Random Graphs and Networks - 10° LECTURE

lunedì 14 febbraio 2022 14:43

A.] Configuration model

- vertex set $[n]$, $\forall n \in \mathbb{N}$
- We consider a degree sequence $\underline{d} = (d_i)_{i\in[n]}$ with d_i = degree of vertex i, s.t.
  - * $d_i \geq 1 \quad \forall i \in [n]$
  - * $\sum_{i\in[n]} d_i =: \ell_n$ <u>even</u> so that $E(G) = \frac{\ell_n}{2}$

  $\forall$ G compatible with $\underline{d}$

- We would like to construct a RG with uniform probability over $\mathcal{G}_n^{\underline{d}} = \{ G \in \mathcal{G}_n : d_G(i) = d_i \, , \, \forall i \in [n] \}$

① $|\mathcal{G}_n^{\underline{d}}| = ?$ ② Is $\mathcal{G}_n^{\underline{d}} \neq \phi$?

$\hookrightarrow$ One can add assumptions on $\underline{d} = (d_i)_{i\in\mathbb{N}}$ so that $\underline{d}$ is <u>graphical</u> $\iff \mathcal{G}_n^{\underline{d}} \neq \phi$

One can address problem ① by considering multigraphs over $[n]$:

$$\mathcal{M}_n^{\underline{d}} = \{ G \text{ multigraph over } [n] : d_G(i) = d_i \ \forall i \in [n] \}$$

The measure over $\mathcal{M}_n^{\underline{d}}$ is constructed as follows:

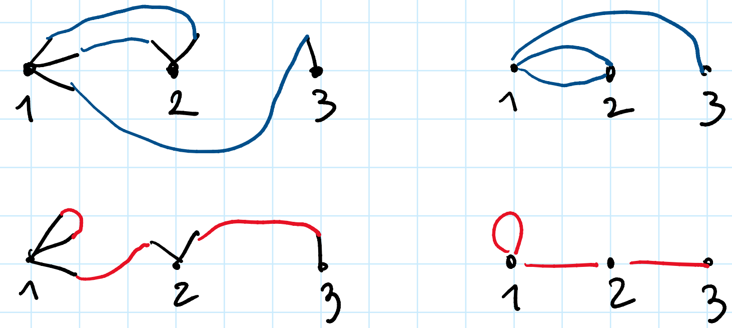
- assign to each vertex i, d_i half-edges
- take a <u>uniform</u> <u>matching</u> of the half-edges:

• take a uniform matching of the half-edges:

1. Couple two half-edges to form an edge, iteratively (no matter of the order), until no more half-edges unpaired

2. every matching is taken with equal probability

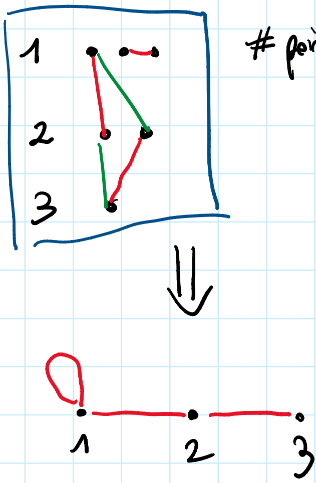
Example



This procedure can be visualized as:

- List all the vertices from 1 to n
- For each vertex i, draw d_i points
- Construct a matching on the total number of points $= l_n$
- Collapse the d_i points into vertex i

this called configuration



$\#$ points $= d_i$

Notice that # matching over $2M$ points is:

$$\frac{\binom{2M}{2}\binom{2M-2}{2}\cdots\binom{2}{2}}{M!} = (2M-1)!!$$

However $\phi : \mathrm{Conf}(\underline{d}) \longrightarrow \mathcal{M}_n^{\underline{d}}$ is surjective but not injective

hence the measure over $\mathcal{M}_n^{\underline{d}}$ that we obtain is not uniform. But it can be written explicitly.

uniform. But it can be written explicitly.

