A smooth introduction to Ergodic Theory

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Foreword. *Ergodic Theory* is a branch of the theory of *Dynamical Systems,* namely the branch in which the reference to a measure plays an essential role. Ergodic theory was started as a mathematical theory at the end of twenties/beginning of thirties of last century by G.D. Birkhoff and J. von Neumann, who provided the good mathematical frame and the first nontrivial results. The roots of the theory, however, and some leading ideas, go back to the work of L. Boltzmann and J.W. Gibbs, in the last decades of 19^{th} century, namely to their program to provide a microscopic dynamical fundation of thermodynamics, through statistical mechanics. Ergodic Theory had a strong development in the sixties and seventies of last century, after the introduction of the notion of Entropy by A.N. Kolmogorov and Ya.G. Sinai (end of fifties), the introduction by Sinai of the first really nontrivial examples of ergodic dynamical systems (Sinai billiards, 1962), and the connection with the theory of Lyapunov Characteristic Exponents (V. Oseledets 1968, Ya. Pesin 1974).

This short note is more a guide to the literature than a real introduction to Ergodic Theory. It includes the mathematical frame of the theory (Sect. 1), the very central notions of "ergodicity", which gives the name to the theory, and of "mixing" (Sect. 2), an introduction to the Kolmogorov-Sinai entropy, with the main ideas and a few hints on selected nontrivial results (Sect. 3), and finally, an introduction to Lyapunov Characteristic Exponents in Ergodic Theory (Sect. 4). Appendix A reports a few comments on the physical roots of Ergodic Theory in Boltzmann and Gibbs, while Appendices B–D are devoted to technical parts.

Books on Ergodic Theory are abundant. Very classical readings include:

I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, *Ergodic Theory*, Springer 1982.

V.I. Arnol'd and A. Avez, *Ergodic problems of classical mechanics*, W.A. Benjamin 1968.

P.R. Halmos, *Ergodic Theory*, Chelsea 1956.

A suggestes very general book on dynamical systems is:

A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press 1995.

1 Introduction

1.1 Dynamical systems

Loosely speaking, a dynamical system is composed by two entities: a space M — a set with some structure, traditionally called *phase space*; a point of M is assumed to define completely the state of a system — and a dynamics on M. The typical entry level is that M is a (separable, complete) metric space, usually assumed to be compact; M has then a topology, and Borel sets form a σ–algebra. The so–called *smooth theory* of dynamical systems, in which the ergodic properties of a system are related to the expansion/contraction rate of tangent dynamics, requires more, namely that M is a smooth manifold of some class $\mathcal{C}^r$. Concerning the dynamics, reference is made to an independent variable t, usually called *time,* which belongs either to $\mathbb{R}$ (continuous dynamical systems) or to $\mathbb{Z}$ (discrete dynamical systems); for any t, a homeomorphism (non smooth case) or a diffeomorphism of class $\mathcal{C}^r$ (smooth case) $\Phi^t : M \to M$ is given, in such a way that

$$\Phi = \{\Phi^t\,,\ t \in \mathbb{R} \text{ or } \mathbb{Z}\}$$

is a one parameter group:

$$\Phi^0 = \mathrm{Id}\ , \qquad (\Phi^t)^{-1} = \Phi^{-t}\ , \qquad \Phi^s \circ \Phi^t = \Phi^{s+t}\ .$$

This is indeed the typical situation one meets in physical systems, for example in Lagrangian or Hamiltonian mechanics, where one is given a vector field X on some manifold M, and $\Phi^t(x)$ is the solution at time t, with initial datum $x \in M$, of the differential equation

$$\dot{x} = X(x)\ .$$

In the discrete case, the group property implies that Φ is formed by the iterates of a map Φ^1 or its inverse Φ^{-1}; Φ^1 is usually shortly denoted Φ. A generalization is given by *non invertible* dynamical systems, where t is nonnegative, $t \in \mathbb{R}^+$ or $\mathbb{N}$, and correspondingly Φ is a semigroup.

Definition 1 *Let M be a compact metric space and*

$$\Phi = \{\Phi^t\,,\ t \in \mathbb{R} \text{ or } \mathbb{Z} \text{ or } \mathbb{R}^+ \text{ or } \mathbb{N}\}$$

be a one–parameter group ($t \in \mathbb{R}$ or $\mathbb{Z}$) or semigrup ($t \in \mathbb{R}^+$ or $\mathbb{N}$) of homeomorphisms ($t \in \mathbb{R}$ or $\mathbb{Z}$) or continuous maps ($t \in \mathbb{R}^+$ or $\mathbb{N}$) $M \to M$. The pair (M, Φ) is called dynamical system: *continuous if $t \in \mathbb{R}$ or $\mathbb{R}^+$, discrete if $t \in \mathbb{Z}$ or $\mathbb{N}$, invertible if $t \in \mathbb{R}$ or $\mathbb{Z}$.*

If in addition M is a manifold of class $\mathcal{C}^r$ and each Φ^t is a diffeomorphism or a differentiable map $M \to M$ of class $\mathcal{C}^r$, the pair (M, Φ) is called smooth dynamical system *of class $\mathcal{C}^r$.*

Occasionally, with some attention, piecewise regular maps are also considered, for example in the study of iterations of maps on the unit interval, or in connection with the so–called *billiards.*

In Ergodic Theory, a further structure on M, namely an invariant measure defined on the Borel sets of M, plays a crucial role.

Definition 2 *Let (M, Φ) be a dynamical system, and let μ be a measure defined on the σ–algebra of the Borel sets of M, normalized in such a way that $\mu(M) = 1$; μ is said to be an* invariant measure *for (M, Φ), if for any t and any measurable $A \subset M$*

$$\mu(\Phi^{-t}(A)) = \mu(A)\ , \tag{1}$$

where
$$\Phi^{-t}(A) = \{x \in M : \Phi^t(x) \in M\} \ .$$

In the non–invertible case, (1) is different from $\mu(\Phi^t(A)) = \mu(A)$.

PROPOSITION 1 (Krylov–Bogoliubov theorem). *Any continuous map on a compact metric space admits at least one invariant measure.*

For the proof, not much interesting for us, see for example [1], sect. 1.8; the idea is to start with any measure, and constsuct from it an invariant one by a suitable averaging in time.

DEFINITION 3 *Let (M, Φ) be a dynamical system and μ an invariant measure for it; the triplet (M, μ, Φ) is called* dynamical system endowed with a measure*; (M, μ, Φ) is said to be smooth, or also* classical*, if (M, Φ) is smooth and moreover in each chart of M it is $\mathrm{d}\mu = \rho\,\mathrm{d}V$, where $\mathrm{d}V$ is the Lebesgue measure and ρ is a smooth density.*

It is common to call (M, μ, Φ) symply a dynamical system, omitting the heavy specification "endowed with a measure". Dynamical systems (M, φ) not endowed with a measure are often called *topological dynamical systems.*

- As we shall see, making reference to a measure is crucial to introduce sensible definitions and formulate interesting propositions. In particular, it will be possible to consider properties which are satisfied *almost everywhere*, i.e. with the possible exception of a zero–measure subset of M. "Almost everywhere" will be abbreviated in "a.e.".

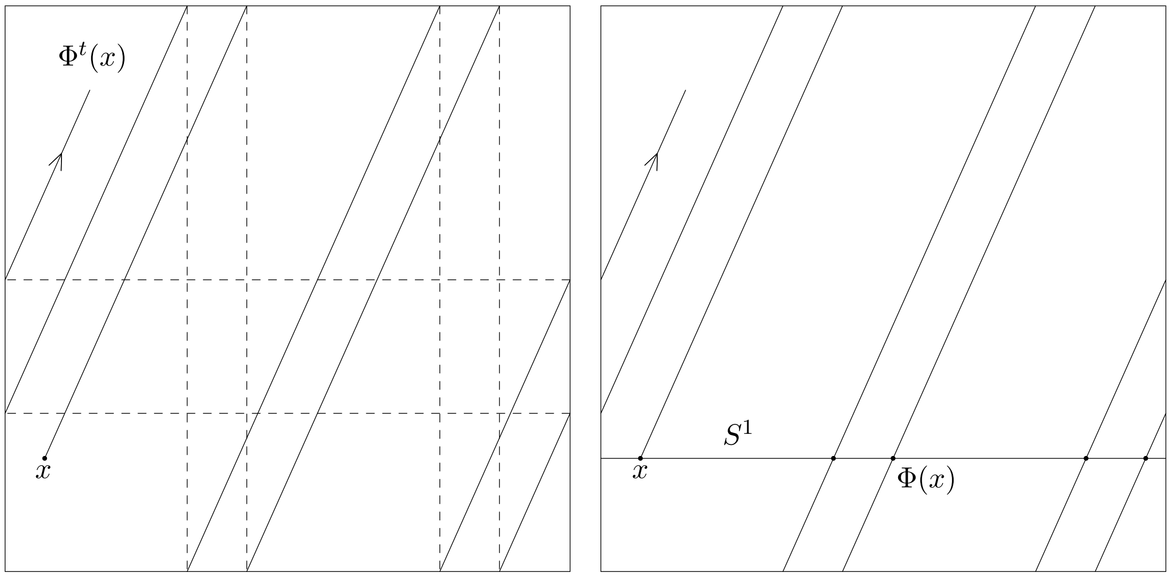
Some advanced results in Ergodic Theory require smoothness. The cornerstones of the theory, however, like the Birkhoff ergodic theorem or the notions of ergodicity and mixing, do not, and in fact are meaningful even without making reference to a metrics on M, only assuming that M is a measure space i.e. an abstract set of points endowed with a σ–algebra of measurable sets and a measure μ on it.[1]

DEFINITION 4 *The triplet (M, μ, Φ), where M is a measure space endowed with normalized measure μ, $\Phi : M \to M$ is as above, and (1) holds, is called* abstract dynamical system.

Some elementary examples will be useful to take familiarity with the notion of dynamical system in Ergodic Theory.

1.2 Examples

EXAMPLE 1 (Free motion on $\mathbb{T}^2$). Let $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mu =$ Lebesgue measure;[2] for any $x \in \mathbb{T}^2$ and $t \in \mathbb{R}$ let
$$\Phi^t(x) = x + vt \pmod 1 \ ,$$
where v is some given velocity, i.e. vector of $\mathbb{R}^2$. This is the system associated to the differential equation $\dot{x} = v$ on $\mathbb{T}^2$; see figure 1a. If the ratio $\alpha = v_1/v_2$ is rational, $\alpha = p/q \in \mathbb{Q}$, then all motions are periodic with period $T = p/v_1 = q/v_2$. If α is irrational, all trajectories are open; as we shall see in a moment (proposition 2) they are dense on the torus.



(a) (b)

Figure 1: *The free motion on $\mathbb{T}^2$ and its section.*

Example 2 (Discrete translation on S^1). Let $M = S^1$, μ = Lebesgue measure, and let Φ be the map defined by

$$\Phi(x) = x + \alpha \pmod 1 \ . \tag{2}$$

The connection with the previous example is immediate: if N is a section of $\mathbb{T}^2$ given by $x_2 = \text{const}$, then, see figure 1b, the free motion on $\mathbb{T}^2$ with velocity v induces a discrete translation of the form (2) on N, with $\alpha = v_1/v_2$, and the Lebesgue measure on N is preserved.

Proposition 2 *If α is irrational, then all trajectories are dense on S^1; correspondingly, trajectories of example 1 are dense on $\mathbb{T}^2$.*

Proof. We show that for any $\varepsilon > 0$ and any $x \in S^1$, the trajectory $\Phi^t(x)$, $t \in \mathbb{Z}$, enters all intervals of length ε ("almost everywhere" here is not necessary). Let $T > \varepsilon^{-1}$; among the T points $x, \Phi(x), \ldots, \Phi^{T-1}(x)$, which for irrational α are all different, two at least, say $\Phi^{t'}(x)$ and $\Phi^{t''}(x)$, $t' > t''$, have distance smaller than ε. But the translation Φ is rigid, i.e. preserves the distance; going back t'' iterations, it follows $\mathrm{dist}\,(\Phi^t(x), x) < \varepsilon$, with $t = t' - t''$. This means the sequence $x, \Phi^t(x), \ldots, \Phi^{kt}(x), \ldots$ proceeds by steps smaller than ε, and thus enters any interval of length ε. $\square$

Example 3 (Free motion on $\mathbb{T}^n$). Example 1 naturally generalizes to higher dimension. Let $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, μ = Lebesgue measure and

$$\Phi^t(x) = x + vt \pmod 1 \ , \qquad v \in \mathbb{R}^n \ .$$

[1] A more accurate notation for the abstract case, frequently found in books, is $(M, \mathfrak{S}, \mu, \Phi)$, where $\mathfrak{S}$ is the σ–algebra on which μ is defined.

[2] Properly speaking, it is the Haar measure on the torus. We shall not make such a distinction.

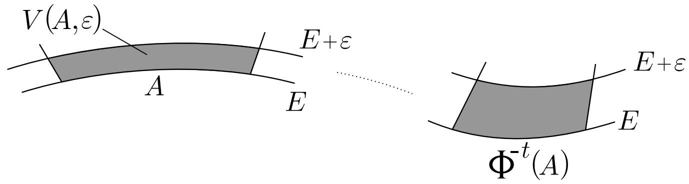


Figure 2: *Illustrating the definition of the Liouville measure.*

The vector v is said to be *resonant* with the integer vector $k \in \mathbb{Z}^n$, $k \neq 0$, if

$$k \cdot v = 0 \ , \tag{3}$$

where the dot denotes the usual scalar product in $\mathbb{R}^n$; v is said to be *nonresonant* if (3) is satisfied only by $k = 0$. The set $\mathcal{L}(v)$ of integer vectors k that satisfy (3) with a given v, is a subgroup of $\mathbb{Z}^n$, called *resonant lattice* or *resonant modulus* of v; its dimension is called the *multiplicity* of the resonance.

Proposition 3 (Jacobi, 1835). *If v is nonresonant, each trajectory is dense in $\mathbb{T}^n$.*

We shall prove the proposition later, as an exercise on the notion of ergodicity.

Proposition 4 *If the dimension of the resonant lattice of v is r, $1 \leq r \leq n-1$, then each trajectory is confined to an invariant manifold N of dimension $n-r$ which is diffeomorphic to the torus $\mathbb{T}^{n-r}$.*

Proof. We provide only a hint. One should preliminarily show that for any lattice $\mathcal{L}$ of dimension r, there exists a matrix J with integer entries and $\det J = 1$ (so that J^{-1} also has integer entries, and $J : \mathbb{Z}^n \to \mathbb{Z}^n$ is invertible), such that for any $k \in \mathcal{L}$ it is $Jk = (k'_1, \ldots, k'_r, 0, \ldots, 0)$. This implies that if $\mathcal{L}$ is the resonant lattice of v, then the change of variable $x = J^T y$ on $\mathbb{T}^n$ (J^T denoting the transposed of J) turns the equation of motion $\dot{x} = v$ into $\dot{y} = u$, with

$$u = J^T v = (0, \ldots, 0, u_{r+1}, \ldots, u_n) \ .$$

So, the trajectory issuing from y is confined to a torus $\mathbb{T}^{n-r}$ and the original trajectory is confined to $N = J^T \mathbb{T}^{n-r}$.

$\square$

- Since $\mathcal{L}$ is assumed to be the resonant lattice of v, no further resonances do exist among $u_{r+1}, \ldots, u_n$. According to Jacobi theorem, $\mathbb{T}^{n-r}$ is densely filled by trajectories; in other words, $\mathbb{T}^n$ is naturally decomposed (foliated) in invariant tori of dimension $n-r$, one for each value of $y_1, \ldots, y_r$ (which are left constant by the dynamics), and this is the finest possible decomposition compatible with the dynamics.

Example 4 (Hamiltonian systems). As is known (Liouville theorem), the Hamiltonian flow Φ^t, $t \in \mathbb{R}$, preserves the Euclidian volume in canonical coordinates, $\mathrm{d}V = \mathrm{d}p_1 \cdots \mathrm{d}p_n \mathrm{d}q_q \cdots \mathrm{d}q_n$. So, if M is the layer between two compact constant energy surfaces Σ_E and $\Sigma_{E'}$ (think, to fix the ideas, they are $(n-1)$-dimensional spheres) and μ is the volume in M normalized to one, (M, μ, Φ) is a smooth dynamical system. However, this is not a deep view, since energy is conserved and so the flow will never mix different constant energy surfaces. It is not difficult, however, to see that the conservation of the volume in the phase space induces an invariant measure on each single

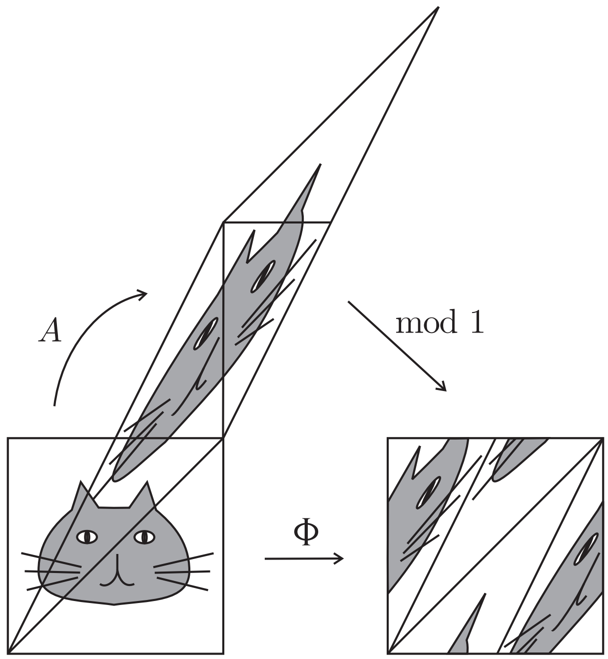


Figure 3: *The Arnol'd cat: the action of A, the folding on $\mathbb{T}^2$.*

constant energy surface Σ_E. Such a measure, called Liouville measure and usually denoted μ_L, can be defined as follows (fig. 2): let A be any disc on Σ_E; consider the thick disc with base A, comprised between Σ_E and $\Sigma_{E+\varepsilon}$, with any choice of the (transversal) lateral walls; let $V(A, \varepsilon)$ be its volume. Then define

$$\mu_L(A) = C \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} V(A, \varepsilon) \tag{4}$$

(the choice of the lateral walls, if transversal, is clearly irrelevant). The measure $\mu_L(A)$ is obviously invariant (both ε and $V(A, \varepsilon)$ are, even before the limit in (4), and transversal lateral walls remain transversal). Once the measure is given on all discs on Σ_E, it naturally extends to all measurable sets of Σ_E. It is worthwhile to observe that μ_L is continuous with respect to the Euclidean area $\mathrm{d}\Sigma$ on Σ_E, namely it is

$$\mathrm{d}\mu_L = C \frac{\mathrm{d}\Sigma}{\|\nabla H\|} \ ,$$

where $\| \, . \, \|$ denotes Euclidian norm; this is an immediate consequence of the fact that the hight $h(x)$ of the thick disc at $x \in A$ is $\varepsilon / \|\nabla H(x)\| + \mathcal{O}(\varepsilon^2)$.

