest of this section, we are working with a fixed object M in a cocomplete abelian category A. We consider the related abelian category $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ and the pair of adjoint functors $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ and $\Phi\colon \mathsf{B} \longrightarrow \mathsf{A}$, as in Section 14. Furthermore, we denote by $\mathsf{G} \subset \mathsf{A}$ the full subcategory formed by all the objects $G \in \mathsf{A}$ for which the adjunction morphism $\Phi(\Psi(G)) \longrightarrow G$ is an isomorphism, and by $\mathsf{H} \subset \mathsf{B}$ the full subcategory of all the objects $H \in \mathsf{B}$ for which the adjunction morphism $H \longrightarrow \Psi(\Phi(H))$ is an isomorphism. One has $\Psi(\mathsf{G}) \subset \mathsf{H}$ and $\Phi(\mathsf{H}) \subset \mathsf{G}$, and the restrictions of the functors Ψ and Φ to the full subcategories G and H are mutually inverse equivalences between them [16, Theorem 1.1],

$$\Psi|_{\mathsf{G}}\colon \mathsf{G} \cong \mathsf{H} :\!\Phi|_{\mathsf{H}}.$$

By construction, we have $\mathsf{Add}(M) \subset \mathsf{G}$ and $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{H}$.

Lemma 15.3. *Let A be a cocomplete abelian category, $M \in \mathsf{A}$ be an object, and $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ be the related abelian category. Suppose that the class of all projective objects in B is closed under (arbitrary or countable) direct limits. Then the class of objects $\mathsf{Add}(M) \subset \mathsf{A}$ is also closed under (arbitrary or countable, resp.) direct limits.*

Proof. Let Γ be a directed poset and $A\colon \Gamma \longrightarrow \mathsf{A}$ be a diagram such that the object $A(\gamma)$ belongs to the class $\mathsf{Add}(M)$ for all $\gamma \in \Gamma$. Applying the functor Ψ, we obtain a diagram $B = \Psi \circ A\colon \Gamma \longrightarrow \mathsf{B}$ such that $B(\gamma)$ is a projective object in B for all $\gamma \in \Gamma$. Applying the functor Φ to get back to the category A, we come to the original diagram $A \cong \Phi \circ B$. Now the functor Φ, being a left adjoint, preserves all colimits, so the natural morphism $\varinjlim_{\gamma\in\Gamma} A(\gamma) \cong \varinjlim_{\gamma\in\Gamma} \Phi(B(\gamma)) \longrightarrow \Phi\bigl(\varinjlim_{\gamma\in\Gamma} B(\gamma)\bigr)$ is an isomorphism in A. Since $\varinjlim_{\gamma\in\Gamma} B(\gamma)$ is a projective object in B by assumption and $\Phi(\mathsf{B}_{\mathsf{proj}}) = \mathsf{Add}(M)$, the desired conclusion follows. $\square$

Now we proceed to discuss the properties of self-pure-projective objects. As usually, we start with the countable case (when our results will be also applicable to Σ-rigid objects).

Proposition 15.4. *Let A be a cocomplete abelian category with exact countable direct limits, $M \in \mathsf{A}$ be an object that is either self-pure-projective or Σ-rigid, $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ be the related abelian category, and $\mathsf{G} \subset \mathsf{A}$ and $\mathsf{H} \subset \mathsf{B}$ be the related two full subcategories. Then one has $\varinjlim_\omega^{\mathsf{A}} \mathsf{Add}(M) \subset \mathsf{G}$ and $\varinjlim_\omega^{\mathsf{B}} \mathsf{B}_{\mathsf{proj}} \subset \mathsf{H}$. The functor Ψ preserves countable direct limits of objects from $\mathsf{Add}(M)$ in A (taking them to countable direct limits of the corresponding projective objects in B).*

Proof. Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \cdots$ be a sequence of objects and morphisms in A with $M_i \in \mathsf{Add}(M)$. Then the short sequence

$$0 \longrightarrow \coprod\nolimits_{i=1}^{\infty} M_i \longrightarrow \coprod\nolimits_{i=1}^{\infty} M_i \longrightarrow \varinjlim\nolimits_{i\ge1} M_i \longrightarrow 0 \tag{5}$$

is pure exact in A. Since the object M is either self-pure-projective or Σ-rigid by assumption and $\bigoplus_{i=1}^{\infty} M_i \in \mathsf{Add}(M)$, the functor $\operatorname{Hom}_{\mathsf{A}}(M,-)$ transforms the short exact sequence (5) into a short exact sequence of abelian groups.

Now, the abelian category $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ is endowed with a faithful exact forgetful functor $\mathbb{T}_M\text{–}\mathsf{mod} \longrightarrow \mathsf{Ab}$, and the composition of the functor Ψ with this forgetful functor is isomorphic to the functor $\mathrm{Hom}_{\mathsf{A}}(M,-)$. It follows that the image of the sequence (5) under the functor Ψ is exact in B.

The functors Ψ and Φ restrict to mutually inverse equivalences between $\mathsf{Add}(M) \subset \mathsf{A}$ and $\mathsf{B}_{\mathsf{proj}} \subset \mathsf{B}$; so, in particular, they transform coproducts of objects from $\mathsf{Add}(M)$ in A to coproducts of projective objects in B and vice versa. The short sequence

$$\coprod\nolimits_{i=1}^{\infty} \Psi(M_i) \longrightarrow \coprod\nolimits_{i=1}^{\infty} \Psi(M_i) \longrightarrow \varinjlim\nolimits_{i\ge1} \Psi(M_i) \longrightarrow 0 \tag{6}$$

is, at least, right exact in B; and the natural morphism from the sequence (6) to the image of the sequence (5) under the functor Ψ is an isomorphism at the leftmost and the middle terms. Hence it is also an isomorphism at the rightmost terms, that is, the natural morphism $\varinjlim_i \Psi(M_i) \longrightarrow \Psi(\varinjlim_i M_i)$ is an isomorphism.

The functor Φ, being a left adjoint, preserves all colimits. Since the adjunction morphism $\Phi\Psi(M_i) \longrightarrow M_i$ is an isomorphism for all i, it follows that the adjunction morphism $\Phi\Psi(\varinjlim_i M_i) \longrightarrow \varinjlim_i M_i$ is an isomorphism, too. Thus $\varinjlim_i M_i \in \mathsf{G}$.

We have shown that $\varinjlim_{\omega}^{\mathsf{A}} \mathsf{Add}(M) \subset \mathsf{G}$, and we have also seen that the functor Ψ transforms countable direct limits of objects from $\mathsf{Add}(M)$ in A to countable direct limits in B. Therefore, we have $\varinjlim_{\omega}^{\mathsf{B}} \mathsf{B}_{\mathsf{proj}} = \Psi(\varinjlim_{\omega}^{\mathsf{A}} \mathsf{Add}(M)) \subset \Psi(\mathsf{G}) = \mathsf{H}$. $\square$

Corollary 15.5. *Let A be a cocomplete abelian category with exact countable direct limits, $M \in \mathsf{A}$ be an object that is either self-pure-projective or Σ-rigid, and $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ be the related abelian category. Consider the following four properties:*

(i) *the class of objects $\mathsf{Add}(M) \subset \mathsf{A}$ is closed under direct limits;*
(ii) *the class of all projective objects in B is closed under direct limits.*
(iii) *the class of objects $\mathsf{Add}(M) \subset \mathsf{A}$ is closed under countable direct limits;*
(iv) *the class of all projective objects in B is closed under countable direct limits;*
(v) *all the objects from $\varinjlim_{\omega} \mathsf{Add}(M)$ have $\mathsf{Add}(M)$-covers in A;*
(vi) *all the objects from $\varinjlim_{\omega} \mathsf{B}_{\mathsf{proj}}$ have projective covers in B.*

Then the implications (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (v) $\Longleftrightarrow$ (vi) *hold.*

Furthermore, assume that the monad $\mathbb{T}_M \colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ is isomorphic to the monad $\mathbb{T}_{\mathfrak{R}}$ for a complete, separated topological associative ring $\mathfrak{R}$ with a base of neighborhoods of zero formed by open right ideals. Consider the property

(vii) *all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.*

Then the implication (vi) $\Longrightarrow$ (vii) *holds. If $\mathfrak{R}$ satisfies the condition (d) of Section 10, then all the properties* (i-vii) *are equivalent to each other.*

Proof. The implications (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) and (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) are obvious. The implication (ii) $\Longrightarrow$ (i) is provided by Lemma 15.3. The equivalences (iii) $\Longleftrightarrow$ (iv) and (v) $\Longleftrightarrow$ (vi) follow immediately from Proposition 15.4.

The implication (iv) $\Longrightarrow$ (vii) is provided by Corollary 2.4, and the implication (vi) $\Longrightarrow$ (vii) by Corollary 3.6.

Assuming that the topological ring $\mathfrak{R}$ satisfies the condition (d), the implication (vii) $\Longrightarrow$ (ii) is provided by Theorem 10.4 (iv) $\Longrightarrow$ (iii), because the direct limits of projective contramodules are always flat. $\square$

Remark 15.6. When A is a Grothendieck abelian category, the condition (i) of Corollary 15.5 implies that the class of objects $\mathsf{Add}(M)$ is covering in A [4, Theorem 1.2]. The same conclusion holds in the more general case of a locally presentable abelian category A [35, Theorem 1.7 or Corollary 4.17] (cf. the proof of Proposition 13.1). Furthermore, if the category A is locally presentable, then the category B is locally presentable by [36, Proposition 8.1] (see [35, Section 1.1 in the introduction] and [33, Section 1] for some background). So, by Theorem 7.1, the condition (ii) implies that all the objects of B have projective covers.

The next proposition is the uncountable version of Proposition 15.4.

Proposition 15.7. *Let A be a cocomplete abelian category with exact direct limits, $M \in \mathsf{A}$ be a self-pure-projective object, $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ be the related abelian category, and $\mathsf{G} \subset \mathsf{A}$ and $\mathsf{H} \subset \mathsf{B}$ be the related two full subcategories. Then one has $\varinjlim^{\mathsf{A}} \mathsf{Add}(M) \subset \mathsf{G}$ and $\varinjlim^{\mathsf{B}} \mathsf{B}_{\mathsf{proj}} \subset \mathsf{H}$. The functor Ψ preserves direct limits of objects from $\mathsf{Add}(M)$ in A (taking them to direct limits of the corresponding projective objects in B).*

Proof. Let Γ be a directed poset and $A\colon \Gamma \longrightarrow \mathsf{A}$ be a diagram in A with $A(\gamma) \in \mathsf{Add}(M)$ for all $\gamma \in \Gamma$. Then the bar-complex $(*)$ from the proof of Proposition 12.5 is pure exact in A. As all the terms of this complex, except perhaps the rightmost one, belong to $\mathsf{Add}(M)$ and the object M is self-pure-projective, it follows that the functor $\mathrm{Hom}_{\mathsf{A}}(M,-)$ takes the complex $(*)$ to an exact sequence of abelian groups. As in the proof of Proposition 15.4, we conclude that the functor Ψ transforms the complex $(*)$ into an exact complex in B.

On the other hand, the similar bar-complex (4) constructed in the abelian category B for the diagram $B = \Psi \circ A$ is exact, at least, at its rightmost term. Furthermore, the natural morphism from the complex (4) to the image of the complex $(*)$ under Ψ is an isomorphism at all the terms, except perhaps the rightmost one. It follows that this morphism of complexes is an isomorphism at the rightmost terms, too; that is, the natural morphism $\varinjlim_{\gamma\in\Gamma} \Psi(A(\gamma)) \longrightarrow \Psi(\varinjlim_{\gamma\in\Gamma} A(\gamma))$ is an isomorphism.

The argument finishes in the same way as the proof of Proposition 15.4. $\square$

Corollary 15.8. *Let A be a Grothendieck abelian category, $M \in \mathsf{A}$ be a self-pure-projective object, and $\mathsf{B} = \mathbb{T}_M\text{–}\mathsf{mod}$ be the related abelian category. Consider the following six properties:*

(i) *the class of objects $\mathsf{Add}(M) \subset \mathsf{A}$ is closed under direct limits;*
(ii) *the class of all projective objects in B is closed under direct limits;*
(iii) *the class of objects $\mathsf{Add}(M)$ is covering in A;*
(iv) *all the objects of B have projective covers;*
(v) *all the objects from $\varinjlim \mathsf{Add}(M)$ have $\mathsf{Add}(M)$-covers in A;*
(vi) *all the objects from $\varinjlim \mathsf{B}_{\mathsf{proj}}$ have projective covers in B.*

Then the equivalences (i) $\Longleftrightarrow$ (ii) *and* (v) $\Longleftrightarrow$ (vi) *hold, and the implications* (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (v) *and* (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (vi) *hold as well.*

Furthermore, assume that the monad $\mathbb{T}_M\colon \mathsf{Sets} \longrightarrow \mathsf{Sets}$ *is isomorphic to the monad* $\mathbb{T}_{\mathfrak{R}}$ *for a complete, separated topological associative ring* $\mathfrak{R}$ *with a base of neighborhoods of zero formed by open right ideals. Consider the property*

(vii) *all the discrete quotient rings of the topological ring* $\mathfrak{R}$ *are left perfect.*

Then the implication (vi) $\Longrightarrow$ (vii) *holds. If* $\mathfrak{R}$ *satisfies the condition (d) of Section 10, then all the properties* (i-vii) *are equivalent to each other.*

Proof. The equivalences (i) $\Longleftrightarrow$ (ii) and (v) $\Longleftrightarrow$ (vi) follow from Proposition 15.7.

The implications (i) $\Longrightarrow$ (iii) and (ii) $\Longrightarrow$ (iv) were explained in Remark 15.6.

The implications (iii) $\Longrightarrow$ (v) and (iv) $\Longrightarrow$ (vi) are obvious.

For the implications (vi) $\Longrightarrow$ (vii) and (vii) $\Longrightarrow$ (ii), see Corollary 15.5. $\square$

16. Matlis Category Equivalences for a Ring Epimorphism

Let $u\colon R \longrightarrow U$ be a ring epimorphism, i. e., in other words, a homomorphism of associative rings such that the multiplication map $U \otimes_R U \longrightarrow U$ an isomorphism of R-R-bimodules. Then one has $U \otimes_R D \cong D \cong \mathrm{Hom}_R(U, D)$ for all left U-modules D, and the functor of restriction of scalars $U\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful. The similar assertions hold for the right modules. We will say that a certain R-module "is a U-module" if it belongs to the image of the functor of restriction of scalars.

We will use the simple notation U/R for the cokernel of the map $u\colon R \longrightarrow U$. So U/R is an R-R-bimodule.

