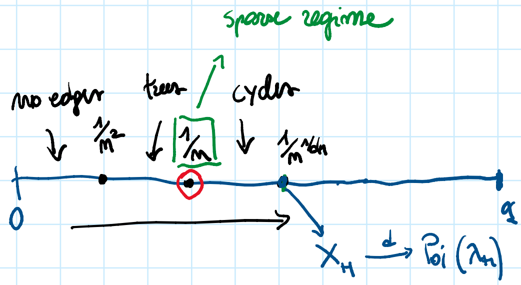
Random Graphs and Networks – 4° LECTURE

lunedì 31 gennaio 2022 23:13

Resume

ER random graph

$G(n,p)$, $n \in \mathbb{N}$, $p = p(n)$



H balanced graph, $d_H = \frac{e_H}{v_H}$ $\longrightarrow$ threshold at $p = n^{-\frac{1}{d_H}}$

A. Connectivity

Theorem 1 Let $p \equiv p(n) = \frac{\lg n + c_n}{n}$. Then

$$\lim_{n\to\infty} \mathbb{P}\big(G(n,p) \text{ is connected}\big) = \begin{cases} 0 & \text{if } c_n \xrightarrow[n\to\infty]{} -\infty \\ e^{-e^{-c}} & \text{if } c_n \xrightarrow[n\to\infty]{} c \\ 1 & \text{if } c_n \xrightarrow[n\to\infty]{} +\infty \end{cases}$$

Preliminary result:

Theorem 2: Let $X_1(G) = \#$ isolated vertices in G. Then:

$$= \sum_{j\in[n]} \mathbb{1}\{j \text{ is isolated vertex}\}$$

a. if $p \ll n^{-2}$ $\Longrightarrow$ $\mathbb{E}_{n,p}(X_1) \longrightarrow +\infty$

b. if $p \gg n^{-2}$ but $np - \lg n \longrightarrow -\infty$ $\quad p \le \frac{\lg n + c_n}{n}$, $c_n \to -\infty$ $\Longrightarrow$

$$\mathbb{E}_{n,p}(X_1) \longrightarrow +\infty, \quad \frac{X_1 - \mathbb{E}_{n,p}(X_1)}{\mathrm{Var}(X_1)^{1/2}} \xrightarrow[n\to\infty]{d} N(0,1)$$

c. if $p = \frac{\lg n + c}{n}$ $\Longrightarrow$ $X_1 \xrightarrow[n\to\infty]{d} \mathrm{Poi}(e^{-c})$

d. if $p \geq \frac{\lg n + c_n}{n}$, $c_n \to +\infty$ $\Longrightarrow$ $X_1 \xrightarrow{d} 0$

Proof Idea of Thm 2:

$\lg(1-p) \approx -p$

$\mathbb{E}(X_1) = \sum \mathbb{P}(j \text{ is isolated vertex}) \ldots \; {}^{n-1} \; \ldots e^{-(n-1)p}$

$$\mathbb{E}_{m,p}(X_1) = \sum_{j\in[m]} \mathbb{P}_{m,p}(\underbrace{j \text{ isolated vertex}}_{(1-p)^{m-1}}) = m\cdot(1-p)^{m-1} \approx m e^{-(m-1)p}$$

$$\xrightarrow[m\to\infty]{} \begin{cases} +\infty & \text{if } p \leq \frac{\lg m + c_m}{m}, \quad c_m \to -\infty \\ e^{-c} & \text{if } p = \frac{\lg m + c}{m} \\ 0 & \text{if } p \geq \frac{\lg m + c_m}{m}, \quad c_m \to +\infty \end{cases}$$

#

Proof of Theorem 1:

Consider $p = \frac{\lg m + c}{m}$ (the other regimes follow by monotonicity)

$$\mathbb{P}(G(m,p) \text{ is not connected}) = \mathbb{P}\Big(\bigcup_{k=1}^{m/2} \{G(m,p) \text{ has components of size } k\}\Big) \overset{*}{=}$$

- For $k \in \{1, \dots, \frac{m}{2}\}$, let $X_k(G) = \#$ components in G of size k

$$\overset{*}{=} \mathbb{P}_{m,p}\Big(\bigcup_{k=1}^{m/2} \{X_k > 0\}\Big) \leq \sum_{k=1}^{m/2} \mathbb{P}_{m,p}(X_k > 0)$$

$$\geq \mathbb{P}_{m,p}(X_1 > 0)$$

Hence:
- $\mathbb{P}(G(m,p) \text{ is not connected}) \leq \mathbb{P}_{m,p}(X_1 > 0) + \underbrace{\sum_{k=2}^{m/2} \mathbb{P}_{m,p}(X_k > 0)}_{\to 0} \quad (*)$
- $\mathbb{P}(G(m,p) \text{ is not connected}) \geq \mathbb{P}_{m,p}(X_1 > 0)$

If $(*)$ is true, we get:

$$\mathbb{P}(G(m,p) \text{ is not connected}) = \mathbb{P}_{m,p}(X_1 > 0) + o(1)$$

$$\xrightarrow[m\to\infty]{} 1 - e^{-e^{-c}}$$

↳ $X_1 \xrightarrow{d} \mathrm{Poi}(e^{-c})$ hence $\mathbb{P}_{m,p}(X_1 > 0) = 1 - \mathbb{P}_{m,p}(X_1 = 0)$

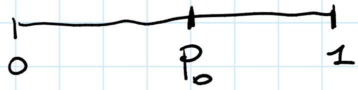
Let us prove $(*)$:

Let us prove ⊛:

$$\sum_{k=2}^{n/2} \mathbb{P}_{n,p}(X_k>0) \underset{\uparrow \ \text{1st moment method}}{\le} \sum_{k=2}^{n/2} \mathbb{E}_{n,p}(X_k) \le \underbrace{\sum_{k=2}^{n/2} \binom{n}{k} k^{k-2} \cdot p^{k-1} (1-p)^{k(n-k)}}_{\to 0}$$

#

B. Dense regime : $p(n) = p_0 \in (0,1)$



1. $$\mathbb{E}\left(|E(G(n,p))|\right) = \binom{n}{2} \cdot p \sim c n^2$$ → DENSE REGIME

because $|E(G(n,p))| \overset{d}{\sim} \text{Bin}\left(\binom{n}{2}, p\right)$

2. $d_{G(n,p)}(j) \overset{d}{=} \text{Bin}(n-1, p)$ with average $(n-1)\cdot p \to \infty$

3. Recall that $\text{diam}(G) = \max_{i,j \in [v]} \text{dist}_G(i,j)$

<u>Theorem</u> : If $p(n) \equiv p_0 \in (0,1)$ then

$$\text{diam}(G(n,p)) \xrightarrow[n\to\infty]{d,P} 2$$

→ Notice that $\text{diam}(K_n) = 1$

<u>Proof</u> : Enough to prove

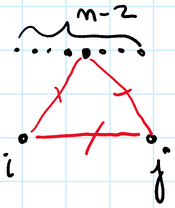
1. $\mathbb{P}_{n,p}\left(\text{dist}_G(i,j) > 2 \text{ for some } i,j \in [n]\right) \to 0$

2. $\mathbb{P}_{n,p}\left(\text{dist}_G(i,j) = 1 \text{ for all } i,j \in [n]\right) \longrightarrow 0$ ✓

$$\shortparallel$$
$$p^{\binom{n}{2}}$$

Let us prove 1. : $$\mathbb{P}_{n,p}\left(\bigcup_{\substack{i,j \in [n] \\ i \neq j}} \{\text{dist}_G(i,j) > 2\} \right) \le \sum_{\substack{i,j \in [n] \\ i \neq j}} \underbrace{\mathbb{P}_{n,p}\left(\text{dist}_G(i,j) > 2\right)}_{\shortparallel}$$

We us prove ... : $\mathbb{P}_{m,p}\big(\ldots_{\substack{i,j\in[m] \\ i\neq j}} \ldots \big) \leq \sum_{\substack{i,j\in[m] \\ i\neq j}} \ldots$ [illegible]



$$= \binom{m}{2}(1-p)\left(1-p^2\right)^{m-2} \xrightarrow[m\to\infty]{} 0$$

$$(1-p)\left(1-p^2\right)^{m-2}$$

$$\sim m^2 e^{-cm}$$

#

C. SPARSE REGIME : $p = \frac{\lambda}{m}$, $\lambda \in \mathbb{R}$

- $G(m,p)$, $p = \frac{\lambda}{m}$ is not scale free . Indeed

$\forall j \in [m]$, then $d_{G(m,p)}(j) \overset{d}{\sim} \text{Bin}\left(m-1, \frac{\lambda}{m}\right) \xrightarrow[m\to\infty]{d} \text{Poi}(\lambda)$

Hence : $\mathbb{P}_{m,p}\left(d(j) = k\right) = \binom{m}{k} p^k (1-p)^{m-k} \approx e^{-\lambda} \cdot \frac{\lambda^k}{k!}$ → exponential decay in k

(no power-law decay : $k^{-\tau}$, $\tau > 1$)

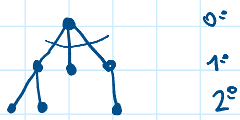
→ Let us focus on high connectivity and small world property.

D. Random trees : Galton-Watson Branching processes

They describes the genealical tree from 1 individual.

Def : Let $\underline{p} = (p_k)_{k\in\mathbb{N}_0}$ a probability density called offspring distr.

We set :



- $Z_0 = 1$
- $Z_m = \#$ individuals in generation m, $\forall m \in \mathbb{N}$

where , if $(X_{m,j})_{\substack{m\in\mathbb{N} \\ j\in\mathbb{N}}}$ iid r.v. s.t. $\mathbb{P}(X_{m,j} = k) = p_k \quad \forall k \in \mathbb{N}_0$,

$$Z_m = \sum_{j=1}^{Z_{m-1}} X_{m,j} \quad .$$

$$Z_m = \sum_{j=1} X_{m,j} \,.$$

Then $(Z_m)_{m\in\mathbb{N}_0}$ is denoted $BP(\underline{p})$

<u>Remark</u>: • Each individual has the same offspring distr

$\longrightarrow$ homogeneous BP.

• $(Z_m)_{m\in\mathbb{N}_0}$ is a Markov Chain.

Q_1: What is the probability that $BP(\underline{p})$ is infinite?

• Let $\quad \eta = P(\exists\, m : Z_m = 0)$ extinction probability

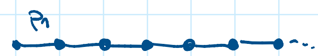
and set $\quad \xi = 1-\eta$ survival probability

<u>Trivial cases</u>:

• $p_0 = 0$ (sure survival)

• $p_0 = 1$ (sure extinction)

• $p_0 + p_1 = 1 \longrightarrow$



$\Longrightarrow \xi = \lim_{m\to\infty} p_1^m = 0$

<u>Theorem</u>: If $\underline{p}$ is not trivial and set $\mu := \sum_{k=1}^{\infty} k\cdot p_k$, then

a. If $\mu \le 1 \Longrightarrow \eta = 1 \quad (\xi = 0)$

b. If $\mu > 1 \Longrightarrow \eta < 1 \quad (\xi > 0)$

and in particular η is the smallest solution of the fixed point eq. $\boxed{\eta = G(\eta)}$

where $\quad G(z) = \sum_{k=0}^{\infty} z^k p_k$

<u>Proof</u>: 1. $\{Z_m = 0\} \subseteq \{Z_{m+1} = 0\} \quad \forall\, m\in\mathbb{N} \quad$ (increasing events)

Proof: 1. $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\} \quad \forall n \in \mathbb{N}$ (events)

$$\Rightarrow \eta = \lim_{n\to\infty} \mathbb{P}(Z_n = 0)$$

2. Set $G_n(z) := \mathbb{E}(z^{Z_n}) \Rightarrow G_n(0) = \mathbb{P}(Z_n = 0)$

$$= \sum_{k=0}^{\infty} z^k \cdot \mathbb{P}(Z_n = k) \quad , \quad 0^0 = 1$$

$\hookrightarrow$ Notice that $G_1(z) = \mathbb{E}(z^{Z_1}) = \sum_{k=0}^{+\infty} z^k \cdot p_k = G(z)$

3. Lemma: $G_n(z) = G(G_{n-1}(z)) \left(= G_{n-1}(G(z)) \right)$

passing to the limit $n \longrightarrow \infty$, for $z = 0$: $\quad \eta = G(\eta)$