PROBABILITY THEORY

3° LECTURE - 29/11/2018

Function of RV's

A. Real RV: let $X : \Omega \to \mathcal{X}_X \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$

Then $Y = g(X) : \Omega \to \mathcal{X}_Y \subseteq \mathbb{R}$ has density that depends on the density of X.

If $g^{-1}(y) := \{x \in \mathcal{X}_X \text{ s.t } g(x) = y\} \quad \forall y \in \mathcal{X}_Y$

* Discrete case:

$$P(Y=y) = \mathbb{P}(X \in g^{-1}(y)) = \sum_{z \in g^{-1}(y)} \mathbb{P}(X=z)$$

* Absolutely continuous case: If g is invertible and differentiable

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } g^{-1}(y) \in \mathcal{X}_X \\ 0 & \text{otherwise} \end{cases}$$

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* If g is not invertible but admits inverses on disjoint sets of $\mathcal{X}_X$ then we can proceed similarly as in the following example.

Ex.: Let $X \sim N(0,1)$ and compute the density and the average of Y^2.

- $Y = g(X)$ with $g(x) = x^2$. g has inverse on the disjoint sets $\mathbb{R}^-$ and $\mathbb{R}^+_0$, given by

$g_1^{-1}(y) = -\sqrt{y}$ and $g_1^{-1}(y) = \sqrt{y}$, with $|g_k^{-1}(y)| = \frac{1}{2\sqrt{y}}$, $k=1,2$ →

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \left|\frac{1}{2\sqrt{y}}\right| \mathbb{1}(y \geq 0) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \left|\frac{1}{2\sqrt{y}}\right| \mathbb{1}(y \geq 0)$$

$$= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} \mathbb{1}(y \geq 0) \qquad \left(\begin{array}{c}\text{density of} \\ \chi^2(1) \equiv \Gamma(\frac{1}{2}, \frac{1}{2})\end{array}\right)$$

Then $\mathbb{E}(Y) = \begin{cases} \mathbb{E}(X^2) = 1 \\ \int_0^\infty y\, f_Y(y)\, dy \end{cases}$

B: Multivariate R.V (or random vector)

Let $(X_1, \dots, X_n) : \Omega \longrightarrow \mathcal{X}_{\underline{X}} \subseteq \mathbb{R}^n$

(i.) Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\underline{x} \longrightarrow g(\underline{x}) = (g_1(\underline{x}), \dots, g_n(\underline{x}))$

and set $\underline{Y} = g(\underline{X}) : \Omega \longrightarrow \mathcal{X}_{\underline{Y}} \subseteq \mathbb{R}^n$ r. vector

Then we set $g^{-1}(\underline{y}) = \{\underline{x} \in \mathcal{X}_{\underline{X}} \text{ s.t. } g(\underline{x}) = \underline{y}\} \quad \forall\, \underline{y} \in \mathcal{X}_{\underline{Y}}$

* discrete case: $\mathbb{P}(\underline{Y} = \underline{y}) = \mathbb{P}(\underline{X} \in g^{-1}(\underline{y}))$

$$= \sum_{\underline{z} \in g^{-1}(\underline{y})} \mathbb{P}(\underline{X} = \underline{z})$$

* absolutely continuous case: If g is invertible and differentiable

Then $f_{\underline{Y}}(\underline{y}) = \begin{cases} f_{\underline{X}}(g^{-1}(\underline{y})) \cdot \left|\det J_{g^{-1}}(\underline{y})\right| & \text{if } g^{-1}(\underline{y}) \in \mathcal{X}_{\underline{X}} \\ 0 & \text{otherwise} \end{cases}$

where $J_{g^{-1}}(\underline{y})$ is the Jacobian of $g^{-1}(\underline{y}) = (h_1(\underline{y}), \dots, h_n(\underline{y}))$

$$J_{g^{-1}}(\underline{y}) = \left(\frac{\partial h_i}{\partial y_j}(\underline{y}) \right)_{i,j=1,\dots n}$$

Ex 1: Let (X,Y) a 2. vector with joint density $f_{(X,Y)}$.
Let $Z = X+Y$ and $W = (X-Y)$.
Compute the joint density $f_{(Z,W)}$ of (Z,W) as a function of $f_{(X,Y)}$.