Example 5 (Algebric automorphism of $\mathbb{T}^2$, better known as *Arnol'd cat*). Let $M = \mathbb{T}^2$, $\mu =$ Lebesgue measure, and

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \mod 1 \ , \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \ ;$$

see figure 3. Matrix A can be replaced by any 2×2 matrix with integer entries, $\det A = 1$ (so that A^{-1} also has integer entries), $\operatorname{Tr} A > 2$. The fact that both A and A^{-1} have integer entries, implies that Φ is a diffeomorphism $\mathbb{T}^2 \to \mathbb{T}^2$; $\det A = 1$ implies that μ is preserved. So, $(\mathbb{T}^2, \mu, \Phi)$ is a discrete invertible smooth dynamical system. The fact that $\operatorname{Tr} A > 2$ implies that the eigenvalues are real and different from 1 and the eigenvectors have irrational slope. In the example, the eigenvalues are $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$, with

$$\lambda = \frac{1}{2}(3 + \sqrt{5}) > 1 \ ;$$

the eigenvectors are

$$u_1 = (1, \lambda - 1) \ , \qquad u_2 = (1, \lambda^{-1} - 1) \ .$$

The map Φ contracts in the direction of u_2 and expands (more precisely: Φ^{-1} contracts) in the direction of u_1, at exponential rate.

The eigenspaces E_1, E_2, reported on the torus by mod 1, thanks to the irrational slope fill it densely (Proposition 2). Now, let B be a no matter how small ball; for simplicity imagine it centered in the origin, which is a fixed point, although this is not relevant. From the very definition of the map Φ, it follows that

$$\Phi^t \begin{pmatrix} x \\ y \end{pmatrix} = A^t \begin{pmatrix} x \\ y \end{pmatrix} \quad \bmod 1$$

(the reduction to the torus can be made at the end), and so

$$\Phi^t(B) = A^t B \quad \bmod 1 \ .$$

But $A^t B$, for large positive t, is a thin strip along E_1 of length $\lambda^t d$ and width $\lambda^{-1} d$, d being the diameter of B; the reduction to the torus via the operation mod 1, scatters $\Phi^t(B)$ everywhere in $\mathbb{T}^2$. This is of course vague and heuristic, but is a good wiew. Later on we shall formalize this vague idea into a precise notion, called *mixing.* This is a toy–model, but is the prototype of a class of systems, called *Anosov's systems*, where contraction–dilatation of sets at exponential rate plays a determinant role. Figure 4 shows the evolution of 20,000 points randomly chosen inside a small box adjacent to the origin, up to 8 iterations.

Exercise 1 Show that in the Arnol'd cat the set of periodic points is dense in $\mathbb{T}^2$. [Hint: all points with rational coordinates are periodic]

Example 6 (Rubber-band map). This is a very simple discrete non–invertible system, with $M = S^1$, $\mu =$ Lebesgue, $\Phi(x) = 2x \bmod 1$ (the common folding of a rubber band). Distances are stretched, without any contracting direction; measure is preserved because $\Phi^{-1}(A)$ is the disjoint union of two subsets, whose measure is one half of the measure of A. See figure 5.

Example 7 (Bernoulli shifts). This is a dynamical system (thus, a deterministic system) which nevertheless is a model for a process as random as the throwing of a dice. Let I be a finite alphabet with n symbols, that we denote here $0, \ldots, n-1$. The phase space, that we shall denote Σ, is $I^{\mathbb{Z}}$, i.e. the points of Σ are the sequences

$$\sigma = (\ldots \sigma_{-2},\ \sigma_{-1},\ \sigma_0,\ \sigma_1,\ \sigma_2,\ \ldots) \ , \qquad \sigma_k \in I$$

(think each of them be an a priori possible exit of an infinite sequence of throwings of a dice with n faces). A distance on Σ can be defined by following the idea that σ, σ' are close, if they coincide for a large interval around $k = 0$; for example

$$\operatorname{dist}(\sigma, \sigma') = \sum_{k \in \mathbb{Z}} 2^{-|k|} \, \delta(\sigma_k, \sigma'_k) \ , \qquad \delta(l, l') = \begin{cases} 0 & \text{if } l = l' \\ 1 & \text{if } l \neq l' \end{cases}$$

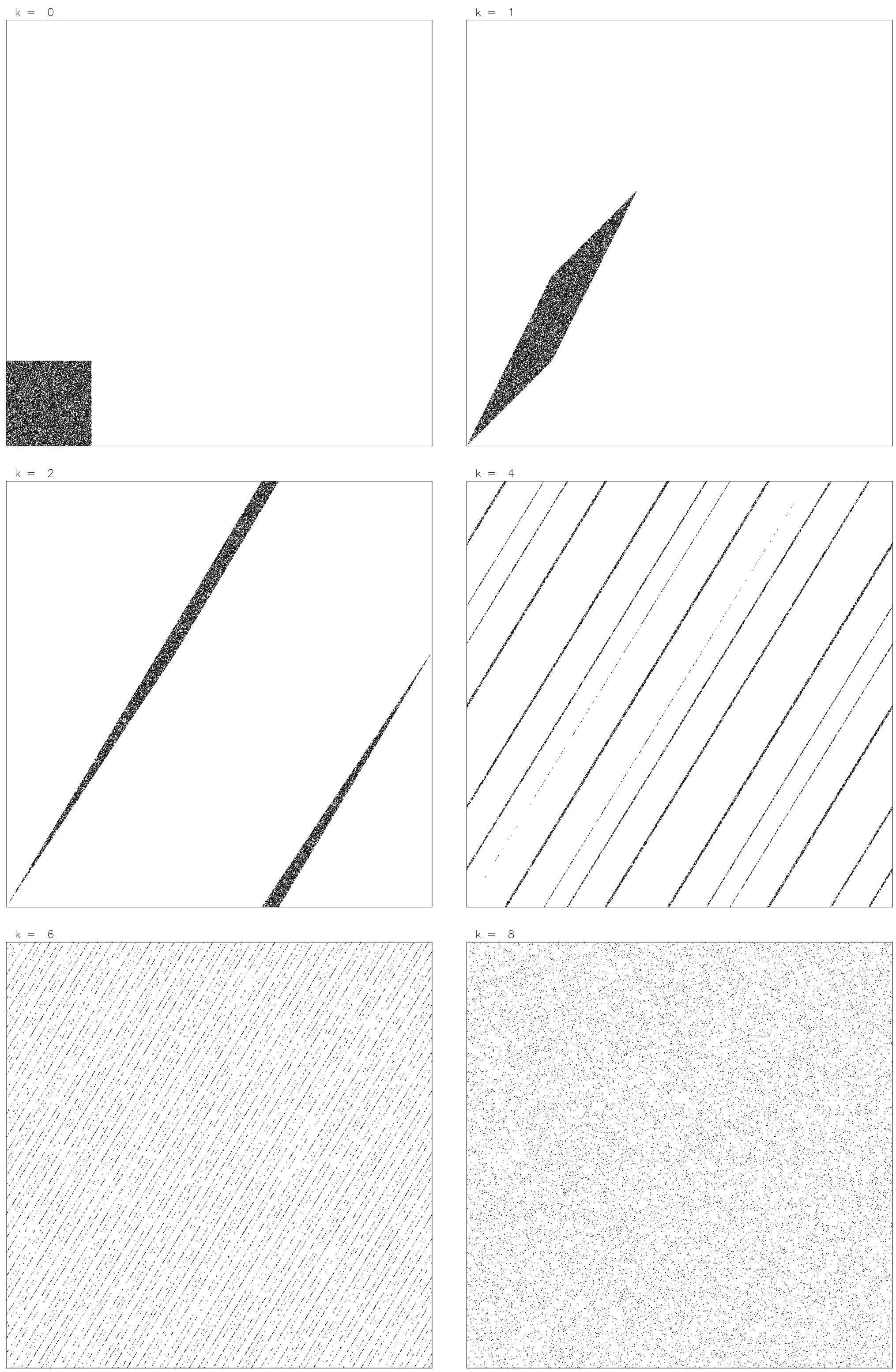


Figure 4: *The evolution of* 20, 000 *points initially confined in a small square near the origin, in the Arnol'd cat, for* $t = 0, 1, 2, 4, 6, 8$.

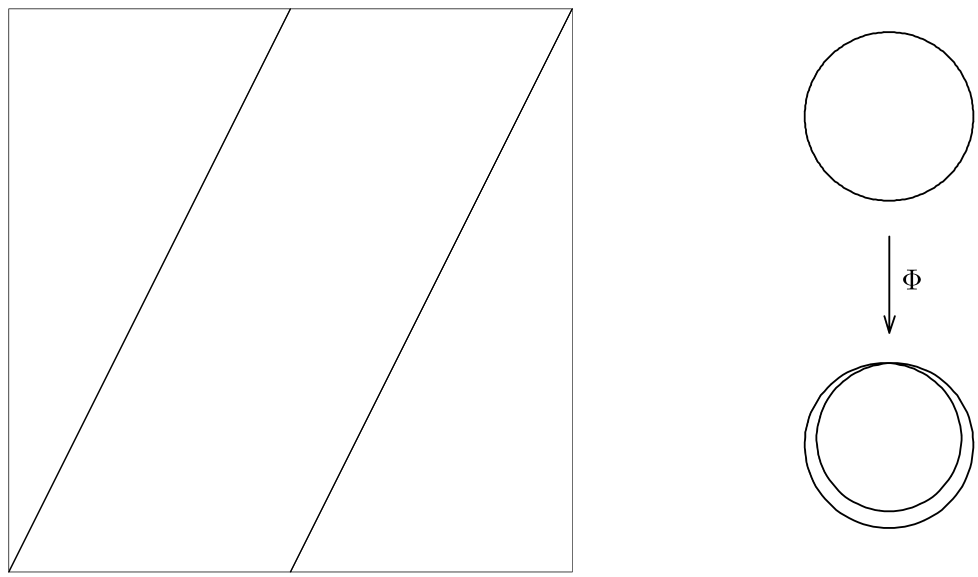


Figure 5: *The rubber-band map.*

The σ–algebra of measurable sets is the one generated by the "elementary cylinders"

$$\mathcal{C}_k^l := \{x \in \Sigma : x_k = l\} \ , \qquad k \in \mathbb{Z} \, , \quad l \in I \ ;$$

the measure is assigned on it as follows: first, each symbol in I is assigned a weight $p_l > 0$, with $\sum_{l \in I} p_l = 1$ (the a priori probability of the faces of the dice). Then one assigns

$$\mu(\mathcal{C}_k^l) = p_l \tag{5}$$

(the time k of the individual throwing is irrelevant) and extends the measure to all "cylinders"

$$\mathcal{C}_{k_1,\ldots,k_m}^{l_1,\ldots,l_m} = \bigcap_{j=1}^{m} \mathcal{C}_{k_j}^{l_j}$$

of "base" $k_1, \ldots, k_m$ and "specification" $l_1, \ldots, l_m$, by posing

$$\mu(\mathcal{C}_{k_1,\ldots,k_m}^{l_1,\ldots,l_m}) = p_{l_1} \cdots p_{l_m}$$

(different throwings are independent). It can be seen that such a perscription is sufficient to define μ on all measurable sets. Concerning the dynamics, we define $\Phi : \Sigma \to \Sigma$ by saying that $\sigma' = \Phi(\sigma)$ is σ itself, left shifted by one position:

$$\sigma'_k = \sigma_{k+1} \ , \qquad k \in \mathbb{Z} \ .$$

Quite clearly, Φ is a homeomorphism and preserves μ. The nonsmooth discrete dynamical system (Σ, μ, Φ) constructed in this way is called *Bernoulli shift*, and is usually denoted $B_{p_0,\ldots,p_{n-1}}$. Bernoulli shifts play a quite relevant role in Ergodic Theory.

Exercise 2 Show that in any Bernoulli shift the periodic points are dense in Σ.

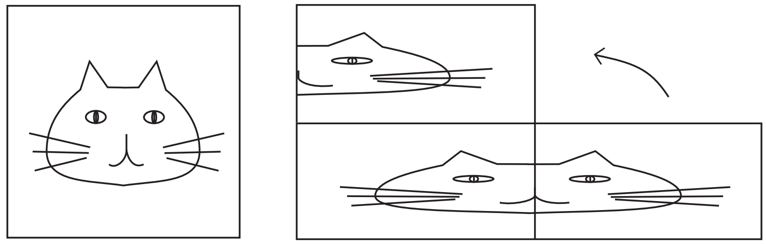


Figure 6: *The Baker's map.*

EXAMPLE 8 (Baker's map). This is piecewise smooth system. Let M be the square $[0,1) \times [0,1)$, $\mu =$ Lebesgue measure, and

$$\Phi(x,y) = \begin{cases} (2x, \frac{1}{2}y) & \text{for } x < \frac{1}{2} \\ (2x-1, \frac{1}{2}y + \frac{1}{2}) & \text{for } x \geq \frac{1}{2} \end{cases} ;$$

see figure 6. Φ contracts vertically and expands horizontally; figure 7 shows what happens to 20,000 points originally filling a disc in the center of the square. Φ^{-1} is the same as Φ, with x and y exchanged.

1.3 Isomorphism

Any interesting notion in Ergodic Theory must be invariant by *isomorphism.*

DEFINITION 5 *The dynamical systems (M, μ, Φ) and (N, ν, Ψ) are said to be* isomorphic, *if there exist a map $h : M \to N$, defined and invertible almost everywhere, such that (i) for any measurable $A \subset M$, $h(A) \subset N$ is measurable and $\nu(h(A)) = \mu(A)$, and conversely; (ii) the two dynamics commute:*

$$h \circ \Phi^t = \Psi^t \circ h \ . \tag{6}$$

An example is the following:

PROPOSITION 5 *The Baker's map and the Bernoully shift $B_{\frac{1}{2},\frac{1}{2}}$ are isomorphic.*

PROOF. Let M be the phase space (the square) of the Baker's map; let $z = (x,y) \in M$, and write x and y in binary digits:

$$x = 0.a_0a_1a_2\ldots \ , \qquad y = 0.b_0b_1b_2\ldots \ ;$$

the isomorphism is $\sigma = h(x,y) \in \Sigma$ with

$$\sigma = (\ldots\sigma_{-3}, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots) = (\ldots b_2, b_1, b_0, a_0, a_1, a_2\ldots) \ .$$

It is not difficult to check that the measures behave appropriately (look at the generators) and (6) holds. Note that h is not one to one on strings that end, in either direction, by a sequence of symbols "1".

$\square$

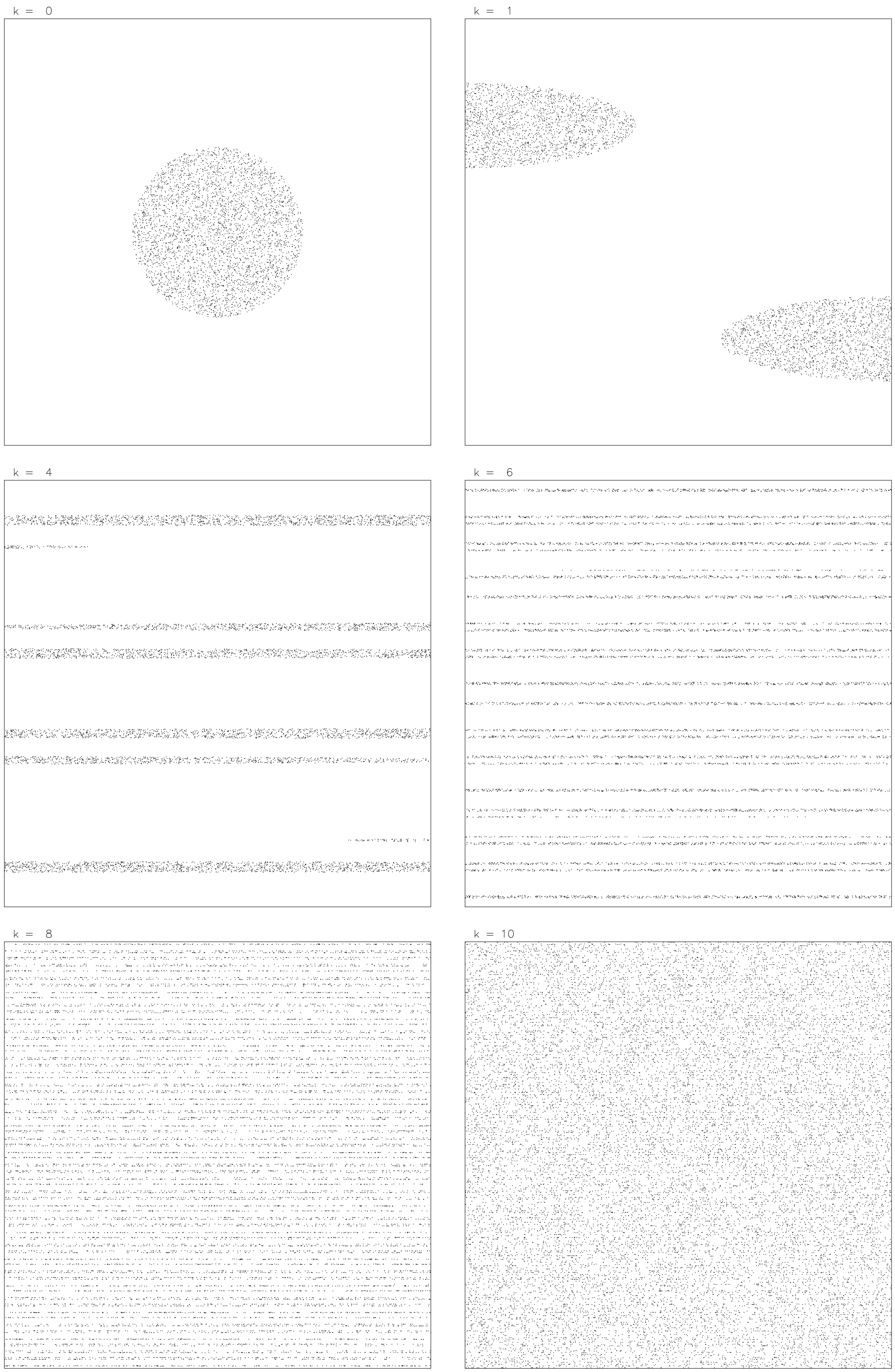


Figure 7: *The evolution of* 20 000 *points initially filling a disc in the center of the square, by applying the Baker's map:* $t = 0, 1, 4, 6, 8, 10$.

