Random Graphs and Networks - 3° LECTURE

mercoledì 26 gennaio 2022 17:55

Comment: $G(n,p)$ $p \in [0,1]$, $n \in \mathbb{N}$

- $|E(G(n,p))| \sim Bin\left(\binom{n}{2}, p\right)$

$$\Rightarrow \mathbb{E}\left(\underbrace{|E(G(n,p))|}\right) = \binom{n}{2} \cdot p \simeq \frac{n^2}{2} \cdot p$$

Imposing that $\simeq \boxed{cn} \iff p = \frac{c}{n}$

In general, we consider $p = p(n)$ $\left(\in [0,1] \; \forall n \in \mathbb{N}\right)$

$\hookrightarrow$ e.g. $\frac{1}{n^2}, \frac{1}{n}, \frac{1}{\sqrt{n}}, \frac{\lg n}{n}, 1 - \frac{1}{n}$

$$\Rightarrow \left(G(n,p)\right)_{n \in \mathbb{N}} \text{ where } \boxed{p = p(n)}$$

Recall:

- If $p \ll \frac{1}{n^2} \Rightarrow G(n,p)$ has no edges w.h.p as $n \to \infty$
- If $p \gg \frac{1}{n^2} \Rightarrow G(n,p)$ is not empty w.h.p as $n \to \infty$

Example 2.

Let $A_1 = \{G \in \mathcal{G}_n : G \text{ has at least a } \Delta\}$

$\Longrightarrow A_1$ has a threshold at $p_c = \frac{1}{n}$

Proof: $Z : \mathcal{G}_n \longrightarrow \mathbb{N}_0$

$$Z(G) = \# \{\text{triangles subgraph of } G\}$$

$$= \sum_{j=1}^{M} \underbrace{\mathbb{1}\{T_j \subseteq G\}}_{Be(p^3)}$$

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where $T_1, \dots, T_M$ is a given enumeration of triangles in K_n

$\longrightarrow M = \binom{n}{3}$

- Let $\underline{p = p(n) \ll \frac{1}{n}}$. Notice that

$$\mathbb{P}_{n,p}(A_1) = \mathbb{P}_{n,p}(Z \geq 1) \leq \mathbb{E}_{n,p}(Z) \xrightarrow[n\to\infty]{} 0$$

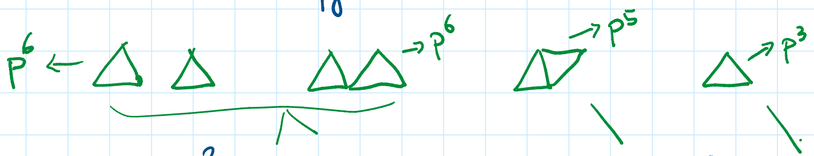
$$\mathbb{E}_{n,p}(Z) = \sum_{j=1}^{M} p^3 = \binom{n}{3} \cdot p^3 \simeq \frac{n^3}{3!} p^3 \xrightarrow[n\to\infty]{} 0$$

- Let $\underline{p = p(n) \gg \frac{1}{n}}$. Notice that

$$\mathbb{P}_{n,p}(A_1^c) = \mathbb{P}_{n,p}(Z = 0) \leq \frac{Var_{n,p}(Z)}{\mathbb{E}_{n,p}(Z)^2}$$

$$Var_{n,p}(Z) := \mathbb{E}_{n,p}(Z^2) - \mathbb{E}_{n,p}(Z)^2$$

$$\mathbb{E}_{n,p}(Z^2) = \sum_{i,j=1}^{M} P(T_i, T_j \subseteq G(n,p)) =$$



$$= \underbrace{\binom{n}{3}^2 p^6}_{= \mathbb{E}_{n,p}(Z)^2} + O(n^5 \cdot p^6) + O(n^4 \cdot p^5) + \underbrace{\binom{n}{3} p^3}_{= \mathbb{E}_{n,p}(Z)}$$

Hence $Var_{n,p}(Z) = \mathbb{E}_{n,p}(Z) + O(n^5 \cdot p^6) + O(n^4 \cdot p^5)$

$$\mathbb{P}_{n,p}(A_1) \leq \frac{\cancel{\mathbb{E}_{n,p}(Z)}\,1}{\mathbb{E}_{n,p}(Z)^{\cancel{2}\,1}} + \underbrace{\frac{O(\cancel{n^5} \cdot \cancel{p^6})}{O(n^{\cancel{6}} \cdot \cancel{p^6})}}_{\to O\left(\frac{1}{n}\right)} + \underbrace{\frac{O(\cancel{n^4} \cdot \cancel{p^5})}{O(n^{\cancel{6}\,2} p^{\cancel{6}})}}_{O\left(\frac{1}{n^2 p}\right) \to 0}$$