Sol: $(Z,W) = g(X,Y)$ with $g(x,y) = (\underbrace{x+y}_{g_1}, \underbrace{x-y}_{g_2})$.

Compute the inverse of g:

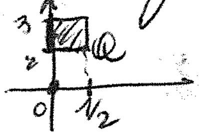
$$\begin{cases} z = x+y \\ w = x-y \end{cases} \Longrightarrow \begin{cases} x = \frac{z+w}{2} \\ y = \frac{z-w}{2} \end{cases} \quad \text{that is } g^{-1}(z,w) = \left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

. Determine the $J_{g^{-1}}(z,w)$ and its determinant.

$$J_{g^{-1}}(z,w) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad |\det J_{g^{-1}}| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

. Then $f_{Z,W}(z,w) = f_{X,Y}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) \cdot \frac{1}{2}$

Ex 2 Let (X,Y) a 2. vector with joint density



$$f_{XY}(x,y) = \frac{\alpha}{2\sqrt{x}} \mathbb{1}((x,y) \in Q)$$

with $Q = \{(x,y) \in \mathbb{R}^2 : x \in [0, \frac{1}{2}],\ y \in [2,3]\}$, $\alpha > 0$

Set $Z = \sqrt{X}$ and $W = Y - X$.

$\longrightarrow$

a. Compute α

b. Determine the marginal densities of X and Y

c. Determine the joint density of (Z,W) and the range of (Z,W).

Sol:

a. Use that $\iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\,dx\,dy = 1 \iff \alpha \cdot \iint_Q \frac{1}{2\sqrt{x}}\,dx\,dy = 1$

Compute $\iint_Q \frac{1}{2\sqrt{x}}\,dx\,dy = \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{x}}\,dx \int_2^3 dy = \frac{1}{2}\int_0^{1/2}\frac{1}{\sqrt{x}}\,dx = \left[\sqrt{x}\right]_0^{1/2} = \frac{1}{\sqrt{2}}$

$\Rightarrow \alpha \cdot \frac{1}{\sqrt{2}} = 1 \Rightarrow \alpha = \sqrt{2}$.

b. • $f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx = \mathbb{1}(y\in(2,3)) \cdot \int_0^{1/2} \frac{1}{\sqrt{2x}}\,dx$

$= \mathbb{1}[y\in(2,3)]\left[\frac{2\sqrt{x}}{\sqrt{2}}\right]_0^{1/2} = \mathbb{1}(y\in(2,3)) \Rightarrow Y \sim \mathcal{U}[2,3]$

• $f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy = \mathbb{1}(x\in(0,\tfrac{1}{2}))\int_2^3 \frac{1}{\sqrt{2x}}\,dy = \frac{1}{\sqrt{2x}}\mathbb{1}(x\in(0,\tfrac{1}{2}))$

c. $(Z,W) = (\sqrt{X}, Y-X) = g(X,Y)$.

* Compute g^{-1}:

$\begin{cases} Z = \sqrt{X} \\ W = Y-X \end{cases} \Rightarrow \begin{cases} X = Z^2 \\ Y = W+Z^2 \end{cases}$ that is $g^{-1}(z,w) = (z^2, w+z^2)$

* Compute the Jacobian of g^{-1} and its determinant

$J_{g^{-1}}(z,w) = \begin{pmatrix} 2z & 0 \\ 2z & 1 \end{pmatrix}, \quad |\det J_{g^{-1}}| = |2z - 0| = |2z|$

* By the transformation formula

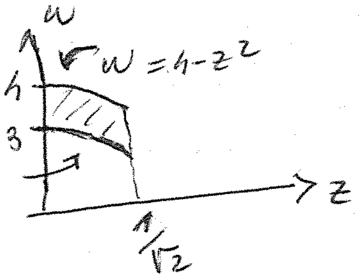
$$f_{Z,W}(z,w) = f_{XY}(z^2, w+z^2) \cdot |2z| \, \mathbb{1}((z^2, w+z^2) \in Q)$$

$$= \frac{|2z|}{\sqrt{2z^2}} \cdot \mathbb{1}((z^2, w+z^2) \in Q)$$

To identify the range of (Z,W) we have to compute $g(Q)$:

$$\{(z,w) \in \mathbb{R}^2 : (z^2, w+z^2) \in Q\} = \{(z,w) \in \mathbb{R}^2 : z^2 \in [0, \tfrac{1}{2}],\ w+z^2 \in [3,4]\}$$

$$\begin{cases} z^2 \in [0, 1/2] \\ w+z^2 \in [3,4] \end{cases} \quad \begin{cases} z \in [0, \frac{1}{\sqrt{2}}] \\ w \in [3-z^2, 4-z^2] \end{cases} \longrightarrow$$