- An equivalent definition of the isomorphism is that $\sigma = h(z)$ is the *story* of z, observed through the natural partition of M

$$M = M_0 \cup M_1 \ , \qquad M_0 = \{z = (x,y) \in M : x < \tfrac{1}{2}\} \ ;$$

this means $\sigma_k = l$ iff $\Phi^k(z) \in M_l$. Indeed, the above introduced map h is such that $z \in M_l$ iff $\sigma = h(z) \in \mathcal{C}_0^l$; it follows

$$\Phi^k(z) \in M_l \iff h(\Phi^k(z)) \in \mathcal{C}_0^l \iff \Psi^k(h(z)) \in \mathcal{C}_0^l \iff h(z) \in \mathcal{C}_k^l \iff \sigma_k = l \ .$$

Quite in general, for any dynamical system (M, μ, Φ), if a partition $M = M_0 \cup \cdots \cup M_{l-1}$ is introduced, then a map $h : M \to \Sigma = (0, \ldots, l-1)^{\mathbb{Z}}$ gets defined, such that $h(z)$ is the story of z through the partition, and the shift Ψ in Σ obviously satisfies (6). But in general it is far from trivial to understand which is the image $h(M) \in \Sigma$ (which stories are effectively realized), if h is invertible in $h(M)$, and which is the measure ν in Σ corresponding to μ in M. The dynamics in a space of (sequences of) symbols, conjugate to the dynamics one is interested to study, is called *symbolic dynamics*. If one is able to find a good partition of M, such that h is invertible on its image and ν is easy (for example: a Markov measure, with well defined transition probabilities), then (M, μ, Φ) is completely understood. Not easy.

1.4 General results

Ergodic Theory deals, basically, with the statistical behavior of typical trajectories; this was indeed the very motivation of Boltzmann and Gibbs, who started it at the end of 19$^{\text{th}}$ century (Appendix A). The most elementary notion, in this perspective, is averaging. Given any function $f : M \to \mathbb{R}$, one wishes to know something about its *time average* $\bar{f} : M \to \mathbb{R}$, defined in the natural way as the limit (if it exists)

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\Phi^t(x))\mathrm{d}t \qquad \text{or} \qquad \bar{f}(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\Phi^t(x))$$

respectively in the continuous and in the discrete case. For invertible systems, the backword time average is also defined in the obvious way, namely

$$\bar{f}_{(-)}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\Phi^{-t}(x))\mathrm{d}t \qquad \text{or} \qquad \bar{f}_{(-)}(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\Phi^{-t}(x)) \ .$$

For example, for any measurable set $A \subset M$, one might be interested in the fraction of time spent by a given trajectory in A, actually

$$\tau_A(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_A(\Phi^t(x))\mathrm{d}t \qquad \text{or} \qquad \tau_A(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \chi_A(\Phi^t(x)) \ , \tag{7}$$

where χ_A denotes the characteristic function of A: $\chi_A(x) = 1$ for $x \in A$, otherwise $\chi_A(x) = 0$. The quantity $\tau_A(x)$ is naturally interpreted as the probability to find the system in A, for the motion with initial datum x, over a long time interval.

Another average, in a sense trivial for it does not involve the dynamics, is the *phase average* $\langle f \rangle$ of f, namely the number defined by

$$\langle f \rangle = \int_M f \, \mathrm{d}\mu \ .$$

The phase average exists, by definition, for summable functions, i.e. $f \in L_1(M,\mu)$. What about the time average? As is not trivial, the time average does exist for *any* dynamical system, for *all* summable functions and *almost all* points $x \in M$. This is guaranteed by a fundamental theorem due to Birkhoff, usually called *Birkhoff* (or Birkhoff-Kinchin) *ergodic theorem*. The theorem does not make reference to distance, so it holds for abstract systems as well; to be definite, in the statement we refer to a discrete system, but the transposition to a continuous system is immediate.

PROPOSITION 6 *Let (M,μ,Φ) be any discrete dynamical system, and f be any function in $L_1(M,\mu)$. For almost any $x \in M$ the limit*

$$\bar{f}(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\Phi^t(x))$$

exists and moreover

$$\bar{f}(\Phi^t(x)) = \bar{f}(x) \ , \qquad \langle \bar{f} \rangle = \langle f \rangle \ . \tag{8}$$

If the system is invertible, $\bar{f}_{(-)}(x)$ also exists almost everywhere, and almost everywhere coincides with $\bar{f}(x)$.

The proof, not difficult but rather long and not much instructive, is deferred to the Appendix.

- One should stress the "almost everywhere" appearing in the statement. For example, for the Bernoulli shift $B_{\frac{1}{2},\frac{1}{2}}$ and for the characteristic function of the cylinder $\mathcal{C}_0^1$, one easily finds initial data for which the foreward average does not exist: for example,

  $$\sigma = (\,\ldots,\, \sigma_{-2},\, \sigma_{-1},\, 0,1,1,0,0,0,0,1,1,1,1,1,1,1,1,\, (16\times 0),\, (32\times 1),\, \ldots\,) \ ,$$

  with any σ_k for $k<0$. Quite clearly, the average truncated at a finite T oscillates, for no matter how large t, between $1/3$ and $2/3$. It is worwhile mentioning that by letting the above special string start in any position $k>0$, rather than in $k=0$, with arbitrary entries σ_j for $j \le k-1$, one obtains a dense set in Σ. So, in this example (and in general) $\bar{f}$ is *not* defined in *any* open set. This shows how crucial is making reference to a measure, and decide to disregard what happens to any set of points of zero measure. By similar tricks, one easily constructs a dense set of points x for which $\bar{f}(x)$ and $\bar{f}_{(-)}(x)$ are both defined, but do not coincide.

We conclude the Section with a second general result, due to Poincaré, showing a nontrivial subtle aspect of dynamics called "recurrence".

DEFINITION 6 *Let (M,μ,Φ) be any dynamical system, and consider a measurable subset A of M; a point $x \in A$ is said to be* recurrent *in A, if for any $T>0$ there exists $t \ge T$ such that $\Phi^t(x) \in A$.*

The set of non-recurrent points of A (points of A which after a certain time leave A forever) is then

$$N_A = \{x \in A :\, \exists T>0 :\, \Phi^t(x) \notin A \ \forall t \ge T\} \ . \tag{9}$$

For a pendulum, all points are recurrent in any open neighborhood (much more than recurrent: periodic), but those on the separatrices.

The following theorem holds:

Proposition 7 (Poincaré recurrence theorem) *For any measurable $A \subset M$, N_A is measurable and has zero measure.*

Correspondingly, almost all points of A are recurrent in A.

- An immediate consequence, when M is a metric space, is that for any $\varepsilon > 0$ and almost any $x \in M$, it is $\mathrm{dist}\,(\Phi^t(x), x) < \varepsilon$ for some arbitrarily large t (consider a finite covering of M with balls of diameter less than ε, and apply the recurrence theorem to each of them).

Proof. We can restrict ourselves to the case of discrete systems: the continuous case indeed reports to the discrete one by condidering the time-one map (the set N_A defined in (9), if one restricts t to integers, possibly grows but does not schrink). For any $T \in \mathbb{N}$, denote

$$N_{A,T} = \{x \in A : \Phi^t(x) \notin A \; \forall t \geq T\}$$

("T-non recurrent" points of A); it is obviously

$$N_A = \bigcup_{T \in \mathbb{N}} N_{A,T} \ ,$$

so it is enough to show that for any measurable A and any $T \in \mathbb{N}$ the set $N_{A,T}$ is measurable and has zero measure. The measurability of $N_{A,T}$ follows from its very definition, which can be rewritten

$$N_{A,T} = A \bigcap \Big[\bigcap_{t \geq T} \Phi^{-t}(M \setminus A) \Big] \ .$$

Consider now the sets

$$N_{A,T} \ , \ \Phi^{-T}(N_{A,T}) \ , \ \Phi^{-2T}(N_{A,T}) \ , \ \ldots$$

They are all disjoint: indeed, should it be

$$x \in \Phi^{-kT}(N_{A,T}) \bigcap \Phi^{-lT}(N_{A,T}) \neq \emptyset \ ,$$

with for example $k < l$, then

$$\Phi^{kT}(x) \in N_{A,T} \bigcap \Phi^{-(l-k)T}(N_{A,T}) \ ,$$

against the definition of $N_{A,T}$. Due to the conservation of measure, all such sets have the same measure. The overall measure of the space being finite, it is necessarily $\mu(N_{A,T}) = 0$. $\square$

2 Ergodicity and mixing

2.1 Ergodicity

Ergodicity is the first nontrivial property one meets in ergodic theory, which discriminates among systems and starts a classification. It can be intruduced in different equivalent ways, which stress different aspects of this deep notion.

E1. For any measurable $A \subset M$, the time spent in A by a generic trajectory coincides with the measure of A:

$$\tau_A(x) = \mu(A) \qquad \text{a.e. in } M \ , \tag{10}$$

$\tau_A(x)$ being defined as in (7).

E2. For any $f \in L_1(M, \mu)$, it is

$$\bar{f}(x) = \langle f \rangle \qquad \text{a.e. in } M \ . \tag{11}$$

E3. No constants of motion exist, but the trivial ones: namely if $f \in L_1(M, \mu)$ and

$$f(\Phi^t(x)) = f(x) \quad \forall t \ , \quad \text{a.e. in } M \ ,$$

then f is constant a.e. in M.

E4. The system is *metrically indecomposable*, namely for any measurable $A \subset M$

$$\Phi^{-t}(A) = A \quad \Longrightarrow \quad \mu(A) = 0 \text{ or } 1$$

(any decomposition $M = A \cup (M \setminus A)$, if measurable and invariant, is metrically trivial).

Properties E1–E4 are immediately seen to be invariant by isomorphism.

- E1 suitably formalizes the Boltzmann ergodic hypothesis, see Appendix A. E2 is proposed as definition of ergodicity in many textbooks of statistical mechanics. E3 formalizes the Gibbs idea, see again Appendix A, that the equilibrium distribution ρ^* is unique (if f is a nontrivial constant of motion, then $\rho^* = cf$, c providing normalization, is a Gibbs equilibrium distribution). Finally, E4 is the definition used by Birkhoff: very "elementary", and very useful in proofs.

PROPOSITION 8 *Properties* E1, ..., E4 *are equivalent.*

PROOF. Many cross implications are easily proved. An easy path is the following:

(a) E2 $\Rightarrow$ E1
(b) E1 $\Rightarrow$ E4
(c) E4 $\Rightarrow$ E2
(d) E4 $\Leftrightarrow$ E3 .

(a) is trivial, as (10) is a particular case of (11). To prove (b), assume by absurd that E4 is violated, i.e. there exists A nontrivial and invariant; then $\tau_A(x) = 1 \neq \mu(A)$ for $x \in A$, against E1. Concerning (c), assume by absurd that E2 is violated; this means that for some function f it is, for example, $\bar{f}(x) > \langle f \rangle$ in a set of positive measure. Denote

$$A = \{x \in M : \bar{f}(x) > \langle f \rangle\} \ .$$

By Birkhof ergodic theorem A is invariant (use the former of (8)) and nontrivial (use the latter: $\bar{f}(x)$ cannot be a.e. larger than $\langle f \rangle$), against E4. Finally (d) follows, in one direction, for, if E4 is violated by a set A, then χ_A is a nontrivial constant of motion and E3 is then violated; in the other direction for, if f is a nontrivial constant of motion, then for some $c \in \mathbb{R}$ the invariant set

$$A = \{x \in M : f(x) \leq c\}$$

is nontrivial, against E4. □

Definition 7 *The dynamical system* (M, μ, Φ) *is said to be* ergodic*, if any of E1–E4, and thus all, are satisfied.*

- In E2 and E3, reference is made to all functions $f \in L_1(M, \mu)$. Quite clearly, it is sufficient that properties are satisfied by all $f \in L_2(M, \mu)$, to ensure they are satisfied by all functions in $L_1(M, \mu)$; the restriction to a much smaller set of functions, like the set of characteristic functions of measurable sets, is in fact sufficient.

Proposition 9 *Any Hamiltonian system with one degree of freedom, restricted to a compact connected line of constant energy which does not contain singular points, is ergodic.*

Proof. Quite trivially, the trajectory runs on the line of constant energy, passing through all points. According to E4, the system is ergodic. □

- Consider a set of $n \geq 2$ independent harmonic oscillators, with hamiltonian

$$H(p, q) = \sum_{i=1}^{n} \tfrac{1}{2}(p_i^2 + \omega_i^2 q_i^2) \ ;$$

  the system is obviously not ergodic (the n energies are separately conserved). Understanding how to perturb the system, in such a way that it becomes ergodic, is among the most important still largely open problems of statistical mechanics (see the "Fermi-Pasta-Ulam problem").

Proposition 10 *The discrete translation on* S^1 (example 2) *is ergodic iff* α *is irrational.*

Proof. If α is rational, trajectories are periodic and the system is obviously non ergodic (or also: if $\alpha = p/q$, then $f(x) = \cos 2\pi q x$ is a non trivial constant of motion). Assume now α is irrational; consider any $f \in L_2(S^1, \mu)$ and let

$$f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k \, e^{2\pi i k x} \ .$$

Quite clearly,

$$f(\Phi(x)) = \sum_{k \in \mathbb{Z}} (\hat{f}_k \, e^{2\pi i k \alpha}) \, e^{2\pi i k x} \ ,$$

and so $f(\Phi(x)) = f(x)$ a.e. if, for any $k \in \mathbb{Z}$, $\hat{f}_k(e^{2\pi i k \alpha} - 1) = 0$. This implies $\hat{f}_k = 0 \ \forall k \neq 0$, i.e. f is a.e. constant on S^1. □

Exercise 3 Prove that if f is continuous, or is the characteristic function of an interval, then equalities (10) and (11) hold everywhere and not only a.e. (this it true in fact for all Riemann integrable functions, see [2]).

Proposition 11 *The free motion on* $\mathbb{T}^n$ *(examples 1, 3) is ergodic iff* v *is nonresonant. In particular (Jacobi's theorem) if* v *is nonresonant, trajectories are dense on* $\mathbb{T}^n$*.*

Proof. The proof exploits the same idea used in Proposition 10; details are left as an exercise. □

k	2^k	k	2^k	k	2^k
0	1	15	32768	30	1073741824
1	2	16	65536	31	2147483648
2	4	17	131072	32	4294967296
3	8	18	262144	33	8589934592
4	16	19	524288	34	17179869184
5	32	20	1048576	35	34359738368
6	64	21	2097152	36	68719476736
7	128	22	4194304	37	137438953472
8	256	23	8388608	38	274877906944
9	512	24	16777216	39	549755813888
10	1024	25	33554432	40	1099511627776
11	2048	26	67108864	41	2199023255552
12	4096	27	134217728	42	4398046511104
13	8192	28	268435456	43	8796093022208
14	16384	29	536870912	44	17592186044416

The Table: *The first 45 powers of* 2.

EXERCISE 4 Find the condition on $\alpha \in \mathbb{R}^n$ such that $(\mathbb{T}^n, \mu, \Phi)$, where μ is the Lebesgue measure and $\Phi : \mathbb{T}^n \to \mathbb{T}^n$ is the discrete translation $\Phi(x) = x + \alpha \pmod 1$, is ergodic. [$k \cdot \alpha \notin \mathbb{Z}$, $\forall k \neq 0$.]

EXERCISE 5 Table 1 reports the first 45 powers of 2; none of the numbers begins with the digit 7 or 9. Study the asymptotic frequencies p_l, $1 \leq l \leq 9$, of the powers of 2 which begin with the digit l: do they exist? How are they ordered? Why do the digits 4 and 8 appear so regularly, every ten powers?

2.2 Mixing

The next relevant notion in ergodic theory is mixing. The underlying idea is that if one looks at sets of points, rather than at individual points, the dynamics is essentially irreversible; equivalently, any macroscopic initial information on the initial datum, is lost for large time. Formally, we can introduce two equivalent notions:

M1. For any pair of measurable sets $A, B \subset M$, it is

$$\lim_{t\to\infty} \quad \mu(\Phi^{-t}(A) \cap B) = \mu(A)\mu(B) \ . \tag{12}$$

M2. For any pair of functions $f, g \in L_2(M, \mu)$, it is

$$\lim_{t\to\infty} \langle (f \circ \Phi^t)\, g \rangle = \langle f \rangle \langle g \rangle \ , \tag{13}$$

that is

$$\lim_{t\to\infty} \int_M (f \circ \Phi^t)\, g \,\mathrm{d}\mu = \int_M f \,\mathrm{d}\mu \int_M g \,\mathrm{d}\mu \ . \tag{14}$$

Properties M1, M2 are immediately seen to be invariant by isomorphirm.

- According to M1, the set $\Phi^{-t}(A)$, though preserving its measure, gets asymptotically diluted uniformly in M, more or less as when we put a drop of ink in a glass of water and stirr. According to M2, correlations between any two functions (or physical observables) are lost with time. In general, the function $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{C}(t) = \langle (f \circ \Phi^t)\, g \rangle - \langle f \rangle \langle g \rangle \ ,$$

  is called the *correlation function* of f and g (*autocorrelation function* of f, if $g = f$). If $\mathcal{C}(t) \neq 0$, then the knowledge of g at time zero conditions the expectation of f at time t.

- M2 formalizes Gibbs' idea of convergence to equilibrium (take $\rho_0 = cg$, c providing normalization, and introduce the change of variables $x = \Phi^{-t}(x')$ at the left hand side of (14)).