Denoting by $CM(n, \underline{d})$ the corresponding RG, we have

$$\mathbb{P}_{n,\underline{d}}(G) := \mathbb{P}(CM(n,\underline{d}) = G) = \frac{N(G)}{(\ell_n - 1)!!}, \quad \forall\, G \in \mathcal{M}_n^{\underline{d}}$$

where $N(G) = \#\{\text{configurations over } \ell_n \text{ mapped to } G\}$

$$= \prod_{i \in [n]} d_i! \cdot \frac{1}{2^{\#\text{loops in } G} \cdot \prod_{e \in E(G)} \text{mult}(e)!} \qquad \left(\text{if } G \text{ is simple } \prod d_i!\right)$$

$\boxed{\text{B.}}$ Random Graph on $\mathcal{G}_n^{\underline{d}}$

$\mathcal{G}_n^{\underline{d}} \cap$

Goal: define a measure over $\mathcal{G}_n^{\underline{d}}$. Starting $(\mathcal{M}_n^{\underline{d}}, \mathbb{P}_{n,\underline{d}})$:

1° METHOD: discard graphs that are not simple

2° METHOD: erase loops and multiple-edges

$\longrightarrow$ this changes the degree sequence: $\underline{d} \longrightarrow \underline{d}^{er}$.

1. Assumptions on degrees:

Let $\underline{d} = (d_i)_{i \in \mathbb{N}}$ s.t. $\forall\, n \in \mathbb{N}$: $\sum_{i \in [n]} d_i =: \ell_n$ even

Let $U \sim \text{Uniform}[n]$ and $D_n := d_U$. Then

a. $\exists$ r.v. D s.t. $D_n \xrightarrow{d} D$. Equivalently:

$$\mathbb{P}(D_n = k) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i = k\}} \xrightarrow[n \to \infty]{} \mathbb{P}(D = k) =: p_k$$

$\hookrightarrow$ If $(d_i)_{i \in \mathbb{N}}$ are i.i.d. r.v., this convergence (in probability or a.s.) is verified by LLN, with $D \overset{d}{=} d_i$.

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b. $E(D_n) \xrightarrow[n\to\infty]{} \mathbb{E}(D)$ ($< \infty$ for having sparse RG)

↳ for $(d_i)_{i\in\mathbb{N}}$ iid r.v., this should understood as a convergence in probability

c. $E(D_n^2) \xrightarrow[n\to\infty]{} \mathbb{E}(D^2)$

<u>Remark</u>: If we take r.v. $(d_i)_{i\in\mathbb{N}}$ iid s.t.

$$\mathbb{P}(d_i \geq k) \sim c \cdot k^{-(\tau-1)} \qquad (\text{with } \tau > 2)$$

$\Longrightarrow$ The resulting graph have typical vertex degree that satisfies the corresponding power law decay (SCALE-FREE PROPERTY)

Consider $CM(n, \underline{d})$, where $\underline{d} = (d_i)_{i\in\mathbb{N}}$ satisfies assumptions a., b., c.

<u>Theorem</u>:

$$\mathbb{P}(\underbrace{CM(n,\underline{d}) \text{ is simple}}_{\in \mathcal{G}_n^{\underline{d}}}) \xrightarrow[n\to\infty]{} e^{-\frac{\nu}{2} - \frac{\nu^2}{4}}$$

where
$$\nu := \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}(D)}.$$

<u>Proof (idea)</u>:

<u>1° STEP</u>:

$$S_n(G) := \#\{\text{loops in } G\} \qquad \forall\, G \in \mathcal{M}_n^{\underline{d}}$$
$$M_n(G) := \#\{\text{multiple edges in } G\}$$

$$\Longrightarrow (S_n, M_n) \xrightarrow{d} (S, M)$$

where S, M are independent: $S \sim \mathrm{Poi}\left(\frac{\nu}{2}\right)$, $M \sim \mathrm{Poi}\left(\frac{\nu^2}{4}\right)$