<u>Ex3</u>: Let X_1, X_2 iid $\sim Exp(1)$

Let $Y_1 = \sqrt{X_1 \cdot X_2}$ and $Y_2 = \sqrt{\frac{X_1}{X_2}}$

Compute the joint density of (Y_1, Y_2)

- $f_{X_1 X_2}(x_1, x_2) = e^{-(x_1+x_2)} \mathbb{1}(x_1 > 0, x_2 > 0)$ (by hypotheses)

Moreover $Y = (Y_1, Y_2) = g(X_1, X_2) = \left(\sqrt{X_1 X_2}, \sqrt{\frac{X_1}{X_2}}\right)$

Then $\begin{cases} Y_1 = \sqrt{X_1 X_2} \\ Y_2 = \sqrt{\frac{X_1}{X_2}} \end{cases} \Longrightarrow \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = \frac{Y_1}{Y_2} \end{cases}$ that is $g^{-1}(y_1, y_2) = \left(y_1 y_2, \frac{y_1}{y_2}\right)$

and $J_{g^{-1}}(y_1, y_2) = \begin{pmatrix} y_2 & y_1 \\ \frac{1}{y_2} & -\frac{y_1}{y_2^2} \end{pmatrix}$ $\quad |\det J_{g^{-1}}| = \left| -\frac{y_1}{y_2} - \frac{y_1}{y_2} \right| = 2\frac{y_1}{y_2}$

$$\Longrightarrow f_{Y_1 Y_2}(y_1, y_2) = 2\frac{y_1}{y_2} e^{-y_1 y_2 - \frac{y_1}{y_2}} \mathbb{1}(y_1 > 0, y_2 > 0)$$

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Ex 4: Let $X \sim Exp(1)$ and $Y = X^{\frac{1}{\theta}}$, for $\theta \geq 0$

Compute the density of Y:

$Y = g(X)$ with $g(x) = x^{1/\theta} \Rightarrow g^{-1}(y) = y^{\theta}$ for $x, y \geq 0$

then $f_Y(y) = e^{-y^{\theta}} \cdot \theta y^{\theta - 1} \mathbb{1}_{y > 0}$ (Weibull distribution with index θ)

(ii) Fcts of lower dimension

Let $X: \Omega \to \mathbb{R}^n$ and $g: \mathbb{R}^n \to \mathbb{R}^d$, with $d < n$

In that case g is not invertible, but we can go back to (*) constructing $\tilde{g} = (g, g^\circ): \mathbb{R}^n \to \mathbb{R}^n$ s.t $\tilde{g}$ is invertible and differentiable (hopefully!).

The construction is not unique (clearly) and one has to guess what is more convenient. We show some examples

Ex 1: Let X, Y iid $\sim N(0,1)$ and set $Z = \frac{X}{Y}$

Then $Z = g(X,Y)$ where $g: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \to \frac{x}{y}$

Consider $\tilde{g}(x,y) = \left(\frac{x}{y}, y\right)$. If $(Z, W) = \tilde{g}(X,Y)$, then

$$\begin{cases} Z = \frac{X}{Y} \\ W = Y \end{cases} \Rightarrow \begin{cases} X = Z \cdot W \\ Y = W \end{cases} \qquad J_{\tilde{g}^{-1}}(z,w) = \begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix}$$

with $|\det J_{\tilde{g}^{-1}}| = |w| \Rightarrow f_{Z,W}(z,w) = \frac{1}{2\pi} e^{-\frac{1}{2}(z^2w^2 + w^2)} \cdot |w|$

then, to compute f_Z, we have to compute the marginal

$$f_Z(z) = \int_{\mathbb{R}} f_{Z,W}(z,w)\,dw = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |w|\, e^{-\frac{1}{2}w^2(z^2+1)}\,dw$$

$$= \frac{1}{\pi}\int_0^{+\infty} w\, e^{-\frac{1}{2}w^2(z^2+1)}\,dw = \frac{1}{\pi}\left[-\frac{e^{-\frac{1}{2}w^2(z^2+1)}}{(z^2+1)}\right]_0^{+\infty}$$

$$= \frac{1}{\pi(z^2+1)}, \quad z\in\mathbb{R} \qquad \text{(Cauchy distribution)}$$

Ex2: Let $X \sim N(0,1)$, $Y \sim \chi^2(n)$, independent.
Let $Z = \dfrac{X}{\sqrt{Y/n}}$ and compute the density f_Z.