PROPOSITION 12 *Properties* M1 *and* M2 *are equivalent.*

PROOF. M1 is a particular case of M2, when $f = \chi_A$ and $g = \chi_B$: indeed, it is $\chi_A \circ \Phi^t = \chi_{\Phi^{-t}(A)}$, $\langle (\chi_A \circ \Phi^t)\, \chi_B \rangle = \mu(\Phi^{-t}(A) \cap B)$, and so M2 turns into M1. To show that M1 implies M2, consider preliminarily the case in which f and g are simple functions, i.e. finite sums of characteristic functions of suitable measurable sets:

$$f = \textstyle\sum_i f_i \chi_{A_i} \ , \qquad g = \sum_j g_j \chi_{B_j} \ ;$$

for such functions, using $\chi_{A_i} \circ \Phi^t = \chi_{\Phi^{-t}(A_i)}$, it follows

$$\begin{aligned} \langle (f \circ \Phi^t) g \rangle \ &= \ \textstyle\sum_{ij} f_i g_j \langle \chi_{\Phi^{-t}(A_i)} \chi_{B_j} \rangle = \sum_{ij} f_i g_j \mu(\Phi^{-t}(A_i) \cap B_j) \\ &\to \ \textstyle\sum_{ij} f_i g_j \mu(A_i) \mu(B_j) = \sum_{ij} f_i g_j \langle \chi_{A_i} \rangle \langle \chi_{B_j} \rangle = \langle f \rangle \langle g \rangle \ . \end{aligned}$$

For generic functions $f, g \in L_2(M, \mu)$, one exploits the fact that for arbitrary ε they can be written $f = f_0 + f_1$, $g = g_0 + g_1$, with f_0, g_0 simple and $\|f_1\|, \|g_1\| < \varepsilon$. One then easily gets

$$|\langle (f \circ \Phi^t) g \rangle - \langle (f_0 \circ \Phi^t) g_0 \rangle| < C\,\varepsilon \ , \qquad |\langle f \rangle \langle g \rangle - \langle f_0 \rangle \langle g_0 \rangle| < C\,\varepsilon$$

with some $C > 0$, and the conclusion follows. □

DEFINITION 8 *The dynamical system* (M, μ, Φ) *is said to be* mixing*, if any of* M1, M2, *and thus both, are satisfied.*

PROPOSITION 13 *Mixing is stronger than ergodicity.*

PROOF. Assume (M, μ, Φ) is mixing, and let A be measurable and invariant. Using M1 with $B = A$, it follows

$$\mu(A) = \mu(\Phi^{-t}(A) \cap A) \to \mu(A)^2 \ .$$

So $\mu(A) = 0$ or 1, and according to E4, the system is ergodic. An example of an ergodic non mixing system is the free motion on $\mathbb{T}^n$ (Φ^{-t} is a rigid motion, mixing cannot occur). □

PROPOSITION 14 *The Arnold's cat* (example 5) *is mixing.*

Proof. The functions $u_k(x) = e^{2\pi i k\cdot x}$, $k \in \mathbb{Z}^2$, are an orthogonal basis for $L_2(M,\mu)$, namely

$$\langle u_k u_l \rangle = 0 \qquad \text{for} \quad k \neq -l \ .$$

From the definition of Φ, it follows

$$u_k \circ \Phi = u_{\tilde{A}k} \ , \qquad u_k \circ \Phi^t = u_{\tilde{A}^t k} \ ,$$

where $\tilde{A}$ denotes the transposed of A (we are not assuming here A is symmetric). Now, for $k \neq 0$, the trajectory of k under the action of $\tilde{A}^t$, i.e. $\{\tilde{A}^t k,\ t \in \mathbb{N}\}$, is open, and with any norm $|\,.\,|$ in $\mathbb{Z}^2$ it is[3]

$$|\tilde{A}^t k| \to \infty \quad \text{for } t \to \infty \ , \qquad \forall k \in \mathbb{Z}^2 \ , \ k \neq 0 \ ;$$

this follows because the contracting direction of $\widetilde{A}$ has irrational slope, and cannot contain any integer $k \neq 0$.

Mixing easily follows, making reference to M2. Consider preliminarily the case in which f, g are Fourier polynomials, say

$$f(x) \ = \sum_{k\in\mathbb{Z}^2,|k|\le K} \hat{f}_k e^{2\pi i k\cdot x} \ , \qquad g(x) \ = \sum_{k\in\mathbb{Z}^2,|k|\le K} \hat{g}_k e^{2\pi i k\cdot x} \ ,$$

for some $K > 0$. For such functions it is

$$\langle (f \circ \Phi^t)\, g \rangle \ = \sum_{\substack{k,l\in\mathbb{Z}^2\\ |k|,|l|\le K}} \hat{f}_k\, \hat{g}_l\, \langle u_{\widetilde{A}^t k} u_l \rangle \ ,$$

but for large enough t and $k \neq 0$, it is certainly $|\widetilde{A}^t k| > K \geq |l|$; the only nonvanishing term in the sum is then $k = l = 0$. This means

$$\langle (f \circ \Phi^t)\, g \rangle = f_0\, g_0 = \langle f \rangle \langle g \rangle \ .$$

The results extends to generic $f, g \in L_2(M,\mu)$, by approximating them, in L_2 norm, with Fourier polynomials.

□

- The idea of the proof is the existence of an orthogonal basis $U = \{u_k,\ k \in \mathbb{Z}\}$ in $L_2(M,\mu)$, such that (i) U is invariant under the dynamics induced by Φ (i.e., composition $u_k \circ \Phi$), (ii) all functions but u_0 have an open non–recurrent behavior in U. The same idea will be used in the next proposition.

Proposition 15 *Bernoulli shifts are mixing.*

Proof. We shall profit of notion M1 of mixing. Consider any cylinder $\mathcal{C}^{l_{-K},\dots,l_K}_{-K,\dots,K}$, with base between $-K$ and K and any specification; it is clearly

$$\Phi^{-t}(\mathcal{C}^{l_{-K},\dots,l_K}_{-K,\dots,K}) = \mathcal{C}^{l_{-K},\dots,l_K}_{-K+t,\dots,K+t} \ ,$$

[3]In fact, the norm $|\tilde{A}^t k|$ grows exponentially for large t; profiting of this, one can understand more, namely that correlations in this example decay exponentially in t.

i.e. cylinders go into cylinders, the base going to infinity for $t \to \infty$. (Cylinders here play the role of the basis functions u_k in the previous proof.) Let now

$$A = \bigcup_{(l_{-K},\dots,l_K)\in L} \mathcal{C}^{l_{-K},\dots,l_K}_{-K,\dots,K} \ , \qquad B = \bigcup_{(l'_{-K},\dots,l'_K)\in L'} \mathcal{C}^{l'_{-K},\dots,l'_K}_{-K,\dots,K} \ ,$$

L and L' being any sets of indices in I^{2K+1}. Quite clearly, it is

$$\mu(\Phi^{-t}(A) \cap B) = \mu(A)\mu(B) \qquad \text{for} \qquad t > 2k+1 \ ;$$

indeed the base of cylinders forming A, after translation, get disjoint from the base of cylinders forming B, so the measure of the intersection factorizes. (Sets A and B play the same role as Fourier polynomials in the previous proof.)

The extension to any pair of measurable sets is straightforward: since the cylinders generate the σ–algebra of measurable sets, then for any $\varepsilon > 0$ there exist A_0, B_0 of the form above, with sufficiently large $K(\varepsilon)$, such that

$$\mu(A - A_0) < \varepsilon, \qquad \mu(B - B_0) < \varepsilon \ ,$$

the symbol "$-$" denoting symmetric difference. One gets immediately

$$\begin{aligned} |\mu(\Phi^{-t}(A) \cap B) - \mu(\Phi^{-t}(A_0) \cap B_0)| &< (\text{const})\,\varepsilon \\ |\mu(A)\,\mu(B) - \mu(A_0)\,\mu(B_0)| &< (\text{const})\,\varepsilon \ , \end{aligned}$$

and so for $t > 2K(\varepsilon) + 1$ it is

$$\mu(\Phi^{-t}(A) \cap B) - \mu(A)\mu(B) < (\text{const})\,\varepsilon \ ;$$

M1 is thus satisfied. $\square$

Exercise 6 Prove that the rubber–band map (exercise 6) is mixing.

Exercise 7 Show that (M, μ, Φ^2) is mixing, if and only if (M, μ, Φ) is mixing; show that if (M, μ, Φ^2) is ergodic, then (M, μ, Φ) is ergodic, but not conversely.

Exercise 8 Show that in the Bernoulli shift $B_{p,1-p}$, for almost all strings the frequency of the symbol "0" is p.

Exercise 9 For Arnol'd cat, find a continuous function f such that $\bar{f}(x) \neq \langle f \rangle$ in a dense set. [Hint: take any function with nonvanishing average, which however vanishes in the origin, for example $f(x_1, x_2) = \sin^2 x_1$; look at $\bar{f}(x)$ for x in the contracting eigenspace E_2 of the origin, reported to the torus via mod 1.]

Besides elementary examples as the above ones, not so many systems are known to be ergodic, or mixing. Among them, there are some *billiards.* Billiards, as dynamical systems, have been first studied by Birkhoff, in his celebrated book *Dynamical Systems.* Quickly: a billiard is a region M of the plain, or of the torus, with piecewise smooth border, on which a point freely moves, bounching on the border according to the laws of elastic reflection. The measure $\mathrm{d}\mu = C\mathrm{d}x\mathrm{d}y\mathrm{d}\vartheta$ is easily seen to be preserved. The ergodic properties entirely depend on the shape of the border, and the question is highly nontrivial. The first billiard for which ergodicity and mixing have been proved (Sinai, 1962) is the *Sinai billiard*: a torus $\mathbb{T}^2$ with a circular obstacle inside. An equivalent problem is that of two discs on a torus, bounching elastically. The proof has been extended to $n \leq 5$ discs; a proof of ergodicity for generic n has been longly searched, but obstacles have been found (a model of n bounching discs, or spheres in three dimensions, is a beautiful model of a gas in classical mechanics).

2.3 Comparing measures

Let M be a compact metric space and Φ a dynamics on it; the Krylov–Bogoliobov theorem (Section 1.1) ensures the existence of at least one invariant measure. In special cases the invariant measure is unique (for example, the rotation $\dot{x} = 1$ on S^1), but in general it is not; an example is the set of the Bernoulli shifts $B_{p,1-p}$, $0 < p < 1$. Whenever there are more than one invariant measures, they are necessarity infinite: quite clearly, if μ' and μ'' are invariant measures, the convex combination

$$\mu = c\,\mu' + (1-c)\,\mu'' \ , \qquad 0 \leq c \leq 1 \ ,$$

is also an invariant measure. This also shows that the set of invariant measures is convex.

The ergodic properties of the dynamical system (M, μ, Φ) obviously depend on the choice of μ. A good nontrivial question, for given M and Φ, is characterizing the set of the invariant measures, and among them the set of the ergodic measures, i.e. the measures such that (M, μ, Φ) is ergodic. It is fairly possible that (M, μ, Φ) and (M, ν, Φ), with $\mu \neq \nu$, are both ergodic; this is precisely the case of $B_{p,1-p}$ and $B_{q,1-q}$ if $q \neq p$.

Proposition 16 *For given M and Φ,*

i) If both μ and ν are ergodic, $\mu \neq \nu$, there exist disjoint subsets M_μ, M_ν such that

$$\mu(M_\mu) = 1 \ , \qquad \nu(M_\nu) = 1 \ , \qquad \mu(M_\nu) = 0 \ , \qquad \nu(M_\mu) = 0 \tag{15}$$

(each measure is supported by a subset which has zero measure for the other).

ii) If μ is ergodic and $\nu \neq \mu$ is invariant, then ν is singular with respect to μ (there exists a measurable A, such that $\mu(A) = 0$, $\nu(A) \neq 0$).

iii) If μ and ν are invariant, $\mu \neq \nu$, any strictly convex linear combination

$$\lambda = c\mu + (1-c)\nu \ , \qquad 0 < c < 1 \ , \tag{16}$$

is invariant but not ergodic. Conversely, if a measure λ is not ergodic, there exist μ and ν invariant, such that (16) holds.

Proof. Point (i): since $\mu \neq \nu$, there exists A measurable such that $\mu(A) \neq \nu(A)$. Let

$$M_\mu = \{x \in M : \tau_A(x) = \mu(A)\} \ , \qquad M_\nu = \{x \in M : \tau_A(x) = \nu(A)\} \ .$$

The two sets are invariant (indeed, $\tau_A(\Phi^t(x)) = \tau_A(x)$) and obviusly disjoint. For the assumed ergodicity of μ and ν, (15) holds.

Point (ii): we can assume ν is not ergodic, otherwise (ii) is already acheived ($\nu(M_\nu) > 0$, $\mu(M_\nu) = 0$). But then, according to E4, there exists A invariant such that both $\nu(A)$ and $\nu(M \setminus A)$

are positive. On the other hand, since μ is assumed to be ergodic and A is invariant, either A or $M \setminus A$ have zero μ–measure.

Point (iii): if both μ and ν are ergodic, then the set M_μ as constructed in point (i) is invariant and $\lambda(M_\mu) = c \neq 0, 1$. If instead (for example) μ is not ergodic, let A be invariant, $0 < \mu(A) < 1$; no matter which $\nu(A)$ is, it is $\lambda(A) \neq 0, 1$. Conversely, let λ be not ergodic, and let A be invariant, $\lambda(A) \neq 0, 1$; denote by μ and ν, respectively, the restrictions of λ to A and $M \setminus A$, defined by

$$\mu(B) = \frac{\lambda(B \cap A)}{\lambda(A)} \ , \qquad \nu(B) = \frac{\lambda(B \cap (M \setminus A))}{\lambda(M \setminus A)}$$

for any measurable B. Both μ and ν are invariant and (16) holds, with $c = \lambda(A)$. □

- The situation (i), which might appear paradoxical, occurs for the Bernoulli shifts. Take for simplicity shifts with only two symbols, and denote by μ_p the measure of $B_{p,1-p}$. Let Σ_p be the subset of Σ, including strings σ for which the frequency of the symbol "0" is defined and is p. For any p (exercise 8) it is
$$\mu_p(\Sigma_p) = 1 \ .$$
Now for $p \neq q$ the sets Σ_p, Σ_q are by definition disjoint, and so
$$\mu_p(\Sigma_q) = 0 \quad \forall q \neq p \ .$$
We see that each μ_p is supported by a set Σ_p, which has zero measure for any other Bernoulli measure μ_q with $q \neq p$.

- According to (iii), ergodic measures cannot be expressed as convex combinations of other invariant measures: they stay at the border of the set of invariant measures. Instead, any non–ergodic measure is internal, namely can be expressed as convex combination of invariant measures. The question arises whether any invariant measure can be expressed as suitable convex combination of ergodic measures. The answere is positive, but infinitely many ergodic measures in general are needed (an infinite sum and/or an integral). Correspondingly, M is decomposed into invariant sets, such that the dynamics restricted to any of them is ergodic with respect to measure supported by that set. See, in textbooks, the *Ergodic decomposition.*

3 The Kolmogorov–Sinai entropy

In this section we shall introduce one of the most fundamental notions of ergodic theory, namely entropy. Entropy is a real number associated to any dynamical system (M, μ, φ), which turns out to be invariant by isomorphism. It has connections with information theory, and leads, in particular for smooth dynamical systems, to quite interesting developments, including an important connection (Pesin's theorem) between entropy and dynamical quantities like the Lyapunov characteristic exponents, see Section 4.

3.1 The Entropy of a partition

Let

$$\alpha = \{A_0, \ldots, A_{n-1}\}$$

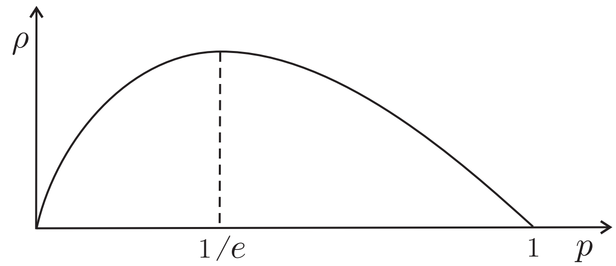


Figure 8: *The function ρ entering the definition of the entropy of a partition*

be a measurable partition of M in n *atoms* $A_0, \dots, A_{n-1}$; in ergodic theory partition means "up to zero measure sets", i.e. different atoms may intersect in a zero measure set and the union of all atoms may differ from M by a zero measure set. Two partitions $\alpha = \{A_0, \dots, A_{n-1}\}$ and $\alpha' = \{A'_0, \dots, A'_{n-1}\}$ are considered to be the same partition if, after suitable ordering, A_i and A'_i, $i = 1, \dots, n$, are identical up to zero measure sets; zero measure sets can always be removed. A partition can be usefully thought of as an experiment, with a finite set of possible exits represented by the symbols $0, \dots, n-1$.

In information theory, it is natural to assign to α the number

$$\eta(\alpha) = -\sum_{i=0}^{n-1} p_i \operatorname{Log} p_i \ , \qquad p_i = \mu(A_i) \ , \qquad 0 \operatorname{Log} 0 = 0 \ ,$$

where Log denotes[4] logarithm in base 2; η is called the *entropy* of the partition α. We shall use the notation

$$\eta = \sum_{i=0}^{n-1} \rho(p_i) \ , \qquad \rho(p) = -p \operatorname{Log} p \ ;$$

the graph of ρ is shown in figure 8. The idea underlying the definition is that of the "a priori uncernitainity" of the exit of the measurement, if the individual probabilities of the exits are $p_0, \dots, p_{n-1}$; such an uncernitainity then represents the expected information produced by the measurement. So, η is zero if the exit is certain (one of the p_i's is one, the other vanish), and is maximal if nothing is known, i.e. all exits have a priori the same probability; for $n = 2$, η increases, left to right, for

$$(p_0, p_1) \ = \ (0, 1) \ , \quad (0.01, 0.99) \ , \quad (\tfrac{1}{3}, \tfrac{2}{3}) \ , \quad (\tfrac{1}{2}, \tfrac{1}{2}) \ .$$

To state a first proposition, which will help us to understand the meaning of η, we need a couple of definitions.

Definition 9 *Two partitions*

$$\alpha = \{A_0, \dots, A_{n-1}\} \ , \qquad \beta = \{B_0, \dots, B_{m-1}\}$$

are said to be independent*, if for any pair of atoms A_i, B_j it is $\mu(A_i \cap B_j) = \mu(A_i)\mu(B_j)$.*

Examples of independent partitions are any partition of a rectangle in vertical and horizontal strips, or for Bernoulli shifts, the partitions $\alpha_k = \{\mathcal{C}_k^0, \dots, \mathcal{C}_k^{n-1}\}$ for different k.

[4]The base of the logarithm plays no role at all, as it simply provides the unit of measure of the entropy. The logarithm in base 2 allows to use the language of "bits", and is natural witin information theory. Nowadays most books in ergodic theory use the natural logarithm.

