Random Graphs and Networks - 6° LECTURE

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Recall: Given a $BP(Poi(\lambda))$ (with offspring distr. $Poi(\lambda)$):

1. $\begin{cases} \bullet \text{ if } \lambda \leq 1 \Rightarrow BP(Poi(\lambda)) \text{ dies out a.s. } (\eta = 1) \\ \bullet \text{ if } \lambda > 1 \Rightarrow BP(Poi(\lambda)) \text{ dies out with prob. } \eta < 1 \\ \qquad \text{and } \eta = e^{\lambda(\eta - 1)} \quad [\zeta = 1 - e^{-\lambda \zeta}] \quad \zeta = 1 - \eta \end{cases}$

2. if $\lambda > 1$, then $BP(Poi(\lambda)) \mid \{\text{dying out}\} \overset{d}{=} BP(Poi(\mu_\lambda))$
   where μ_λ is conjugate of λ: $\mu_\lambda e^{-\mu_\lambda} = \lambda \cdot e^{-\lambda}$ (duality)

3. Exploration process on $BP(Poi(\lambda))$:
   Size of total progeny $T \overset{d}{=} \min\{n : S_n = 0\}$

A. Exploration of the components of $G(n,p)$ $\left[p = \frac{\lambda}{n}, \lambda > 0\right]$

- Let $j \in [n]$ and explore $\mathcal{C}(j) = \{k \in [n] : j \longleftrightarrow k\}$

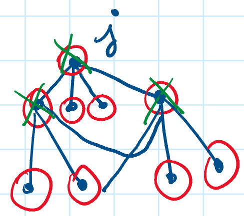
<u>Procedure</u>. It is a dynamics along which vertices may be live, dead or neutral:

- At step $k = 0$: j is live, while all vertices $i \neq j$ are neutral
- At step $k \geq 1$, we choose a live vertex i, then

  * i : live $\rightarrow$ dead

  * If $l \sim i$: $\begin{cases} \text{neutral} \rightarrow \text{live} \\ \text{live} \rightarrow \text{live} \\ \text{dead} \rightarrow \text{dead} \end{cases}$

  * Vertices $l \not\sim i$ keep their value



Then set $S_k = \#$ live vertices at step k, $\forall k \geq 0$

$D_k = \#$ dead vertices at step k $\longrightarrow D_k = k$

$N_k = \#$ neutral vertices at step k

with $S_k + N_k = n - k$

Remark: $|\mathcal{C}(j)| =: T = \min\{k : S_k = 0\}$

1. Let $X_i = \#\{\text{vertices becoming live at step } i\}$, $\forall i \in \mathbb{N}$:

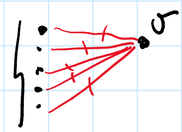
$$S_k = S_{k-1} + X_k - 1 = \dots = \sum_{i=1}^{k} X_i - (k-1)$$

with $S_0 = 1$

Hence $(S_k)_{k\in\{0,\dots,T\}}$ is a RW with increments $(X_i - 1)_{i\in\mathbb{N}}$

1e FACT: $X_i \sim \mathrm{Bin}(N_{i-1}, p)$

2e FACT: $N_i \sim \mathrm{Bin}(n-1, (1-p)^i)$

i 

$$\Rightarrow S_k = n - k - N_k = \underbrace{(n-1) - N_k}_{\mathrm{Bin}(n-1,\, 1-(1-p)^k)} - k + 1$$

$$\stackrel{d}{=} \mathrm{Bin}(n-1, 1-(1-p)^k) - k + 1$$

Heuristics: If i is "small" (w.r.t. n) then:

$N_i \approx n-1$. Thus $X_i \approx \mathrm{Bin}(n-1, p) \underset{p=\frac{\lambda}{n}}{\approx} \mathrm{Poi}(\lambda)$.

But we know that $BP(\mathrm{Poi}(\lambda))$ undergoes a phase transition w.r.t. to λ $\left(\begin{array}{l}\text{if } \lambda \leq 1 \Rightarrow \text{a.s. finite} \\ \text{if } \lambda > 1 \Rightarrow \text{infinite with prob } q\end{array}\right)$

B. Phase transition

Recall $|\mathcal{C}_{max}|$ = size of maximal component in $G(n,p)$,

while $|\mathcal{C}_{(2)}|$ = size of the second maximal component

Recall $|C_{max}|$ = size of maximal component in $G(n,p)$,
while $|\mathcal{C}_2|$ = size of the second maximal component.

