Random Graphs and Networks - 11° LECTURE

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Recall: • Configuration model denoted by $CM(n, \underline{d})$

$\underline{d} = (d_i)_{i \in [n]}$, with law $\mathbb{P}_{n,\underline{d}}$

↳ it is a multigraph

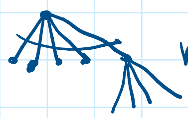
• Uniform model $G(n, \underline{d})$

with uniform law over $\mathcal{G}_n^{\underline{d}}$

↳ $P(G(n, \underline{d}) = G) = \dfrac{1}{|\mathcal{G}_n^{\underline{d}}|}$

• Unimodular BP with offspring distr $\underline{P} = (P_k)_{k \in \mathbb{N}_0} \overset{d}{\sim} X$

↳ Unim BP $(\underline{P})$



k with prob P_k

• root has offspring distr. $\underline{P}$

• other vertices have off. distr $\underline{P}^*$ where

$$P_k^* = \frac{(k+1) \cdot P_{k+1}}{\mathbb{E}(X)}$$

A. Results on Configuration model

1. Local structure: Under assumptions a., b., c. (over the $\underline{d}$) the $CM(n, \underline{d})$, as well as the $G(n, \underline{d})$, converge locally in probability to a Unim BP $(\underline{P})$ where $P_k := P(D = k)$ → D is the asymptotic degree of a typical vertex

2. Phase transition:

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Assume that $p_2 = \mathbb{P}(D=2) < 1$.

Then, under assumptions a., b., c., it holds in $CM(n, \underline{d})$ and in $G(n, \underline{d})$, that

- if $\boxed{\nu = \dfrac{\mathbb{E}(D(D-1))}{\mathbb{E}(D)} > 1}$ then (giant component, hence high connectivity)

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{\mathbb{P}} \zeta, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow{\mathbb{P}} 0$$

where ζ = survival probability of $\text{Unim}\,BP(\underline{p})$ and $p_k = \mathbb{P}(D=k)$

- if $\boxed{\nu \leq 1}$ then $\dfrac{|\mathcal{C}_{max}|}{n} \xrightarrow{\mathbb{P}} 0$

3. Small-world property:

Under the assumptions a., b., c., and assuming that $\nu > 1$ we get that: if U_1, U_2 are independent and Uniform$[n]$, conditionally on the event that $\{U_1 \longleftrightarrow U_2\}$, the $CM(n, \underline{d})$ (as well $G(n, \underline{d})$) are s.t.

$$\frac{\text{dist}(U_1, U_2)}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{\log \nu}$$

[small-world property]

Comment: if $\nu = \infty$, which happens for example when $\mathbb{E}(D^2) = \infty$ $\left(\text{e.g. } (d_i)_{i \in \mathbb{N}} \text{ i.i.d. with } \mathbb{P}(d_i \geq k) \sim c k^{-(\tau-1)} \text{ and } \tau \in [2,3)\right)$,

The result suggests that $\text{dist}(U_1, U_2) = o(\log n)$. Indeed if $(d_i)_{i}$ i.i.d. with $\mathbb{P}(d_i \geq k) \sim c k^{-\tau-1}$ and $\tau \in (2,3)$

if $(d_i)_{i \in \mathbb{N}}$ i.i.d. with $\mathbb{P}(d_i \geq k) \sim c k^{-\tau-1}$, and $\tau \in (2,3)$, Then

$$\frac{\mathrm{dist}_{CM(n,\underline{d})}(U_1, U_2)}{\lg \lg n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{|\lg(\tau-2)|}$$

$\boxed{B.}$ Connected models

- Erased configuration model: $CM^{er}(n, \underline{d})$ is obtained from $CM(n, \underline{d})$ erasing loops and multiple edges → hence it is a RG with value on $\mathcal{G}_n$, where the degree sequence is denoted by $\underline{d}^{er}$ and obtained from $\underline{d}$. Under assumption a. and b. (over $\underline{d}$) it holds

$$\underbrace{\frac{1}{n}\sum_{j\in[n]} \mathbb{1}\{d_j^{er}=k\}}_{P(d_U^{er}=k)} \xrightarrow[n\to\infty]{\mathbb{P}} p_k = \mathbb{P}(D=k) \quad \forall k \in \mathbb{N}$$

As a conclusion, the results stated for $CM(n,\underline{d})$ apply also to $CM^{er}(n,\underline{d})$ (under corresponding assumptions).

- Let $G(n, \underline{w})$ a generalized RG. Then it holds

$$P(G(n,\underline{w})=G \mid \underline{D}=\underline{d}) = \mathbb{P}(CM(n,\underline{d})=G \mid CM(n,\underline{d}) \text{ is simple}) = \mathbb{P}(G(n,\underline{d})=G)$$

where $\underline{D} = (D_i)_{i\in\mathbb{N}}$, $D_i = d_{G(n,\underline{w})}(i)$

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Comment on $G(n,p)$, $G(n,K) \leftrightarrow G(n,\underline{w})$, $CM(n,\underline{d})$:

- Parameters can be settled to obtain
  - sparse regime $(|E(G)| \sim n)$
  - high connectivity $(|\mathcal{C}_{max}| \sim n)$
  - small-world behavior $(\text{dist}(U_1, U_2) \sim \log n)$
- The scale-free property is obtained by specific choices of $\underline{w}$ in $G(n,\underline{w})$, of $\underline{d}$ in $CM(n,\underline{d})$

$\boxed{C.}$ Definition of Preferential Attachment model

Define a dynamical procedure to give rise to a sequence of random graphs $(PA_n^{(m,\delta)})_{n\in\mathbb{N}}$ for $m\in\mathbb{N}$ and $\delta \geq -m$, s.t. at each step $n\in\mathbb{N}$:

- we add one vertex, denoted $v_n^{(m)}$
- we add m edges from $v_n^{(m)}$ to any possible vertex

Hence $|\underbrace{V(PA_n^{m,\delta})}_{\{v_1^m, v_2^m, \dots, v_n^m\}}| = n$, $|E(PA_n^{m,\delta})| = m \cdot n$ $\hookrightarrow$ sparse regime

The random part of the procedure concerns with the choice of the m vertices to be connected with v_n^m:

(This is where δ enters into the def.).

- Let $D_i(n) :=$ degree of v_i^m at time n

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Then set $$P\left(v_n^m \longleftrightarrow v_i^m \mid PA_{n-1}^{m,\delta}\right) \propto m \cdot D_i(n) + \delta$$

Comment: Model also called "Rich-get-richer" model, because vertices with high degree have higher probability of being connected to new vertices.

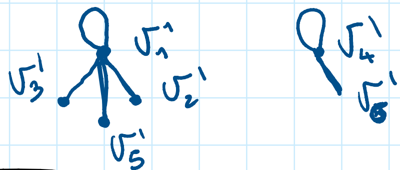
Formally:

$\boxed{\text{Let } m = 1}$ For $\delta \geq -1$ let

$$P\left(v_{n+1}^1 \longrightarrow v_j^1 \mid PA_n^{1,\delta}\right) = \begin{cases} \dfrac{D_j(n)+\delta}{n(2+\delta)+(1+\delta)} & \text{if } j \leq n \\[2ex] \dfrac{1+\delta}{n(2+\delta)+(1+\delta)} & \text{if } j = n+1 \end{cases}$$

Remarks:

1. Self-loops are present (when $j = n+1$) if $\delta \neq -1$
2. Since $m = 1$, the resulting graph is a tree with a loop at the root or a collection of trees:



$\boxed{\text{Let } m \geq 2}$. For $\delta \geq -m$, we consider an auxiliar $PA_{m \cdot n}^{1,\delta/m}$, $\forall\, n \in \mathbb{N}$, with vertices $v_1^1, \dots, v_{m \cdot n}^1$.

We collapse to a unique vertex the group of vertices

$v_1^1, \dots, v_m^1 \longrightarrow v_1^m$

$v_{m+1}^1, \dots, v_{2m}^1 \longrightarrow v_2^m$

$$v^1_{m+1}, \dots, v^1_{2m} \longrightarrow v^m_2$$

$$\vdots$$

$$v^1_{km+1}, \dots, v^1_{(k+1)m} \longrightarrow v^m_k \qquad \forall\, k \in [0, \dots, m]$$

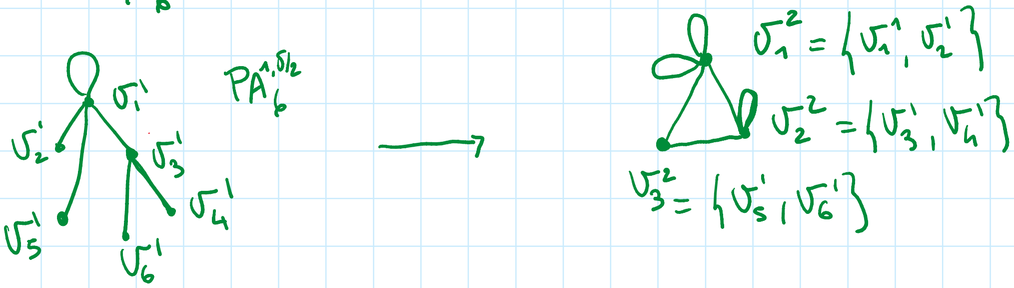
Keeping the edge connections into each pair of vertices

↳ in this way we obtain:

$$\mathbb{P}\left(v^m_{m+1} \longleftrightarrow v^m_j \mid PA^{m,\delta}_m\right) = \begin{cases} \dfrac{m+\delta}{cost} & j = m+1 \\[2ex] \dfrac{m D_j(n) + \delta}{cost} & \forall\, j \le m \end{cases}$$

Example:

$PA^{2,\delta}_3 \longrightarrow PA^{1,\delta/2}_6$



Notice that: 1. $PA^{m,\delta}$ is a multigraphs

2. $PA^{m,\delta}$ is monotone increasing (deterministically)

→ add edges at each step, not removed!