A left R-module M is called a *u-comodule* (or a *left u-comodule*) if

$$U \otimes_R M = 0 = \mathrm{Tor}_1^R(U, M).$$

Similarly, a right R-module N is said to be a u-comodule (or a right u-comodule) if $N \otimes_R U = 0 = \mathrm{Tor}_1^R(N, U)$.

A left R-module C is called a *u-contramodule* (or a *left u-contramodule*) if

$$\mathrm{Hom}_R(U, C) = 0 = \mathrm{Ext}_R^1(U, C).$$

By [18, Proposition 1.1], the class of all left u-comodules is closed under direct sums, cokernels of morphisms, and extensions in $R\text{–}\mathsf{mod}$. The class of all left u-contramodules is closed under products, kernels of morphisms, and extensions.

We will use the notation $\mathrm{pd}\,{}_RE$ for the projective dimension of a left R-module E and $\mathrm{fd}\,E_R$ for the flat dimension of a right R-module E.

Borrowing the terminology going back to Harrison [20] and Matlis [25], we will say that a left R-module A is *u-torsion-free* if it is an R-submodule of a left U-module, or equivalently, if the map $A \longrightarrow U \otimes_R A$ induced by the ring homomorphism u is injective. Similarly, we will say that a left R-module B is *u-h-divisible* if it is a quotient module of a left U-module, or equivalently, if the map $\mathrm{Hom}_R(U, B) \longrightarrow B$ induced by u is surjective.

Clearly, the class of all u-torsion-free left R-modules is closed under subobjects, direct sums, and products in R–mod. Any left R-module A has a unique maximal u-torsion-free quotient module, which can be computed as the image of the natural R-module morphism $A \longrightarrow U \otimes_R A$. The class of all u-h-divisible left R-modules is closed under quotients, direct sums, and products. Any left R-module B has a unique maximal u-h-divisible submodule, which can be computed as the image of the natural R-module morphism $\mathrm{Hom}_R(U, B) \longrightarrow B$.

A left R-module A is said to be *u-torsion* if its maximal u-torsion-free quotient module vanishes, or equivalently, if $U \otimes_R A = 0$. A left R-module B is said to be *u-h-reduced* if its maximal u-h-divisible submodule vanishes, or equivalently, if $\mathrm{Hom}_R(U, B) = 0$.

Let us warn the reader that our terminology is slightly confusing: a left R-module with no nonzero u-torsion submodules does *not* need to be u-torsion-free (unless $\mathrm{fd}\, U_R \le 1$, as we will see below). Similarly, a left R-module with no u-h-reduced quotient modules does not need to be u-h-divisible (unless $\mathrm{pd}\, {}_RU \le 1$). The problem is that, unless such homological dimension conditions are imposed on the R-R-bimodule U or the ring homomorphism u, the classes of u-torsion-free and u-h-divisible left R-modules do not need to be closed under extensions.

The following theorem provides what appears to be the maximal natural generality for the first of the two classical *Matlis category equivalences* [25, Theorem 3.4] (going back to Harrison's [20, Proposition 2.1]).

Theorem 16.1. *Assume that $\mathrm{Tor}_1^R(U, U) = 0$. Then the restrictions of the adjoint functors $M \longmapsto \mathrm{Hom}_R(U/R, M)$ and $C \longmapsto (U/R) \otimes_R C$ are mutually inverse equivalences between the additive categories of u-h-divisible left u-comodules M and u-torsion-free left u-contramodules C.*

Before proceeding to prove the theorem, let us formulate and prove a lemma.

Lemma 16.2. *If $\mathrm{Tor}_1^R(U, U) = 0$, then*

(a) *for any left R-module M, the left R-module $\mathrm{Hom}_R(U/R, M)$ is a u-torsion-free u-contramodule;*

(b) *for any left R-module C, the left R-module $(U/R) \otimes_R C$ is a u-h-divisible u-comodule.*

Proof. Part (a): the left R-module $\mathrm{Hom}_R(U/R, M)$ is u-torsion-free as an R-submodule of the left U-module $\mathrm{Hom}_R(U, M)$. Furthermore, since $U \otimes_R U = U$, we have $(U/R) \otimes_R U = 0$, and therefore $\mathrm{Hom}_R(U, \mathrm{Hom}_R(U/R, M)) = 0$.

To show that $\mathrm{Ext}_R^1(U, \mathrm{Hom}_R(U/R, M)) = 0$, one observes that our assumptions $U \otimes_R U = U$ and $\mathrm{Tor}_1^R(U, U) = 0$ imply $\mathrm{Tor}_1^R(U/R, U) = 0$, because the map $(R/\ker(u)) \otimes_R U \longrightarrow U$ is an isomorphism.

For any left R-module L, R-R-bimodule E, and left R-module M such that $\mathrm{Tor}_1^R(E, L) = 0$, there is a natural injective map of abelian groups

$$\mathrm{Ext}_R^1(L, \mathrm{Hom}_R(E, M)) \longrightarrow \mathrm{Ext}_R^1(E \otimes_R L,\ M).$$

In particular, $\mathrm{Ext}_R^1(U, \mathrm{Hom}_R(U/R, M))$ is a subgroup of $\mathrm{Ext}_R^1((U/R) \otimes_R U,\ M) = 0$.

The proof of part (b) is dual-analogous. The left R-module $(U/R) \otimes_R C$ is u-h-divisible as a quotient R-module of the left U-module $U \otimes_R C$. Since $U \otimes_R (U/R) = 0$, we have $U \otimes_R (U/R) \otimes_R C = 0$.

For any right R-module B, R-R-bimodule E, and left R-module C such that $\operatorname{Tor}_1^R(B, E) = 0$, there is a natural surjective map of abelian groups

$$\operatorname{Tor}_1^R(B \otimes_R E,\ C) \longrightarrow \operatorname{Tor}_1^R(B,\ E \otimes_R C).$$

In particular, $\operatorname{Tor}_1^R(U,\ (U/R) \otimes_R C)$ is a quotient group of $\operatorname{Tor}_1^R(U \otimes_R (U/R),\ C) = 0$. For a more high-tech derived category/spectral sequence presentation of the same argument, see Lemmas 16.6–16.7 below. □

Proof of Theorem 16.1. By Lemma 16.2, the functor $M \longmapsto \operatorname{Hom}_R(U/R, M)$ take u-h-divisible left u-comodules to u-torsion-free left u-contramodules and back (in fact, they take arbitrary left R-modules to left R-modules from these two classes). It remains to show that the restrictions of these functors to these two full subcategories in R–mod are mutually inverse equivalences between them.

Let M be a u-h-divisible left u-comodule. We will show that the adjunction morphism $(U/R) \otimes_R \operatorname{Hom}_R(U/R, M) \longrightarrow M$ is an isomorphism. Since M is u-h-divisible, we have a natural short exact sequence of left R-modules

$$0 \longrightarrow \operatorname{Hom}_R(U/R, M) \longrightarrow \operatorname{Hom}_R(U, M) \longrightarrow M \longrightarrow 0.$$

Since the left R-module $\operatorname{Hom}_R(U/R, M)$ is u-torsion-free, we also have a natural short exact sequence of left R-modules

$$0 \longrightarrow \operatorname{Hom}_R(U/R, M) \longrightarrow U \otimes_R \operatorname{Hom}_R(U/R, M) \longrightarrow U/R \otimes_R \operatorname{Hom}_R(U/R, M) \longrightarrow 0.$$

Since M is a u-comodule, applying the functor $U \otimes_R -$ to the former short exact sequence produces an isomorphism $U \otimes_R \operatorname{Hom}_R(U/R, M) \cong U \otimes_R \operatorname{Hom}_R(U, M) = \operatorname{Hom}_R(U, M)$. Now we have a natural morphism from the latter short exact sequence to the former one, which is the identity on the leftmost terms and an isomorphism on the middle terms. Therefore, it is an isomorphism on the rightmost terms, too.

Let C be a u-torsion-free left u-contramodule. Let us show that the adjunction morphism $C \longrightarrow \operatorname{Hom}_R(U/R,\ (U/R) \otimes_R C)$ is an isomorphism. Since C is u-torsion-free, we have a natural short exact sequence of left R-modules

$$0 \longrightarrow C \longrightarrow U \otimes_R C \longrightarrow (U/R) \otimes_R C \longrightarrow 0.$$

Since the left R-module $(U/R) \otimes_R C$ is u-h-divisible, we also have a natural short exact sequence of left R-modules

$$0 \longrightarrow \operatorname{Hom}_R(U/R,\ (U/R) \otimes_R C) \longrightarrow \operatorname{Hom}_R(U,\ (U/R) \otimes_R C) \longrightarrow (U/R) \otimes_R C \longrightarrow 0.$$

Since C is a u-contramodule, applying the functor $\operatorname{Hom}_R(U, -)$ to the former short exact sequence produces an isomorphism $U \otimes_R C = \operatorname{Hom}_R(U,\ U \otimes_R C) \cong \operatorname{Hom}_R(U,\ (U/R) \otimes_R C)$. We have a natural morphism from the former short exact sequence to the latter one, which is the identity on the rightmost terms and an isomorphism on the middle terms. Therefore, it is an isomorphism on the leftmost terms, too. □

Let $K^\bullet$ denote the two-term complex $R \longrightarrow U$, with the term R placed in the cohomological degree -1 and the term U in the cohomological degree 0. We will view $K^\bullet$ as an object of the bounded derived category of R-R-bimodules $\mathsf{D^b}(R\text{–}\mathsf{mod}\text{–}R)$. So, there is a distinguished triangle

$$R \longrightarrow U \longrightarrow K^\bullet \longrightarrow R[1] \tag{7}$$

in the triangulated category $\mathsf{D^b}(R\text{–}\mathsf{mod}\text{–}R)$.

Alternatively, the complex $K^\bullet$ can be considered as an object of the bounded derived category of left R-modules $\mathsf{D^b}(R\text{–}\mathsf{mod})$ endowed with a right action of the ring R by its derived category object endomorphisms, or as an object of the bounded derived category of right R-modules $\mathsf{D^b}(\mathsf{mod}\text{–}R)$ endowed with a left action of R. Then (7) is viewed as a distinguished triangle in $\mathsf{D^b}(R\text{–}\mathsf{mod})$ or $\mathsf{D^b}(\mathsf{mod}\text{–}R)$.

By an abuse of notation, given a left R-module B, we will denote simply by

$$\mathrm{Ext}^n_R(K^\bullet, B) = H^n(\mathbb{R}\mathrm{Hom}_R(K^\bullet, B)) = \mathrm{Hom}_{\mathsf{D^b}(R\text{–}\mathsf{mod})}(K^\bullet, B[n])$$

the abelian group of all morphisms $K^\bullet \longrightarrow B[i]$ in the derived category of left R-modules. The right action of R in the object $K^\bullet \in \mathsf{D^b}(R\text{–}\mathsf{mod})$ induces a left R-module structure on the groups $\mathrm{Ext}^i_R(K^\bullet, B)$.

Similarly, we set

$$\mathrm{Tor}^R_i(K^\bullet, A) = H^{-i}(K^\bullet \otimes^{\mathbb{L}}_R A)$$

for any left R-module B. Here $K^\bullet$ is viewed as an object of the bounded derived category of right R-modules for the purpose of computing the derived tensor product $K^\bullet \otimes^{\mathbb{L}}_R A$, and then the left action of R in the object $K^\bullet \in \mathsf{D^b}(\mathsf{mod}\text{–}R)$ induces a left R-module structure on the groups $\mathrm{Tor}^R_i(K^\bullet, A)$.

Lemma 16.3. *For every left R-module C, there are natural isomorphisms of left R-modules*

(a) $\mathrm{Tor}^R_n(K^\bullet, C) = 0 = \mathrm{Ext}^n_R(K^\bullet, C)$ *for* $n < 0$*;*

(b) $\mathrm{Tor}^R_n(K^\bullet, C) = \mathrm{Tor}^R_n(U, C)$ *and* $\mathrm{Ext}^n_R(K^\bullet, C) = \mathrm{Ext}^n_R(U, C)$ *for all* $n > 1$*;*

(c) $\mathrm{Tor}^R_0(K^\bullet, C) = (U/R) \otimes_R C$ *and* $\mathrm{Ext}^0_R(K^\bullet, C) = \mathrm{Hom}_R(U/R, C)$*.*

Proof. All the assertions follow immediately from the (co)homology long exact sequences obtained by applying the functors $\mathbb{R}\mathrm{Hom}_R(-, C)$ and $- \otimes^{\mathbb{L}}_R C$ to the distinguished triangle (7). $\square$

Furthermore, for any left R-modules A and B there are five-term exact sequences of low-dimensional Tor and Ext induced by the distinguished triangle (7):

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}^R_1(U, A) \longrightarrow \mathrm{Tor}^R_1(K^\bullet, A) \\ \longrightarrow A \longrightarrow U \otimes_R A \longrightarrow \mathrm{Tor}^R_0(K^\bullet, A) \longrightarrow 0 \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}^0_R(K^\bullet, B) \longrightarrow \mathrm{Hom}_R(U, B) \longrightarrow B \\ \longrightarrow \mathrm{Ext}^1_R(K^\bullet, B) \longrightarrow \mathrm{Ext}^1_R(U, B) \longrightarrow 0. \end{aligned} \tag{9}$$

Both (8) and (9) are exact sequences of left R-modules.

Borrowing the terminology of Matlis [25], we will say that a left R-module A is *u-special* if the map $A \longrightarrow U \otimes_R A$ is surjective. Equivalently (in view of the exact sequence (8) or Lemma 16.3(c)), this means that $\mathrm{Tor}_0^R(K^\bullet, A) = 0$. Similarly, a left R-module B is *u-cospecial* if the map $\mathrm{Hom}_R(U, B) \longrightarrow B$ is injective. Equivalently (by the exact sequence (9) or Lemma 16.3(c)), this means that $\mathrm{Ext}_R^0(K^\bullet, B) = 0$.

The next lemma provides another characterization of u-special and u-cospecial modules.

Lemma 16.4. (a) *A left R-module A is u-special if and only if its maximal u-torsion-free quotient module is a U-module.*

(b) *A left R-module B is u-cospecial if and only if its maximal u-h-divisible submodule is a U-module.*

Proof. Part (b): if B is u-cospecial, then the morphism $\mathrm{Hom}_R(U, B) \longrightarrow B$ is injective, so $\mathrm{Hom}_R(U, B)$ is the maximal u-h-divisible submodule of B. Conversely, if the maximal u-h-divisible submodule of B is a U-module D, then $\mathrm{Hom}_R(U/R, B) = \mathrm{Hom}_R(U/R, D) = 0$.

Part (a): if A is u-special, then the morphism $A \longrightarrow U \otimes_R A$ is surjective, so $U \otimes_R A$ is the maximal u-torsion-free quotient module of A. Conversely, if the maximal u-torsion-free quotient module of A is a U-module D, then $\mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(U/R, \mathbb{Q}/\mathbb{Z})) = \mathrm{Hom}_{\mathbb{Z}}(D, \mathrm{Hom}_R(U/R, \mathbb{Q}/\mathbb{Z})) = 0$ (since $\mathrm{Hom}_{\mathbb{Z}}(U/R, \mathbb{Q}/\mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{Z}}(U, \mathbb{Q}/\mathbb{Z})$ is a u-torsion-free left R-module), hence $U/R \otimes_R A = 0$. ☐

The following theorem is our version of the second Matlis category equivalence [25, Theorem 3.8] (going back to Harrison's [20, Proposition 2.3]).

Theorem 16.5. *Assume that $\mathrm{Tor}_1^R(U, U) = 0 = \mathrm{Tor}_2^R(U, U)$. Then the restrictions of the functors $M \longmapsto \mathrm{Ext}_R^1(K^\bullet, M)$ and $C \longmapsto \mathrm{Tor}_1^R(K^\bullet, C)$ are mutually inverse equivalences between the additive categories of u-cospecial left u-comodules M and u-special left u-contramodules C.*

Before proving the theorem, we formulate two lemmas, which extend the result of Lemma 16.2.

Lemma 16.6. (a) *If $\mathrm{Tor}_1^R(U, U) = 0$, then the left R-module $\mathrm{Tor}_0^R(K^\bullet, A)$ is a u-comodule for any left R-module A.*

(b) *If $\mathrm{Tor}_1^R(U, U) = 0 = \mathrm{Tor}^2_R(U, U) = 0$, then the left R-module $\mathrm{Tor}_1^R(K^\bullet, A)$ is a u-comodule for any left R-module A such that $\mathrm{Tor}_0^R(K^\bullet, A) = 0$.*

(c) *If $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{fd}\, U_R \leq 1$, then the left R-module $\mathrm{Tor}_1^R(K^\bullet, A)$ is a u-comodule for any left R-module A.*

Proof. There is a spectral sequence

$$E_{pq}^2 = \mathrm{Tor}_p^R(U, \mathrm{Tor}_q^R(K^\bullet, A)) \Longrightarrow E_{pq}^\infty = \mathrm{gr}_p \, \mathrm{Tor}_{p+q}^R(U \otimes_R^{\mathbb{L}} K^\bullet,\ A),$$

where $\mathrm{Tor}_n^R(U \otimes_R^{\mathbb{L}} K^\bullet,\ A) = H^{-n}(U \otimes_R^{\mathbb{L}} K^\bullet \otimes_R^{\mathbb{L}} A) = 0$ whenever $H^{-i}(U \otimes_R^{\mathbb{L}} K^\bullet) = 0$ for all $0 \leq i \leq n$. Since $U \otimes_R U = U$, the latter condition holds whenever $\mathrm{Tor}_i^R(U, U) = 0$

for all $1 \le i \le n$. Thus $E^\infty_{pq} = 0$ whenever $p + q \le 1$ in the assumptions of part (a), whenever $p + q \le 2$ in the assumptions of part (b), and for all p, $q \in \mathbb{Z}$ in the assumptions of part (c).

The differentials are $\partial^r_{pq}\colon E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$, $r \ge 2$. Now all the differentials involving $E^r_{0,0}$ and $E^r_{0,1}$ vanish for the dimension reasons, so $E^\infty_{0,0} = 0 = E^\infty_{0,1}$ implies $E^2_{0,0} = 0 = E^2_{0,1}$. This proves part (a). Furthermore, the only possibly nontrivial differentials involving $E^r_{0,1}$ and $E^r_{1,1}$ are

$$\partial^2_{2,0}\colon E^2_{2,0} \longrightarrow E^2_{0,1} \quad \text{and} \quad \partial^2_{3,0}\colon E^2_{3,0} \longrightarrow E^2_{1,1}.$$

When $\operatorname{Tor}^0_R(K^\bullet, A) = 0$, one has $E^2_{p,0} = 0$ for all $p \in \mathbb{Z}$. When $\operatorname{fd} U_R \le 1$, one has $E^2_{pq} = 0$ for $p \ge 2$ and all q. In both cases, $E^\infty_{0,1} = 0 = E^\infty_{1,1}$ implies $E^2_{0,1} = 0 = E^2_{1,1}$, proving parts (b) and (c). ☐

Lemma 16.7. (a) *If* $\operatorname{Tor}^R_1(U, U) = 0$*, then the left* R*-module* $\operatorname{Ext}^0_R(K^\bullet, B)$ *is a* u*-contramodule for any left* R*-module* B*.*

(b) *If* $\operatorname{Tor}^R_1(U, U) = 0 = \operatorname{Tor}^2_R(U, U) = 0$*, then the left* R*-module* $\operatorname{Ext}^1_R(K^\bullet, B)$ *is a* u*-contramodule for any left* R*-module* B *such that* $\operatorname{Tor}^0_R(K^\bullet, B) = 0$*.*

(c) *If* $\operatorname{Tor}^R_1(U, U) = 0$ *and* $\operatorname{pd}_R U \le 1$*, then the left* R*-module* $\operatorname{Ext}^1_R(K^\bullet, B)$ *is a* u*-contramodule for any left* R*-module* B*.*

Proof. Dual-analogous to Lemma 16.6 (and similar to [32, Lemma 1.7]). ☐

Proof of Theorem 16.5. Let M be a u-cospecial left u-comodule. By Lemma 16.7(b), the left R-module $\operatorname{Ext}^1_R(K^\bullet, M)$ is a u-contramodule. Furthermore, the exact sequence (9) for the R-module M reduces to a four-term sequence

$$0 \longrightarrow \operatorname{Hom}_R(U, M) \longrightarrow M \longrightarrow \operatorname{Ext}^1_R(K^\bullet, M) \longrightarrow \operatorname{Ext}^1_R(U, M) \longrightarrow 0.$$

Denoting by E the image of the map $M \longrightarrow \operatorname{Ext}^1_R(K^\bullet, M)$, we have two short exact sequences of left R-modules $0 \longrightarrow \operatorname{Hom}_R(U, M) \longrightarrow M \longrightarrow E \longrightarrow 0$ and $0 \longrightarrow E \longrightarrow \operatorname{Ext}^1_R(K^\bullet, M) \longrightarrow \operatorname{Ext}^1_R(U, M) \longrightarrow 0$.

The assumptions that $U \otimes_R U = U$ and $\operatorname{Tor}^R_i(U, U) = 0$ for $i = 1$ and 2 imply that $U \otimes_R D = D$ and $\operatorname{Tor}^R_i(U, D) = 0$ for all left U-modules D and $i = 1$, 2. Hence (by Lemma 16.3 and the exact sequence (8) for the R-module D) we have $\operatorname{Tor}^R_i(K^\bullet, D) = 0$ for $-1 \le i \le 2$. In particular, this applies to the left U-modules $D = \operatorname{Hom}_R(U, M)$ and $\operatorname{Ext}^1_R(U, M)$.

Now from the long exact sequences of $\operatorname{Tor}^R_*(K^\bullet, -)$ related to our two short exact sequences of left R-modules we see that both the maps $\operatorname{Tor}^R_i(K^\bullet, M) \longrightarrow \operatorname{Tor}^R_i(K^\bullet, E) \longrightarrow \operatorname{Tor}^R_i(K^\bullet, \operatorname{Ext}^1_R(K^\bullet, M))$ are isomorphisms for $i = 0$ and 1. In particular, $\operatorname{Tor}^R_0(K^\bullet, \operatorname{Ext}^1_R(K^\bullet, M)) \cong \operatorname{Tor}^R_0(K^\bullet, M) = (U/R) \otimes_R M = 0$, since $U \otimes_R M = 0$. Hence the left R-module $\operatorname{Ext}^1_R(K^\bullet, M)$ is u-special.

Furthermore, the map $\operatorname{Tor}^R_1(K^\bullet, M) \longrightarrow M$ in the short exact sequence (8) is an isomorphism, since $U \otimes_R M = 0 = \operatorname{Tor}^1_R(U, M)$. Thus we obtain a natural isomorphism $\operatorname{Tor}^R_1(K^\bullet, \operatorname{Ext}^1_R(K^\bullet, M)) \cong M$.

The dual-analogous argument shows that the left R-module $\operatorname{Tor}^1_R(K^\bullet, C)$ is a u-cospecial u-comodule for any u-special u-contramodule C, and provides a natural isomorphism $\mathrm{Ext}^1_R(K^\bullet, \mathrm{Tor}_1^R(K^\bullet, C)) \cong C$. One has to observe that $\mathrm{Hom}_R(U, D) = D$ and $\mathrm{Ext}^i_R(U, D) = 0$ for for all left U-modules D and $i = 1, 2$, hence $\mathrm{Ext}^i_R(K^\bullet, D) = 0$ for $-1 \le i \le 2$, etc. $\square$

In the rest of this section we discuss how our theory simplifies and improves in the assumptions that the projective dimension of the left R-module U and/or the flat dimension of the right R-module U do not exceed 1.

Lemma 16.8. (a) *Assume that* $\mathrm{Tor}_1^R(U, U) = 0$ *and* $\mathrm{fd}\, U_R \le 1$. *Then a left R-module A is u-torsion-free if and only if* $\mathrm{Tor}_1^R(K^\bullet, A) = 0$.

(b) *Assume that* $\mathrm{Tor}_1^R(U, U) = 0$ *and* $\mathrm{pd}\,{}_R U \le 1$. *Then a left R-module B is u-h-divisible if and only if* $\mathrm{Ext}^1_R(K^\bullet, B) = 0$.

Proof. This is similar to [32, Lemma 5.1(b)]. Let us prove part (a). The "if" claim follows immediately from the exact sequence (8). To prove the "only if", assume that A is u-torsion-free. Then the exact sequence (8) implies that the left R-module morphism $\mathrm{Tor}_1^R(U, A) \longrightarrow \mathrm{Tor}_1^R(K^\bullet, A)$ is an isomorphism. Since $\mathrm{Tor}_1^R(K^\bullet, A)$ is a left u-comodule by Lemma 16.6(c) and $\mathrm{Tor}_1^R(U, A)$ is a left U-module, they can only be isomorphic when both of them vanish. $\square$

It is clear from the definition and Lemma 16.8(a) that, when $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{fd}\, U_R \le 1$, the full subcategory of u-torsion-free R-modules is closed under extensions, subobjects, direct sums, and products. So u-torsion-free R-modules form the torsion-free class of a certain torsion pair in R–mod. The related torsion class is the class of all u-torsion R-modules, that is, all left R-modules A such that $U \otimes_R A = 0$.

Similarly, it is clear from the definition and Lemma 16.8(b) that, whenever $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{pd}\,{}_R U \le 1$, the full subcategory of u-h-divisible R-modules is closed under extensions, quotients, direct sums and products. So u-h-divisible R-modules form the torsion class of a certain torsion theory in R–mod. The related torsion class is the class of all u-h-reduced R-modules, that is, all left R-modules B such that $\mathrm{Hom}_R(U, B) = 0$.

It is clear from the definition that the full subcategory of u-special left R-modules is closed under extensions, quotients, and direct sums. Hence it is the torsion class of a torsion pair in R–mod. When $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{fd}\, U_R \le 1$, the related torsion-free class can be described as the class of all u-torsion-free u-h-reduced left R-modules.

Similarly, the full subcategory of u-cospecial left R-modules is closed under extensions, subobjects, direct sums, and products. Hence it is the torsion-free class of a torsion pair in R–mod. When $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{pd}\,{}_R U \le 1$, the related torsion class can be described as the class of all u-h-divisible u-torsion left R-modules.

Remark 16.9. Notice that every left u-comodule is u-torsion, but the converse implication does not need to be true. The torsion class of all u-torsion left R-modules does *not* need to be hereditary, i. e., a submodule of a u-torsion R-module does not need to be u-torsion. In fact, if $\mathrm{Tor}_1^R(U, U) = 0$ and $\mathrm{fd}\, U_R \le 1$, then any one of these two properties holds if and only if U is a flat right R-module.

Indeed, if the classes of left u-comodules and u-torsion left R-modules coincide, then the class of u-torsion left R-modules is closed under kernels and quotients, hence it is also closed under submodules. In particular, from the exact sequence (8) we see that for any left R-module A the left R-module $\operatorname{Tor}_1^R(U,A)$ is a submodule of the left R-module $\operatorname{Tor}_1^R(K^\bullet,A)$. By Lemma 16.6(c), the latter is a u-torsion module (and even a u-comodule). If the class of all u-torsion left R-modules is closed under submodules, then $\operatorname{Tor}_1^R(U,A)$ is a u-torsion left R-module. Being simultaneously a left U-module, it follows that $\operatorname{Tor}_1^R(U,A)=0$.

Examples of noncommutative homological ring epimorphisms of projective dimension 1 (on both sides) that are not flat (on either side) do exist. We are grateful to J. Šťovíček for bringing the following one to our attention. Let k be a field, $k[x]$ be the polynomial ring in one variable x with the coefficients in k, and $kx\subset k[x]$ be the one-dimensional k-vector subspace spanned by x. Then the embedding of matrix rings $R=\left(\begin{smallmatrix} k & k\oplus kx \\ 0 & k\end{smallmatrix}\right)\longrightarrow\left(\begin{smallmatrix} k[x] & k[x] \\ k[x] & k[x]\end{smallmatrix}\right)=U$ is an injective ring epimorphism such that $\operatorname{Tor}_1^R(U,U)=0$ and $\operatorname{pd}{}_RU=\operatorname{pd}U_R=\operatorname{fd}{}_RU=\operatorname{fd}U_R=1$.

On the other hand, if $u\colon R\longrightarrow U$ is an epimorphism of commutative rings such that $\operatorname{Tor}_1^R(U,U)=0$ and $\operatorname{pd}{}_RU\le 1$, then U is a flat R-module. Indeed, assume first that u is injective. Then $U\oplus U/R$ is a 1-tilting R-module [2, Theorem 3.5], hence $C=\operatorname{Hom}_{\mathbb{Z}}(U\oplus U/R,\ \mathbb{Q}/\mathbb{Z})$ is a 1-cotilting R-module of cofinite type [19, Theorems 15.2 and 15.18]. The 1-cotilting class associated with C consists of all the R-submodules of U-modules; in other words, it is what we call the class of all u-torsion-free R-modules. Hence the torsion class in the 1-cotilting torsion pair associated with C is the class of all u-torsion R-modules. According to [21, Proposition 3.11], any 1-cotilting torsion pair of cofinite type in the category of modules over a commutative ring is hereditary. By the above argument, it follows that $\operatorname{fd}{}_RU=0$.

More generally, let $u\colon R\longrightarrow U$ be a (not necessarily injective) homological epimorphism of commutative rings such that $\operatorname{pd}{}_RU\le 1$. Then $U\oplus U/R$ is a 1-silting R-module [24, Example 6.5], and a 2-term projective resolution of the complex $U\oplus K^\bullet$ is the related silting complex. Hence $C=\operatorname{Hom}_{\mathbb{Z}}(U\oplus U/R,\ \mathbb{Q}/\mathbb{Z})$ is a cosilting R-module of cofinite type [1, Corollary 3.6]. The cosilting class associated with C consists of all the u-torsion-free R-modules, and the torsion class in the cosilting torsion pair is the class of all u-torsion R-modules. By [1, Lemma 4.2], any cosilting torsion pair of cofinite type in the category of modules over a commutative ring is hereditary. Once again, we can conclude that U is a flat R-module.

17. Abelian Categories of u-Comodules and u-Contramodules

In this section, as in the previous one, $u\colon R\longrightarrow U$ is an associative ring epimorphism. For most of the results, we will have to assume that u is a *homological* ring epimorphism, that is, $\operatorname{Tor}_i^R(U,U)=0$ for $i\ge 1$. In fact, we will mostly have to assume either that the flat dimension of the right R-module U does not exceed 1 (when discussing left u-comodules), or that the projective dimension of the left R-module U does not exceed 1 (when considering left u-contramodules).

Let us denote the full subcategory of left u-comodules by $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}} \subset R\text{–}\mathsf{mod}$, and the full subcategory of left u-contramodules by $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \subset R\text{–}\mathsf{mod}$. For any left R-module C, we set $\Gamma_u(C) = \mathrm{Tor}_1^R(K^\bullet, C)$ and $\Delta_u(C) = \mathrm{Ext}_R^1(K^\bullet, C)$. The natural left R-module morphisms (occuring in the exact sequences (8–9)) are denoted by $\gamma_{u,C}\colon \Gamma_u(C) \longrightarrow C$ and $\delta_{u,C}\colon C \longrightarrow \Delta_u(C)$.

Proposition 17.1. *Assume that* $\mathrm{fd}\, U_R \le 1$. *Then*

(a) *the full subcategory* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ *is closed under the kernels, cokernels, extensions, and direct sums in* $R\text{–}\mathsf{mod}$. *So* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ *is an abelian category and the embedding functor* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}} \longrightarrow R\text{–}\mathsf{mod}$ *is exact;*

(b) *assuming also that* $\mathrm{Tor}_1^R(U,U) = 0$, *the functor* $\Gamma_u\colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ *is right adjoint to the fully faithful embedding functor* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}} \longrightarrow R\text{–}\mathsf{mod}$.

Proof. Part (a) is a particular case of [18, Proposition 1.1] or [31, Theorem 1.2(b)]. To prove part (b), notice that $\Gamma_u(A) \in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ for any $A \in R\text{–}\mathsf{mod}$ by Lemma 16.6(c). We have to show that for every left R-module A, every left u-comodule M, and an R-module morphism $M \longrightarrow A$ there exists a unique R-module morphism $M \longrightarrow \Gamma_u(A)$ making the triangle diagram $M \longrightarrow \Gamma_u(A) \longrightarrow A$ commutative.

Indeed, looking on the exact sequence (8), the composition $M \longrightarrow A \longrightarrow U \otimes_R A$ vanishes, since $U \otimes_R M = 0$. Now the obstruction to lifting the morphism $M \longrightarrow A$ to a morphism $M \longrightarrow \mathrm{Tor}_1^R(K^\bullet, A)$ lies in the group $\mathrm{Ext}_R^1(M, \mathrm{Tor}_1^R(U,A))$, and the obstruction to uniqueness of such a lifting lies in the group $\mathrm{Hom}_R(M, \mathrm{Tor}_1^R(U,A))$. Once again, $\mathrm{Tor}_1^R(U,A)$ is a left U-module, and any R-module morphism from M into a left U-module D vanishes, since $U \otimes_R M = 0$. Finally, we have $\mathrm{Ext}_R^1(M,D) = \mathrm{Ext}_U^1(U \otimes_R M,\, D) = 0$ for any such D, since $\mathrm{Tor}_1^R(U,M) = 0$ and $U \otimes_R M = 0$. $\square$

Proposition 17.2. *Assume that* $\mathrm{pd}\,{}_R U \le 1$. *Then*

(a) *the full subcategory* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ *is closed under the kernels, cokernels, extensions, and products in* $R\text{–}\mathsf{mod}$. *So* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ *is an abelian category and the embedding functor* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$ *is exact;*

(b) *assuming also that* $\mathrm{Tor}_1^R(U,U) = 0$, *the functor* $\Delta_u\colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ *is left adjoint to the fully faithful embedding functor* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$.

Proof. Part (a) is a particular case of [18, Proposition 1.1] or [31, Theorem 1.2(a)]. The proof of part (b) is based on Lemma 16.7(c) and dual-analogous to the proof of Proposition 17.1(b); cf. [32, Theorem 3.4]. $\square$

Lemma 17.3. *Assume that* $\mathrm{fd}\, U_R \le 1$, $\mathrm{pd}\,{}_R U \le 1$, *and* $Tor_1^R(U,U) = 0$. *Then*

(a) *for any u-h-divisible left R-module B, the left R-module* $\Gamma_u(B)$ *is also u-h-divisible;*

(b) *for any u-torsion-free left R-module A, the left R-module* $\Delta_u(A)$ *is also u-torsion-free.*

Proof. Let us prove part (a). Following Lemma 16.8(b), we have to check that $\mathrm{Ext}_R^1(K^\bullet, \mathrm{Tor}_1^R(K^\bullet, B)) = 0$. Since B is u-h-divisible, we have $\mathrm{Tor}_0^R(K^\bullet, B) =$

$(U/R)\otimes_R B=0$, so the five-term exact sequence (8) reduces to a four-term sequence. Furthermore, $\mathrm{Ext}^*_R(K^\bullet,D)=0$ for any left U-module D. Thus it follows from (8) that $\mathrm{Ext}^1_R(K^\bullet,\mathrm{Tor}^R_1(K^\bullet,B))=\mathrm{Ext}^1_R(K^\bullet,B)=0$ (cf. the proof of Theorem 16.5). The proof of part (b) is dual-analogous. ☐

Lemma 17.4. *Assume that* $\mathrm{Tor}^R_1(U,U)=0$ *and* $\mathrm{fd}\,U_R\le 1$*. Then* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ *is a Grothendieck abelian category. If J is an injective cogenerator of the abelian category* $R\text{–}\mathsf{mod}$*, then* $\Gamma_u(J)$ *is an injective cogenerator of* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$*.*

Proof. By Proposition 17.1(a), the full subcategory $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ is closed under direct limits in $R\text{–}\mathsf{mod}$; it is also an abelian category with an exact embedding functor $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}\longrightarrow R\text{–}\mathsf{mod}$. Hence the direct limit functors in $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ are exact, and it remains to show that this category has a set of generators.

By Proposition 17.1(b), the functor $\Gamma_u=\mathrm{Tor}^R_1(K^\bullet,-)$ is right adjoint to the embedding functor $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}\longrightarrow R\text{–}\mathsf{mod}$. Viewed as a functor $R\text{–}\mathsf{mod}\longrightarrow R\text{–}\mathsf{mod}$, the functor $\mathrm{Tor}^R_1(K^\bullet,-)$ clearly preserves direct limits; hence it follows the functor $\Gamma_u\colon R\text{–}\mathsf{mod}\longrightarrow R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ preserves direct limits, too.

Now let G denote the set of all u-comodule left R-modules of the form $\Gamma_u(G)$, where G ranges over (representatives of the isomorphism classes of) all the finitely presented left R-modules. We claim that G is a set of generators of $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$.

Indeed, let M be a u-comodule left R-module; then we have $M\cong\Gamma_u(M)$. Let (G_α) be a diagram of finitely presented left R-modules, indexed by some directed poset, such that $M\cong\varinjlim_\alpha G_\alpha$. Then we have $M\cong\Gamma_u(M)\cong\varinjlim_\alpha\Gamma_u(G_\alpha)$. So M is the direct limit of a diagram of objects from G in $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$, hence it is also a quotient of a coproduct of copies of objects from G.

The functor Γ_u takes injective objects in $R\text{–}\mathsf{mod}$ to injective objects in $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$, since it is right adjoint to an exact functor. To show that $\Gamma_u(J)$ is an injective cogenerator of $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ when J is an injective cogenerator of $R\text{–}\mathsf{mod}$, it suffices to compute $\mathrm{Hom}_R(M,\Gamma_u(J))=\mathrm{Hom}_R(M,J)\ne 0$ when $0\ne M\in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$. ☐

Lemma 17.5. *Assume that* $\mathrm{Tor}^R_1(U,U)=0$ *and* $\mathrm{pd}\,{}_RU\le 1$*. Then* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ *is a locally presentable abelian category with a projective generator* $\Delta_u(R)\in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$*.*

Proof. Following [35, Example 4.1(1-2)], if λ is a regular cardinal such that the left R-module U is λ-presentable (i. e., isomorphic to the cokernel of a morphism of free left R-modules with less than λ generators), then the category $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ is locally λ-presentable. Since the functor Δ_u is left adjoint to an exact (fully faithful) functor $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}\longrightarrow R\text{–}\mathsf{mod}$, it takes projective left R-modules to projective u-contramodule left R-modules. Finally, one has $\mathrm{Hom}_R(\Delta_u(R),C)=\mathrm{Hom}_R(R,C)=C\ne 0$ for any object $0\ne C\in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$. ☐

According to the discussion in [35, Section 1.1 in the introduction], [36, Section 6.1], and [33, Examples 1.2(4) and 1.3(4)], the abelian category $\mathsf{B}=R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ with its natural projective generator $P=\Delta_u(R)$ can be described as the category of modules over an additive monad $\mathbb{T}_u$ on the category of sets. For any set X, the coproduct $P^{(X)}$ of X copies of the object P in the category B can be computed as $P^{(X)}=\Delta_u(R^{(X)})$, where $R^{(X)} = R[X]$ is the free R-modules with generators indexed by X. The monad $\mathbb{T}_u$ assigns to every set X the set $\mathrm{Hom}_{\mathsf{B}}(P, P^{(X)}) = \Delta_u(R^{(X)})$. In particular, to a one-element set $*$, the monad $\mathbb{T}_u$ assigns the underlying set of the R-module $P = \Delta_u(R)$. In fact $P = \mathbb{T}_u(*) \in \mathbb{T}_u\text{–}\mathsf{mod} \cong \mathsf{B}$ is the free $\mathbb{T}_u$-module with one generator.

For any additive monad $\mathbb{T}$ on the category of sets, the set $\mathbb{T}(*)$ has a natural associative ring structure. This is the ring of endomorphisms of the forgetful functor $\mathbb{T}\text{–}\mathsf{mod} \longrightarrow \mathsf{Ab}$. In particular, the ring $\mathfrak{R} = \mathbb{T}_u(*)$ can be computed as the opposite ring to the ring of endomorphisms

$$\Delta_u(R) = \mathrm{Ext}^1_R(K^\bullet, R) = \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})}(K^\bullet, R[1]) \cong \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})}(K^\bullet, K^\bullet).$$

of the derived category object $K^\bullet \in \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$. Notice that the right action of the ring R by endomorphisms of the derived category object $K^\bullet \in \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ induces a natural ring homomorphism $R \longrightarrow \mathfrak{R}$.

Lemma 17.6. *Let $u\colon R \longrightarrow U$ be an epimorphism of commutative rings such that $\mathrm{Tor}^1_R(U, U) = 0$. Then the ring $\mathfrak{R} = \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})}(K^\bullet, K^\bullet)$ is commutative. In particular, if u is injective, then the ring $\mathfrak{R} = \mathrm{Hom}_R(U/R, U/R)$ is commutative.*

Proof. This is a generalization of [32, Proposition 3.1]. Let us prove the equivalent assertion that the ring $\mathfrak{R} = \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})}(K^\bullet[-1], K^\bullet[-1])$ is commutative (where $K^\bullet[-1]$ is the complex $R \longrightarrow U$ with the term R placed in the cohomological degree 0 and the term U placed in the cohomological degree 1). Denote by K the full subcategory in $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ consisting of the single object $K^\bullet[-1]$ (and all the objects isomorphic to it). Then the functor of truncated tensor product

$$L^\bullet \mathbin{\bar\otimes} M^\bullet = \tau_{\ge -1}(L^\bullet \otimes^{\mathbb{L}}_R M^\bullet)$$

defines a unital tensor (monoidal) category structure on the category K with the unit object $K^\bullet[-1]$. In other words, there is a natural isomorphism $K^\bullet[-1] \mathbin{\bar\otimes} K^\bullet[-1] \cong K^\bullet[-1]$ transforming both the endomorphisms $f \mathbin{\bar\otimes} \mathrm{id}$ and $\mathrm{id} \mathbin{\bar\otimes} f$ into the endomorphism f for any $f\colon K^\bullet[-1] \longrightarrow K^\bullet[-1]$. The commutativity of endomorphisms follows formally from that (see the computation in [32]).

When u is a homological epimorphism, one does not need to truncate the tensor product, so one can use the functor $\otimes^{\mathbb{L}}_R$ instead of $\bar\otimes$. When u is an injective epimorphism, it suffices to consider the full subcategory spanned by the object $K = U/R$ in $R\text{–}\mathsf{mod}$ and the functor $\mathrm{Tor}^R_1(-,-)$ in the role of the tensor product operation. Then one has to use the natural isomorphism $\mathrm{Tor}^R_1(K, K) \cong K$. ☐

The next lemma shows that the second assertion of Lemma 17.6 also holds for noninjective ring epimorphisms u of projective dimension ≤ 1.

Lemma 17.7. *Let $u\colon R \longrightarrow U$ be an epimorphism of associative rings such that $\mathrm{Tor}^1_R(U, U) = 0$ and $\mathrm{pd}\,_R U \le 1$. Then the associative ring homomorphism*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})}(K^\bullet, K^\bullet) \longrightarrow \mathrm{Hom}_R(U/R, U/R)$$

produced by applying the degree-zero cohomology functor $H^0\colon \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}) \longrightarrow R\text{–}\mathsf{mod}$ to the complex $K^\bullet \in \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ is surjective. In particular, if the ring R is commutative, then so is the ring $\mathrm{Hom}_R(U/R, U/R)$.

