Random Graphs and Network - 2° LECTURE

martedì 25 gennaio 2022 14:44

Recall: $(G_n)_{n \in \mathbb{N}}$ s.t. $G_n = ([n], E)$

- $n \to \infty$ (large graphs)
- sparse: $E(G_n) \simeq c \cdot n$, $c > 0$
- high connection: $|\mathcal{C}_{max}| \simeq c' n$, $c' > 0$
- small-world: $d_{G_n}(v,w) \leq k \log n$ for (a.e.) couple of (v,w)
- Scale free property: Let $U \sim \text{Uniform}([n])$

$$d_{G_n}(U) \xrightarrow[n \to \infty]{d} D \quad \text{r.v. s.t.}$$

$$\mathbb{P}(D \geq k) \simeq c \cdot k^{-(\tau-1)}, \quad \text{for } \tau > 1$$

$\boxed{A.}$ Random setting

- Let $\mathcal{G}_n := \{\text{graphs } G : V(G) = [n]\}$, $\forall n \in \mathbb{N}$

Note: $\mathcal{G}_n \xleftrightarrow[\text{su}]{1-1} \{0,1\}^{\mathcal{E}_n} \ni (x_j)_{j \in \mathcal{E}_n}$

$$\to \quad |\mathcal{G}_n| = 2^{\binom{n}{2}}$$

- Let us consider $(\mathcal{G}_n, \mathcal{F}, \mathbb{P}_n)$ probability space

Power set of $\mathcal{G}_n$

is given, and may depend on other parameters

Def: A random graph is an element of $\mathcal{G}_n$ chosen with probability $\mathbb{P}_n$, and denoted G_n.

Remark. Sometime is useful to think at G_n as a

Remark. Sometime is useful to think of $\mathbb{G}_n$ as a r.v. taking vale on $\mathcal{G}_n$ with law $\mathbb{P}_n$:

$$\mathbb{G}_n : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow \mathcal{G}_n \qquad \text{s.t}$$
$$\omega \longrightarrow \mathbb{G}_n(\omega)$$

$$\mathbb{P}(\mathbb{G}_n = G) = \mathbb{P}_n(G).$$

Example 1: Uniform model (defined Erdös-Rényi '60)

- Fix $m \in \mathbb{N}$ and let $\mathcal{G}_n^m := \{ G \in \mathcal{G}_n : |E(G)| = m \}$

$\forall\, G \in \mathcal{G}_n^m \subseteq \mathcal{G}_n$

$$\mathbb{P}_{n,m}(G) = \frac{1}{|\mathcal{G}_n^m|}, \quad \text{where } |\mathcal{G}_n^m| = \binom{\binom{n}{2}}{m}$$

The corresponding RG is denoted $\mathbb{G}(n,m)$.

Example 2: Binomial model (Erdös-Rényi RG)

- Fix $p \in [0,1]$ and consider the measure on $\mathcal{G}_n$ s.t

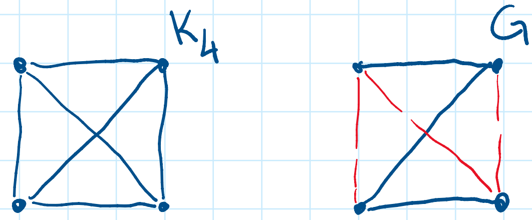
$$\mathbb{P}_{n,p}(G) := p^{|E(G)|} \cdot (1-p)^{\binom{n}{2} - |E(G)|} \qquad \left(\text{introduced by Gilbert '59} \right)$$

The corresponding RG is denoted $\mathbb{G}(n,p)$.

- Equivalently we can say that an edge $e \in \mathcal{E}_n$ appears in $\mathbb{G}(n,p)$ with prob. p, independently of the other edges:

other edges:

e.g. $n=4$



$\longrightarrow \mathbb{P}_{n,p}(G) = p^3(1-p)^3$

Hence, there is a bijection $\mathcal{G}(n,p) \xleftrightarrow[\text{su}]{H} \{X_e\}_{e\in\mathcal{E}_n}$

where $\{X_e\}_{e\in\mathcal{E}_n}$ i.i.d. $\sim Be(p)$.

Then, $\forall\ i \in [n]$:

$$d_{\mathcal{G}(n,p)}(i) = \sum_{j\in[n]} \overbrace{\mathbb{1}_{\{i\sim j \text{ in } \mathcal{G}(n,p)\}}}^{X_{(i,j)}} \overset{d}{\sim} Bin(n-1, p)$$

$$\mathbb{E}\left(d_{\mathcal{G}(n,p)}(i)\right) = (n-1)\cdot p \quad , \forall\ i \in [n]$$

B. Comparison between $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$

<u>Lemma</u>: $\mathcal{G}(n,p)$ conditioned to have m edges is distributed has $\mathcal{G}(n,m)$.