Definition 10 *The partition β is said to be* finer *than α, or to be a* refinement *of α, denoted $\beta \succeq \alpha$, if (up to zero measure sets) each of the atoms of β is entirely contained in one of the atoms of α: for any j there exists i such that $\mu(B_j \cap A_i) = \mu(B_j)$ (and consequently $\mu(B_j \cap A_{i'}) = 0$ for $i' \neq i$).*

Definition 11 *The composition $\alpha \vee \beta$ of two partitions*

$$\alpha = \{A_0, \ldots, A_{n-1}\} \ , \qquad \beta = \{B_0, \ldots, B_{m-1}\}$$

is the partition

$$\{A_i \cap B_j \neq \emptyset,\ 0 \leq i < n,\ 0 \leq j < m\} \ .$$

Proposition 17 *The function η has the following properties:*

i) *$\eta = 0$ iff one of the p_i is one, and consequently the remaining ones are zero.*

ii) *For given n, η is maximal if $p_0 = \ldots = p_{n-1} = 1/n$, and $\eta_{\max} = \operatorname{Log} n$.*

iii) *If α and β are independent, then*

$$\eta(\alpha \vee \beta) = \eta(\alpha) + \eta(\beta) \ .$$

iv) *If $\beta \succeq \alpha$, then $\eta(\beta) \geq \eta(\alpha)$.*

In the proof we shall use an elementary inequality valid for concave functions, known as Jensen's inequality:

Lemma 18 (Jensen's inequality). *If $f : \mathbb{R} \to \mathbb{R}$ is concave, then*

$$f(\textstyle\sum_{i=1}^n c_i x_i) \ \geq \ \sum_{i=1}^n c_i\, f(x_i) \qquad \text{for} \quad \sum_{i=1}^n c_i = 1 \ .$$

Proof of the Lemma. For $n = 2$ it is just the definition of concave function; for larger n it is an easy induction. □

Proof of the Proposition. Point i) depends only on the fact that ρ vanishes in $0, 1$ and is positive in between. Point ii) follows from the concavity of ρ, using Jensen's inequality with $c_i = 1/n$, $x_i = p_i$ (also recalling $\sum_i p_i = 1$). Point iii) uses the explicit form of ρ, with the logarithm inside: if $p_0, \ldots, p_{n-1}$ and $q_0, \ldots, q_{m-1}$ are respectively the measures of the atoms of α and β, for independent partitions it is

$$\eta(\alpha \vee \beta) = -\textstyle\sum_{ij} p_i q_j (\operatorname{Log} p_i + \operatorname{Log} q_j) = \eta(\alpha) \sum_j q_j + \eta(\beta) \sum_i p_i = \eta(\alpha) + \eta(\beta) \ .$$

Finally, for point iv), it is enough to show that if an atom A of α of measure p is the union of two atoms B and B' of β, of measure respectively cp and $(1-c)p$, then $\rho(cp) + \rho((1-c)p) \geq \rho(p)$. This follows from concavity of ρ together with $\rho(0) = 0$: indeed,

$$\rho(cp) = \rho(cp + (1-c)0) \geq c\rho(p) + (1-c)\rho(0) = c\rho(p) \ ;$$

similarly $\rho((1-c)p) \geq (1-c)\rho(p)$, and the conclusion follows. □

- It can be proved[5] that properties i) — iv) completely characterize η, up to a multiplicative constant (corresponding to the arbitrary choice of the base of the logarithm).

[5] See for example A.I. Kinchin, *Mathematical foundations of information theory,* Dover, New York (1957).

- If there are only two possible exits of a measurement, with equal a priori probability, then $\eta = 1$: this is the elementary information corresponding to one binary digit, or *bit*. Four, eight,... equally probable exits, give two, three... bits of information. If there are three exits A, B, C with a priori probability $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, then with probability $\frac{1}{2}$ one gains one bit, with probability $\frac{1}{4}$ one gains two bits, and again with probability $\frac{1}{4}$ one gains two bits. Correspondingly it is $\eta = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2 = \frac{3}{2}$ bits (the uniform probability would give $\eta = \operatorname{Log} 3 \simeq 1.58$).

3.2 The entropy of a system

A. Entropy of (M, Φ) *relative to a partition* α. Let (M, μ, Φ) be a discrete dynamical system and $\alpha = \{A_0, \ldots, A_{n-1}\}$ be any measurable partition of M. Let

$$\beta_t = \alpha \vee \Phi^{-1}(\alpha) \vee \cdots \vee \Phi^{-t+1}(\alpha) \ ,$$

where of course

$$\Phi^{-1}(\alpha) = \{\Phi^{-1}(A_0), \ldots, \Phi^{-1}(A_{n-1})\} \ .$$

Definition 12 *We shall call* entropy of (M, Φ) relative to the partition α *the limit, which will be proved to exist,*

$$h(\Phi, \alpha) = \lim_{t \to \infty} \frac{1}{t} \eta(\beta_t) \ . \tag{17}$$

- Each atom of β can be interpreted is as one of the possible exits of a sequence of t measurements of α, at times $0, 1, \ldots, t-1$; indeed, if x belongs to the atom

  $$\bigcap_{s=0}^{t-1} \Phi^{-s}(A_{l_s})$$

  of β, for some $l_0, l_1, \ldots, l_{t-1}$, then

  $$x \in A_{l_0} \ , \qquad \Phi(x) \in A_{l_1} \ , \quad \ldots \ , \quad \Phi^{t-1}(x) \in A_{l_{t-1}} \ ,$$

  and $l_0, l_1, \ldots, l_{t-1}$ is the sequence of results. The entropy $h(\Phi, \alpha)$ appears then as the expected information per measurement, in an infinite sequence of measurements.

- A positive value of $h(\Phi, \alpha)$ indicates that β_t fragments, in a sense, with a rate "on the average, exponential" (remember the Log inside η). Indeed, let $B(x, t)$ be the atom of β_t that contains x; one easily checks that

  $$h(\Phi, \alpha) = \lim_{t \to \infty} -\frac{1}{t} \int_M \operatorname{Log} \mu(B(x, t))\, \mathrm{d}\mu \ . \tag{18}$$

  In this (weak) sense, it appears that the measure of atoms contracts, on the average, exponentially (this does not mean that all of them effectively do). A stronger result, in which we shall not enter, is provided by the *Shannon–McMillan–Brian theorem.*

Proposition 19 *The limit (17) exists.*

To prove the proposition, we need a definition and a couple of lemmas.

DEFINITION 13 *The quantity*

$$\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}$$

is called measure of A conditioned to B.

Observe that for atoms A and B of two independent partitions, it is $\mu(A|B) = \mu(A)$.

LEMMA 20 *For any pair of measurable partitions*

$$\alpha = \{A_0, \dots, A_{n-1}\} \ , \qquad \beta = \{B_0, \dots, B_{m-1}\} \ ,$$

it is

$$\eta(\alpha \vee \beta) \leq \eta(\alpha) + \eta(\beta) \ , \tag{19}$$

the equality holding iff the partitions are independent.

PROOF. From the definition of η, one easily writes

$$\begin{aligned} \eta(\alpha \vee \beta) &= -\textstyle\sum_{ij}\mu(A_i \cap B_j)\mathrm{Log}\,(\mu(B_j)\mu(A_i|B_j)) \\ &= -\textstyle\sum_{ij}\mu(A_i \cap B_j)\mathrm{Log}\,\mu(B_j) - \sum_{ij}\mu(A_i \cap B_j)\mathrm{Log}\,\mu(A_i|B_j) \\ &= \eta(\beta) - \textstyle\sum_{ij}\mu(B_j)\mu(A_i|B_j)\mathrm{Log}\,\mu(A_i|B_j) \end{aligned}$$

(use is made of $\sum_i \mu(A_i \cap B_j) = \mu(B_j)$). The second term on the r.h.s. has the form

$$\textstyle\sum_i \left[\sum_j c_j\, \rho(p_{ij})\right] ,$$

with

$$c_j = \mu(B_j) \ , \qquad p_{ij} = \mu(A_i|B_j) \ , \qquad \rho(p) = -p\mathrm{Log}\,p \ ;$$

using Jensen's inequality for each index i, and then summing over i, one immediately sees that such a term does not exceed $\eta(\alpha)$, i.e. (19) is satisfied. On the other hand, due to the strict concavity of ρ, Jensen's inequality is strict, unless all p_{ij}, for any i, do not depend on j; this means $\mu(A_i|B_j)$ is independent of j, and this in turn implies that partitions are independent.

LEMMA 21 *If the sequence $h_1, h_2, \dots$ is bounded from below and satisfies the sub–additivity condition*

$$h_{t+s} \leq \frac{t}{t+s}\, h_t + \frac{s}{t+s}\, h_s \ ,$$

the limit $\lim_{t\to\infty} h_t$ exists and is equal to $h = \inf\{h_t,\ t \in \mathbb{N}\}$.

The lemma extends to sub–additive functions the well known property of monotone non increasing functions.[6]

PROOF. We need to show that for large t, the difference $h_t - h$ is arbitrarily small. From the definition of h, one has that for any $\varepsilon > 0$ there exists $s = s(\varepsilon)$ such that

$$h_s < h + \varepsilon \ ;$$

[6] Monotone non increasing functions are immediately seen to be sub–additive; an example of sub–additive non monotone function is $h_t = 0$ for even t, $h_t = t^{-1}$ for odd t.

from sub–additivity it follows immediately $h_{js} \leq h_s$, and so

$$h_{js} < h + \varepsilon \qquad \forall j > 0 \ .$$

Let now t stay between js and $(j+1)s$, say $t = js + k$ with $0 < k < s$. From sub–additivity one has

$$h_t \leq \frac{js}{js+k}\, h_{js} \ + \ \frac{k}{js+k}\, h_k < h + \varepsilon + \frac{1}{j}\, h_k \ ;$$

on the other hand, once more for additivity, it is $h_k < h_1$: as a consequence,

$$h_t < h + 2\varepsilon$$

for large enough $j \geq h_1/\varepsilon$, and so for large enough $t \geq T(\varepsilon) = h_1 s(\varepsilon)/\varepsilon$. □

The proof of the proposition now gets trivial.

Proof of Proposition 19. From the definition of β_t and from Lemma 20, it follows

$$\eta(\beta_{t+s}) \leq \eta(\beta_t) + \eta(\Phi^{-t}(\beta_s)) = \eta(\beta_t) + \eta(\beta_s) \ .$$

The sequence $h_t = \frac{1}{t}\eta(\beta_t)$ then satisfies the assumptions of Lemma 21, and so the limit (17) does exist. □

Exercise 10 Show that for the Bernoulli shift $B_{p_0,\ldots,p_{n-1}}$, denoting by α the partition in elementary cylinders $\alpha = \{\mathcal{C}_0^l \, , \ l \in I\}$, it is

$$h(\Phi, \alpha) = \eta(\alpha) = -\sum_{i=0}^{n-1} p_i \operatorname{Log} p_i \ .$$

Exercise 11 Let (M, μ, Φ) be any discrete dynamical system and α be any measurable partition of M; let $\beta_s = \alpha \vee \Phi^{-1}(\alpha) \vee \cdots \vee \Phi^{-s+1}(\alpha)$, for some $s > 1$. Show that $h(\Phi, \beta_s) = h(\Phi, \alpha)$. (In particular, for a Bernoulli shift $B_{p_0,\ldots,p_{n-1}}$, denoting $\beta_s = \{\mathcal{C}_{0,\ldots,s-1}^{l_0,\ldots,l_{s-1}} \, , \ l_0, \ldots, l_{s-1} \in I\}$, it is $h(\Phi, \beta_s) = h(\Phi, \alpha)$.) In the interpretation of measurements: by measuring β_s several times, we repeat some already done measures, and do not acheive new information.

B. The entropy of a system. We are now ready to define the entropy of a system.

Definition 14 *Let (M, μ, Φ) be any discrete dynamical system. The quantity*

$$h(\Phi) = \sup_{\alpha \text{ measurable}} h(\Phi, \alpha)$$

is called Kolmogorov–Sinai entropy *of the system.*

Exercise 12 Show that h is invariant by isomorphism

Exercise 13 Show that, for a discrete invertible system, $h(\Phi^{-1}) = h(\Phi)$. *Hint*: use $\eta(\Phi^{t-1}(\beta_t) = \eta(\beta_t)$.

Exercise 14 Show that for any s it is $h(\Phi^s) = |s| h(\Phi)$. *Hint*: study $h(\Phi^s, \alpha^{(s)})$, $\alpha^{(s)} = \alpha \vee \ldots \vee \Phi^{-s+1}(\alpha)$.

Let us now come to continuous dynamical systems, that till now we disregarded. Consider any continuous dynamical system (M, μ, Φ) and let (M, μ, Φ^s) denote the discrete system corresponding to the time–s map of the continuous flow. It can be shown that for any $s \in \mathbb{R}$, the entropy of (M, μ, Φ^s) is $h(\Phi^s) = |s| h(\Phi^1)$. This motivates the following definition:

Definition 15 *Let (M, μ, Φ) be a continuous dynamical system. Its Kolmogorov–Sinai entropy is defined as the entropy $h(\Phi^1)$ of the discretized system (M, μ, Φ^1).*

Definition 14 might appear obscure: taking the supremum among all *measurable* partitions — a quite huge set, far beyond the intuition — is apparently abstract and non constructive at all; moreover, since $\eta(\alpha)$ and thus $\eta(\beta_t)$ grow by taking finer and finer α, one could be afraid the supremum is infinite. Some relevant results, shortly sketched in the next papagraph, will help to make clear the situation. Here we only show, on the basis of an example, that taking a finer and finer α, does not obviously produce a growth of $h(\Phi, \alpha)$. The example is that of a Bernoulli shift. Let α be the partition in elementary cylinders based on zero,

$$\alpha = \{\mathcal{C}_0^l\,,\ l \in I\}\ ,$$

and denote

$$\alpha_k = \{\mathcal{C}_{-k,\dots,k}^{l_{-k},\dots,l_k}\ ,\ l_{-k}, \dots, l_k \in I\}$$

$(\alpha_0 = \alpha)$. Quite clearly, α_k gets finer for larger k, namely $\eta(\alpha_k) = (2k+1)\eta(\alpha)$; however,

$$\begin{aligned} \alpha_k \vee \cdots \vee \Phi^{-t+1}(\alpha_k) &= \Phi^k(\alpha) \vee \cdots \vee \Phi^{-k-t+1}(\alpha) \\ \eta(\alpha_k \vee \cdots \vee \Phi^{-t+1}(\alpha_k)) &= (2k+t)\eta(\alpha) \end{aligned}$$

and consequently $h(\Phi, \alpha_k) = h(\Phi, \alpha) = \eta(\alpha)$.

3.3 Some results on entropy

The set $\mathcal{P}$ of all measurable partitions is naturally endowed with a distance.

Definition 16 *Given any two measurable partitions $\alpha = \{A_0, \dots, A_{n-1}\}$, $\beta = \{B_0, \dots, B_{m-1}\}$, the quantity*

$$\eta(\alpha\,|\,\beta) = -\textstyle\sum_j \mu(B_j) \sum_i \mu(A_i|B_j) \operatorname{Log} \mu(A_i|B_j)$$

is called entropy of α relative to β*.*

An equivalent definition, as is immediately checked, is

$$\eta(\alpha\,|\,\beta) = \eta(\alpha \vee \beta) - \eta(\beta)\ . \tag{20}$$

By means of relative entropy, we can introduce a distance in $\mathcal{P}$, according to

$$\operatorname{dist}(\alpha, \beta) = \eta(\alpha\,|\,\beta) + \eta(\beta\,|\,\alpha)\ .$$

Lemma 22

i. The above defined quantity $\operatorname{dist}(\alpha, \beta)$ *is a distance in $\mathcal{P}$.*

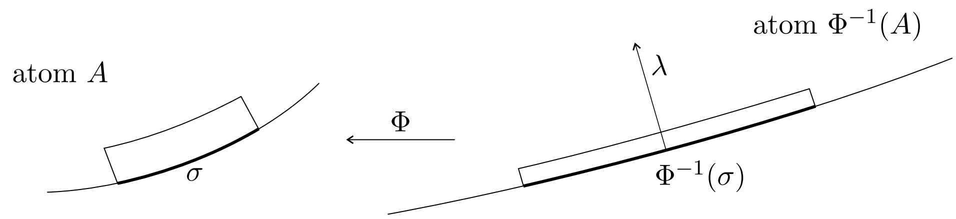


Figure 9: *An atom A and its pre-image $\Phi^{-1}(A)$. Due to volume conservation, a portion σ of the surface cannot grow more than the maximal dilatation λ of lengths in the foreward dynamics.*

ii. For any dynamical system (M, μ, Φ), $h(\Phi, \alpha)$ is continuous in α, more precisely it is

$$|h(\Phi, \alpha) - h(\Phi, \beta)| \leq \operatorname{dist}(\alpha, \beta) \ .$$

The proof is in Appendix C.

The above lemma is important: it allows to approximate partitions with "easier" ones, and the search of the supremum, in the definition 15 of entropy, can be restricted to any subset $\mathcal{P}_0$ dense in $\mathcal{P}$.

Definition 17 *Let M be a compact manifold. A partition of M is said to be smooth, if its atoms are polyhedra having as border a piecewise smooth submanifold of M of codimension* 1*.*

Lemma 23 *The subset $\mathcal{P}_0$ of smooth partitions of M is dense in the space $\mathcal{P}$ of measurable partitions of M.*

The proof is in Appendix C.

Proposition 24 (Kouchnirenko's theorem). *The entropy of smooth dynamical systems is finite.*

We provide only a sketch of the proof, assuming Φ is a diffeomorphism and M is endowed with a Riemannian metrics such that $\mathrm{d}\mu = \rho(x)\mathrm{d}V$ with smooth positive ρ.

Proof. Thanks to Lemma 23, it is enough to take into consideration smooth partitions of M. One proceeds as follows:

i) The metrics can by adapted to the measure, so as $\mathrm{d}\mu = \mathrm{d}V$ (multiply the metric tensor by $\rho(x)^{-1/n}$). Let $\mathcal{S}(A)$ be the surface of atom A in such a metrics, and $\mathcal{S}(\alpha)$ be the sum of the surfaces of all atoms of partition α. Then the surface of $\beta_t = \alpha \vee \cdots \vee \Phi^{-t+1}(\alpha)$ grows at most exponentially with t:

$$\mathcal{S}(\beta_t) \leq (\mathrm{const})\, \lambda^t\, \mathcal{S}(\alpha) \ , \qquad (21)$$

λ being the maximal dilatation coefficient of lengths produced by Φ (the operator norm of the derivative $D\Phi$).

Indeed, take a thin disc with base on the surface of A (Figure 9); since the volume is preserved by Φ, the base of the disc grows, by applying Φ^{-1}, only if the thickness of the disc decreases. But the maximal contraction produced by Φ^{-1} is the maximal dilatation produced by Φ. It follows

$$\mathcal{S}(\Phi^{-t}(A)) \leq \lambda^t\, \mathcal{S}(A) \ ,$$

and then (also using the obvious inequality $\mathcal{S}(\alpha \vee \alpha') \leq \mathcal{S}(\alpha) + \mathcal{S}(\alpha')$)

$$\mathcal{S}(\beta_t) \leq \frac{\lambda^t - 1}{\lambda - 1}\, \mathcal{S}(\alpha) \ ; \tag{22}$$

(21) follows. For $\lambda = 1$, i.e. rigid transformations in the adapted metrics, (22) is be replaced by

$$\mathcal{S}(\beta_t) \leq t\, \mathcal{S}(\alpha) \ . \tag{23}$$

ii) In turn, a bound on the surface of a partition β turns into a bound on $\eta(\beta)$, namely

$$\eta(\beta) \leq (\mathrm{const}) \ + \ m \,\mathrm{Log}\, \mathcal{S}(\beta) \ , \qquad m = \dim M \tag{24}$$

(fragmenting M in small atoms, so as to increase $\eta(\beta)$, has a minimal cost in surface).