<u>2° STEP</u>: $\mathbb{P}(CM(n,\underline{d}) \text{ is simple}) = \mathbb{P}_{n,\underline{d}}(S_n = 0, M_n = 0)$

2° STEP: $\mathbb{P}(\,CM(m,\underline{d}) \text{ is simple}) = \mathbb{P}_{m,\underline{d}}\left(S_m = 0,\ M_m = 0\right)$

$$\approx \mathbb{P}(S=0)\,\mathbb{P}(M=0) = e^{-\frac{\nu}{2}-\frac{\nu^2}{4}}$$

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CONSEQUENCES:

1. Size of $\mathcal{G}_m^{\underline{d}}$: Under assumptions a., b., c.:

$$\left|\mathcal{G}_m^{\underline{d}}\right| = \frac{(\ell_m-1)!!}{\prod_{i\in[m]} d_i!}\cdot e^{-\frac{\nu}{2}-\frac{\nu^2}{4}}\,(1+o(1)) \quad \text{as } m\to\infty$$

Proof: Let $G \in \mathcal{G}_m^{\underline{d}} \subset \mathcal{M}_m^{\underline{d}} \Rightarrow \mathbb{P}_{m,\underline{d}}(G) = \dfrac{\prod_{i\in[m]} d_i!}{(\ell_m-1)!!}$

$$e^{-\frac{\nu}{2}-\frac{\nu^2}{4}}(1+o(1)) = \mathbb{P}(CM(m,\underline{d}) \text{ is simple}) = \sum_{G\in\mathcal{G}_m^{\underline{d}}} \mathbb{P}_{m,\underline{d}}(G) = \frac{\prod d_i!}{(\ell_m-1)!!}\left|\mathcal{G}_m^{\underline{d}}\right|$$

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Example: Take $d_i = r$ $\forall\, i\in[m]$ so that $\nu = \frac{r(r-1)}{r} = r-1$

$$\Rightarrow \left|\mathcal{G}_m^{r}\right| = \frac{(rm-1)!!}{(r!)^m}\cdot e^{-\frac{r-1}{2}-\frac{(r-1)^2}{4}}\,(1+o(1))$$

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2. Under assumptions a., b., c., it holds:

Proposition: Let $\mathcal{E}_m \subseteq \mathcal{G}_m^{\underline{d}} \subset \mathcal{M}_m^{\underline{d}}$ s.t. $\mathbb{P}(CM(m,\underline{d})\in\mathcal{E}_m) \xrightarrow[m\to\infty]{} 1$, Let $G(m,\underline{d})$ be a uniform RG chosen from $\mathcal{G}_m^{\underline{d}}$. Then

$$\mathbb{P}(G(m,\underline{d})\in\mathcal{E}_m) \xrightarrow[m\to\infty]{} 1.$$

Proof: By hyp. $\mathbb{P}(CM(m,\underline{d})\in\mathcal{E}_m^c) \xrightarrow[m\to\infty]{} 0$.

Then $\mathbb{P}(G(m,\underline{d})\in\mathcal{E}_m^c) = \mathbb{P}(CM(m,\underline{d})\in\mathcal{E}_m^c \mid \underbrace{CM(m,\underline{d}) \text{ is simple}}_{\in\,\mathcal{G}_m^{\underline{d}}})$

Then $\mathbb{P}\left(G(n,\underline{d}) \in \mathcal{E}_n^c\right) = \mathbb{P}\left(CM(n,\underline{d}) \in \mathcal{E}_n^c \,\middle|\, \overbrace{CM(n,\underline{d}) \text{ is simple}}^{\Leftarrow\, \mathcal{G}_n}\right)$

$$= \frac{\mathbb{P}\left(CM(n,\underline{d}) \in \mathcal{E}_n^c \cap \mathcal{G}_n^{\underline{d}}\right)}{\mathbb{P}\left(CM(n,\underline{d}) \text{ is simple}\right)} \leq \frac{\mathbb{P}\left(CM(n,\underline{d}) \in \mathcal{E}_n^c\right)}{e^{-\frac{\nu}{2}-\frac{\nu^2}{4}}} \xrightarrow[n\to\infty]{} 0$$

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$\boxed{C.}$ Unimodular Branching Process

It is a random tree, similar to Galton-Watson BP$(\underline{p})$

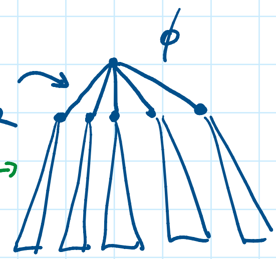
but with a different branching rule between root and other vertices:

- the root has offspring distribution $\underline{p} = (p_k)_{k\geq 0} \overset{d}{\sim} X$
- a vertex $\neq$ root has offspring distr. $\underline{p}^* = (p_k^*)_{k\geq 0}$ s.t.

$$\boxed{p_k^* := \frac{(k+1)\cdot p_{k+1}}{\sum_{k\geq 0} k\cdot p_k} = \frac{(k+1)\cdot p_{k+1}}{\mathbb{E}(X)}}$$

k with prob p_k

iid BP$(\underline{p}^*)$



Hence:

Let $\eta := \sum_{k\geq 0} p_k^* \cdot \eta^k =: G^*(\eta)$ (extinction probability of BP(p^*))

Then $$\boxed{\zeta = \sum_{k\geq 0} p_k \cdot (1-\eta^k)}.$$

- Notice that $\eta = 1 \iff \sum_{k\geq 0} k\cdot p_k^* \leq 1$

$$\iff \sum_{k\geq 0} \frac{k\cdot(k+1)\, p_{k}}{\mathbb{E}(X)} = \underbrace{\frac{\mathbb{E}\left(X(X-1)\right)}{\mathbb{E}(X)}}_{=:\,\nu} \leq 1$$

In conclusion, it holds:

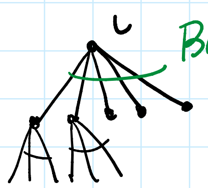
<u>Theorem</u>: if $\nu \leq 1 \Rightarrow$ Unim $BP(\underline{P})$ dies out a.s. $(\zeta = 0)$

if $\nu > 1 \Rightarrow$ Unim $BP(\underline{P})$ survives with positive prob. $(\zeta > 0)$

Informal explanation of unimodularity

- In ER random graph $G(n,p)$ if we look at the neighborhood of a vertex:

$p = \frac{\lambda}{n}$



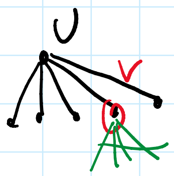
$Bin(n-1,p) \approx Poi(\lambda)$

$Poi(\lambda) \approx Bin(n,p) \overset{\text{if } k=o(n)}{\approx} Bin(n-k,p)$

- The same fact holds for inhomogeneous RG
- In $CM(n, \underline{d})$ the degree sequence is assigned:

  Let us pick a vertex $U \sim$ uniform $[n]$ as a root-vertex

  We know:



k vert. with prob p_k

- $\mathbb{P}(d_U = k) \approx p_k = \mathbb{P}(D = k)$
- Let V vertex at distance 1 from U

$$\mathbb{P}(\#\text{ offspring of } V = j) = \mathbb{P}(d_V = j+1, \text{ and } V \text{ is connected to } U)$$

$$\underset{\text{uniform matching}}{\approx} \frac{(j+1)\#\{v : d_v = j+1\}/n}{\sum_{\ell \geq 1} \ell \cdot \#\{v : d_v = \ell\}/n} \approx \frac{(j+1)\cdot p_{j+1}}{\sum_{\ell \geq 1} \ell \cdot p_\ell} = p_j^*$$