Consider $(Z,W) = \left(\dfrac{X}{\sqrt{Y/n}}, Y\right) = \tilde{g}(X,Y)$

then $\tilde{g}^{-1}(z,w) = \left(z\cdot\sqrt{\frac{w}{n}}, w\right)$ and $J_{g^{-1}}(z,w) = \begin{pmatrix} \sqrt{\frac{w}{n}} & \frac{z}{2\sqrt{nw}} \\ 0 & 1 \end{pmatrix}$

with $\left|\det J_{g^{-1}}\right| = \sqrt{\frac{w}{n}}$

Then, recalling that $f_{X,Y}(x,y) = \dfrac{1}{\sqrt{2\pi}\, 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{1}{2}(x^2+y)} \mathbb{1}(y>0)$

we get

$$f_{Z,W}(z,w) = \frac{1}{\sqrt{2\pi n}\, 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} w^{\frac{n-1}{2}} e^{-\frac{1}{2}\left(\frac{z^2 w}{n}+w\right)} \mathbb{1}(w>0)$$

$$\Rightarrow f_Z(z) = \frac{1}{\sqrt{2\pi n}\, 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \underbrace{\int_0^{+\infty} w^{\frac{n-1}{2}} \cdot e^{-w\cdot\frac{1}{2}\left(\frac{z^2}{n}+1\right)}\,dw}_{\Gamma\left(\frac{n+1}{2},\, \frac{1}{2}\left(\frac{z^2}{n}+1\right)\right)}$$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2\pi n}\, 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \cdot \left(\frac{1}{2}\left(\frac{z^2}{n}+1\right)\right)^{-\frac{n+1}{2}} \qquad \text{t-student law } (n)$$

Ex3: Let $X_1, X_2, \dots, X_n$ iid $\sim Exp(1)$.

Let $Y_k = \dfrac{X_k}{X_1+\dots+X_n}$, $\forall k = 1, \dots, n-1$ and compute the joint density

then $(Y_1, \dots, Y_{n-1}) \in \underbrace{\{(y_1, \dots, y_{n-1}) \in (0,1)^{n-1} : \sum_{j=1}^{n-1} y_j \leq 1\}}_{= S_{n-1}} \subseteq \mathbb{R}^{n-1}$

Consider $\underline{Y} = \tilde{g}(\underline{X}) \in \mathbb{R}^n$ s.t. $\tilde{g}_k(\underline{X}) = \dfrac{X_k}{X_1+\dots+X_n}$ $\forall k \leq n-1$

and $\tilde{g}_n(\underline{X}) = X_1 + \dots + X_n$, so that

$$\tilde{g}^{-1}(\underline{Y}) = \left(Y_1 \cdot Y_n, \dots, Y_{n-1} \cdot Y_n, (1 - Y_1 - \dots - Y_{n-1}) Y_n\right)$$

with $J_{\tilde{g}^{-1}}(\underline{y}) = \begin{pmatrix} y_n & 0 & \cdots & \cdots & 0 & y_1 \\ 0 & y_n & 0 & \cdots & 0 & y_2 \\ \vdots & & \ddots & & & \vdots \\ 0 & & 0 & y_n & & y_{n-1} \\ -y_n & -y_n & \cdots & -y_n & & 1 - \sum_{i=1}^{n-1} y_i \end{pmatrix}$ and

with $\left| \det J_{\tilde{g}^{-1}}(\underline{y}) \right| = y_n^{n-1}$.

then $f_{\underline{Y}}(\underline{y}) = y_n^{n-1} e^{-y_n}$

with $\underline{y} = (y_1, \dots, y_n)$ s.t. $y_1, \dots, y_{n-1} \in S_{n-1}$ and $y_n \in \mathbb{R}^+$