Proof. Let $I \subset R$ be the kernel of the map u. Then we have a natural distinguished triangle

$$I[1] \longrightarrow K^\bullet \longrightarrow U/R \longrightarrow I[2]$$

in $\mathsf{D^b}(R\text{–}\mathsf{mod}\text{–}R)$, and we can also consider it as a distinguished triangle in $\mathsf{D^b}(R\text{–}\mathsf{mod})$. Applying the functor $\operatorname{Hom}_{\mathsf{D^b}(R\text{–}\mathsf{mod})}(K^\bullet, -[*])$ to this triangle, we see that the map $\operatorname{Hom}_{\mathsf{D^b}(R\text{–}\mathsf{mod})}(K^\bullet, K^\bullet) \longrightarrow \operatorname{Hom}_{\mathsf{D}(R\text{–}\mathsf{mod})}(K^\bullet, U/R)$ is surjective, because $\operatorname{Hom}_{\mathsf{D}(R\text{–}\mathsf{mod})}(K^\bullet, I[2]) = \operatorname{Ext}^2_R(K^\bullet, I) \cong \operatorname{Ext}^2_R(U, I) = 0$ Lemma 16.3(b) and since $\operatorname{pd}_R U \le 1$. Finally, we have $\operatorname{Hom}_{\mathsf{D}(R\text{–}\mathsf{mod})}(K^\bullet, U/R) = \operatorname{Ext}^0_R(K^\bullet, U/R) \cong \operatorname{Hom}_R(U/R, U/R)$ by Lemma 16.3(c).

This proves the first assertion of the lemma. The second one follows from the first one together with the first assertion of Lemma 17.6. □

18. Triangulated Matlis Equivalence

In this section we reproduce some of the results of Chen and Xi [12, Section 4.1]. The approach in [12] is based on the technique of complete Ext-orthogonal pair in abelian categories, which was introduced by Krause and Šťovíček in [23] (see also [7]).

In this section, we restrict ourselves to the case of a homological ring epimorphism, following the approach by the second-named author in [32]. The aim is to show what can be asserted when some of the assumptions of [12] are not made. Besides, we work with arbitrarily bounded or unbounded derived categories, while the assertions of [12, Corollary 4.4] are formulated for bounded derived categories only.

Let $u\colon R \longrightarrow U$ be a homological epimorphism of associative rings, that is a ring homomorphism such that the natural map of U-U-bimodules $U \otimes_R U \longrightarrow U$ is an isomorphism and $\operatorname{Tor}^R_i(U, U) = 0$ for all $i > 0$. Then, according to [18, Theorem 4.4], [26, Theorem 3.7], the restriction of scalars with respect to u is a fully faithful functor between the unbounded derived categories $\mathsf{D}(U\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(R\text{–}\mathsf{mod})$. We denote this functor, acting between the bounded or unbounded derived categories, by

$$u_*\colon \mathsf{D}^\star(U\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod}),$$

where $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$ is a derived category symbol.

In the case of the unbounded derived categories ($\star = \varnothing$), the functor u_* has a left adjoint functor $\mathbb{L}u^*\colon \mathsf{D}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(U\text{–}\mathsf{mod})$ and a right adjoint functor $\mathbb{R}u^!\colon \mathsf{D}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(U\text{–}\mathsf{mod})$. When U is a right R-module of finite flat dimension, the functor $\mathbb{L}u^*$ also acts between bounded derived categories,

$$\mathbb{L}u^*\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(U\text{–}\mathsf{mod}).$$

When U is a left R-module of finite projective dimension, the functor $\mathbb{R}u^!$ acts between bounded derived categories,

$$\mathbb{R}u^!\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(U\text{–}\mathsf{mod}).$$

Since the trianguated functor u_* is fully faithful, its left and right adjoints $\mathbb{L}u^*$ and $\mathbb{R}u^!$ are Verdier quotient functors.

Theorem 18.1. (a) *Assume that* $\operatorname{fd} U_R \le 1$. *Then the kernel of the functor* $\mathbb{L}u^*\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(U\text{–}\mathsf{mod})$ *coincides with the full subcategory* $\mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$ *of all complexes of left R-modules with u-comodule cohomology modules. Hence for every symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, we have a triangulated equivalence*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod})/u_*\mathsf{D}^\star(U\text{–}\mathsf{mod}) \;\cong\; \mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}).$$

(b) *Assume that* $\operatorname{pd}{}_RU \le 1$. *Then the kernel of the functor* $\mathbb{R}u^!\colon \mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(U\text{–}\mathsf{mod})$ *coincides with the full subcategory* $\mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod})$ *of all complexes of left R-modules with u-contramodule cohomology modules. Hence for every symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, we have a triangulated equivalence*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod})/u_*\mathsf{D}^\star(U\text{–}\mathsf{mod}) \;\cong\; \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}).$$

Proof. Part (a): the functor $\mathbb{L}u^*$ is constructed as the derived tensor product $\mathbb{L}u^*(A^\bullet) = U \otimes_R^{\mathbb{L}} A^\bullet$ for any complex of left R-modules $A^\bullet$. In particular, when $\operatorname{fd} U_R \le 1$, we have short exact sequences of cohomology

$$0 \longrightarrow U \otimes_R H^n(A^\bullet) \longrightarrow H^n(\mathbb{L}u^*(A^\bullet)) \longrightarrow \operatorname{Tor}_1^R(U, H^{n+1}(A^\bullet)) \longrightarrow 0$$

for any complex $A^\bullet \in \mathsf{D}^\star(R\text{–}\mathsf{mod})$ and all $n \in \mathbb{Z}$. It follows immediately that $\mathbb{L}u^*(A^\bullet) = 0$ if and only if $H^n(A^\bullet) \in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ for all $n \in \mathbb{Z}$.

Part (b): the functor $\mathbb{R}u^!$ is constructed as the derived homomorphisms $\mathbb{R}u^!(B^\bullet) = \mathbb{R}\operatorname{Hom}_R(U, B^\bullet)$ for any complex of left R-modules $B^\bullet$. In particular, when $\operatorname{pd}{}_RU \le 1$, we have short exact sequences of cohomology

$$0 \longrightarrow \operatorname{Ext}^1_R(U, H^{n-1}(B^\bullet)) \longrightarrow H^n(\mathbb{R}u^!(B^\bullet)) \longrightarrow \operatorname{Hom}_R(U, H^n(B^\bullet)) \longrightarrow 0$$

for any complex $B^\bullet \in \mathsf{D}^\star(R\text{–}\mathsf{mod})$ and all $n \in \mathbb{Z}$. It follows immediately that $\mathbb{R}u^!(B^\bullet) = 0$ if and only if $H^n(B^\bullet) \in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ for all $n \in \mathbb{Z}$. ☐

Corollary 18.2. *Assume that* $\operatorname{fd} U_R \le 1$ *and* $\operatorname{pd}{}_RU \le 1$. *Then for every symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$ there is a triangulated equivalence*

$$\mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}) \;\cong\; \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod})$$

provided by the mutually inverse functors $\mathbb{R}\operatorname{Hom}_R(K^\bullet[-1], -)\colon \mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod})$ *and* $K^\bullet[-1] \otimes_R^{\mathbb{L}} -\colon \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod})$.

Proof. More generally, in the context of Theorem 18.1(a), the functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod})$ right adjoint to the embedding $\mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ is computed as $K^\bullet[-1] \otimes_R^{\mathbb{L}} -$. Similarly, in the context of Theorem 18.1(b), the functor $\mathsf{D}^\star(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod})$ left adjoint to the embedding $\mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$ is computed as $\mathbb{R}\operatorname{Hom}_R(K^\bullet[-1], -)$ (cf. [32, Proposition 4.4]). ☐

In addition to the assumptions on the projective and flat dimension of the left and right R-module U that we used above, the results below in this section require certain assumptions about the properties of injective and projective left R-modules vis-à-vis the homological ring homomorphism $u\colon R \longrightarrow U$. Specifically, these are the assumptions that injective left R-modules are u-special and projective left R-modules are u-cospecial, or in other words, the left R-modules $\operatorname{Tor}_0^R(K^\bullet, J) = U/R \otimes_R J$

and $\mathrm{Ext}^0_R(K^\bullet, F) = \mathrm{Hom}_R(U/R, F)$ vanish for all injective left R-modules J and projective left R-modules F (cf. Lemmas 16.3(c) and 16.4).

Theorem 18.3. (a) *Assume that* $\mathrm{fd}\, U_R \le 1$ *and* $(U/R) \otimes_R J = 0$ *for all injective left R-modules J. Then, for any conventional derived category symbol* $\star = \mathsf{b}, +, -,$ *or* $\varnothing$, *the triangulated functor*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$$

induced by the exact embedding of abelian categories $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}} \longrightarrow R\text{–}\mathsf{mod}$ *is fully faithful, and its essential image coincides with the full subcategory*

$$\mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod}),$$

providing an equivalence of triangulated categories

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}) \cong \mathsf{D}^\star_{u\text{-}\mathsf{co}}(R\text{–}\mathsf{mod}).$$

(b) *Assume that* $\mathrm{pd}\,{}_R U \le 1$ *and* $\mathrm{Hom}_R(U/R, F) = 0$ *for all projective left R-modules F. Then, for any conventional derived category symbol* $\star = \mathsf{b}, +, -,$ *or* $\varnothing$, *the triangulated functor*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}) \longrightarrow \mathsf{D}^\star(R\text{–}\mathsf{mod})$$

induced by the exact embedding of abelian categories $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$ *is fully faithful, and its essential image coincides with the full subcategory*

$$\mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}) \subset \mathsf{D}^\star(R\text{–}\mathsf{mod}),$$

providing an equivalence of triangulated categories

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}) \cong \mathsf{D}^\star_{u\text{-}\mathsf{ctra}}(R\text{–}\mathsf{mod}).$$

Proof. This is an application of the general technique formulated in [32, Theorem 6.4 and Proposition 6.5]. Let us explain part (b). The pair of functors $\mathrm{Ext}^i_R(K^\bullet, -)$, $i = 0, 1$, is a cohomological functor between the abelian categories $R\text{–}\mathsf{mod}$ and $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$, that is, for every short exact sequence of left R-modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ there is a short exact sequence of left u-contramodules (cf. Lemmas 16.3(a-b) and 16.7(c))

$$0 \longrightarrow \mathrm{Ext}^0_R(K^\bullet, A) \longrightarrow \mathrm{Ext}^0_R(K^\bullet, B) \longrightarrow \mathrm{Ext}^0_R(K^\bullet, C) \\ \longrightarrow \mathrm{Ext}^1_R(K^\bullet, A) \longrightarrow \mathrm{Ext}^1_R(K^\bullet, B) \longrightarrow \mathrm{Ext}^1_R(K^\bullet, C) \longrightarrow 0.$$

Since, by our assumption, the functor $\mathrm{Ext}^0_R(K^\bullet, -)$ annihilates projective left R-modules, it follows that our cohomological functor $\mathrm{Ext}^*_R(K^\bullet, -)$ is the left derived functor of the functor $\Delta = \Delta_u = \mathrm{Ext}^1_R(K^\bullet, -)\colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$, that is $\mathbb{L}_1\Delta_u = \mathrm{Ext}^0_R(K^\bullet, -)$ and $\mathbb{L}_i\Delta_u = 0$ for $i > 1$.

By Lemma 17.2(b), the functor Δ_u is left adjoint to the exact, fully faithful embedding functor $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \longrightarrow R\text{–}\mathsf{mod}$, so we are in the setting of [32, Theorem 6.4]. It remains to point out that $\mathbb{L}_1\Delta_u(B) = \mathrm{Ext}^0_R(K^\bullet, B) = 0$ for all left u-contramodules B. Notice that the class $R\text{–}\mathsf{mod}_{\Delta\text{-}\mathsf{adj}} = \mathrm{Ker}(\mathbb{L}_{>0}\Delta)$ of Δ-adjusted left R-modules, playing a key role in the argument in [32, Section 6], is nothing but the class of u-cospecial left R-modules in our context, according to Lemma 16.4.

Similarly, in part (a) one observes that the pair of functors $\operatorname{Tor}_i^R(K^\bullet, -)$, $i = 0, 1$ is a homological functor between the abelian categories $R\text{–}\mathsf{mod}$ and $R\text{–}\mathsf{mod}_{u\text{-co}}$, hence, whenever the functor $\operatorname{Tor}_1^R(K^\bullet, -)$ annihilates injective left R-modules, it is the right derived functor of the functor $\Gamma = \Gamma_u = \operatorname{Tor}_1^R(K^\bullet, -)\colon R\text{–}\mathsf{mod} \longrightarrow R\text{–}\mathsf{mod}_{u\text{-co}}$, that is $\mathbb{R}^1\Gamma_u = \operatorname{Tor}_0^R(K^\bullet, -)$ and $\mathbb{R}^i\Gamma = 0$ for $i > 1$. It remains to point out that $\mathbb{R}^1\Gamma_u(A) = \operatorname{Tor}_0^R(K^\bullet, A) = 0$ for all left u-comodules A. As above, we notice that the class $R\text{–}\mathsf{mod}_{\Gamma\text{-adj}} = \operatorname{Ker}(\mathbb{R}^{>0}\Gamma)$ of Γ-adjusted left R-modules is just the class of u-special left R-modules discussed in Section 16. $\square$

Remark 18.4. Conversely, if $\operatorname{fd} U_R \le 1$ and the triangulated functor $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}_{u\text{-co}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ is fully faithful, then $(U/R) \otimes_R J = 0$ for all injective left R-modules J. A proof of this can be found in [12, Lemma 3.9 and Proposition 4.2] (cf. [32, Remark 6.8]). Similarly, if $\operatorname{pd}_R U \le 1$ and the triangulated functor $\mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod}_{u\text{-ctra}}) \longrightarrow \mathsf{D}^{\mathsf{b}}(R\text{–}\mathsf{mod})$ is fully faithful, then $\operatorname{Hom}_R(U/R, F) = 0$ for all projective left R-modules F [12, Lemma 3.9 and Proposition 4.1].

The following corollary is the main result of this section. It is our (unbounded derived) version of [12, Corollary 4.4].

Corollary 18.5. *Let $u\colon R \longrightarrow U$ be a homological ring epimorphism. Assume that $\operatorname{fd} U_R \le 1$ and $\operatorname{pd}_R U \le 1$. Suppose further that $(U/R) \otimes_R J = 0$ for all injective left R-modules J and $\operatorname{Hom}_R(U/R, F) = 0$ for all projective left R-modules F. Then for every conventional derived category symbol $\star = \mathsf{b}$, $+$, $-$, or $\varnothing$, there is a triangulated equivalence between the derived categories of the abelian categories $R\text{–}\mathsf{mod}_{u\text{-co}}$ and $R\text{–}\mathsf{mod}_{u\text{-ctra}}$ of left u-comodules and left u-contramodules,*

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-co}}) \cong \mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-ctra}}).$$

Proof. According to Corollary 18.2 and Theorem 18.3(a-b), we have a chain of triangulated equivalences

$$\mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-co}}) \cong \mathsf{D}^\star_{u\text{-co}}(R\text{–}\mathsf{mod}) \cong \mathsf{D}^\star_{u\text{-ctra}}(R\text{–}\mathsf{mod}) \cong \mathsf{D}^\star(R\text{–}\mathsf{mod}_{u\text{-ctra}}).$$

$\square$

Example 18.6. The conditions that $(U/R) \otimes_R J = 0$ and $\operatorname{Hom}_R(U/R, F) = 0$ hold for any injective ring epimorhism $u\colon R \longrightarrow U$. Indeed, if u is injective and J is an injective left R-module, then any left R-module morphism $R \longrightarrow J$ can be extended to a left R-module morphism $U \longrightarrow J$. Hence the left R-module J is u-h-divisible (i. e., a quotient R-module of a left U-module). Thus $U/R \otimes_R U = 0$ implies $U/R \otimes_R J = 0$. Similarly, the map $F \longrightarrow U \otimes_R F$ is injective for any flat left R-module F, so F is u-torsion-free (i. e., an R-submodule of a left U-module). Therefore, $\operatorname{Hom}_R(U/R, U) = 0$ implies $\operatorname{Hom}_R(U/R, F) = 0$.

19. Injective Ring Epimorphisms of Projective Dimension 1

In this section we discuss the covering properties of the tilting modules and objects related to an injective homological ring epimorphism $u\colon R \longrightarrow U$. Since u is injective, the two-term complex of R-R-bimodules $K^\bullet = (R \to U)$ is naturally quasi-isomorphic to the quotient bimodule U/R; so we set $K = U/R$. Recall that, assuming $\operatorname{pd}_R U \le 1$, the left R-module $U \oplus K$ is 1-tilting [2, Theorem 3.5]. Another 1-tilting-cotilting correspondence situation associated with an injective ring epimorphism u is described in the following two theorems.

Theorem 19.1. *Assume that* $\operatorname{fd} U_R \le 1$ *and* $\operatorname{pd}_R U \le 1$*. Then the two abelian categories* $\mathsf{A} = R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ *and* $\mathsf{B} = R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ *are connected by the* 1*-tilting-cotilting correspondence in the following way. The injective cogenerator is* $J = \Gamma_u(\operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) \in \mathsf{A}$*, and the* 1*-tilting object is* $T = K \in \mathsf{A}$*. The projective generator is* $P = \Delta_u(R) \in \mathsf{B}$*, and the* 1*-cotilting object is* $W = \operatorname{Hom}_{\mathbb{Z}}(K, \mathbb{Q}/\mathbb{Z}) \in \mathsf{B}$*. The functor* $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ *is* $\Psi = \operatorname{Hom}_R(K, -)$*, and the functor* $\Phi\colon \mathsf{B} \longrightarrow \mathsf{A}$ *is* $\Phi = K \otimes_R -$*. The* 1*-tilting class* $\mathsf{E} \subset \mathsf{A}$ *is the class of all* u*-h-divisible* u*-comodule left* R*-modules, and the* 1*-cotilting class* $\mathsf{F} \subset \mathsf{B}$ *is the class of all* u*-torsion-free* u*-contramodule left* R*-modules. The equivalence of exact categories* $\mathsf{E} \cong \mathsf{F}$ *is the first Matlis category equivalence of Theorem 16.1.*

Consider the topological ring $\mathfrak{R} = \operatorname{Hom}_R(K, K)^{\mathrm{op}}$ opposite to the ring of endomorphisms of the left R-module K, as defined in Section 1.13. Then (as we already mentioned in Section 17) the right action of the ring R in the R-R-bimodule K induces a homomorphism of associative rings $R \longrightarrow \mathfrak{R}$.

Theorem 19.2. *Assume that* $\operatorname{pd}_R U \le 1$*. Then the forgetful functor* $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ *is fully faithful, and its essential image coincides with the full subcategory of* u*-contramodule left* R*-modules* $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \subset R\text{–}\mathsf{mod}$*. So we have an equivalence of abelian categories* $\mathfrak{R}\text{–}\mathsf{contra} \cong R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$*.*

Proof of Theorems 19.1 and 19.2. We discuss the proofs of the two theorems simultaneously, because they are closely related (even though the assumptions in Theorem 19.1 are slightly more restrictive than in Theorem 19.2). We start with the following proposition, which may be of independent interest.

Proposition 19.3. *Let* A *be a complete, cocomplete abelian category with an injective cogenerator* J*, and let* B *be a complete, cocomplete abelian category with a projective generator* P*. Suppose that there is a derived equivalence* $\mathsf{D}^{\mathsf{b}}(\mathsf{A}) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{B})$ *taking the object* $J \in \mathsf{A}$ *to an object* $W \in \mathsf{B} \subset \mathsf{D}^{\mathsf{b}}(\mathsf{B})$ *and the object* $P \in \mathsf{B}$ *to an object* $T \in \mathsf{A} \subset \mathsf{D}^{\mathsf{b}}(\mathsf{A})$*. Then, for any integer* $n \ge 0$*, the following conditions are equivalent:*

- (I) *the projective dimension of the object* T *in the category* A *does not exceed* n*;*
- (II) *the injective dimension of the object* W *in the category* B *does not exceed* n*;*
- (III) *the standard t-structures on the derived categories* $\mathsf{D}^{\mathsf{b}}(\mathsf{A})$ *and* $\mathsf{D}^{\mathsf{b}}(\mathsf{B})$*, viewed as two t-structures on the same triangulated category* D *using the triangulated equivalence* $\mathsf{D}^{\mathsf{b}}(\mathsf{A}) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{B})$*, satisfy the inclusion* $\mathsf{D}^{\mathsf{b},\le 0}(\mathsf{A}) \subset \mathsf{D}^{\mathsf{b},\le n}(\mathsf{B})$*, or equivalently,* $\mathsf{D}^{\mathsf{b},\ge n}(\mathsf{B}) \subset \mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{A})$*.*

If any one of these conditions is satisfied, then the object $T \in \mathsf{A}$ is n-tilting; the object $W \in \mathsf{B}$ is n-cotilting; and the abelian category A with the injective cogenerator J and the n-tilting object T and the abelian category B with the n-cotilting object W are connected by the n-tilting-cotilting correspondence. The n-tilting class $\mathsf{E} \subset \mathsf{A}$ is the intersection $\mathsf{A} \cap \mathsf{B} \subset \mathsf{D}$ viewed as a full subcategory in A, and the n-cotilting class $\mathsf{F} \subset \mathsf{B}$ is the same intersection $\mathsf{B} \cap \mathsf{A} \subset \mathsf{D}$ viewed as a full subcategory in B (hence the equivalence of exact categories $\mathsf{E} \cong \mathsf{F}$). The functor $\Psi\colon \mathsf{A} \longrightarrow \mathsf{B}$ assigns to an object $A \in \mathsf{A}$ the degree-zero cohomology of the related complex in $\mathsf{D}^{\mathsf{b}}(\mathsf{B})$, and the functor $\Phi\colon \mathsf{B} \longrightarrow \mathsf{A}$ assigns to an object $B \in \mathsf{B}$ the degree-zero cohomology of the related complex in $\mathsf{D}^{\mathsf{b}}(\mathsf{A})$, that is, $\Psi(A) = H^0_{\mathsf{B}}(A)$ and $\Phi(B) = H^0_{\mathsf{A}}(B)$.

Proof. This is essentially the material of [36, Sections 1 and 3] (the description of the functors Φ and Ψ can be found in the beginning of [36, Section 4]). So we only give a brief sketch of the argument.

Notice, first of all, that the inclusions $\mathsf{D}^{\mathsf{b},\le 0}(\mathsf{B}) \subset \mathsf{D}^{\mathsf{b},\le 0}(\mathsf{A})$ and $\mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{A}) \subset \mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{B})$ always hold in our assumptions, because an object $Z \in \mathsf{D}$ belongs to $\mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{B})$ if and only if $\operatorname{Hom}_{\mathsf{D}}(P, Z[i]) = 0$ for all $i < 0$, and $\operatorname{Hom}_{\mathsf{D}}(S, Z[i]) = 0$ for $Z \in \mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{A})$, all $i < 0$, and all $S \in \mathsf{A}$ (in particular, for $S = T$). Similarly one shows that the two inclusions in (III) (which are obviously equivalent to each other) are equivalent to (I) on the one hand and to (II) on the other hand, (I) $\Longleftrightarrow$ (III) $\Longleftrightarrow$ (II).

The inclusion $\mathsf{A} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{A})$ preserves coproducts, because the coproduct functors are exact in A; and the inclusion $\mathsf{B} \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{B})$ preserves products, because the product functors are exact in B. Furthermore, we have $\mathsf{A} \cap \mathsf{B} = \mathsf{A} \cap \mathsf{D}^{\mathsf{b},\le 0}(\mathsf{B}) \subset \mathsf{D}$, since $\mathsf{B} = \mathsf{D}^{\mathsf{b},\le 0}(\mathsf{B}) \cap \mathsf{D}^{\mathsf{b},\ge 0}(\mathsf{B})$ and $A \subset \mathsf{D}^{\ge 0}(\mathsf{A}) \subset \mathsf{D}^{\ge 0}(\mathsf{B})$. The full subcategory $\mathsf{D}^{\mathsf{b},\le 0}(\mathsf{B})$ is closed under coproducts in D (those coproducts that exist in D), because the left part of any t-structure is closed under coproducts. Hence the full subcategory $\mathsf{A} \cap \mathsf{B}$ is closed under coproduct in D, and consequently in A and B. Similary, the full subcategory $\mathsf{A} \cap \mathsf{B}$ is closed under products in D, and consequently in A and B. So the products and coproducts of objects of E computed in A agree with the products and coproducts of objects of F computed in B. (Cf. [36, Lemma 4.3 and Remark 4.4].)

Now we can see that $\operatorname{Ext}^i_{\mathsf{A}}(T, T^{(X)}) = \operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{A})}(T, T^{(X)}[i]) = \operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathsf{B})}(P, P^{(X)}[i])$ $= 0$ for all $i > 0$, and similarly $\operatorname{Ext}^i_{\mathsf{B}}(W^X, W) = 0$ for all $i > 0$ and all sets X. This proves the n-tilting axiom (ii) for T and the n-cotilting axiom (ii*) for W; while the axioms (i) and (i*) are provided by the conditions (I) and (II). It remains to apply [36, Proposition 1.5 and Corollary 3.4(b)] in order to conclude that the object $T \in \mathsf{A}$ is n-tilting and the object $W \in \mathsf{B}$ is n-cotilting. It is also clear from the construction of the n-tilting-cotilting correspondence in [36, Theorems 3.10-3.11 and Corollary 3.1] that the triples (A, J, T) and (B, P, W) are connected by such. $\square$

Now Theorem 19.1 is simplest obtained by applying Proposition 19.3 (for $n = 1$) to the derived equivalence of Corollary 18.5. To be more precise, the latter derived equivalence, obtained from the "recollement" of Theorem 18.1, needs to be shifted by [1] before it becomes a tilting derived equivalence. The triangulated equivalence in Corollary 18.2 is provided by the functors $\mathbb{R}\operatorname{Hom}_R(K^\bullet[-1], -)$ and $K^\bullet[-1] \otimes^{\mathbb{L}}_R -$, while in our present context one has to consider the equivalence provided by the functors $\mathbb{R}\operatorname{Hom}_R(K,-)$ and $K \otimes_R^{\mathbb{L}} -$.

The assumptions of Corollary 18.5 hold in our case by Example 18.6. The left R-module $J = \Gamma_u(\operatorname{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z}))$ is an injective cogenerator of $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ by Lemma 17.4, and the left R-module $P = \Delta_u(R)$ is a projective generator of $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ by Lemma 17.5. Furthermore, the R-R-bimodule K is both a left and a right u-comodule, and consequently $\operatorname{Hom}_{\mathbb{Z}}(K,\mathbb{Q}/\mathbb{Z})$ is a left u-contramodule.

Now we can can compute that $\mathbb{R}\operatorname{Hom}_R(K,T) = \operatorname{Hom}_R(K,K) = \operatorname{Ext}^1_R(K,R) = P$, since $\operatorname{Ext}^1_R(K,K) = \operatorname{Ext}^2_R(K,R) = 0$. Similarly, $\mathbb{R}\operatorname{Hom}_R(K,J) = \operatorname{Hom}_R(K,J) = \operatorname{Hom}_R(K,\operatorname{Hom}_{\mathbb{Z}}(R,\mathbb{Q}/\mathbb{Z})) = \operatorname{Hom}_R(K,\mathbb{Q}/\mathbb{Z}) = W$, since $\operatorname{Ext}^1_R(K,J) = \operatorname{Ext}^1_{\mathsf{A}}(K,J) = 0$ (as $\mathsf{A} = R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}} \subset R\text{–}\mathsf{mod}$ is a full subcategory closed under extensions). Finally, any one of the conditions (I–III) of Proposition 19.3 is easily verified. This finishes the proof of Theorem 19.1.

Alternatively, one can check that $K \in R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ is a 1-tilting object in the way similar to the argument in [36, Example 5.1]. Following [36, Corollary 7.2], the abelian category B corresponding this tilting object in the abelian category $\mathsf{A} = R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ can be described as $\mathsf{B} = \mathfrak{R}\text{–}\mathsf{contra}$. The functor Ψ is then still computed as $\Psi = \operatorname{Hom}_R(K,-)$ [36, Corollary 7.4], while Φ is the functor of contratensor product $\Phi = K \odot_{\mathfrak{R}} -$ with the discrete right $\mathfrak{R}$-module K [36, Corollary 7.6] (which is the same thing as the tensor product $K \otimes_R -$ provided that the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ is fully faithful, cf. [36, Lemma 7.9]). Comparing this approach to the previous one yields $\mathfrak{R}\text{–}\mathsf{contra} \cong \mathsf{B} \cong R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$, that is the assertion of Theorem 19.2 (in the assumpions of Theorem 19.1).