Indeed, in any Riemannian manifold an isoperimetric inequality holds: there exist constants C and $\overline{V}$ such that, if $\mathrm{Vol}\,(B) \leq \overline{V}$, then

$$\mathrm{Vol}\,(B) \leq C \mathcal{S}(B)^{\frac{m}{m-1}} \ .$$

It is not restrictive to assume the atoms of β have volume smaller than $\overline{V}$ (if not, replace β with a convenient refinement); it is then, denoting $p_i = \mathrm{Vol}\,(B_i)$,

$$\begin{aligned}
\eta(\beta) \ &= \ -\textstyle\sum_i p_i \mathrm{Log}\, p_i = m \sum_i p_i \mathrm{Log}\, p_i^{-\frac{1}{m}} \\
&\leq \ m \,\mathrm{Log} \left(\textstyle\sum_i p_i p_i^{-\frac{1}{m}} \right) = m \,\mathrm{Log} \left(\textstyle\sum_i p_i^{\frac{m-1}{m}} \right) \\
&\leq \ m \,\mathrm{Log} \left(C^{\frac{m-1}{m}} \textstyle\sum_i \mathcal{S}(B_i) \right) = m \,\mathrm{Log}\, (\mathcal{S}(\beta)) + (m-1)\, \mathrm{Log}\, C \ .
\end{aligned}$$

Putting together (20) and (21) one finds $\eta(\beta_t) \leq \mathrm{const} + m\mathcal{S}(\alpha) + tm\mathrm{Log}\,\lambda$ and thus, for any smooth α, $h(\Phi, \alpha) \leq m\mathrm{Log}\,\lambda$. As a consequence,

$$h(\Phi) \leq m \mathrm{Log}\,\lambda \ . \tag{25}$$

$\square$

- Equation (25), concluding the proof, establishes a connection between entropy and dilatation of distances, here found as an upper bound to entropy. Far beyond this inequality, for smooth systems there exists an exact formula, called *Piesin's formula*, connecting entropy to the so-called *Lyapunov characteristic exponents*; see the next section. For the Arnol'd cat, (25) provides the inequality $h \leq 2\,\mathrm{Log}\,\lambda$, where $\lambda = \frac{1}{2}(3+\sqrt{5})$ is the maximal eigenvalue of matrix A; the exact value is $h = \mathrm{Log}\,\lambda$.

- For rigid transformations, the growth of $\mathcal{S}(\beta_t)$ is sub-exponential, see (23), and thus $h(\Phi) = 0$. In particular, *for the translations on the torus (examples 1–3) it is* $h(\Phi) = 0$.

We close the section by mentioning one further results, the so-calles Sinai's *Generator Theorem*; the proof is in Appendix D.

Definition 18 *A partition* $\alpha = (A_0, \dots, A_{n-1})$ *is said to be* generating *for a dynamical system* (M, μ, Φ)*, if the atoms of the partition, together with all their iterated* $\Phi^{-t}(A_i)$ *for any* t *and* i*, generate the* σ*–algebra of the measurable sets of* M*.*

This is the case, for example, of the partition $\alpha = \{\mathcal{C}_0^l\,,\ l \in I\}$ for a Bernoulli shift.

Proposition 25 *If α is a generating partition, then $h(\Phi) = h(\Phi, \alpha)$.*

So, in presence of a generating partition, the supremum in Definition 14 is in fact a maximum.

Corollary 26 *For a Bernoulli shift it is $h(\Phi) = -\sum_i p_i \mathrm{Log}\, p_i$.*

This means that, for example, $B_{\frac{1}{2},\frac{1}{2}}$ and $B_{\frac{1}{3},\frac{2}{3}}$ cannot be isomorphic. Instead, $B_{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}}$ and $B_{\frac{1}{2},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}}$ have the same entropy, so it is not excluded they are isomorphic. In fact, they are (Ornstein, 1970: Bernoulli shifts are isomorphic iff they have the same entropy.)

4 The Lyapunov Characteristic Exponents

The study of the Lyapunov Characteristic Exponents (LCE's) as indicators of stability of equilibrium points or periodic orbits goes back to the work of Lyapunov, in the early 20th century. Modern theory started in 1968, with an important paper by V. Oseledets (also spelled Oseledec), where the general existence of LCE's is proved, within ergodic theory, for almost all orbits of a dynamical system.[7] The role of LCE's in ergodic theory become definitely clear in 1975, when Ya. Pesin drew the exact connection between LCE's and entropy, in the so-called Pesin's formula.[8]

LCE's, in the very essence, are a way to introduce formally the notion of exponential divergence of nearby trajectories, with a sufficiently weak definition which allows to prove that such quantities do exist generically. Much beyond their interest in Ergodic Theory, LCE's become soon important in a variety of applications: celestial mechanics, statistical mechanics, turbolence, plasma physics, economy, ecology, biomedicine... indeed, on the one hand they are mathematically well defined quantities, on the other hand it became soon clear they can be computed numerically, and even be investigated experimentally. Nowadays, the common notion of "chaotic" system is based on the positivity of LCE's.

4.1 Exponential divergence of nearby trajectories

Although, since Oseledets paper, there exists an abstract notion of LCE's, we shall consider here only the smooth case, where the notion is more easily understood.

Consider a smooth dynamical system (M, Φ); the invariant measure will be introduced later, when needed. Let M be endowed with any Riemannian metrics, denote by T_xM the tangent space to M in $x \in M$ and let $\|\,.\,\|$ be the norm induced in T_xM by the metrics on M.

For $x \in M$, let $y(s)$, with $s \in (-\varepsilon, \varepsilon)$, be a curve on M, such that $y(0) = x$. The image of the curve at time t is the curve $y_t(s) = \Phi^t(y(s))$, and $y_t(0) = \Phi^t(x)$. If $\xi \in T_xM$ is tangent to $y(s)$ in x, then ξ is mapped linearly in $D\Phi^t_x\xi \in T_{\Phi^t(x)}$, tangent to y_t in $\Phi^t(x)$, where $D\Phi^t_x\xi$ denotes the by the so-called tangent application to Φ^t in x. The "dilatation coefficient" at point x, at time t, in the direction of ξ, is then

$$\gamma(t, x, \xi) \;=\; \lim_{s\to 0} \frac{\mathrm{dist}\,(y_t(s), x_t)}{\mathrm{dist}\,(y(s), x)} \;=\; \frac{\|D\Phi^t_x\,\xi\|}{\|\xi\|}\ .$$

[7] The result is contained in the Ph.D. thesis of Oseledets, at the Moscow State University; the advisor was Ya. Sinai.

[8] The result is contained in the Ph.D. thesis of Pesin, at the Moscow State University, Ya. Sinai and D.V. Anosov being the advisors.

It is spontaneous to say there is exponential divergence of trajectories close to $\Phi^t(x)$, with initial datum shifted in the direction of ξ, if for large t the coefficient $\gamma(t,x,\xi)$ grows exponentially in time, $\gamma \sim e^{\chi t}$ with positive χ; exponential contraction if $\chi < 0$. A convenient weak way to define χ is the following:[9]

DEFINITION 19 *Let (M,Φ) be a smooth dynamical system, let $x \in M$ and $\xi \in T_xM$. The quantity*

$$\chi(x,\xi) \;=\; \lim_{t\to\infty} \frac{1}{t} \log \frac{\|D\Phi_x^t\,\xi\|}{\|\xi\|} \;, \tag{26}$$

provided the limit exists, is called the Lyapunov characteristic exponent *of ξ in x.*

The denominator $\|\xi\|$ in (26) could be omitted. From the very definition it follows that equivalent metrics give the same value of $\chi(x,\xi)$.

4.2 The "filtration" for a periodic motion

Let us consider the simple case of a periodic motion of period τ, so that $\Phi^\tau(x) = x$; the tangent application $D\Phi_x^\tau$ is then an ordinary linear operator: $T_xM \to T_xM$, and it makes sense to discuss of its eigenvalues and eigenvectors. Assume for simplicity $D\Phi_x^\tau$ has $n = \dim M$ real eigenvalues $\lambda_1,\ldots,\lambda_n$ such that

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| \;; \tag{27}$$

let $e_1,\ldots,e_n$ be the corresponding eigenvectors. Then:

a) It is

$$\chi(x,e_i) = \tau^{-1}\log|\lambda_i|$$

(this is immediate if one takes the limit on the subsequence $t = k\tau$, $k\in\mathbb{N}$, but the general existence of the limit easily follows).

b) Consider a vector $\xi = \sum_{i\ge r} c_i e_i$, with $c_r \neq 0$. Then the dilatation of e_r dominates and

$$\chi(x,\xi) = \chi(x,e_r) \;.$$

Consequently, by varying ξ in T_xM, $\chi(x,\xi)$ assumes only $n = \dim M$ different values.

c) Denote by $[a,b,c,\ldots]$ the linear subspace of T_xM generated by a, b, $c,\ldots \in T_xM$, and let

$$\begin{aligned} L_1 &= [e_1,\ldots,e_n] \;=\; T_xM \\ L_2 &= [e_2,\ldots,e_n] \\ &\;\vdots \\ L_n &= [e_n] \;; \end{aligned}$$

it is clearly

$$T_xM = L_1 \supset L_2 \supset \ldots \supset L_n \tag{28}$$

and

$$\chi(x,\xi) = \chi(x,e_i) \qquad \text{for} \qquad \xi \in L_i \setminus L_{i+1} \;, \qquad i = 1,\ldots,n \;,$$

having denoted $L_{n+1} = \{0\}$.

[9] A stronger definition of exponential divergence would be: there exist $C,\chi > 0$ such that $\gamma(t,x,\xi) > Ce^{\chi t}$. Such a uniform exponential divergence characterizes a very special class of dynamical systems, called *Anosov systems*.

Let us introduce the following general definition:

DEFINITION 20 *Let E be any vector space of finite dimension n. A sequence of linear subspaces*

$$E = L_1 \supset \cdots \supset L_m \ , \qquad m \leq n \ ,$$

of strictly decreasing dimension, is called a filtration *of E.*

So, (28) is an example of filtration. It is easy to see that the special assumption of real eigenvalues satisfying (27) can be released: in any case for a periodic orbit, even in presence of multiple possibly complex eigenvalues $\lambda_1, \ldots, \lambda_m$, $m \leq n$, with multiplicities $\nu_1, \ldots, \nu_m$, a filtration

$$T_xM = L_1 \supset \ldots \supset L_m \ , \qquad m \leq n \ , \qquad \dim L_i \setminus L_{i+1} = \nu_i \ , \tag{29}$$

remains defined, such that

$$\chi(x, \xi) = \tau^{-1} \log |\lambda_i| \qquad \text{for} \qquad \xi \in L_i \setminus L_{i+1} \tag{30}$$

(it is enough to represent $D\Phi^t_x$ by a matrix in Jordan form).

4.3 Generalization to any motion; Oseledets' theorem

For a non periodic motion, the notion of eigenvalue and eigenvector of $D\Phi^t_x$ does not make sense anymore, since there is no way, in general, to identify T_xM and $T_{\Phi^t(x)}M$. Instead, from the very definition of $\chi(x, \xi)$, and even from the weaker temporary definition

$$\chi(x, \xi) = \limsup_{t \to \infty} \frac{1}{t} \, \log \| D\Phi^t_x \, \xi \| \ ,$$

which allows to postpone the question of the existence of the limit, a filtration of T_xM satisfying (29) and (30) remains defined. Precisely:

PROPOSITION 27 *Let $x \in M$.*

i) By varying ξ in T_xM, $\chi(x, \xi)$ assumes a finite number $m \leq n$ different values

$$\chi^*_1(x) > \chi^*_2(x) > \cdots > \chi^*_m(x) \ .$$

ii) Ther exists a filtration of T_x in m subspaces,

$$T_xM = L_1 \supset L_2 \supset \cdots \supset L_m \ ,$$

such that

$$\chi(x, \xi) = \chi(x, e_i) \qquad \textit{for} \qquad \xi \in L_i \setminus L_{i+1} \ , \qquad i = 1, \ldots, m \ ,$$

where $L_{m+1} = \{0\}$.

iii) Let $(e_1, \ldots, e_n)$ be a basis of T_x obtained by taking $\nu_i = \dim L_i - \dim L_{i+1}$ independent vectors in $L_i \setminus L_{i+1}$ (normal basis), and $(f_1, \ldots, f_n)$ any basis of T_xM. Then

$$\sum_{i=1}^{n} \chi(x, e_i) \, \leq \, \sum_{i=1}^{n} \chi(x, f_i) \ ,$$

the equality holding iff $(f_1, \ldots, f_n)$ is also a normal basis.

Definition 21 *The number $\nu_i = \dim L_i - \dim L_{i+1}$ is called the* multiplicity *of χ_i^*; the set*

$$\mathrm{Sp}\,(x) = \{\chi_1(x), \ldots, \chi_n(x)\}\ ,$$

obtained by repeating each χ_i^* as many times as its multiplicity, is called the spectrum *in x.*

Equivalently, one could define

$$\chi_i(x) = \chi(x, e_i)\ , \qquad i = 1, \ldots, n\ ,$$

$(e_1, \ldots, e_n)$ being any (suitably ordered) normal basis of T_xM.

Proof. Let $x \in M$. For any $c > 0$ and any ξ, $\xi' \in T_xM$ it is

$$\chi(x, c\xi) = \chi(x, \xi)\ , \qquad \chi(x, \xi + \xi') \le \max\left(\chi(x, \xi), \chi(x, \xi')\right)\ .$$

The former relation is trivial, the latter is easily deduced from the definition of lim sup (the inequality takes into account the possibility that the dominant divergence of ξ and ξ' cancel). This shows that for any ϑ, the set

$$L(\vartheta) = \{\xi \in T_xM : \chi(x, \xi) \le \vartheta\}$$

is a linear subspace of T_xM (let $\chi(x, 0) = -\infty$, so as to include in $L(\vartheta)$ the null vector). But $L(\vartheta') \subset L(\vartheta)$ for $\vartheta' < \vartheta$, and if there exists $\xi \in L(\vartheta)$ which realizes $\chi(x, \xi) = \vartheta$, then $\xi \notin L(\vartheta')$, i.e. the inclusion is proper and $\dim L(\vartheta') < \dim L(\vartheta)$ strictly. As a conclusion, $\chi(x, \xi)$ assumes at most $m \le n$ distinct values $\chi_1^* > \cdots > \chi_m^*$, and the subspaces $L_k = L(\chi_k^*)$, $1 \le k \le m$, provide the desired filtration.

The last point follows because the normal basis has as many vectors as possible in subspaces with high index and so small χ.

□

The trivial property $\chi(x, c\xi) = \chi(x, \xi)$, used in the proof, shows that χ depends only the one-dimensional linear space $E \in T_xM$ including ξ. It is then spontaneous to generalize to subspaces of any dimension.

Definition 22 *Let $E \subset T_xM$, $\dim E = p \le n$. The limit (if existing)*

$$\chi^{(p)}(x, E) = \lim_{t\to\infty} \frac{1}{t} \log \frac{\mathrm{Vol}^{\,p}(D\Phi_x^t(\xi_1), \ldots, D\Phi_x^t(\xi_p))}{\mathrm{Vol}^{\,p}(\xi_1, \ldots, \xi_p)}\ , \tag{31}$$

where $(\xi_1, \ldots, \xi_p)$ is any basis of E and $\mathrm{Vol}^{\,p}(\xi_1 \ldots \xi_p)$ is the p-dimensional volume of the parallelepiped generated by $\xi_1, \ldots, \xi_p$, is called Lyapunov Characteristic Exponent *of order p of the subspace E.*

The existence of the Lyapunv exponents of any order as exact limits is guaranted by a nontrivial theorem proved in 1968 by Oseledets within Ergodic Theory, that is with reference to a conserved measure μ and thus to a dynamical system (M, μ, Φ).[10]

Proposition 28 *Let (M, μ, Φ) be any dynamical system, M being a Riemannian manifold.*

[10]It should be stressed that (possibly piecewise) smoothness of M is essential to define LCE's, but no smoothness assumption is needed on μ. So, the theorem holds also for dissipative systems, having a possibly non smooth attractor. Such systems play an important role in the description of turbolence.

i) *For almost any $x \in M$, and any subspace $E \in T_xM$, $\dim E = p$, the limit (31) exists finite. In particular, the exact limit (26) exists a.e. for any tangent vector $\xi \in T_xM$.*

ii) *The spectrum $\mathrm{Sp}\,(x)$ is a summable function of x.*

iii) *For any subspace $E \in T_xM$ there exists a normal basis $(e_1, \ldots, e_p)$, such that*

$$\sum_{i=1}^{p} \chi(x, e_i) \le \sum_{i=1}^{p} \chi(x, f_i) \ ,$$

$(f_1, \ldots, f_p)$ being any basis of E. Moreover it is

$$\chi^{(p)}(x, E) = \sum_{i=1}^{p} \chi(x, e_i) \ .$$

We shall not prove the theorem.

4.4 Further properties of LCE's; Pesin's formula

With some attention, but without real difficulties, it is possible to work out some useful properties of LCE's, once their existence is known.

PROPOSITION 29

a) *LCE's are constants of motion: $\mathrm{Sp}\,(\Phi^t(x)) = \mathrm{Sp}\,(x)$; in an ergodic system $\mathrm{Sp}\,(x)$ is constant a.e.*

b) *For an invertible system, if the preserved measure is equivalent to the volume, then*

$$\chi^{(n)}(x, T_xM) = \sum_{i=1}^{n} \chi_i(x) = 0 \ .$$

c) *For a continuous system for which Φ is solution of a differential equation $\dot{x} = X(x)$ on M, if $\Phi^t(x)$ does not converge to an equilibrium point for $t \to \infty$, then*

$$\chi(x, X(x)) = 0 \ .$$

d) *For a Hamiltonian system (M, μ, Φ) with n degrees of freedom, M being a compact constant energy surface ($\dim M = 2n - 1$), the spectrum is symmetric:*

$$\mathrm{Sp}\,(x) = \{\chi_1(x), \ldots, \chi_{n-1}(x), \chi_n(x), -\chi_{n-1}(x), \ldots, -\chi_1(x)\} \ ;$$

according to point c), if $\Phi^t(x)$ does not converge to an equilibrium point, $\chi_n(x) = 0$.

For the corresponding Hamiltonian system for which M is the layer between two compact constant energy surfaces ($\dim M = 2n$),

$$\mathrm{Sp}\,(x) = \{\chi_1(x), \ldots, \chi_{n-1}(x), \chi_n(x), -\chi_n(x), -\chi_{n-1}(x), \ldots, -\chi_1(x)\} \tag{32}$$

(χ_n, in the middle, is repeated twice; generically, there are two zeros in the middle).