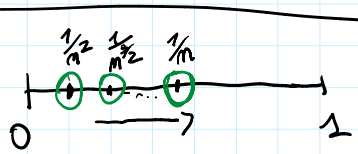
$\hookrightarrow O\left(\frac{1}{n}\right) \qquad\qquad O\left(\frac{1}{n^2 p}\right)$

$$\leq O(1) \cdot \frac{1}{\binom{n}{3} p^3} \xrightarrow[n\to\infty]{} 0$$

Example 3. <u>Trees containment</u>

Let $k \geq 3$ and $A_k = \{ G \in \mathcal{G}_n : k\text{-vertex Tree is in } G \}$

The threshold for A_k is $p_c = \frac{1}{n^{k/k-1}}$ $\left(\ll \frac{1}{n} \right)$



- $p \ll \frac{1}{n^2} \longrightarrow$ empty graph
- $p \ll \frac{1}{n^{3/2}} \longrightarrow$ no wedge 
- $p \ll \frac{1}{n^{k/k-1}} \longrightarrow$ no k-vertex trees
- $p \ll \frac{1}{n} \longrightarrow$ no $\triangle$ (no cycles)

What's next:

A. General subgraph containment

B. Inside the windows (of threshold)

C. Connectivity and "dense regime"

$\boxed{4.}$ Small graph containment

Let H be a given graph ()

Notation: $v_H := |V(H)|$, $e_H := |E(H)|$ and

$$\boxed{d_H := \frac{e_H}{v_H}}$$

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Def: H is <u>balanced graph</u> if $d_H \geq d_{H'}$, $\forall H' \subseteq H$

$\hookrightarrow$ e.g.  H not balanced

Example 4. Let H be a balanced graph and

$$A_H = \{ G \in \mathcal{G}_n : H \text{ subgraph of } G \}$$

Then A_H has a threshold at $p_c = \frac{1}{n^{1/d_H}}$.

Example: Take $H = C_k$ = Cycle with k vertices

$$\Rightarrow d_{C_k} = \frac{k}{k} = 1$$

Proof: Let $H_1, \dots, H_M$ be an enumeration os subgraph H in K_n , and set $X_H : \mathcal{G}_n \to \mathbb{N}_0$

$$X_H(G) = \sum_{j=1}^{M} \underbrace{\mathbb{1}_{\{H_j \subseteq G\}}}_{Be(p^{e_H})}$$

$$\mathbb{E}(X_H) = M \cdot p^{e_H} \qquad M = \binom{n}{v_H} \cdot \frac{v_H!}{|aut(H)|}$$

$$\simeq c\, n^{v_H} \cdot p^{e_H}$$

If $\boxed{p \ll \frac{1}{n^{1/d_H}}}$ then

$$\mathbb{P}_{n,p}(A_H) = \mathbb{P}_{n,p}(X_H \geq 1) \leq \mathbb{E}_{n,p}(X_H) \simeq n^{v_H} \cdot p^{e_H} \xrightarrow[n \to \infty]{} 0 .$$

If $p \gg \frac{1}{n^{1/d_H}}$ then

$$\mathbb{P}_{n,p}(A_H^c) = \mathbb{P}_{n,p}(X_H = 0) \leq Var_{n,p}(X_H) \leq$$

$$\mathbb{P}_{n,p}(A_M^c) = \mathbb{P}_{n,p}(X_M = 0) \leq \frac{\mathrm{Var}_{n,p}(X_M)}{\mathbb{E}_{n,p}(X_M)^2} \leq$$

$$\mathbb{E}_{n,p}(X_M^2) = \sum_{i,j=1}^{M} \mathbb{P}(H_i, H_j \in G(n,p)) = \dots$$

$$\leq \frac{1}{\mathbb{E}_{n,p}(X_M)} + o(1) \longrightarrow 0$$

$\boxed{B.}$ Inside small windows [Poisson paradigm]

Preliminaries:

- Let $X_n \overset{d}{\sim} \mathrm{Bin}(n,p)$, $n \in \mathbb{N}$

$\boxed{1^{\circ}\text{ FACT}}$ If $p = p(n) = \frac{\lambda}{n}$ (or more generally if $n.p(n) \xrightarrow[n\to\infty]{} \lambda$)

Then $X_n \xrightarrow[n\to\infty]{d} \mathrm{Poi}(\lambda)$

Proof: $\mathbb{P}(X_n = k) = \underbrace{\binom{n}{k} p^k}_{\sim \frac{\cancel{n^k}}{k!} \cdot \frac{\lambda^k}{\cancel{n^k}}} \underbrace{(1-p)^{n-k}}_{\text{"}\left(1-\frac{\lambda}{n}\right)^{n-k} \sim e^{-\lambda}} \xrightarrow[n\to\infty]{} \frac{\lambda^k}{k!} e^{-\lambda}$

$\forall k \in \mathbb{N}_0$

$\hookrightarrow$ An alternative proof is based on the following Lemma.