A direct proof of Theorem 19.2 (in full generality) can be given based on [33, Proposition 2.1]. For any set X, we have to construct a natural isomorphism of left R-modules $\Delta_u(R[X]) \simeq \mathfrak{R}[[X]]$. Indeed,

$$\Delta_u(R[X]) = \operatorname{Ext}^1_R(K,R[X]) \cong \operatorname{Hom}_R(K,K[X]) \cong \mathfrak{R}[[X]]$$

by [36, proof of Theorem 7.1].

Let us spell out this argument a bit more explicitly. There are enough projective objects of the form $P = \Delta_u(R[X])$ in $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$, and these are precisely the images of the free $\mathfrak{R}$-contramodules $\mathfrak{R}[[X]]$ under the forgetful functor. To show that the whole image of the forgetful functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}$ lies inside $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$, observe that the forgetful functor preserves cokernels, the full subcategory $R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}} \subset R\text{–}\mathsf{mod}$ is closed under cokernels, and every left $\mathfrak{R}$-contramodule is the cokernel of a morphism of free left $\mathfrak{R}$-contramodules.

As an abelian category with enough projective objects is determined by its full subcategory of projective objects, in order to prove that the functor $\mathfrak{R}\text{–}\mathsf{contra} \longrightarrow R\text{–}\mathsf{mod}_{u\text{-}\mathsf{ctra}}$ is an equivalence of categories it suffices to show that it is an equivalence in restriction to the full subcategories of projective objects. In other words, we have to check that the natural map $\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]],\mathfrak{R}[[Y]]) \longrightarrow \operatorname{Hom}_R(\mathfrak{R}[[X]],\mathfrak{R}[[Y]])$ is isomorphism for all sets X and Y. Indeed, we have

$$\operatorname{Hom}^{\mathfrak{R}}(\mathfrak{R}[[X]],\mathfrak{R}[[Y]]) \cong \mathfrak{R}[[Y]]^X \cong \operatorname{Hom}_R(\mathfrak{R}[[X]],\mathfrak{R}[[Y]]),$$

where the second isomorphism holds because, by Theorem 16.1,

$$\mathrm{Hom}_R(\mathfrak{R}[[X]], \mathfrak{R}[[Y]]) \cong \mathrm{Hom}_R(K[X], K[Y]) \cong \mathrm{Hom}_R(K, K[Y])^X \cong \mathfrak{R}[[Y]]^X$$

as $K[X]$ is a u-h-divisible left u-comodule and $\mathrm{Hom}_R(K, K[X]) \cong \mathfrak{R}[[X]]$.

The proof of Theorems 19.1 and 19.2 is finished. □

Remark 19.4. Let $u\colon R \longrightarrow U$ be an injective ring epimorphism such that U is a flat left R-module. Then the set of all right ideals $I \subset R$ such that $R/I \otimes_R U = 0$ is a base of neighborhoods of zero in a topological ring structure on R (in fact, it is the set of all open right ideals in this topological ring structure). This is called the *perfect right Gabriel topology* associated with a left flat ring epimorphism [42, Sections XI.2–3]. Let us denote by $\mathfrak{T}$ the completion of the ring R with respect to this topology (see Section 1.3). Then U/R is a discrete right R-module (since U is a flat left R-module and $U/R \otimes_R U = 0$), and consequently, also a discrete right $\mathfrak{T}$-module. The right action of $\mathfrak{T}$ in U/R commutes with the left action of R, since the right action of R does. It follows that the right action of $\mathfrak{T}$ in U/R induces a continuous homomorphism of topological rings $\mathfrak{T} \longrightarrow \mathfrak{R} = \mathrm{Hom}_R(U/R, U/R)^{\mathrm{op}}$.

Hence for every set X we have the induced map of sets $\mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$. In fact, we have a commutative triangle diagram of ring homomorphisms $R \longrightarrow \mathfrak{R} \longrightarrow \mathfrak{T}$; so the map $\mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$ is a left R-module morphism. Now let us assume additionally that U is a left R-module of projective dimension not exceeding 1. Then, by [34, Proposition 9.2], $\mathfrak{T}[[X]]$ is a u-contramodule left R-module. By the adjunction property of the functor Δ_u (see Proposition 17.2(b)), there exists a unique left R-module morphism $\Delta_u(R[X]) \longrightarrow \mathfrak{T}[[X]]$ forming a commutative triangle diagram with the adjunction map $\delta_{u,R[X]}\colon R[X] \longrightarrow \Delta_u(R[X])$ and the natural map $R[X] \longrightarrow \mathfrak{T}[[X]]$. The composition $\Delta_u(R[X]) \longrightarrow \mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$ is the isomorphism $\Delta_u(R[X]) \simeq \mathfrak{R}[[X]]$ from the above proof of Theorem 19.2.

There is a further set of additional assumptions listed in [34, Theorem 9.6] under which one can claim that the map $\Delta_u(R[X]) \longrightarrow \mathfrak{T}[[X]]$ is an isomorphism, too. One needs the perfect Gabriel topology on the ring R to satisfy the condition (T_ω) from [34, Section 2] and to be ω-cofaithful in the sense of [34, Section 9]. Then it follows that the map $\mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$ is bijective for every set X. In particular, the associative ring homomorphism $\mathfrak{T} \longrightarrow \mathfrak{R}$ is an isomorphism. It still does *not* seem to follow from anything that it is an isomorphism *of topological rings* (that is, that the topologies on $\mathfrak{T}$ and $\mathfrak{R}$ are the same); but it is a bijective continuous ring homomorphism inducing a bijective map $\mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$ for every set X.

When the perfect Gabriel topology on the ring R associated with an injective left flat ring epimorphism $u\colon R \longrightarrow U$ has a base of neighborhoods of zero consisting of centrally generated ideals (e. g., the ring R is commutative), the above-mentioned two additional assumptions concerning this topology hold automatically [34, Corollary 9.7]. Thus, if $\mathrm{pd}\,_R U \le 1$, then the continuous ring homomorphism $\mathfrak{T} \longrightarrow \mathfrak{R}$ is bijective and induces bijective maps $\mathfrak{T}[[X]] \longrightarrow \mathfrak{R}[[X]]$ for all sets X.

In the rest of this section, we discuss the covering and direct limit closedness properties of the tilting objects $U \oplus K \in R\text{–}\mathsf{mod}$ and $K \in R\text{–}\mathsf{mod}_{u\text{-co}}$ in connection with the perfectness properties of the related rings.

Assuming that $\operatorname{pd}_R U \le 1$, denote by (N, G) the 1-tilting cotorsion pair in $R\text{–}\mathsf{mod}$ associated with the 1-tilting left R-module $U \oplus K$. Then G is the class of all u-h-divisible left R-modules (cf. Lemma 16.8(b)).

Assuming that $\operatorname{fd} U_R \le 1$ and $\operatorname{pd}_R U \le 1$, we also have the 1-tilting cotorsion pair in the abelian category $\mathsf{A} = R\text{–}\mathsf{mod}_{u\text{-co}}$ associated with the 1-tilting object K. The right class E in this pair is the class of all u-h-divisible left u-comodules $\mathsf{A} \cap \mathsf{G}$ (because the functors Ext^1_R and $\operatorname{Ext}^1_{\mathsf{A}}$ agree). Moreover, the left class L in the 1-tilting cotorsion pair in A coincides with $\mathsf{A} \cap \mathsf{N}$, as one can see by comparing its descriptions as the left Ext^1-orthogonal class to the right class in the pair, on the one hand, and as the class of all finitely $\mathsf{Add}(K)$-coresolved objects, on the other hand (see [36, Theorem 2.4] or the beginning of Section 12). Thus we have $\mathsf{E} = \mathsf{A} \cap \mathsf{G}$ and $\mathsf{L} = \mathsf{A} \cap \mathsf{N}$.

Let us start with the 1-tilting object $K \in R\text{–}\mathsf{mod}_{u\text{-co}}$. We keep the notation F for the 1-cotilting class in the abelian category $R\text{–}\mathsf{mod}_{u\text{-ctra}} = \mathsf{B} = \mathfrak{R}\text{–}\mathsf{contra}$ (so the exact category F is equivalent to $\mathsf{E} = \mathsf{A} \cap \mathsf{G}$).

Proposition 19.5. *Assume that* $\operatorname{fd} U_R \le 1$ *and* $\operatorname{pd}_R U \le 1$*. Then the following nine conditions are equivalent:*

(i) *every left R-module has an* $\mathsf{A} \cap \mathsf{N}$*-cover;*
(ii) *every module from* G *has an* $\mathsf{A} \cap \mathsf{N}$*-cover;*
(iii) *every module from* A *has an* $\mathsf{A} \cap \mathsf{N}$*-cover;*
(iv) *every module from* $\mathsf{A} \cap \mathsf{G}$ *has an* $\mathsf{A} \cap \mathsf{N}$*-cover;*
(v) *every left R-module has an* $\mathsf{Add}(K)$*-cover;*
(vi) *every module from* G *has an* $\mathsf{Add}(K)$*-cover;*
(vii) *every module from* A *has an* $\mathsf{Add}(K)$*-cover;*
(viii) *every module from* $\mathsf{A} \cap \mathsf{G}$ *has an* $\mathsf{Add}(K)$*-cover;*
(ix) *every (contra)module from* F *has a projective cover in the abelian category* $R\text{–}\mathsf{mod}_{u\text{-ctra}} = \mathsf{B} = \mathfrak{R}\text{–}\mathsf{contra}$.

If any one of these equivalent conditions holds, then all the discrete quotient rings of the topological ring $\mathfrak{R}$ *are left perfect.*

Proof. The implications (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (vi), (vii) $\Longrightarrow$ (viii) are obvious. The equivalence of the four conditions (iii), (iv), (vi), and (ix) is a particular case of the equivalence of the four conditions (i-iv) in Proposition 13.2.

The implication (iii) $\Longrightarrow$ (i) holds because the embedding functor $\mathsf{A} \longrightarrow R\text{–}\mathsf{mod}$ has a right adjoint Γ_u. Given a left R-module C, let $L \longrightarrow \Gamma_u(C)$ be an $\mathsf{A} \cap \mathsf{N}$-cover of the the module $\Gamma_u(C) \in \mathsf{A}$; then the composition $L \longrightarrow \Gamma_u(C) \longrightarrow C$ is an $\mathsf{A} \cap \mathsf{N}$-cover of C.

To check the implication (vi) $\Longrightarrow$ (v), recall that G is the class of all u-h-divisible left R-modules and $\mathsf{Add}(K) \subset \mathsf{G}$. Every left R-module C has a unique maximal u-h-divisible R-submodule $h(C)$. Let $M \longrightarrow h(C)$ be an $\mathsf{Add}(K)$-cover of $h(C)$; then the composition $M \longrightarrow h(C) \longrightarrow C$ is an $\mathsf{Add}(K)$-cover of C.

Finally, the implication (viii) $\Longrightarrow$ (vi) follows from Lemma 17.3(a). Let C be a u-h-divisible left R-module; then the left R-module $\Gamma_u(C)$ belongs to $\mathsf{A}\cap\mathsf{G}$. If $M\longrightarrow\Gamma_u(C)$ is an $\mathsf{Add}(K)$-cover of $\Gamma_u(C)$, then the composition $M\longrightarrow\Gamma_u(C)\longrightarrow C$ is an $\mathsf{Add}(K)$-cover of C. □

Theorem 19.6. *Assume that* $\operatorname{fd}U_R\le 1$ *and* $\operatorname{pd}{}_RU\le 1$*, and assume further that the topological ring* $\mathfrak{R}$ *satisfies the condition (d) of Section 10. Then the following conditions are equivalent:*

(i) *all left R-modules have $\mathsf{A}\cap\mathsf{N}$-covers;*
(ii) *all left R-modules from $\varinjlim_\omega \mathsf{Add}(K)$ have $\mathsf{A}\cap\mathsf{N}$-covers;*
(iii) *the class of left R-modules $\mathsf{A}\cap\mathsf{N}$ is closed under (countable) direct limits;*
(iv) *all left R-modules have $\mathsf{Add}(K)$-covers;*
(v) *all left R-modules from $\varinjlim_\omega \mathsf{Add}(K)$ have $\mathsf{Add}(K)$-covers;*
(vi) *the class of left R-modules $\mathsf{Add}(K)$ is closed under (countable) direct limits;*
(vii) *all the objects of B have projective covers;*
(viii) *all the objects from $\varinjlim_\omega^{\mathsf{B}} \mathsf{B}_{\mathsf{proj}}$ have projective covers in B;*
(ix) *the class of objects $\mathsf{B}_{\mathsf{proj}}$ is closed under (countable) direct limits in B;*
(x) *all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.*

In particular, if the ring R is commutative and $\operatorname{pd}{}_RU\le 1$, then the ten conditions (i-x) are equivalent. The condition (x) can be rephrased by saying that the topological ring $\mathfrak{R}$ is pro-perfect in this case.

Proof. The implications (i) $\Longleftrightarrow$ (iv) $\Longrightarrow$ (x) hold by Proposition 19.5.

The conditions (iii), (vi), and (ix) are equivalent to each other, for countable direct limits, by Corollary 12.4, and for uncountable ones, by Corollary 12.8. Notice that $\mathsf{A}=R\text{–}\mathsf{mod}_{u\text{-}\mathsf{co}}$ is a Grothendieck abelian category by Lemma 17.4.

The conditions (ii), (v), and (viii) are equivalent to each other by Proposition 13.2.

Alternatively, the conditions (v) and (viii) are equivalent by Corollary 15.5 (and their uncountable versions are equivalent by Corollary 15.8). These two corollaries also provide another proof of the equivalence of (vi) and (ix). Notice that the left R-module K is always self-pure-projective by Lemma 15.1(c,e), as a direct summand of a 1-tilting left R-module $U\oplus K$ (and K is also Σ-rigid, of course).

All the conditions (vii-x) are equivalent to each other (in our assumptions) by Theorem 10.4. This also establishes the equivalence of the countable and uncountable versions of the condition (viii). The condition (x) implies (i), (iii), (iv), and (vi) by Proposition 13.1 (in view of Proposition 19.5 (i) $\Longleftrightarrow$ (iii) and (v) $\Longleftrightarrow$ (vii)).

If the ring R is commutative, then so is the ring $\mathfrak{R}$ by Lemma 17.6. So condition (a) of Section 8 is satisfied. (It is worth recalling that $\operatorname{pd}{}_RU\le 1$ implies $\operatorname{fd}{}_RU=0$ for commutative rings R, by Remark 16.9.) □

Now let us discuss the 1-tilting left R-module $U\oplus K$. We denote by $\mathfrak{S}$ the topological ring $\operatorname{Hom}_R(U\oplus K,\,U\oplus K)^{\mathrm{op}}$, and denote by by $\mathsf{H}\subset\mathfrak{S}\text{–}\mathsf{contra}$ the 1-cotilting class associated with the 1-cotilting left $\mathfrak{S}$-contramodule $\operatorname{Hom}_{\mathbb{Z}}(U\oplus K,\,\mathbb{Q}/\mathbb{Z})$. So the exact category H is equivalent to G.

