Random Graphs and Networks - 12° LECTURE

mercoledì 16 febbraio 2022 15:54

Recall: $\left(PA_n^{m,\delta}\right)_{n\in\mathbb{N}}$ where

- $|V(PA_n^{m,\delta})| = n$, $|E(PA_n^{m,\delta})| = m\cdot n$
- $D_i(n)$ = degree v_i^m at step n is a r.v. , $\forall n \geq i$

A. Vertex-degree sequence

Theorem: $\forall m \geq 1$, $\forall \delta > -m$ it holds

1. $D_i(n) \sim n^{\frac{1}{2+\delta/m}}$ as $n \to \infty$

$\forall i \leq n$: $$\frac{D_i(n)}{n^{\frac{1}{2+\delta/m}}} \xrightarrow[n\to\infty]{a.s.} \xi_i \quad \text{r.v.}$$

2. $\mathbb{E}(D_i(n)) \sim c(m,\delta)\, n^{\frac{1}{2+\delta/m}} \cdot i^{-\left(\frac{1}{2+\delta/m}\right)}$

$\forall i \leq n$ $$\frac{\mathbb{E}(D_i(n))}{n^{\frac{1}{2+\delta/m}}} \xrightarrow[n\to\infty]{} c(m,\delta)\cdot i^{-\frac{1}{2+\delta/m}}$$

3. Let $U \sim \text{Uniform}[n]$, then
$$D_U(n) \xrightarrow[n\to\infty]{\mathbb{P}} D$$
where $$\mathbb{P}(D=k) = p_k = c(m,\delta)\cdot k^{-\overbrace{\left(3+\frac{\delta}{m}\right)}^{\tau}}$$

Equivalently: $$\mathbb{P}(D_U(n)=k) = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{D_i(n)=k\}} \xrightarrow[n\to\infty]{\mathbb{P}} p_k$$

$$\boxed{\tau = 3+\frac{\delta}{m} > 2}\,.$$

Idea of Scale-free property 3.

Idea of scale-free property :

$$P(D_U(n) \geq k) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}\{D_i(n) \geq k\}$$

$$\approx \frac{1}{n} \sum_{i \in [n]} \mathbb{1}\{\mathbb{E}(D_i(n)) \geq k\}$$

$$\approx \frac{1}{n} \sum_{i \in [n]} \mathbb{1}\left\{ c(m,\delta)\left(\frac{n}{i}\right)^{\frac{1}{2+\delta/m}} \geq k \right\}$$

$$= \frac{1}{n} \sum_{i \in [n]} \mathbb{1}\left\{ i \leq n \cdot \underbrace{(c(m,\delta))^{2+\delta/m}}_{=\,c'} \cdot k^{-(2+\delta/m)} \right\}$$

$$= \frac{\cancel{n} \cdot c' \cdot k^{-(2+\frac{\delta}{m})}}{\cancel{n}} = c' k^{-(\tau-1)}$$

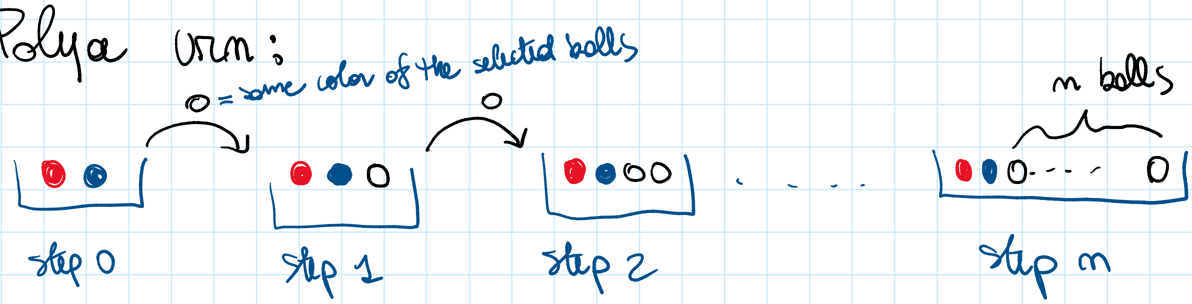
#

$\boxed{B.}$ Results on connectivity

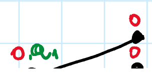
1. Local structure : Let $m \geq 1$ and $\delta > -m$. Then $(PA_n^{m,\delta})_{n \in \mathbb{N}}$ converges locally in probability to a multi-type BP, with types taking value on a continuous setting, called $\boxed{\text{Polya point Tree}}$.

Comment : It is an inhomogeneous BP with different branching distributions depending on the type of a vertex.

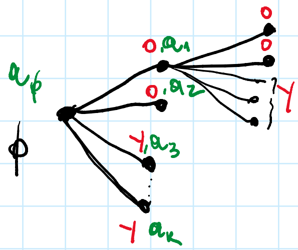
Polya urn :



m=2



$m=2$



Where age is chosen in $[0,1]$. Every vertex has a type $t=(\ell, a)$ where $\ell \in \{0, Y\}$, $a \in [0,1]$

2. If $m \geq 2 \implies \exists\, T < \infty$ s.t.

$$\mathbb{P}\left(PA_n^{(m,\delta)} \text{ is connected } \forall\, n > T\right) \xrightarrow[n\to\infty]{} 1$$

Idea: Since $m \geq 2$, the new vertex will connect two distinct components with a probability $\sim 1-p^m$.

3. a. If $m=1$ and $\delta > -1$, or $m \geq 2$ and $\delta > 0$

$$\frac{dist_{PA_n^{1,\delta}}(U_1, U_2)}{\log n} \xrightarrow{\mathbb{P}} \frac{2(1+\delta)}{2+\delta}$$

b. If $m \geq 2$ and $\delta \in (-m, 0)$, then

$$\frac{dist_{PA_n^{m,\delta}}(U_1, U_2)}{\log \log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau-2)|}$$

c. If $m \geq 2$ and $\delta = 0$, then

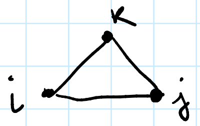
$$\frac{dist_{PA_n^{m,0}}(U_1, U_2)}{\log n / \log\log n} \xrightarrow{\mathbb{P}} 1$$

Further directions

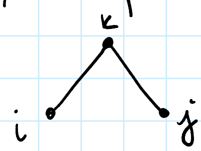
1. Network statistics

a. Cluster coefficient

If $i \sim k$ and $j \sim k$, we expect that $i \sim j$:



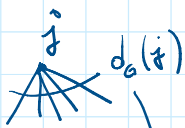
versus



To measure this phenomena, one can consider the <u>cluster coefficient</u> as follows:

For a graph G:

(local) $$\overline{CC}_G = \frac{1}{n} \sum_{j \in [n]} CC_G(j)$$



where $$CC_G(j) = \frac{\Delta_G(j)}{W_G(j)} = \frac{\Delta_G(j)}{d_G(j)(d_G(j)-1)}$$

<u>Def</u>: $(G_n)_{n \in \mathbb{N}}$ is <u>highly clustered</u> if

$$\liminf_{n \to \infty} \overline{CC}_{G_n} > 0$$

<u>FACT</u>: None of the models discussed is highly clustered. Indeed it holds the following:

<u>Theorem</u>: If $(G_n)_{n \in \mathbb{N}}$ converges locally in probability to (G, o) (infinite rooted random graph) then

$$\overline{CC}_{G_n} \xrightarrow[n \to \infty]{\mathbb{P}} \mathbb{E}\left(\frac{\Delta_G(o)}{d_G(o) \cdot (d_G(o) - 1)} \right).$$

Consequences: If the local limit is a random tree, then $\overline{CC}_{G_m} \xrightarrow[m\to\infty]{\mathbb{P}} 0$

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2. Related models

a. Spatial random graphs

Assume that vertices are embedded in a geometric space with a notion of distance that rules their connection probability.

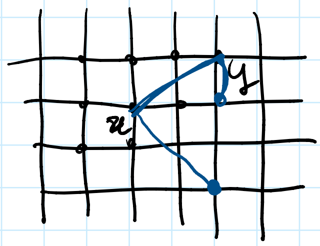
True geometry: people living close have higher probability to know each other

Latent geometry: people with common interests have higher probability to know each other.

Example: Scale-free percolation

↳ modification of the GRG

- vertex are placed in $\mathbb{Z}^d \subseteq \mathbb{R}^d$, where we have the Euclidean dist. $\|x-y\|$



- We assign a weight W_x to every $x \in \mathbb{Z}^d$, so that $(W_x)_{x\in\mathbb{Z}^d}$ iid
- We construct a random graph G s.t., indep. $\forall x, y \in \mathbb{Z}^d$

$$P(x,y) = f(\underline{w}, \|x-y\|)$$

$$= 1 - e^{-\lambda \frac{W_x \cdot W_y}{\|x-y\|^\alpha}} \quad , \quad \lambda, \alpha > 0 \text{ parameter}$$

$$\approx \lambda \cdot \frac{W_x \cdot W_y}{\|x-y\|^\alpha}$$

3. Limit of graphs

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4. Stochastic models on random graphs