Theorem: Let $p = \frac{\lambda}{n}$. Then

1. if $\lambda < 1$: $\frac{|\mathcal{C}_{max}|}{\lg n} \xrightarrow{d,P} c_\lambda = (\lambda - 1 - \lg\lambda)^{-1} = [I_\lambda(1)]^{-1}$

   $(|\mathcal{C}_{max}| \sim c_\lambda \lg n)$

2. if $\lambda > 1$: i. $\frac{|\mathcal{C}_{max}|}{n} \xrightarrow{d,P} \zeta$ (where ζ is sol: $\zeta = 1 - e^{-\lambda\zeta}$)

   $(|\mathcal{C}_{max}| \sim \zeta \cdot n)$

   ii. $\frac{|\mathcal{C}_2|}{n} \xrightarrow{d,P} 0$ (hence $\mathcal{C}_{max}$ is unique)

Comments:

- $\mathcal{C}_{max}$ is called giant component
- $G(n,p)$ with $p = \frac{\lambda}{n}$, $\lambda > 1$ is highly connected
- This abrupt change while passing $\lambda = 1$ is called phase transition and:
  - $\lambda < 1$ is called subcritical
  - $\lambda = 1$ ... critical
  - $\lambda > 1$ is supercritical
- Result 2.ii. can be improved to show that $\exists\, c > 0$

  $$P(|\mathcal{C}_2| \leq c \lg n) \xrightarrow[n \to \infty]{} 1$$

C. Tools: Chernoff bounds

Recall the Markov inequality:

if $X \geq 0$ r.v., $a > 0$: $P(X \geq a) \leq \frac{E(X)}{a}$

Chernoff bounds are obtained as an application of this bound, that applies to general X r.v. (with finite average):

* $a > \mathbb{E}(X)$: $\mathbb{P}(X \geq a) \overset{\forall t>0}{=} \mathbb{P}(e^{Xt} \geq e^{at}) \overset{M.I}{\leq} e^{-at}\,\mathbb{E}(e^{Xt})$
* $a < \mathbb{E}(X)$: $\mathbb{P}(X \leq a) \overset{\forall t>0}{=} \mathbb{P}(e^{-Xt} \geq e^{-at}) \overset{M.I}{\leq} e^{at}\,\mathbb{E}(e^{-Xt})$

Thus, minimizing over $t>0$, we get

$$\mathbb{P}(X \geq a) \leq \inf_{t>0} \{e^{-at}\,\mathbb{E}(e^{Xt})\} = e^{-\sup_{t>0}\{at - \lg \mathbb{E}(e^{Xt})\}}$$

- Let $(X_k)_{k\in\mathbb{N}}$ iid r.v. s.t $\mathbb{E}(X_k) = \mu < \infty$

  Let $S_n = \sum_{k=1}^{n} X_k \implies \mathbb{E}(S_n) = n\cdot\mu$

  For large n (by LLN + CLT) : $S_n \approx n\mu + O(\sqrt{n})$

  If $a > \mu$: $\mathbb{P}(S_n \geq an) = \mathbb{P}(S_n \geq \mu n + (a-\mu)\cdot n)$

Theorem [Chernoff bounds]

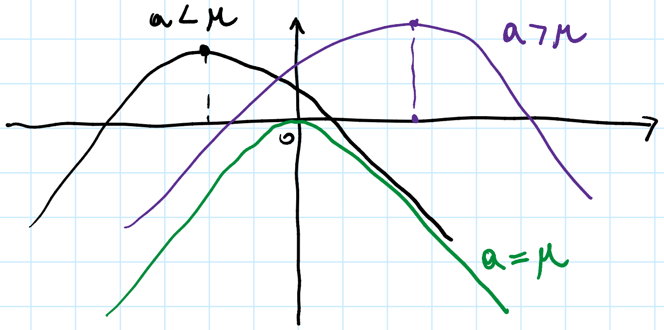
- If $a > \mu \implies \mathbb{P}(S_n \geq an) \leq e^{-nI(a)}$
- If $a < \mu \implies \mathbb{P}(S_n \leq an) \leq e^{-nI(a)}$

where $I : \mathbb{R} \to \overline{\mathbb{R}}$ given by $I(a) = \sup_{t\in\mathbb{R}} \{ta - \lg \mathbb{E}(e^{tX_1})\}$

Comments :

- I is the Legendre transform of $\lg \mathbb{E}(e^{tX_1})$
- Let $g(t) = t\cdot a - \lg \mathbb{E}(e^{tX_1})$. It turns out that
  * $g(t)$ is concave
  * $g'(t)|_{t=0} = a - \mathbb{E}(X_1)^{\mu} = \begin{cases} <0 & \text{if } a<\mu \\ =0 & \text{if } a=\mu \end{cases}$

$*\quad g'(t)\big|_{t=0} = a - \mathbb{E}(X_1) = \begin{cases} =0 & \text{if } a=\mu \\ >0 & \text{if } a>\mu \end{cases}$



$\Downarrow$

- if $a=\mu$, the bound are useless
- if $a \neq \mu$, then $I(a) > 0$.