At last, to compute the density of $(Y_1, \dots, Y_{n-1})$: $\forall (y_1, \dots, y_{n-1}) \in S_{n}$

$$\int_{\mathbb{R}} dy_n \, f_{\underline{Y}}(\underline{y}) = \int_0^{+\infty} y_n^{n-1} e^{-y_n} \, dy_n = \Gamma(n) = (n-1)! = \underline{f_{Y_1,\dots,Y_{n-1}}(y_1, \dots, y_{n-1})}$$

$$\Longrightarrow (Y_1, \dots, Y_{n-1}) \sim \mathcal{U}(S_{n-1})$$

Ex 4) let $X_1, \dots, X_n$ iid s.t $f_X(x) = \theta x^{\theta-1} \mathbb{1}_{(x\in(0,1))}$ $\boxed{y}$
~~for~~ $\theta > 0$. let $Y_1 = -\frac{1}{n} \sum_{j=1}^{n} \lg X_j$.

Then $Y_1 \in \mathbb{R}^+$. Consider $\underline{Y} = (Y_1, \dots, Y_n) = \tilde{g}(\underline{X})$
s.t. $\tilde{g}_1(\underline{X}) = \frac{-1}{n} \sum_{j=1}^{n} \lg X_j$ and $\tilde{g}_j(\underline{X}) = -\lg X_j$ $\forall j = 2 \dots n$

$$\Downarrow$$

$$\begin{cases} Y_1 = -\frac{1}{n} \sum \lg X_j \\ Y_2 = -\lg X_2 \\ \vdots \\ Y_n = -\lg X_n \end{cases} \Rightarrow \begin{cases} X_1 = e^{-nY_1 + (Y_2 + \dots + Y_n)} \\ X_2 = e^{-Y_2} \\ \vdots \\ X_n = e^{-Y_n} \end{cases} = g^{-1}(\underline{Y})$$

Then we have

$$J_{\tilde{g}^{-1}}(\underline{y}) = \begin{pmatrix} -n e^{-nY_1 + (-)} & e^{-nY_1 + (-)} & \dots & e^{-nY_1 + (-)} \\ 0 & -e^{-Y_2} & 0 & \dots \quad 0 \\ 0 \dots & 0 & \ddots & 0 \\ 0 & \dots & 0 & -e^{-Y_n} \end{pmatrix}$$

$$|\det J_{\tilde{g}^{-1}}(\underline{y})| = n e^{-ny_1}$$

Then $f_{\underline{Y}}(\underline{y}) = \theta^n e^{-n(\theta-1)y_1} \cdot n e^{-ny_1} = n\theta^n e^{-n\theta y_1}$

$\forall \underline{y}$ s.t. $y_j > 0$ $\forall j = 1 \dots n$ and $y_1 n > y_2 + \dots + y_n$.

The marginal over y_1 is then:

$$f_{Y_1}(y_1) = n\theta^n e^{n\theta y_1} \int_0^{y_1 n} \dots \int_0^{y_1 n - y_n - \dots - y_3} dy_n \dots dy_2 = n\theta^n e^{-n\theta y_1} \frac{(ny_1)^{n-1}}{(n-1)!} \quad \#$$

Important special case: Sum of random variables

- If X and Y are discrete r.v. with joint density $P(X=k, Y=j)$, $(k,j) \in \mathcal{X}_{(X,Y)}$ and $Z = X+Y$

$$\Longrightarrow \mathbb{P}(Z=n) = \sum_{k \in \mathcal{X}_X} \mathbb{P}(X=k, Y=n-k) = \sum_{j \in \mathcal{X}_Y} \mathbb{P}(X=n-j, Y=j)$$

- If X and Y are absol. cont. r.v. with joint density $f_{(X,Y)}(x,y)$, $(x,y) \in \mathbb{R}^2$, and $Z = X+Y$

$\Longrightarrow$ Z is abs. cont. r.v. with density

$$f_Z(z) = \int_{\mathbb{R}} f_{X,Y}(x, z-x)\,dx = \int_{\mathbb{R}} f_{X,Y}(z-y, y)\,dy$$

Applications:

1) $X \sim Bi(n,p)$, $Y \sim Bi(m,p)$ independent

$$\Longrightarrow X+Y \sim Bi(n+m, p)$$

2) $X \sim Poi(\lambda)$, $Y \sim Po(\mu)$ independent

$$\Longrightarrow X+Y \sim Po(\lambda+\mu)$$

3) $X \sim \Gamma(\alpha,\lambda)$, $Y \sim \Gamma(\beta,\lambda)$ independent

$$\Longrightarrow X+Y \sim \Gamma(\alpha+\beta, \lambda)$$

$\hookrightarrow X_1, \dots, X_n \sim Exp(\lambda)$ indep $\Longrightarrow \sum_{i=1}^{n} X_i \sim \Gamma(n,\lambda)$

4) $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ independent

$$\Longrightarrow X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

Ex 1: Let $X_1, \dots, X_n$ iid $\sim N(0,1)$

Compute the density of $Y = X_1^2 + \dots + X_n^2$ and the average of Y.

Sol: We've already proved that $X_i^2 \sim \chi^2(1) \equiv \Gamma(\frac{1}{2}, \frac{1}{2})$

If we set $Y_i = X_i^2 \quad \forall i = 1, \dots n$, then

$Y = Y_1 + \dots + Y_n$, with $Y_j \sim \Gamma(\frac{1}{2}, \frac{1}{2})$ and indep-

$\Rightarrow Y \sim \Gamma(\frac{n}{2}, \frac{1}{2})$ that is

$$f_Y(y) = \frac{(\frac{1}{2})^{n/2}}{\Gamma(\frac{n}{2})} \cdot y^{\frac{n}{2}-\frac{1}{2}} \cdot e^{-\frac{y}{2}} \cdot \mathbb{1}(y \geq 0)$$

To compute $\mathbb{E}(Y)$, either we recall that if $Z \sim \Gamma(\alpha, \lambda)$

$\Rightarrow \mathbb{E}(Z) = \frac{\alpha}{\lambda}$ (that in our case provide $\frac{n}{2} \cdot \frac{2}{1} = n$),

either we use linearity of the average:

$$\mathbb{E}(Y) = \sum_{i=1}^{n} \mathbb{E}(X_i^2) = \sum_{i=1}^{n} 1 = n \qquad \#$$

Ex 2 Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ independent.

Compute the density of $Z = X + Y$

Sol: $$f_Z(z) = \int_{\mathbb{R}} f_{(X,Y)}(x, z-x)\, dx = \int_{\mathbb{R}} f_X(x) f_Y(z-x)\, dx$$

$$= \int_{\mathbb{R}} \lambda e^{-\lambda x} \mathbb{1}(x \geq 0) \mu e^{-\mu(z-x)} \mathbb{1}(z-x \geq 0)\, dx =$$

$$= \lambda \cdot \mu e^{-\mu z} \int_0^z e^{-(\lambda-\mu)x} dx \; \mathbb{1}(z \geq 0)$$

$$= \frac{\lambda \cdot \mu}{\lambda - \mu} e^{-\mu z} \left(1 - e^{-(\lambda-\mu)z}\right) \mathbb{1}(z \geq 0)$$

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Ex3 Let $X \sim Bi(n,p)$, $Y \sim Bi(m,p)$ indep.

Compute $Cov(X,Y)$ and variance of $Z = X+Y$.

Sol. X, Y indep $\Longrightarrow Cov(X,Y) = 0$

- $Var(Z) = Cov(Z,Z) = Cov(X+Y, X+Y) =$

$$= Cov(X,X) + Cov(Y,Y) + 2 \underbrace{Cov(X,Y)}_{=0}$$

$$= Var(X) + Var(Y).$$

On the other hand $X = X_1 + \dots + X_n$ for iid $X_i \sim Be(p)$

and similarly $Y = Y_1 + \dots + Y_m$ for iid $Y_j \sim Be(p)$

$$\Longrightarrow Var(X) = \sum_{i=1}^n Var(X_i) = \sum_{i=1}^n p(1-p) = np(1-p)$$

and $Var(Y) = mp(1-p)$

$$\Longrightarrow Var(Z) = (n+m)p(1-p)$$

We could have also used that Z

and then $Z = Z_1 + \dots + Z_{n+m}$, with $Z_j \sim Be(p)$ iid

and $Var(Z) = \sum_{j=1}^{n+m} Var(Z_j) = (n+m)p(1-p)$ #