$\left(\text{hence} \quad \mathcal{G}(n,p) \,\middle|\, \mathcal{G}_n^m \overset{d}{=} \mathcal{G}(n,m)\right)$

<u>Proof</u>: Let $G \in \mathcal{G}_n^m$:

$$\mathbb{P}_{n,p}(G \mid \mathcal{G}_n^m) = \frac{\mathbb{P}_{n,p}(\overbrace{G \cap \mathcal{G}_n^m}^{"G})}{\mathbb{P}_{n,p}(\mathcal{G}_n^m)} = \frac{\cancel{p^m (1-p)^{\binom{n}{2}-m}}}{|\mathcal{G}_n^m|\cdot \cancel{p^m(1-p)^{\binom{n}{2}-m}}}$$

$$= \mathbb{P}_{n,m}(G)$$

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$\boxed{C.}$ Monotonicity of $\mathcal{G}(n,p)$ $\left(\overset{\text{or}}{\mathcal{G}(n,m)}\right)$ w.r.t. to p (or m)

[C.] Monotonicity of $G(n,p)$ ($G(n,m)$) w.r.t. p (or m)

Goal: compare
- $G(n,p)$ and $G(n,p')$ for $p < p'$
- $G(n,m)$ and $G(n,m')$ for $m < m'$

Def: An event $A \subseteq \mathcal{G}_n$ is <u>increasing</u> if s.t.:

$$\text{if } G \in A \Rightarrow G \cup \{e\} \in A, \quad \forall e \in \mathcal{E}_n$$

Def: A real r.v. $X : \mathcal{G}_n \to \mathbb{R}$ is increasing if

$\forall x \in \mathbb{R}$: $\{X \geq x\} = \{G \in \mathcal{G}_n : X(G) \geq x\}$ is increasing event

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Examples:
- $A_1 = \{G \in \mathcal{G}_n : \Delta \subseteq G\}$ is incr.
- $A_2 = \{G \in \mathcal{G}_n : H \subseteq G\}$ for fixed H
- $A_3 = \{G \in \mathcal{G}_n : i \longleftrightarrow j\}$ for fixed $i \neq j \in [n]$
- $A_4 = \{G \in \mathcal{G}_n : G \text{ is connected}\}$

$\hookrightarrow$ $A_5 = \{G \in \mathcal{G}_n : H \subseteq_{ind} G\}$ not increasing or decreasing

$A_6 = \{G \in \mathcal{G}_n : d_G(i) = k\}$, $i \in [n]$, $k \in \mathbb{N}$

Lemma (Monotonicity)

If A is an increasing event of $\mathcal{G}_n$ then

1. $\mathbb{P}_{n,p_1}(A) \leq \mathbb{P}_{n,p_2}(A) \quad \forall p_1 \leq p_2$

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2. $\mathbb{P}_{n,m_1}(A) \leq \mathbb{P}_{n,m_2}(A) \qquad \forall\ m_1 \leq m_2$

Proof of 1.

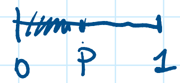
- We want to construct a coupling of $\mathbb{P}_{n,p_1}$ and $\mathbb{P}_{n,p_2}$.

Def: Given μ, ν probability measures on $(\Omega, \mathcal{F})$, a coupling of μ and ν is a prob. measure γ on $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ s.t.

$$\gamma(A \times \Omega) = \mu(A), \qquad \forall A \in \mathcal{F}$$
$$\gamma(\Omega \times A) = \nu(A)$$

Let us consider $\underline{U}^n = (U_e)_{e \in E_n}$ iid $\sim$ Uniform $[0,1]$

For any $p \in [0,1]$ we consider



$$G(\underline{U}^n, p) := \{e \in E_n : U_e \leq p\} \overset{?}{\overset{d}{=}} G(n,p)$$

- $$\mathbb{P}(G(\underline{U}^n, p) = G) = \prod_{e \in E(G)} \underbrace{\mathbb{P}(U_e \leq p)}_{= p} \prod_{e \notin E(G)} \underbrace{\mathbb{P}(U_e > p)}_{= 1-p}$$

$$= p^{|E(G)|} \cdot (1-p)^{\binom{n}{2} - |E(G)|} = \mathbb{P}(G(n,p) = G)$$

Since this holds for all $p \in [0,1]$, in fact we found a coupling $(\mathbb{P}_{n,p})_{p \in [0,1]}$.