For a symplectic diffeomorphism $M \to M$, where M is a compact symplectic manifold of dimension $2n$, the spectrum is also symmetric and satisfies (32).

Points a–c) are easy. Point d) instead is not completely trivial and requires some work. The idea is to compare, for each t, $D\Phi^t_x : T_x M \to T_{\Phi^t(x)} M$ with its inverse and its adjoint, both $T_{\Phi^t(x)} M \to T_x M$.

We now consider the completely smooth case, that is we assume the measure μ is also smooth. Pesin's theorem then holds (1975):

PROPOSITION 30 *For a smooth dynamical system* (M, μ, Φ) *it is*

$$h(\Phi) = C \int_M \textstyle\sum_i^+ \chi_i(x) \, \mathrm{d}\mu \ , \qquad C = 1/\log 2 \ , \tag{33}$$

where $\sum^+$ *denotes the sum restricted to* $\chi_i(x)$ *positive.*

The theorem is definitely not elementary and we shall not prove it. There exist extensions to piecewise smooth systems (including billiards) and, with appropriate formulation, to certain nonsmooth systems with attractors. The constant C in (33) disappears, if natural logarithm is used in the definition of entropy in place of logarithm in base 2. Equation (33) is known as *Pesin's formula.*

APPENDICES

A The roots of Ergodic Theory in Boltzmann and Gibbs

As already remarked in the Foreword, the roots of Ergodic Theory go back to the work of the founding fathers of Statistical Mechanics, in particular Boltzmann and Gibbs, who understood that the key idea to connect thermodynamics and microscopic dynamics, in systems with many degrees of freedom, is a statistical treatment, in which the volume of the phase space acquires the meaning of probability.

A.1 Macroscopic vs. microscopic description

Thermodynamics is a macroscopic experimental science, in which every quantity entering the game is conceptually defined with reference to an ideal experimental procedure. Take for simplicity a gas in a box of volume V; the thermodynamical state, at equilibrium, is completely characterized by specifying the density $\varrho = n/V$, n being the amount of gas in moles, the pressure p and the temperature T; such variables however are not independent but related by an "equation of state", namely a relation (depending on the substance at hand) of the form $F(\varrho, p, T) = 0$. For ideal gases it is, as is well known,

$$pV = nRT \ , \qquad \text{i.e.} \qquad p/\varrho = RT \ ,$$

R being the so-called universal constant of gases. For real gases the equation of state is more complicated, but does exist; more complex systems (mixtures, coexistence of different phases...) require more variables to define the state, but conceptually the situation is not different. Non equilibrium situations can also be considered, in which ϱ, p and T are not uniform in the box and an equation of state $F(\varrho(x), p(x), T(x)) = 0$ is satisfied locally in any point x of the box.

Besides such variables, thermodynamics introduces further important quantities depending on the state of the system, among them the *internal energy* U and the *entropy* S, and describes what happens to such quantities during a thermodinamical transformation, namely a change of the thermodynamical state, the system possibly interacting with nearby systems (making work, absorbing heat). The statements include the fact that in any isolated system U is conserved (1^{st} principle), while S can only grow, and does grow in any irreversible (or spontaneous, or natural) process (2^{nd} principle). An easy example to have in mind is a gas initially confined in a corner of a box and let free to expand: it will expand, and so raise its entropy, while the opposite process is not natural and cannot take place. More generally, if ϱ, p and T are not initially uniform in an isolated system, they become, and correspondingly entropy increases.

On the other hand, one knows or believes[11] that thermodynamic systems have a microscopic internal structure, namely are composed of a huge number of very small subsystems, which individually obey the laws of mechanics. The highly relevant and deep question then arises, *whether it is*

[11] This was still a little conjectural throughout the 19^{th} century, before the determination of Avogadro's number and so of the mass of atoms and molecules. Avogadro's number is approximately 6×10^{23}; this is indeed the number of molecules in 2 grams of hydrogen or 18 grams of water.

possible to deduce the macroscopic behavior of a system from the microscopic mecanical laws governing its elementary constituents. A quite ambitious goal: reducing thermodynamics to mechanics. Boltzmann was certainly the person who more strongly felt, and even suffered, the necessity of such a reduction.

Apparently the purpose is hopless, while instead some paradoxes suggest that thermodynamics and mechanics are incompatible.

- *The Loschmidt paradox* (or reversibility paradox): mechanical processes are reversible, namely for each process the reversed one is possible as well, while instead, as remarked above, thermodynamical processes in general are irreversible and entropy definitely grows. How is it then possible the reduction?
- *The Zermelo paradox* (or recurrence paradox): conservative mechanical systems, like Hamiltonian systems, are recurrent, that is for most initial data, after a convenient time — possibly huge but finite — the system comes back near the initial state (Poincaré recurrence theorem, Section 1.1). Nothing similar, however, is expected in thermodynamics. How does recurrence disappear, passing from microcopic to macroscopic?

The question is indeed subtle, and a crucial role is played by the distinction, to be stressed, between a microscopic mechanical state (a point in a suitable phase space) and a macroscopic or thermodynamical state, defined with reference to experimental procedures, to be better understood in a theoretical frame. The ideas of Boltzmann and Gibbs can be regarded, in a sense, as attempts to give a meaning to the notion of macroscopic state, such that the paradoxes in principle solve and the reduction is al least conceptually possible.

It is of course impossible to enter here the not easy work and thougt of Boltzmann and Gibbs, and we shall (drastically) limit ourselves to a few comments, in which the ideas of these authors are oversimplified and also a little reinterpreted.

A.2 Boltzmann's equiprobability of microscopic states

Let us follow Boltzmann and consider a gas in a box composed of N identical weakly interacting molecules, each having l degrees of freedom, so that the complete system has $n = lN$ degrees of freedom; denote by[12] γ and Γ, respectively, the phase space of the single molecule and of the complete system, $\Gamma = \gamma^N$ (for a gas of point masses in a box B, it is $n = 3$, $\gamma = B \times \mathbb{R}^3$). Let

$$(p^{(i)}, q^{(i)}) = (p_1^{(i)}, \ldots, p_l^{(i)}, q_1^{(i)}, \ldots, q_l^{(i)}) \in \gamma$$

denote the canonical coordinates of the i-th molecule. The state of the whole system is then described by N (ordered) points in γ or equivalently by a single point

$$x = (p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in \Gamma \ ;$$

the evolution of the system appears then equivalently as a single movement in Γ, or an evolving cloud of N points in γ. The evolution is supposed to be governed by a Hamiltonian of the form

$$H(p, q) = \sum_{i=1}^{N} h(p^{(i)}, q^{(i)}) + V(q) \ ,$$

[12]The traditional notation for the phase of a single molecule is μ; we use here γ to avoid the conflict with the measure.

h being the individual Hamiltonian of a molecule, while V is some interaction potential; V is supposed to be small and not much relevant in the energy balance (almost free molecules).

Boltzmann's thaught then develops more or less as follows:

i. A macroscopic state of the system is identified with a distribution of the cloud of the N points in γ. In a rough but deep analysis, Boltzmann imagines γ is partitioned into very small cells of identical volume ω — so small that moving molecules inside their cell is not appreciable physically — and identifies a macroscopic state with the "occupation numbers" N_1, N_2, ... of the cells, $\sum_j N_j = N$. Each cell has an energy ε_i (that of any chosen point in it) and so, denoting by E the total energy, it is

$$E \simeq \textstyle\sum_j N_j \, \varepsilon_j \ .$$

The macroscopic state does not change by moving molecules inside a cell, or exchanging molecules among different cells. In place of the numbers N_j one could use the densities[13]

$$f_j = \frac{N_j}{N\omega} \ , \qquad \textstyle\sum_j f_j \omega \ = \ 1 \ .$$

ii. A macroscopic state (given occupation numbers) occupies a volume $W(N_1, N_2, \ldots)$ in Γ, actually in a layer $\Gamma_{E \pm \Delta E}$ around the constant energy surface Σ_E, with ΔE related to the size of the cells in γ. One immediately gets

$$W(N_1, N_2, \ldots) = \frac{N!}{N_1! \, N_2! \, \ldots} \, \omega^N \ , \tag{A.1}$$

where ω^N accounts for the displacements of the N molecules inside their cell, while the combinatorial coefficient counts the exchanges among different cells.

The volume W of a state (up to a normalization) *is given the strong meaning of the a priori probability that the state is realized.* This is often referred to as the "a priori equiprobability of the microscopic states (points) in Γ": it is not important where points are, only their overall volume is relevant.

A standard computations then shows that

iii. The maximum W^* of W, for fixed N and E, is found for[14]

$$N_j^* = C \ N \, e^{-\beta \varepsilon_j} \ , \qquad C^{-1} = \textstyle\sum_j e^{-\beta \varepsilon_j} \ ,$$

where β is a Lagrange multiplier determined by the energy per molecule E/N; equivalently for $f_j = f_j^*$, with

$$f_j^* = \frac{C}{\omega} \, e^{-\beta \varepsilon_j} \ . \tag{A.2}$$

[13] The underlying idea, difficult unfortunately to formalize in well defined limit procedure, is to make a finer and finer partition of γ in smaller and smaller cells, letting correspondingly N go to infinity. Boltzmann in fact did not care too much of the continuum limit. Also the microscopic dynamics in Γ is replaced, if useful, by a discretized dynamics: Γ itself is discretized (decomposed into cells), and a finite time step is introduced, more or less as we do today in computer simulations. The fundamental intuition is that the details of the dynamics should be irrelevant, provided a few essential features, like the conservation of the volume in phase space and the conservation of energy, are saved.

[14] Stirling approximation for the factorials, the N_j's being treated as real numbers; computing the constrained maximum of $\log W$ is straightforward.

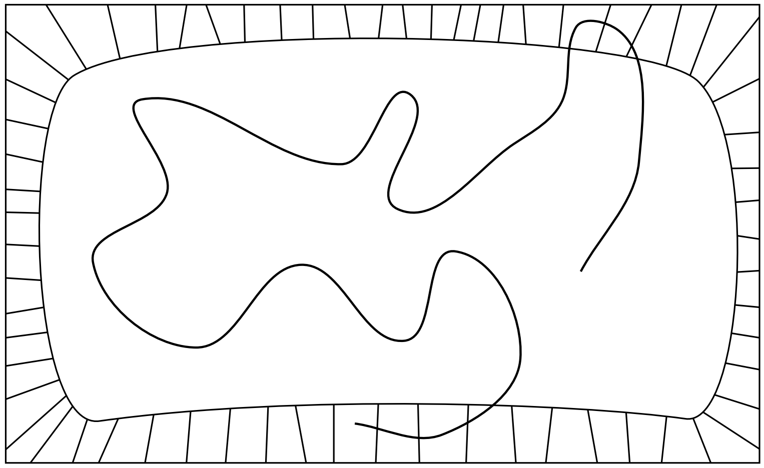


Figure 10: *The Maxwell-Boltzmann state dominates in* $\Gamma_{\pm E}$.

Such a state is commonly called *the Maxwell-Boltzmann state*; the exponential is called *the Boltzmann exponential factor*.

Moreover: small changes δN_j of the occupation numbers drastically reduce W, namely[15]

$$W(N_1^* + \delta N_1, N_2^* + \delta N_2, \ldots) \simeq W^* \prod_j e^{-\frac{1}{2}\frac{(\delta N_j)^2}{N_j^*}} \;;$$

for example, for minor changes of the occupation numbers, say

$$\delta N_j = \sqrt{N_j^*} \ll N_j \;,$$

the volume W reduces by a huge factor, actually the exponential of the number of the occupied cells; for δN_j proportional to N (assigned fluctuations δf_j), W reduces as much as the exponential of N.

Finally, and this is the most important point for us, the interpretation of the volume as probability is supported by a fundamental *dynamical assumption*, known as the *Boltzmann ergodic hypothesis*:[16]

iv. A typical trajectory, observed over a long time interval, wanders erratically in $\Gamma_{E \pm \Delta E}$, spending asymptotically in any subset of volume W a time proportional to W. In this sense, the chance a macroscopic state is dynamically realized, in a long time while, is proportional to W, in agreement with the idea of equiprobability of microscopic states.

The overall picture is summarized in figure 10, representing symbolically the layer $\Gamma_{E \pm \Delta E}$. The big set, of overwhelming volume, is the Maxwell-Boltzmann state, together with the states similar to it within $\delta N_j = (N_j^*)^{1/2}$. Small sets correspond instead to states well distinguishable from

[15]For this, it is enough to compute the second derivatives of $\log W$ at N_j^* (the first derivatives obviously vanish).

[16]The notion of ergodicity in Boltzmann has been often misinterpreted, even by higly qualified authors like P. Ehrenfest. For a critical discussion, see G. Gallavotti, *L'hypothèse ergodique et Boltzmann,* in *Dictionnaire Phylosophique des Presses Univ. Francaises,* p. 1081–1085 (Paris, 1988); *Ergodicity, Ensembles, Irreversibility and beyond,* Journ. Stat. Phys. **78**, 1571–1589 (1995).

it. Practically, no matter how the initial state is chosen, the trajectory is expected to reach soon the Maxwell-Boltzmann state, and spend there the overwhelming majority of time, up to very short stays, unlike but possible, again in small sets. The transient, i.e. the time needed to enter the Maxwell-Boltzmann state, represents, in this view, the approach to equilibrium, starting from an exceptional state far from it. Properly speaking, however, *the equilibrium is not the Maxwell-Boltzmann state, but the collection of all macroscopic states, each having probability to be realized proportional to its volume.* Equilibrium is the equiprobability of microscopic states in Γ, or equivalently the distribution (A.2) in γ. *Equilibrium includes fluctuations.*

- Among the fluctuations, there is the one that reports a system close to the initial datum in Γ: extremely unlike, and so not expected in any physically conceivable time interval, but dynamically not excluded; this solves, in principle, the recurrence paradox. Concerning the reversibility paradox, the answer is similar: reversed trajectories are included in the equilibrium state, but like any other trajectory, they spend the overwhelming majority of time in the Maxwell-Boltzmann state, and their transit in the extremely small set corresponding to the chosen non–uniform initial conditions, is too unlike to be observed.

Since Boltzmann, the equiprobability of microscopic states is the very basis of statistical mechanics. Boltzmann himself could deduce from it beautiful results. He was indeed able to provide a mechanical interpretation of the fundamental thermodynamical variables p, T, U, S, thus constructing a model of thermodynamics, in which the second principle of thermodynamics, in the form $\mathrm{d}U + p\mathrm{d}V \leq T\mathrm{d}S$, is satisfied. Let us recall that T is connected to the multiplier β via

$$\beta = \frac{1}{k_B T} \ ,$$

where $k_B > 0$ is the *Boltzmann constant.*[17] In elementary models, $\frac{1}{2}k_B T$ is also the average kinetic energy per degree of freedom. Concerning S, it turns out to be defined, microscopically, by $S = k_B \log W$; so, up to an inessential additive constant, it is

$$S = -k_B \, N \, \omega \sum_j f_j \log f_j \ .$$

A.3 Gibbs' statistical ensembles

Let us shortly describe an alternative view, commonly referred to as Gibbs' view. Gibbs' view is somehow more abstract, and probability plays a more primitive role in it. The space γ of single molecules does not play a role, and attention is addressed only to Γ.

The basic idea is that a macroscopic state is any given probability distribution ρ in Γ. While Boltzmann focuses the attention on a single trajectory, which assigns the probability to a set W of states through the fraction of time spent in it in a long time interval, Gibbs instead imagines to deal, at any time, with a family, or *ensemble* of evolving points, independent mental replicas of the same system, distributed in Γ with some density ρ_t evolving in time. One should think that in each experiment the way the system is prepared does not correspond to a single mechanical initial datum, rather to a spot of initial data, more precisely to a convenient initial distribution ρ_0 in Γ (the initially prepared macroscopic state). Each replica of the system then evolves independently

[17] It is $k_B = 1.380649 \times 10^{23}$ Joules/Kelvin degree: an exact value fixed conventionally, nowadays used in the "International System of units" as a primitive value; the Kelvin degree, not anymore primitive, remains consequently defined.

according to Hamilton equations, and correspondingly ρ_t evolves in time, as in a fluid of non interacting particles. From the conservation of the volume of the phase space one immediately deduces the evolution law[18]

$$\rho_t(x) = \rho_0(\Phi^{-t}(x)) \ , \qquad x \in \Gamma \tag{A.3}$$

(ρ_t is constant along trajectories).

It is now natural to search for equilibrium states, namely states such that $\rho_t(x)$ at any x is independent of t. An easy example is

$$\rho^*(x) = \begin{cases} c & \text{in } \Gamma_{E\pm\Delta E} \\ 0 & \text{elsewhere} \end{cases} \ ,$$

with c such as to ensure normalization; so, Boltzmann's equiprobability of microscopic states is, in Gibbs' view, an equilibrium state. It is obviously not unique: for the conservation of energy, any $\rho(x) = F(H(x))$, with any $F : \mathbb{R} \to \mathbb{R}$ up to the normalization, is an equilibrium state.

Let us then focus the attention on a single constant energy surface Σ_E, passing from volume densities to surface densities without changing the notation; a macroscopic state is then a surface probability distribution ρ_t, evolving in time, such that the probability of any subset $A \in \Sigma_E$ at time t is

$$\int_A \rho_t(x)\,\mathrm{d}\mu_L \ ,$$

μ_L being the Liouville measure introduced in Section 1.2, Example 4. The measure μ_L being preserved, ρ_t evolves in time according to (A.3). Quite clearly,

$$\rho^*(x) = 1 \qquad \forall\, x \in \Sigma_E$$

is an equilibrium state (this is indeed Boltzmann's equilibrium, rewritten in a more transparent mathematical notation). Two nontrivial question now are well posed:

i. Whether ρ^* is unique, or there is a multiplicity of equilibria;

ii. Whether, in addition, the system, prepared in a non equilibrium state ρ_0, does reach asymptotically equilibrium:

$$\lim_{t\to\infty} \rho_t \ = \ \rho^* \ .$$

Due to (A.3), the limit cannot be pointwise but should be understood in the weak sense

$$\int_{\Sigma_E} f(x)\rho_t(x)\,\mathrm{d}\mu_L \ \longrightarrow \ \int_{\Sigma_E} f(x)\,\mathrm{d}\mu_L \ ,$$

for some class of functions.

In Section 2 it is shown how the ideas of Boltzmann and Gibbs have been formalized in Ergodic Theory, through the basic notions of ergodicity and mixing.