Proof: Application of the exponential Markov inequality:

$$\mathbb{E}\left[e^{tS_n}\right] = \prod_{j=1}^{n} \mathbb{E}\left[e^{tX_j}\right] = e^{n \log \mathbb{E}(e^{tX_1})}$$

#

Examples:

1. Let $S_n \sim Poi(n\lambda)$. $S_n \overset{d}{=} \sum_{k=1}^{n} X_k$, (X_k) iid $\sim Poi(\lambda)$

Recall $\mathbb{E}\left[e^{tX_1}\right] = e^{-\lambda(1-e^t)}$. Hence:

$$I_\lambda(a) = \sup_{t\in\mathbb{R}} \{ta + \lambda(1-e^t)\} = \lambda - a - a\log\frac{\lambda}{a}$$

Notice that $I_\lambda(1) = \lambda - 1 - \log\lambda$

$\Longrightarrow$ If $a > \lambda$: $\mathbb{P}(S_n \geq an) \leq e^{-nI_\lambda(a)}$

2. Let $S_n \sim Bin(n,p)$. $S_n \overset{d}{=} \sum_{k=1}^{n} X_k$, with $(X_k)_{k\in\mathbb{N}}$ iid $\sim Be(p)$.

But $\mathbb{E}\left[e^{tX_1}\right] = pe^t + (1-p) = 1 + p(e^t - 1)$ $\left(\leq e^{-p(1-e^t)}\right)$

hence

$$I(a) = \sup_{t\in\mathbb{R}} \{ta - \log(1+p(e^t-1))\} \geq I_p(a)$$

$\hookrightarrow$ rate function of $Poi(p)$

$$= a\log\frac{p}{a} + (1-a)\log\frac{1-p}{1-a}$$

$$= a \log \frac{p}{a} + (1-a) \log \frac{1-p}{1-a}$$

↳ rate function of $\mathrm{Poi}(p)$

D. Sub-critical regime ($p = \frac{\lambda}{n}$, $\lambda < 1$)

The proof amounts to show:

- Upper bound: $\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \geq c \log n) \longrightarrow 0$ If $c > c_\lambda$
- Lower bound: $\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \leq c' \log n) \longrightarrow 0$ If $c' < c_\lambda$

Upper bound: Let $c > c_\lambda$

$$\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \geq c \log n) \leq \sum_{j \in [n]} \mathbb{P}_{n,p}(|\mathcal{C}(j)| > c \log n) = o(1)$$

$$\boxed{\text{GOAL: } \mathbb{P}_{n,p}(|\mathcal{C}(j)| > c \log n) = o(n^{-1})}$$

If $(S_k)_{k=1,\dots,T}$ is the exploration process of $\mathcal{C}(j)$:

$$\mathbb{P}_{n,p}(|\mathcal{C}(j)| > c \log n) = \mathbb{P}_{n,p}(T > c \log n)$$

- For given $k = k(n) = o(n)$, we have

$$\mathbb{P}_{n,p}(T > k) = \mathbb{P}_{n,p}(S_k \geq 1) = \mathbb{P}\Big(\mathrm{Bin}\big(n-1, \underbrace{1-(1-p)^k}_{k \cdot p = k\frac{\lambda}{n}}\big) \geq k\Big)$$

$$\approx \mathbb{P}(\mathrm{Poi}(k\lambda) \geq k \cdot 1) \leq e^{-k \cdot I_\lambda(1)} = n^{-c \underbrace{I_\lambda(1)}_{>1}} = o(n^{-1})$$

↳ $\lambda < 1$ ↳ Chernoff bounds

Taking $k = c \log n$ we get #

Remark: this argument breaks:

- when $c \leq c_\lambda$ ($\Rightarrow P(T > k) \leq n^{-a}$, $a \leq 1$)

- when $\check{c} \leq c_\lambda$ ($\Rightarrow P(T > k) \leq n^{-a}$, $a \leq 1$)
- when $k = \gamma \cdot n$ ($\Rightarrow 1 - (1-p)^k \sim 1 - e^{-\lambda\gamma}$)
- when $\lambda > 1$ (Chernoff bound).