Lemma 19.7. (a) *All the discrete quotient rings of the topological ring $\mathfrak{S}$ are left perfect if and only if the ring U is left perfect and all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.*

(a) *If the topological ring $\mathfrak{R}$ satisfies the condition (d) of Section 10, then so does the topological ring $\mathfrak{S}$.*

Proof. We have $\mathrm{Hom}_R(U,U)^{\mathrm{op}} = U$, $\mathrm{Hom}_R(K,K)^{\mathrm{op}} = \mathfrak{R}$, and $\mathrm{Hom}_R(U/R,U) = 0$. So $\mathfrak{S}$ is the matrix ring (cf. Example 10.1)

$$\begin{pmatrix} U & \mathfrak{K} \\ 0 & \mathfrak{R} \end{pmatrix}$$

where $\mathfrak{K} = \mathrm{Hom}_R(U, U/R)$ is a nilpotent strongly closed two-sided ideal in $\mathfrak{S}$ (obviously, $\mathfrak{K}^2 = 0$). Now we have $\mathfrak{S}/\mathfrak{K} = U \times \mathfrak{R}$, so part (a) follows from Lemma 10.3. Furthermore, the discrete ring U trivially has a countable base of neighborhoods of zero and satisfies the condition (b) of Section 8, hence it remains to apply Lemma 10.6 in order to prove part (b). $\square$

Theorem 19.8. *Assume that $\mathrm{pd}_R U \le 1$ and that the topological ring $\mathfrak{R}$ satisfies the condition (d) of Section 10. Then the following conditions are equivalent:*

- (i) *all left R-modules have N-covers;*
- (ii) *all left R-modules from $\varinjlim_\omega \mathsf{Add}(U \oplus K)$ have N-covers;*
- (iii) *the class of left R-modules N is closed under (countable) direct limits;*
- (iv) *all left R-modules have $\mathsf{Add}(U \oplus K)$-covers;*
- (v) *all left R-modules from $\varinjlim_\omega \mathsf{Add}(U \oplus K)$ have $\mathsf{Add}(U \oplus K)$-covers;*
- (vi) *the class of left R-modules $\mathsf{Add}(U \oplus K)$ is closed under (countable) direct limits;*
- (vii) *the left R-module $U \oplus K$ is Σ-pure-split;*
- (viii) *all the objects of $\mathfrak{S}\text{–}\mathsf{contra}$ have projective covers;*
- (ix) *all the objects from $\varinjlim_\omega \mathfrak{S}\text{–}\mathsf{contra}_{\mathsf{proj}}$ have projective covers in $\mathfrak{S}\text{–}\mathsf{contra}$;*
- (x) *the class of all projective left $\mathfrak{S}$-contramodules is closed under (countable) direct limits in $\mathfrak{S}\text{–}\mathsf{contra}$;*
- (xi) *all the discrete quotient rings of the topological ring $\mathfrak{S}$ are left perfect;*
- (xii) *the ring U is left perfect and all the discrete quotient rings of the topological ring $\mathfrak{R}$ are left perfect.*

In particular, if the ring R is commutative and $\mathrm{pd}_R U \le 1$, then the twelve conditions (i-xii) are equivalent.

Proof. The condition (iii) (for uncountable direct limits) is equivalent to (vii) by [19, Proposition 13.55]. All the seven conditions (i-vii) are equivalent to each other by [3, Theorem 3.6, Theorem 5.2, and Corollary 5.5].

The conditions (xi) and (xii) are equivalent by Lemma 19.7. The equivalence of all the ten conditions (i-vi, viii-xi) is provable in the same way as the equivalence of the ten conditions in Theorem 19.6. Alternatively, the implications (vii) $\Longrightarrow$ (vi), (x), (xi) hold by Propositions 14.2 and 14.4.

The last assertion of the theorem is essentially the same as in Theorem 19.6. $\square$

Example 19.9. Let R be a commutative ring and $S \subset R$ be a multiplicative subset consisting of regular elements. Denote the multiplicative subset of all regular elements in R by $S \subset S_{reg} \subset R$. Set $U = S^{-1}R$; then the localization map $u\colon R \longrightarrow U$ is an injective flat epimorphism of commutative rings. The topological ring $\mathfrak{R} = \operatorname{Hom}_R(U/R, U/R)$ is naturally topologically isomorphic to the S-completion $\varprojlim_{s\in S} R/sR$ of the ring R (viewed as the topological ring in the projective limit topology), which was discussed in Example 9.2.

Assume that $\operatorname{pd}_R S^{-1}R \le 1$, and set $K = U/R$. Then the homomorphism of commutative rings $R \longrightarrow S^{-1}R = U$ satisfies the assumptions of Theorems 19.6 and 19.8. By Theorem 19.6, the class of R-modules $\mathsf{A} \cap \mathsf{N}$ is covering (if and only if the class $\mathsf{Add}(K) \subset R\text{–}\mathsf{mod}$ is covering and) if and only if the ring R/sR is perfect for every $s \in R$. By Theorem 19.8, the class of R-modules N is covering (if and only if the class $\mathsf{Add}(U \oplus K) \subset R\text{–}\mathsf{mod}$ is covering and) if and only if two conditions hold: the ring R/sR is perfect for every $s \in R$, *and* the ring $S^{-1}R$ is perfect.

The latter two conditions are equivalent to the following two: one has $S^{-1}R = S_{reg}^{-1}R$, and the ring R is almost perfect (in the sense of the paper [17]). It is worth noticing that the condition that all the rings R/sR are perfect already implies $\operatorname{pd}_R S^{-1}R \le 1$ [17, Lemma 3.4], [6, Corollary 6.13].

For example, let $R = \mathbb{Z}$ be the ring of integers, p be a prime number, and $S = \{1, p, p^2, p^3, \dots\} \subset R$ be the multiplicative subset in $\mathbb{Z}$ generated by p. Then the class of abelian groups $\mathsf{A} \cap \mathsf{N} \subset \mathsf{Ab}$ is covering, but the class $\mathsf{N} \subset \mathsf{Ab}$ is not. Alternatively, let $S' \subset \mathbb{Z}$ be the multiplicative subset of all integers *not* divisible by p. Then, once again, the related class $\mathsf{A} \cap \mathsf{N}' \subset \mathsf{Ab}$ is covering, but the class of abelian groups $\mathsf{N}' \subset \mathsf{Ab}$ is not.

References

[1] L. Angeleri Hügel, M. Hrbek. Silting modules over commutative rings. *Internat. Math. Research Notices* **2017**, #13, p. 4131–4151. `arXiv:1602.04321 [math.RT]`

[2] L. Angeleri Hügel, J. Sánchez. Tilting modules arising from ring epimorphisms. *Algebras and Represent. Theory* **14**, #2, p. 217–246, 2011. `arXiv:0804.1313 [math.RT]`

[3] L. Angeleri Hügel, J. Šaroch, J. Trlifaj. Approximations and Mittag–Leffler conditions—the applications. Electronic preprint `arXiv:1612.01140 [math.RA]`, to appear in *Israel Journ. of Math.*

[4] R. El Bashir. Covers and directed colimits. *Algebras and Represent. Theory* **9**, #5, p. 423–430, 2006.

[5] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. of the Amer. Math. Soc.* **95**, #3, p. 466–488, 1960.

[6] S. Bazzoni, L. Positselski. S-almost perfect commutative rings. Electronic preprint `arXiv:1801.04820 [math.AC]`.

[7] S. Bazzoni, J. Šťovíček. Smashing localizations of rings of weak global dimension at most one. *Advances in Math.* **305**, p. 351–401, 2017. `arXiv:1402.7294 [math.AC]`

[8] A. Beilinson. Remarks on topological algebras. *Moscow Math. Journ.* **8**, #1, p. 1–20, 2008. `arXiv:0711.2527 [math.QA]`

[9] T. Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Phil. Soc.* **129**, #3, p. 447–475, 2000.

[10] L. Bican, R. El Bashir, E. Enochs. All modules have flat covers. *Bull. of the London Math. Soc.* **33**, #4, p. 385–390, 2001.

[11] N. Bourbaki. Topological vector spaces. Translated from the 1981 French edition by H. G. Eggleston and S. Madan. Springer-Verlag, 1987–2003.

[12] H.-X. Chen, C.-C. Xi. Derived decompositions of abelian categories I. Electronic preprint `arXiv:1804.10759 [math.RT]`.

[13] P. C. Eklof, J. Trlifaj. How to make Ext vanish. *Bull. of the London Math. Soc.* **33**, #1, p. 41–51, 2001.

[14] E. Enochs. Injective and flat covers, envelopes and resolvents. *Israel Journ. of Math.* **39**, #3, p. 189–209, 1981.

[15] A. Facchini, Z. Nazemian. Equivalence of some homological conditions for ring epimorphisms. Electronic preprint `arXiv:1710.00097 [math.RA]`, to appear in *Journ. of Pure and Appl. Algebra.*

[16] A. Frankild, P. Jørgensen. Foxby equivalence, complete modules, and torsion modules. *Journ. of Pure and Appl. Algebra* **174**, #2, p. 135–147, 2002.

[17] L. Fuchs, L. Salce. Almost perfect commutative rings. *Journ. of Pure and Appl. Algebra* **222**, #12, p. 4223–4238, 2018.

[18] W. Geigle, H. Lenzing. Perpendicular categories with applications to representations and sheaves. *Journ. of Algebra* **144**, #2, p. 273–343, 1991.

[19] R. Göbel, J. Trlifaj. Approximations and endomorphism algebras of modules. Second revised and extended edition. De Gruyter, 2012.

[20] D. K. Harrison. Infinite abelian groups and homological methods. *Annals of Math.* **69**, #2, p. 366–391, 1959.

[21] M. Hrbek. One-tilting classes and modules over commutative rings. *Journ. of Algebra* **462**, p. 1–22, 2016. `arXiv:1507.02811 [math.AC]`

[22] A. J. de Jong et al. The Stacks Project. Available from `https://stacks.math.columbia.edu/`

[23] H. Krause, J. Šťovíček. The telescope conjecture for hereditary rings via Ext-orhtogonal pairs. *Advances in Math.* **225**, #5, p. 2341–2364, 2010. `arXiv:0810.1401 [math.RT]`

[24] F. Marks, J. Šťovíček. Universal localizations via silting. Electronic preprint `arXiv:1605.04222 [math.RT]`. To appear in *Proc. of the Royal Soc. of Edinburgh, Sect. A.*

[25] E. Matlis. Cotorsion modules. *Memoirs of the American Math. Society* **49**, 1964.

[26] D. Pauksztello. Homological epimorphisms of differential graded algebras. *Communic. in Algebra* **37**, #7, p. 2337–2350, 2009. `arXiv:0705.0118 [math.RA]`

[27] L. Positselski. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendix C in collaboration with D. Rumynin; Appendix D in collaboration with S. Arkhipov. Monografie Matematyczne vol. 70, Birkhäuser/Springer Basel, 2010. xxiv+349 pp. `arXiv:0708.3398 [math.CT]`

[28] L. Positselski. Weakly curved A_∞-algebras over a topological local ring. Electronic preprint `arXiv:1202.2697 [math.CT]`.

[29] L. Positselski. Contraherent cosheaves. Electronic preprint `arXiv:1209.2995 [math.CT]`.

[30] L. Positselski. Contramodules. Electronic preprint `arXiv:1503.00991 [math.CT]`.

[31] L. Positselski. Contraadjusted modules, contramodules, and reduced cotorsion modules. *Moscow Math. Journ.* **17**, #3, p. 385–455, 2017. `arXiv:1605.03934 [math.CT]`

[32] L. Positselski. Triangulated Matlis equivalence. *Journ. of Algebra and its Appl.* **17**, #4, article ID 1850067, 2018. `arXiv:1605.08018 [math.CT]`

[33] L. Positselski. Abelian right perpendicular subcategories in module categories. Electronic preprint `arXiv:1705.04960 [math.CT]`.

[34] L. Positselski. Flat ring epimorphisms of countable type. Electronic preprint `arXiv:1808.00937 [math.RA]`.
[35] L. Positselski, J. Rosický. Covers, envelopes, and cotorsion theories in locally presentable abelian categories and contramodule categories. *Journ. of Algebra* **483**, p. 83–128, 2017. `arXiv:1512.08119 [math.CT]`
[36] L. Positselski, J. Šťovíček. The tilting-cotilting correspondence. Electronic preprint `arXiv:1710.02230 [math.CT]`.
[37] L. Positselski, J. Šťovíček. ∞-tilting theory. Electronic preprint `arXiv:1711.06169 [math.CT]`.
[38] L. Salce. Cotorsion theories for abelian groups. *Symposia Math.* **XXIII**, Academic Press, London–New York, 1979, p. 11–32.
[39] J. Šaroch. Approximations and Mittag–Leffler conditions—the tools. Electronic preprint `arXiv:1612.01138 [math.RA]`, to appear in *Israel Journ. of Math.*
[40] J. Šaroch, J. Šťovíček. The countable telescope conjecture for module categories. *Advances in Math.* **219**, #3, p. 1002–1036, 2008. `arXiv:0801.3936 [math.RA]`
[41] I. Shapiro. On the anti-Yetter-Drinfeld module-contramodule correspondence. Electronic preprint `arXiv:1704.06552 [math.KT]`.
[42] B. Stenström. Rings of quotients. An introduction to methods of ring theory. Springer-Verlag, Berlin–Heidelberg–New York, 1975.
[43] C. E. Watts. Intrinsic characterizations of some additive functors. *Proc. of the Amer. Math. Soc.* **11**, #1, p. 5–8, 1960.
[44] J. Xu. Flat covers of modules. *Lecture Notes in Math.* **1634**, Springer, 1996.
[45] A. Yekutieli. On flatness and completion for infinitely generated modules over noetherian rings. *Communicat. in Algebra* **39**, #11, p. 4221–4245, 2010. `arXiv:0902.4378 [math.AC]`
[46] A. Yekutieli. Flatness and completion revisited. *Algebras and Represent. Theory* **21**, #4, p. 717–736, 2018. `arXiv:1606.01832 [math.AC]`

(Silvana Bazzoni) Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova, Via Trieste 63, 35121 Padova (Italy)
E-mail address: `bazzoni@math.unipd.it`

(Leonid Positselski) Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa 31905 (Israel); and
Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 119048 (Russia); and
Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127051 (Russia); and
Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1 (Czech Republic)
E-mail address: `positselski@yandex.ru`