Lemma: If $(X_n)_{n\in\mathbb{N}}$ r.v on $\mathbb{N}_0$ st $\forall r \in \mathbb{N}$:

$$\mathbb{E}[(X_n)_r] = \mathbb{E}[X_n (X_n - 1) \cdots (X_n - r + 1)] \to \lambda^r$$

for some $\lambda > 0$, then $X_n \xrightarrow{d} \mathrm{Poi}(\lambda)$.

Remark: If $X_n(G) = \sum_{j=1}^{M} \mathbb{1}_{\{H_j \subseteq G\}}$

$$\mathbb{E}[(X_n)_r] = \mathbb{E}\left[\sum^{M} \mathbb{1}_{\{H \dots \subseteq G\}}\right]$$

$$\Rightarrow \mathbb{E}[(X_n)_r] = \mathbb{E}\Big[\sum_{\substack{i_1,\dots,i_r=1 \\ \text{distinct}}}^{M} \mathbb{1}\{H_{i_1},\dots,H_{i_r} \subseteq G\}\Big]$$

Theorem: If H is balanced graph and $p = p(n) = \left(\frac{c}{n}\right)^{1/d_H}$, $c > 0$, then

$$X_H \xrightarrow{d} \mathrm{Poi}(\lambda_H), \quad \lambda_H = \frac{c^{v_H}}{|\mathrm{aut}(H)|}$$

$$\left[\begin{array}{l} p \ll \frac{1}{n^{1/d_H}} \Rightarrow X_H = 0 \text{ a.s. } (\text{as } n\to\infty) \\ p \gg \frac{1}{n^{1/d_H}} \Rightarrow \mathbb{E}(X_H) = \infty \quad (\text{as } n \to \infty) \end{array}\right.$$

__Proof (idea)__: From Lemma above, we analyze

$$\mathbb{E}_{n,p}((X_H)_r) = \sum_{\substack{i_1,\dots,i_r=1 \\ \text{distinct indexes}}}^{M} \mathbb{P}(\underline{H_{i_1},\dots,H_{i_r}} \subseteq G(n,p)),$$

with $M = \binom{n}{v_H}\frac{v_H!}{|\mathrm{aut}(H)|}$

intermediate steps

$$= \binom{n}{\underbrace{v_H, v_H, \dots, v_H}_{r \text{ times}}}\left(\frac{v_H!}{|\mathrm{aut}(H)|}\cdot p^{e_H}\right)^r (1+o(1))$$

$$\simeq (n^{v_H})^r \cdot \left(\frac{p^{e_H}}{|\mathrm{aut}(H)|}\right)^r (1+o(1))$$

$$= \frac{\cancel{(n^{v_H})^r}}{\cancel{(n^{v_H})^r}} \cdot \Bigg(\underbrace{\frac{c^{v_H}}{|\mathrm{aut}(H)|}}_{\lambda_H}\Bigg)^r (1+o(1)) \xrightarrow[n\to\infty]{} \lambda_H^r$$

Theorem: Let $X_d : \mathcal{G}_n \to \mathbb{N}_0$ s.t.

$$X_d(G) = |\{i \in [n] : d_G(i) = d\}|, \quad d \in \mathbb{N} \text{ fixed}$$

Then:

Then:

a. if $p \ll n^{-\frac{d+1}{d}}$ $\left(\ll n^{-1}\right) \Longrightarrow X_d \xrightarrow[n\to\infty]{a.s} 0$ (1° moment method)

b. if $p = c\, n^{-\frac{d+1}{d}}$, $c>0$ $\Longrightarrow X_d \xrightarrow[n\to\infty]{d} \mathrm{Poi}(\lambda_d)$, $\lambda_d = \frac{c^d}{d!}$

c. if $p \gg n^{-\frac{d+1}{d}}$:

c1. If p is s.t. $np - \log n - d \log\log n \xrightarrow[n\to\infty]{} -\infty$ $\left(p = \frac{\log n}{n} + d\frac{\log\log n}{n} - \frac{c_n}{n}\right)$, $c_n \to +\infty$

$\Longrightarrow$

- $\mathbb{E}_{np}(X_d) \xrightarrow[n\to\infty]{} \infty$
- $\dfrac{X_d - \mathbb{E}_{n,p}(X_d)}{\left(\mathrm{Var}_{n,p}(X_d)\right)^{1/2}} \xrightarrow[n\to\infty]{d} N(0,1)$

c2. If p is s.t. $np - \log n - d\log\log n \xrightarrow[n\to\infty]{} c$

$$\Longrightarrow \quad X_d \xrightarrow[n\to\infty]{d} \mathrm{Poi}\left(\frac{e^{-c}}{d!}\right)$$

c3. If p is s.t. $np - \log n - d\log\log n \xrightarrow[n\to\infty]{} +\infty$ $\left(p = \frac{\log n}{n} + d\frac{\log\log n}{n} + \frac{c_n}{n}\right)$, $c_n \to +\infty$

$$\Longrightarrow \quad X_d \xrightarrow[n\to\infty]{a.s.} 0$$

Altogether: