Random Graphs and Networks - 5° LECTURE

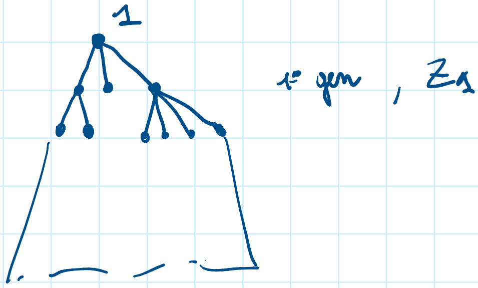
mercoledì 2 febbraio 202 0:54

- Focus on sparse regime $G(n,p)$, $\boxed{p = \frac{\lambda}{n}}$, $\lambda \in \mathbb{R}$

- $BP(\underline{P})$ $\quad \underline{P} = (P_k)_{k \in \mathbb{N}_0}$ offspring distr.

$\parallel$

$(Z_n)_{n \in \mathbb{N}_0}$ s.t. $Z_n = \#$ individuals in generation n :



$$Z_0 = 1$$

$$Z_n = \sum_{j=1}^{Z_{n-1}} X_{n,j}$$

with $(X_{n,j})_{\substack{n \in \mathbb{N} \\ j \in \mathbb{N}}}$ i.i.d. $\overset{d}{\sim} \underline{P}$

- Let $\mu := \sum_{k=0}^{\infty} k \cdot P_k$

- Let $G(z) = \sum_{k=0}^{\infty} z^k \cdot P_k$, $\forall z \in [0,1]$

- let $\eta = \mathbb{P}(\exists n : Z_n = 0) = \lim_{n \to \infty} \underbrace{\mathbb{P}(Z_n = 0)}_{= G_n(0)}$

Theorem:

a. If $\mu \leq 1 \Rightarrow \eta = 1$

b. If $\mu > 1 \Rightarrow \eta < 1$

Moreover η is the smallest sol. of : $z = G(z)$

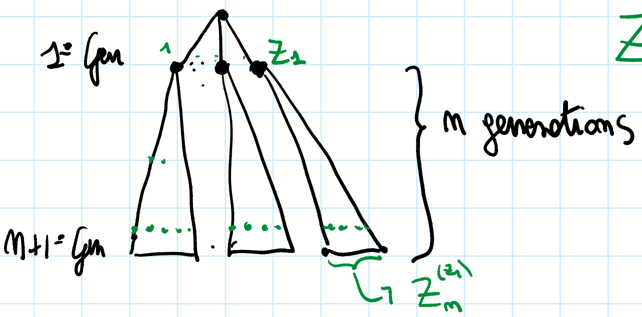
Proof: We already seen that $\eta = G(\eta)$. This comes

Proof: We already seen that $\eta = G(\eta)$. This comes from the following:

Lemma: $\forall z \in [0,1]$:

$$(*) \quad G_{m+1}(z) = G(G_m(z)) \quad \left(= G_m(G(z))\right)$$

Proof: $G_{m+1}(z) = \mathbb{E}\left[z^{Z_{m+1}}\right] = \mathbb{E}\left[\mathbb{E}\left[z^{Z_{m+1}} \mid Z_1\right]\right] \overset{*}{=}$



$$Z_{m+1} \overset{d}{=} Z_m^{(1)} + Z_m^{(2)} + \dots + Z_m^{(Z_1)}$$

where $Z_m^{(j)}$ are all independent

$$\overset{*}{=} \mathbb{E}\left[z^{Z_m^{(1)} + \dots + Z_m^{(Z_1)}}\right] = \mathbb{E}\left[\prod_{j=1}^{Z_1} \overbrace{\mathbb{E}\left(z^{Z_m^{(j)}}\right)}^{G_m(z)}\right] = \mathbb{E}\left(G_m(z)^{Z_1}\right)$$

$$= G(G_m(z)) \qquad \#$$

As a consequence, taking $z=0$, and passing to limit $m \to \infty$, eq. (*) brings:

$$\underset{\substack{\downarrow \\ \eta}}{P(Z_{m+1}=0)} = G_{m+1}(0) = G(G_m(0)) = G(P(Z_m=0)) \;\; \underset{\substack{\| \\ G(\eta)}}{}$$

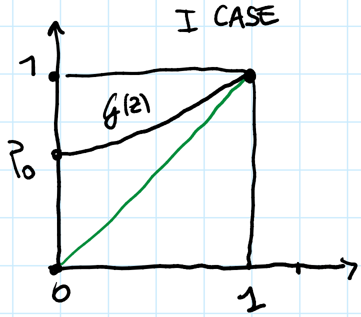
- At last let us study $z = G(z)$

1. $G : [0,1] \to [0,1]$ $\qquad G(0) = P_0 \qquad G(1) = 1$

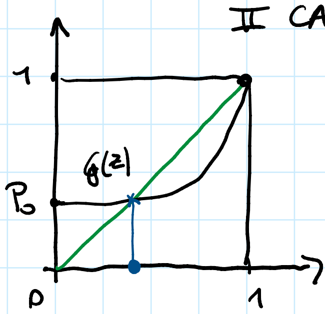
G is strictly increasing and convex in $(0,1)$:

$$G'(z) = \sum_{k=1}^{\infty} k z^{k-1} p_k > 0 \qquad G''(z) = \sum_{k=2}^{+\infty} k(k-1) z^{k-2} p_k > 0$$

I CASE



II CASE



I CASE $\iff \lim_{z\to 1^-} G'(z) \le 1 \iff \mu \le 1$

In that case $z = G(z)$ has only one solution at $z=1$, then $\eta = 1$ (which proves a.)

II CASE $\iff \lim_{z\to 1^-} G'(z) > 1 \iff \mu > 1$

In that case $z = G(z)$ has two solutions: we want to show that η is the smallest one. By induction we show that

$P(Z_n = 0) \le z$ if $z = G(z)$, $\forall n \in \mathbb{N}$:

$$P(Z_{n+1}=0) = G_{n+1}(0) = G(G_n(0)) = G(\overbrace{P(Z_n=0)}^{\le z \text{ by induction}}) \le z$$

$\hookrightarrow$ by monotonicity of G

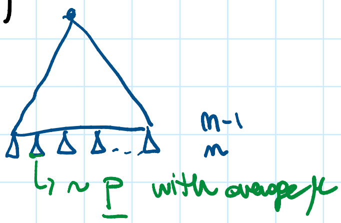
Taking $n \to \infty$; $\eta \le z$ if $z = G(z)$

#

BP($\underline{p}$) with mean μ is $\begin{cases} \text{subcritical} & \text{if } \mu < 1 \\ \text{critical} & \text{if } \mu = 1 \\ \text{supercritical} & \text{if } \mu > 1 \end{cases}$

A. Size of the total progeny

Let $T := \sum_{n=0}^{\infty} Z_n$ $\quad$ (Notice $\mathbb{P}(T<\infty) = \eta$)



1. $\mathbb{E}(Z_n) = \mathbb{E}\left(\mathbb{E}(Z_n \mid Z_{n-1})\right)$

$= \mathbb{E}[\mu \cdot Z_{n-1}]$ $\qquad Z_n = \sum_{j=1}^{Z_{n-1}} X_{n,j}$

$= \mu^2\, \mathbb{E}[Z_{n-2}] = \dots = \mu^n$

Then $\mathbb{E}(T) = \sum_{n=0}^{+\infty} \mathbb{E}(Z_n) = \sum_{n=0}^{+\infty} \mu^n = \begin{cases} \frac{1}{1-\mu} & \text{if } \mu<1 \\ +\infty & \text{if } \mu \geq 1 \end{cases}$

Def : Exploration process in $BP(\underline{p})$

- We think at vertices of $BP(\underline{p})$ as alive, dead or neutral, and that they may change their value as following :
- At step $n=0$: root is live (1 live), all the others are neutral ($T-1$ neutral)
- At each step we chose a live vertex :
  - * the chosen vertex : live $\longrightarrow$ dead
  - * the offspring of the chosen vertex : neutral $\rightarrow$ live

Let
- S_n = # live vertices at step n $\quad (S_0 = 1)$
- K_n = # dead vertices at step n $\quad (K_n = n \ \forall n)$
- N_n is s.t. $N_n + K_n + S_n = T$

It turns out that S_n can be defined iteratively as

$S_0 = 1$ $\qquad S_n = S_{n-1} - 1 + X_n$ $\qquad$ where $(X_k)_{k\in\mathbb{N}}$

$= S_{n-2} + X_n + X_{n-1} - 2$ $\qquad$ all i.i.d $\overset{d}{\sim} \underline{P}$

$= X_1 + \dots + X_n - (n-1)$

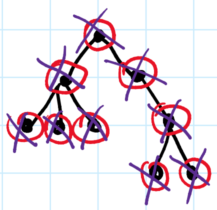
hence $(S_n)_{n\in\mathbb{N}}$ is a RW with increment $(X_k - 1)_{k\in\mathbb{N}}$

and $S_0 = 1$.

Notice that $T' := \inf\{n : S_n = 0\}$

$$\Rightarrow \boxed{T' \overset{d}{=} T}.$$

Idea :



$S_0 = 1$

$S_1 \overset{d}{=} X_1$

$S_2 \overset{d}{=} X_1 + X_2 - 1$

$S_3 \overset{d}{=} X_1 + X_2 + X_3 - 2$

$S_4 \overset{d}{=} X_1 + X_2 + X_3 + X_4 - 3$

Main point: $\quad \mathbb{P}(T = k) = \mathbb{P}(S_k = 0\,,\ S_j > 0\ \forall j \in \{1,\dots,k-1\})$

Theorem (hitting time for RW).

Let $(S_n)_{n\in\mathbb{N}_0}$ be a RW on $\mathbb{R}$ with $S_0 = 0$ and i.i.d integer increments $(X_k)_{k\geq 0}$ s.t. $\mathbb{P}(X_n \geq -1) = 1$

For $k > 0$, let $\quad T_{-k} = \inf\{n : S_n = -k\}$

$$\Rightarrow \mathbb{P}(T_{-k} = N) = \frac{k}{N} \cdot \mathbb{P}(S_N = -k)\,, \quad \forall N \in \mathbb{N}$$

→ Applying this result to the total progeny of $BP(\underline{p})$, T:

$$\mathbb{P}_{BP}(T=N) = \mathbb{P}_0(T'_{-1}=N) = \frac{1}{N}\mathbb{P}(X_1+\dots+X_N = N-1)$$

B. Duality principle

Consider a $BP(\underline{p})$ s.t. $\mu = \sum_k k p_k > 1$ so that $\eta = \mathbb{P}(T<\infty) < 1$.

Def: We say that $\underline{p}' = (p'_k)_{k\in\mathbb{N}_0}$ - probability density over $\mathbb{N}_0$ - is conjugate of $\underline{p}$ $(\mu>1)$ if

$$\boxed{p'_k = \eta^{k-1}\cdot p_k}\quad , \forall k\in\mathbb{N}$$

Theorem [duality principle]: If $\underline{p}$ and $\underline{p}'$ are conjugate, then

$$BP(\underline{p}) \mid \{T<\infty\} \stackrel{d}{=} BP(\underline{p}')$$

Example:

Let $\underline{p} \stackrel{d}{=} Poi(\lambda)$: $p_k = \frac{\lambda^k}{k!}\cdot e^{-\lambda}$

Hence: $\sum_{k=1}^{\infty} k\cdot p_k = \lambda > 1$; $G_{Poi(\lambda)}(z) = e^{\lambda(z-1)}$

Then we have: $\eta = G(\eta) \iff \eta = e^{\lambda(\eta-1)}$

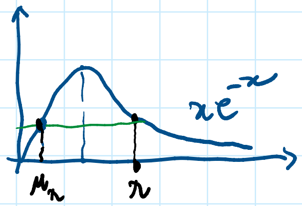
Thus, a conjugate $\underline{p}'$ of $\underline{p}$ is s.t., $\forall k\in\mathbb{N}$

$$p'_k = \eta^{k-1}\cdot p_k = \eta^k\cdot e^{-\lambda(\eta-1)}\frac{\lambda^k}{k!}e^{-\lambda} = \boxed{\frac{(\eta\lambda)^k}{k!}e^{-\lambda\eta}}$$

$$\underbrace{P'_k = \eta^{k-1} \cdot P_k}_{k=1\,:\; P'_1 = P_1} = \eta^k \cdot e^{-\lambda(\eta-1)} \frac{\lambda^k}{k!} e^{-\lambda} = \boxed{\frac{(\eta\lambda)^k e^{-\lambda\eta}}{k!}}$$

$$\Longrightarrow \quad \underline{P}' \overset{d}{=} \mathrm{Poi}(\overset{\mu_\lambda}{\overset{\shortparallel}{\lambda\eta}}) = \mathrm{Poi}(\mu_\lambda)$$

Moreover (from eq. at $k=1$) : $\mu_\lambda e^{-\mu_\lambda} = \lambda e^{-\lambda}$



$$\Longrightarrow \quad \mu_\lambda < 1$$