- Note that if $p_1 \le p_2$

$$\underset{\overset{\|d}{G(n,p_1)}}{G(\underline{U}^n, p_1)} = \{e \in \mathcal{E}_n : U_e \le p_1\} \subseteq \{e \in \mathcal{E}_n : U_e \le p_2\} = \underset{\overset{\|d}{G(n,p_2)}}{G(\underline{U}^n, p_2)}$$

Thus $\mathbb{P}_{n,p_1}(A) \le \mathbb{P}_{n,p_2}(A)$ #

$\boxed{D.}$ Asymptotic equivalence $G(n,p)$, $G(n,p)$

<u>Comment</u>: if we want to infer <u>sparsity</u> in our models, it is reasonable to use:

- $G(n,m)$ s.t. $m = m(n) \simeq c \cdot n$, $c > 0$
- $G(n,p)$ s.t. $\mathbb{E}_{n,p}(|E(G)|) = p \cdot \binom{n}{2} \simeq cn$, $c > 0$

$$p \simeq \frac{c}{n} \Leftrightarrow$$

<u>Theorem</u>:

Let $m \equiv m(n) \xrightarrow[n \to \infty]{} \infty$ and A an increasing property.

Then if "$p \equiv p(n) \simeq \frac{m}{\binom{n}{2}}$" it holds that

$$\text{if } \mathbb{P}_{n,m}(A) \xrightarrow[n \to \infty]{} p_0 \implies \mathbb{P}_{n,p}(A) \xrightarrow[n \to \infty]{} p_0$$

"Similarly, if $p = p(n)$ is given and $m \equiv m(n) = p \cdot \binom{n}{2}$ then it holds the viceversa."

$\boxed{E.}$ Threshold: it refers to monotone events of $\mathcal{G}_n$

[E.] Threshold: it refers to monotone events of $\mathcal{G}_n$ that, in the limit $n \to \infty$, exhibit a sudden transition w.r.t. the parameter of the model.

Def: $p_c = p_c(n) \in [0,1]$ is a threshold in $G(n,p)$ for an increasing event A if:

$$\lim_{n\to\infty} \mathbb{P}_{n,p}(A) = \begin{cases} 0 & \text{if } p \ll p_c \qquad p/p_c \to 0 \\ 1 & \text{if } p \gg p_c \qquad p/p_c \to \infty \end{cases}$$

Remark: p_c is not unique because $p' = c \cdot p_c$, $\forall c > 0$ is also a threshold.

FACT (theorem by Bollobas, Thomason):
Every (non-trivial) increasing event (or property) has a threshold.

Examples:

[1] Let $A_0 = \{G \in \mathcal{G}_n : G \text{ has at least one edge}\}$

$$\Rightarrow \quad \mathbb{P}_{n,p}(A_0) \xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{if } p \ll \frac{1}{n^2} \\ 1 & \text{if } p \gg \frac{1}{n^2} \end{cases}$$

hence A_0 has threshold in at $\frac{1}{n^2}$.

Proof: Let $X(G) := |E(G)| \qquad (X : \mathcal{G}_n \to \mathbb{R})$

$$= \sum_{e \in E_n} \underbrace{\mathbb{1}_{\{e \in G\}}}_{\text{iid} \sim Be(p)}$$

$$\stackrel{d}{=} Bin\left(\binom{n}{2}, p\right)$$

$$\stackrel{d}{=} \mathrm{Bin}\left(\binom{n}{2}, p\right)$$

$$\mathbb{E}_{n,p}(X) = \binom{n}{2} \cdot p \qquad \mathrm{Var}_{n,p}(X) = \binom{n}{2} \cdot p(1-p)$$

1° MOMENT METHOD

$$\mathbb{P}_{n,p}(A_0) = \mathbb{P}_{n,p}(X \geq 1) \overset{\text{Markov inequalities}}{\leq} \mathbb{E}_{n,p}(X) = \binom{n}{2} \cdot p$$

hence, if $p \ll \frac{1}{n^2}$, $\mathbb{P}_n(A_0) \xrightarrow[n\to\infty]{} 0$.

2° MOMENT METHOD

$$\mathbb{P}_{n,p}(A_0^c) = \mathbb{P}_{n,p}(\underbrace{X = 0}_{|X - \mathbb{E}(X)| = \mathbb{E}(X)}) \leq \frac{\mathrm{Var}_{n,p}(X)}{\mathbb{E}_{n,p}(X)^2} =$$

$$= \frac{\cancel{\binom{n}{2}}\, \cancel{p}(1-p)}{\binom{n}{2}^{\cancel{2}} \cdot p^{\cancel{2}}} \leq \frac{1}{p\binom{n}{2}} \xrightarrow[n\to\infty]{\text{if } p \gg \frac{1}{n^2}} 0$$

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