[18] The system is in W at time t iff it is in $\Phi^{-t}(W)$ at $t = 0$, so $\int_W \rho_t(x)\mathrm{d}V = \int_{\Phi^{-t}(W)} \rho_0(x)\mathrm{d}V$. Introduce now at the r.h.s. the change of variable $x = \Phi^{-t}(x')$, which reports the inegration volume to W; for the conservation of volume in Hamiltonian dynamics the Jacobian is 1, and so, dropping the prime, $\int_W \rho_t(x)\mathrm{d}V = \int_W \rho_0(\Phi^{-t(x)})\mathrm{d}V$. The conclusion follows from the arbitrarity of W.

B Proof of the Birkhoff ergodic theorem

The proof of Birkhoff's ergodic theorem is based on a technical Lemma, known as *maximal ergodic theorem*:

Lemma 31 (Maximal ergodic Theorem) *For $f \in L_1(M,\mu)$, let*

$$F_t(x) = f(x) + \cdots + f(\Phi^{t-1}(x)) \ , \qquad A = \{x \in M : \sup_{t \geq 1} F_t \geq 0\} \ .$$

Then A is measurable and

$$\int_A f \, \mathrm{d}\mu \ \geq \ 0 \ .$$

Proof. Let

$$\mathcal{F}_t(x) = \max_{1 \leq s \leq t} F_s(x) \ , \qquad A_t = \{x \in M : \mathcal{F}_t \geq 0\} \ ,$$

so that

$$A_t \subset A_{t+1} \ , \qquad A \ = \bigcup_{0 \leq t < \infty} A_t \ .$$

It is then

$$\int_A f \, \mathrm{d}\mu \ = \ \lim_{t \to \infty} \int_{A_t} f \, \mathrm{d}\mu$$

and to prove the lemma it is enough to show that for any $t \geq 0$ it is

$$\int_{A_t} f \, \mathrm{d}\mu \ \geq \ 0 \ .$$

To this purpose, observe that

$$\mathcal{F}_t(x) = \max\left(f(x), \ldots, f(x) + \cdots + f(\Phi^{t-1}(x))\right) = f(x) + \max\left(0, \mathcal{F}_{t-1}(\Phi(x))\right) \ ,$$

i.e.

$$f(x) = \mathcal{F}_t(x) - \mathcal{F}^+_{t-1}(\Phi(x)) \ ,$$

having denoted $\mathcal{F}^+_t(x) = \max(0, \mathcal{F}_t(x))$. It follows

$$f(x) \geq \mathcal{F}_t(x) - \mathcal{F}^+_t(\Phi(x))$$

and consequently

$$\begin{aligned}
\int_{A_t} f \, \mathrm{d}\mu \ &\geq \ \int_{A_t} \mathcal{F}_t \, \mathrm{d}\mu - \int_{A_t} \mathcal{F}^+_t \circ \Phi \, \mathrm{d}\mu \\
&\geq \ \int_M \mathcal{F}^+_t \, \mathrm{d}\mu - \int_M \mathcal{F}^+_t \circ \Phi \, \mathrm{d}\mu \ = \ 0
\end{aligned}$$

(for the latter inequality: $\mathcal{F}^+_t = \mathcal{F}_t$ in $\mathcal{A}_t$, $\mathcal{F}^+_t = 0$ in the complement). □

We can now prove Birkhoff's theorem.

Proof. (a) *Existence of the limit a.e.* For $a, b \in \mathbb{R}$, $a < b$, let

$$E_{a,b} = \left\{ x \in M : \liminf_{t \to \infty} \frac{1}{t} F_t(x) < a < b < \limsup_{t \to \infty} \frac{1}{t} F_t(x) \right\} \ ;$$

$E_{a,b}$ is measurable (lim inf and lim sup of sequences of measurable functions are measurable) and invariant. The set for which the time average of f *does not* exist is then

$$E = \bigcup_{a,b} E_{a,b} \ ,$$

but since a denumerable union is sufficient (sets are conveniently nested), it is enough to show that any of the $E_{a,b}$ has zero measure. To this purpose, let us show that, thanks to the lemma, the "reversed" inequality

$$b\,\mu(E_{a,b}) \ \le \ \int_{E_{a,b}} f\,\mathrm{d}\mu \ \le \ a\,\mu(E_{a,b})$$

holds, which implies $\mu(E_{a,b}) = 0$. To prove, for example, the left inequality, apply the lemma to

$$g(x) = \begin{cases} f(x) - b & \text{per } x \in E_{a,b} \\ -1 & \text{per } x \notin E_{a,b} \end{cases} ;$$

it is easy to see that the set A appearing in the statement of the lemma, namely the set such that $G_t(x) = \sum_{s=0}^{t-1} g(\Phi^s(x))$ is non negative for at least one $t > 0$, is precisely $E_{a,b}$: indeed, if $x \in E_{a,b}$, then (definition of $E_{a,b}$) there exists $t > 0$ such that $F_t(x)/t > b$, and so $G(x) > 0$; conversely, if $x \notin E_{a,b}$, then $\Phi^t(x) \notin E_{a,b}$ for any $t > 0$ ($E_{a,b}$ is invariant), and correspondingly $G_t(x) < 0$. The lemma than says that

$$\int_{E_{a,b}} g\,\mathrm{d}\mu \ = \int_{E_{a,b}} f\,\mathrm{d}\mu \ - \ b\,\mu(E_{a,b}) \ \ge \ 0 \ ,$$

and the left inequality is satisfied. Similarly, using

$$g(x) = \begin{cases} a - f(x) & \text{per } x \in E_{a,b} \\ -1 & \text{per } x \notin E_{a,b} \end{cases} ,$$

the right inequality can be proved.

(b) *Proof that* $\bar f(\Phi^t(x)) = \bar f(x)$. This is a trivial consequence of the definition of $\bar f(x)$, whenever the limit exists.

(c) *Proof that* $\langle \bar f \rangle = \langle f \rangle$. First of all, let us observe that $\bar f \in L_1(M, \mu)$, as follows from

$$\int_M \Big| \frac{1}{t} \sum_{s=0}^{t-1} f(\Phi^s(x)) \Big| \,\mathrm{d}\mu(x) \le \frac{1}{t} \sum_{s=0}^{t-1} \int_M |f(\Phi^s(x))| \,\mathrm{d}\mu(x) = \int_M |f| \,\mathrm{d}\mu \ .$$

Let now

$$C_{a,b} = (\bar f)^{-1}[a,b) = \{x \in M : a \le \bar f(x) < b\} \ ;$$

it is then

$$a\,\mu(C_{a,b}) \ \le \int_{C_{a,b}} \bar f\,\mathrm{d}\mu \ \le \ b\,\mu(C_{a,b}) \ ,$$

while from the lemma, proceeding as above with a convenient g, one deduces

$$a\,\mu(C_{a,b}) \ \le \int_{C_{a,b}} f\,\mathrm{d}\mu \ \le \ b\,\mu(C_{a,b}) \ ;$$

it follows

$$\Big| \int_{C_{a,b}} \bar f \,\mathrm{d}\mu - \int_{C_{a,b}} f \,\mathrm{d}\mu \Big| \;\le\; (b-a)\,\mu(C_{a,b}) \ .$$

From this inequality, thanks to the arbitrarity of a abd b, it is not difficult to conclude that

$$\int_M \bar f \,\mathrm{d}\mu - \int_M f \,\mathrm{d}\mu = 0 \ .$$

Indeed, given $\varepsilon > 0$, divide $\mathbb{R}$ in intervals $[k\varepsilon, (k+1)\varepsilon)$, $k \in \mathbb{Z}$; it is clearly $\cup_k C_{k\varepsilon,(k+1)\varepsilon} = M$, and so

$$\Big| \int_M \bar f \,\mathrm{d}\mu - \int_M f \,\mathrm{d}\mu \Big| \le \sum_{k\in\mathbb{Z}} \Big| \int_{C_{k\varepsilon,(k+1)\varepsilon}} \bar f \,\mathrm{d}\mu - \int_{C_{k\varepsilon,(k+1)\varepsilon}} f \,\mathrm{d}\mu \Big| \le \varepsilon \sum_{k\in\mathbb{Z}} \mu(C_{k\varepsilon,(k+1)\varepsilon}) \;=\; \varepsilon \ ,$$

and this is enough.

(d) *Proof that in the invertible case $\bar f_-$ exists and coincides a.e. with $\bar f$.* The existence is obvious (just replace Φ with Φ^{-1}). Suppose now it is, for example, $\bar f > \bar f_-$ in a set of positive measure. Denoting

$$A := \{x \in M : \bar f - \bar f_- > 0\} \ ,$$

it is

$$\int_A (\bar f - \bar f_-) \,\mathrm{d}\mu > 0 \ .$$

But since (point b) both $\bar f$ and $\bar f_-$ are constants of motion, A is invariant: so, denoting $g(x) = \chi_A(x) f(x)$, it is $\bar g(x) = \bar f(x)$ for $x \in A$, $\bar g(x) = 0$ elsewere, and similarly for the backwards average. It follows

$$\langle \bar g \rangle - \langle \bar g_- \rangle = \int_A (\bar f - \bar f_-) \,\mathrm{d}\mu > 0 \ .$$

But this is a contradiction because, according to point (c), $\langle \bar g \rangle = \langle \bar g_- \rangle = \langle g \rangle$. This concludes point (d) and the proof of the theorem. □

C Proof of Lemmas 22 and 23

The proof of Lemma 22 is based on a few rather obvious properties of relative entropy, collected in the following lemma (the easy proof is left as an exercise):

Lemma 32

i. $\beta \succeq \alpha \quad \Longleftrightarrow \quad \eta(\alpha\,|\,\beta) = 0$.

ii. $\eta(\alpha\,|\,\beta) \le \eta(\alpha)$, *the equality holding iff partitions are independent.*

iii. $\beta \succeq \alpha \implies \eta(\beta\,|\,\gamma) \ge \eta(\alpha\,|\,\gamma)$ *and conversely* $\eta(\gamma\,|\,\beta) \le \eta(\gamma\,|\,\alpha)$*: as in the numerator-denominator game.*

iv. $\eta(\alpha \vee \beta\,|\,\gamma) \le \eta(\alpha\,|\,\gamma) + \eta(\beta\,|\,\gamma)$.

We now prove Lemma 22.

Proof. Concerning statement (i), quite clearly it is $\mathrm{dist}\,(\alpha,\beta) \geq 0$, $\mathrm{dist}\,(\alpha,\alpha) = 0$, while conversely, if $\mathrm{dist}\,(\alpha,\beta) = 0$, then both $\eta(\alpha\,|\,\beta)$ and $\eta(\beta\,|\,\alpha)$ vanish; it follows $\beta \succeq \alpha$ and $\alpha \succeq \beta$, that is $\alpha = \beta$.

The symmetry of $\mathrm{dist}\,(\alpha,\beta)$ is obvious.

Finally, the triangular inequality follows from

$$\begin{aligned} \eta(\alpha\,|\,\gamma) &= \eta(\alpha \vee \gamma) - \eta(\gamma) \\ &\leq \eta(\alpha \vee \beta \vee \gamma) - \eta(\beta \vee \gamma) + \eta(\beta \vee \gamma) - \eta(\gamma) \ = \ \eta(\alpha\,|\,\beta \vee \gamma) + \eta(\beta\,|\,\gamma) \\ &\leq \eta(\alpha\,|\,\beta) + \eta(\beta\,|\,\gamma) \ , \end{aligned}$$

and similarly $\eta(\gamma\,|\,\alpha) \leq \eta(\gamma\,|\,\beta) + \eta(\beta\,|\,\alpha)$; the conclusion is immediate.

Concerning statement (ii), it is enough to prove that for any $t > 0$ it is

$$|\eta(\alpha \vee \cdots \vee \Phi^{-t+1}(\alpha)) - \eta(\beta \vee \cdots \vee \Phi^{-t+1}(\beta))| \ \leq \ t\,\mathrm{dist}\,(\alpha,\beta) \ .$$

To this purpose, assume for example that for a given t it is $\eta(\alpha \vee \cdots \vee \Phi^{-t+1}(\alpha)) > \eta(\beta \vee \cdots \vee \Phi^{-t+1}(\beta))$; it follows

$$\begin{aligned} \eta(\alpha \vee \cdots \vee \Phi^{-t+1}(\alpha)) &- \eta(\beta \vee \cdots \vee \Phi^{-t+1}(\beta)) \\ &\leq \eta(\alpha \vee \cdots \vee \Phi^{-t+1}(\alpha) \vee \beta \vee \cdots \vee \Phi^{-t+1}(\beta)) - \eta(\beta \vee \cdots \vee \Phi^{-t+1}(\beta)) \\ &= \eta(\alpha \vee \cdots \vee \Phi^{-t+1}(\alpha)\,|\,\beta \vee \cdots \vee \Phi^{-t+1}(\beta)) \\ &\leq \textstyle\sum_s \eta(\Phi^{-s}(\alpha)\,|\,\beta \vee \cdots \vee \Phi^{-t+1}(\beta)) \\ &\leq \textstyle\sum_s \eta(\Phi^{-s}(\alpha)\,|\,\Phi^{-s}(\beta)) = \sum_s \eta(\alpha\,|\,\beta) = t\,\eta(\alpha\,|\,\beta) \ , \end{aligned}$$

and this is enough (use was made of point (iv) of Proposition 16, of equation (20), and of points (iv) and (iii) of the above lemma 32). □

Let us pass to the proof of Lemma 23.

Proof. Any measurable set A can be approximated externally by a finite union of balls, thus with a polyhedron $\tilde{A}$, such that $\mathrm{mes}\,(\tilde{A} \setminus A)$ is arbitrarily small. Given any measurable partition $\alpha = \{A_0, \ldots, A_{n-1}\}$, let $\tilde{A}_0, \ldots \tilde{A}_{n-1}$ be polyhedra such that $\tilde{A}_i \supset A_i$, $\mathrm{mes}\,(\tilde{A} \setminus A) < \varepsilon$; by posing recursively

$$B_0 = \tilde{A}_0 \ , \qquad B_i = \tilde{A}_i \setminus \bigcup_{0 \leq j < i} B_j \ , \quad i = 1, \ldots, n-1 \ ,$$

one clearly obtains a smooth partition $\beta = \{B_0, \ldots, B_{n-1}\}$, such that

$$\mu(A_i - B_i) < (\mathrm{const})\,\varepsilon \ , \qquad \mu(A_i \cap B_j) < (\mathrm{const})\,\varepsilon \quad \text{for } i \neq j \ .$$

It follows

$$|\mu(A_i|B_j) - \delta_{i,j}| \ < \ (\mathrm{const})\,\varepsilon \ , \qquad |\mu(B_i|A_j) - \delta_{i,j}| \ < \ (\mathrm{const})\,\varepsilon \ ,$$

and the conclusion is immediate. □

D Proof of the Generator Theorem

Definition 23 *Let M be endowed with a measure μ. We shall say the sequence of measurable partitions of M*

$$\gamma_0 \preceq \gamma_1 \preceq \cdots \preceq \gamma_k \preceq \cdots$$

converges to the partition in points $\widehat{\varepsilon}$, if for any $\varepsilon > 0$ and any measurable A there exist $k \geq 0$ and $C_1, \cdots, C_m \in \gamma_k$, such that

$$\mu\Big(A - \bigcup_{i=1}^{m} C_i\Big) < \varepsilon \ .$$

For example, if M is a square of side L endowed with the Lebesgue measure, and γ_k denotes the partition in small squares of side $2^{-k}L$, then $\gamma_k \to \widehat{\varepsilon}$ (any rectangle, and thus any measurable set, can be arbitrarily well approximated by a union of sufficiently small squares). From the very definition of generating partition, it follows that if α is generating for the dynamical system (M, μ, Φ), supposed for example to be invertible, then the sequence

$$\gamma_k = \Phi^k(\alpha) \vee \cdots \vee \alpha \vee \cdots \vee \Phi^{-k}(\alpha) \ , \qquad k \in \mathbb{N} \ ,$$

converges to $\widehat{\varepsilon}$.

Lemma 33 *Let $\mathcal{P}$ denote the set of all measurable partitions of M. If the sequence γ_k, $k \in \mathbb{N}$, converges to $\widehat{\varepsilon}$, then the set*

$$\mathcal{P}_0 = \{\beta \in \mathcal{P} \, ; \, \exists k \geq 0 : \beta \preceq \gamma_k\} \tag{D.1}$$

is dense in $\mathcal{P}$.

Proof. We show that for any $\alpha \in \mathcal{P}$ there exists $\beta \in \mathcal{P}_0$ such that $\mathrm{dist}\,(\alpha, \beta)$ is arbitrarily small. Since $\gamma_k \to \widehat{\varepsilon}$, for any $\varepsilon > 0$ and any $A_i \in \alpha$ there exist k and A'_i such that

$$A'_i = \text{suitable union of atoms of } \gamma_k \ , \qquad \mu(A_i - A'_i) < \varepsilon$$

(the same k can be taken for all i's). A partition β of M is then obtained by posing

$$B_0 = A'_0 \ , \qquad B_i = A'_i \setminus \bigcup_{0 \leq j < i} B_j \quad \text{for } 0 < i < n-1 \ , \qquad B_{n-1} = M \setminus \bigcup_{0 \leq j < n-1} B_j \ .$$

A little reflection shows that $\beta \preceq \gamma_k$ and so $\beta \in \mathcal{P}_0$. On the same time,

$$\mu(A_i - B_i) < (\text{const})\, \varepsilon \ , \qquad \mathrm{dist}\,(\alpha, \beta) < (\text{const})\, \varepsilon \ .$$

□

We can now prove proposition 25.

Proof. We show that if α is a generating partition, and β is any measurable partition, then

$$h(\Phi, \alpha) \geq h(\Phi, \beta) \ . \tag{D.2}$$

To this purpose, given α generating, let

$$\gamma_k = \Phi^k(\alpha) \vee \cdots \vee \alpha \vee \cdots \vee \Phi^{-k}(\alpha) \ ;$$

each partition is a refinement of the previous one and the sequence converges to $\widehat{\varepsilon}$, so we can apply the above lemma and say that $\mathcal{P}_0$, as defined in (D.1), is dense in $\mathcal{P}$. It is then enough to verify (D.2) for $\beta \in \mathcal{P}_0$. This is easy: indeed if $\beta \preceq \gamma_k$, then

$$\begin{aligned}
h(\Phi, \beta) \leq h(\Phi, \gamma_k) &= \lim_{t\to\infty} \frac{1}{t}\, \eta(\gamma_k \vee \cdots \vee \Phi^{-t+1}(\gamma_k)) \\
&= \lim_{t\to\infty} \frac{1}{t}\, \eta(\Phi^{k}(\alpha) \vee \cdots \vee \Phi^{-k-t+1}(\alpha)) \\
&= \lim_{t\to\infty} \frac{2k+t}{t}\, \frac{1}{2k+t}\, \eta(\alpha \vee \cdots \vee \Phi^{-2k-t+1}(\alpha)) \\
&= h(\Phi, \alpha)\ .
\end{aligned}$$

∎