Random Graphs and Networks - 7° LECTURE

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<u>Sub-critical regime</u> : $p = \frac{\lambda}{n}$, $\lambda < 1$

1. (of theorem) $\frac{|\mathcal{C}_{max}|}{\lg n} \xrightarrow[n\to\infty]{\mathbb{P}} c_\lambda = (I_\lambda(1))^{-1} = \frac{1}{\lambda - 1 - \lg \lambda}$

$\searrow \mathbb{P}_{n,p}\left(\left|\frac{|\mathcal{C}_{max}|}{\lg n} - c_\lambda\right| > \varepsilon\right) \to 0$

We already proved: $\forall\, c > c_\lambda$

$$\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \geq c \lg n) \xrightarrow[n\to\infty]{} 0 \qquad \text{(upper bound)}$$

<u>Rough idea on the lower bound:</u> $\forall\, c' < c_\lambda$

$$\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \leq c' \lg n) \xrightarrow[n\to\infty]{} 0$$

Notice that taking $c = c_\lambda + \varepsilon$, $c' = c_\lambda - \varepsilon$, we obtain the precise convergence in probability of $\frac{|\mathcal{C}_{max}|}{\lg n}$

For $k \in [n]$, let

$$Z_{\geq k} := \sum_{j\in[n]} \mathbb{1}_{\{|\mathcal{C}(j)| \geq k\}}$$

so that $\mathbb{P}_{n,p}(|\mathcal{C}_{max}| < k) = \mathbb{P}_{n,p}(Z_{\geq k} = 0) \leq \frac{\mathrm{Var}_{n,p}(Z_{\geq k})}{\mathbb{E}_{n,p}(Z_{\geq k})^2}$ (↑ 2° moment method)

$\cdot\ \mathbb{E}_{n,p}(Z_{\geq k}) = \sum_{j\in[n]} \mathbb{P}_{n,p}(|\mathcal{C}(j)| \geq k) = n\,\mathbb{P}_{n,p}(T \geq k)$

$= n\,\mathbb{P}_{n,p}(S_k \geq 0)$

→ Stochastic domination

→ Branching process approximation #

A. Super-critical regime: $p = \frac{\lambda}{n}$, $\lambda > 1$

Recall the result we aim to prove:

i. $\frac{|\mathcal{C}_{max}|}{n} \xrightarrow[n\to\infty]{d,\mathbb{P}} \zeta_\lambda$ ii. $\frac{|\mathcal{C}_2|}{n} \xrightarrow[n\to\infty]{d,\mathbb{P}} 0$

where ζ_λ is the survival probability of $BP(Poi(\lambda))$

↳ $\zeta_\lambda = 1 - \eta_\lambda$, where $\eta_\lambda = e^{\lambda(\eta_\lambda - 1)} \Leftrightarrow \boxed{\zeta_\lambda = 1 - e^{-\lambda \zeta_\lambda}}$

Proof:

1. a. First we show that $\mathbb{P}_{n,p}(|\mathcal{C}_{max}| = o(n)) \xrightarrow[n\to\infty]{} 0$

   b. Then we prove upper and lower bounds on $\frac{|\mathcal{C}_{max}|}{n}$

Step a.: Let $k = k(n)$ s.t. $k(n) \xrightarrow[n\to\infty]{} \infty$ but $k(n) = o(n)$

$$\mathbb{P}_{n,p}(|\mathcal{C}_{max}| \le k) = \mathbb{P}_{n,p}\Big(\bigcap_{j\in[n]} \{|\mathcal{C}(j)| \le k\}\Big)$$

$$\le \mathbb{P}_{n,p}(|\mathcal{C}(j)| \le k)$$

But $\mathbb{P}_{n,p}(|\mathcal{C}(j)| \le k) = \mathbb{P}_{n,p}(T \le k) = \mathbb{P}_{n,p}(S_k \le 0)$

$$= \mathbb{P}_{n,p}\big(Bin(n-1, \underbrace{1-(1-p)^k}_{\approx kp = \frac{k\lambda}{n}}) < k\big)$$

$k = k(n) \to \infty$

$\dots - I_\lambda(1)\cdot k$

$\approx kp = \frac{\lambda k}{m}$ $\qquad k = k(m) \to \infty$

$$\approx \mathbb{P}\left(\text{Poi}(\lambda k) < k \right) \leq e^{-I_\lambda(1)\cdot k} \xrightarrow[m\to\infty]{} 0$$

since $\lambda > 1$, by Chernoff bounds

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b. $\mathbb{P}_{mp}\left(|\mathcal{C}(j)| \leq k \right)$ as $k = c\cdot m$, $c > 0$

$$\mathbb{P}_{mp}\left(T \leq k \right) = \mathbb{P}\left(S_k \leq 0 \right) = \mathbb{P}\left(\text{Bin}\left(m-1, 1-(1-p)^k\right) < k \right)$$

$(1-p)^k \approx 1 - e^{-\lambda c}$

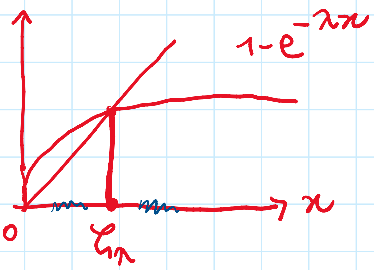
$$\approx \mathbb{P}\left(\text{Bin}\left(m, 1-e^{-\lambda c} \right) < cm \right)$$

- If $1 - e^{-\lambda c} > c$: $\iff$ $c < \zeta_\lambda$

$$\mathbb{P}_{mp}\left(|\mathcal{C}(j)| \leq cm \right) \leq e^{-m I_{\text{Bin}}(c)} \xrightarrow[m\to\infty]{} 0$$

$\hookrightarrow$ Chernoff bounds

But



Then , $\forall\, c < \zeta_\lambda$

$$\mathbb{P}\left(|\mathcal{C}_{max}| \leq c\cdot m \right) \leq \mathbb{P}_{mp}\left(|\mathcal{C}(j)| \leq cm \right) \xrightarrow[m\to\infty]{} 0$$

- If $1 - e^{-\lambda c} < c$ $\iff$ $c > \zeta_\lambda$

$$\mathbb{P}_{mp}\left(|\mathcal{C}(j)| \leq cm \right) = 1 - \mathbb{P}_{mp}\left(|\mathcal{C}(j)| > cm \right)$$

$$\geq 1 - e^{-I_{\text{Bin}}(c)\cdot m} \xrightarrow[m\to\infty]{} 1$$

Then, $\forall\, c > c_\lambda$

$$\mathbb{P}_{n,p}\left(|\mathcal{C}_{max}| > cn\right) \leq n\, \mathbb{P}_{n,p}\left(|\mathcal{C}(j)| > cn\right)$$

$$\leq n e^{-I_{Bin}(c)\cdot n} \longrightarrow 0 \qquad \#$$

[ii] Iterate the exploration procedure:

- select $j_1 \in [n]$ and compute $|\mathcal{C}(j_1)|$
- select $j_2 \in [n] \setminus \mathcal{C}(j_1)$ and compute $|\mathcal{C}(j_2)|$

and so on until we find a giant component.
(so to analyze the second largest component)

1° FACT: Notice that at each iteration, the remaining graph has distribution $G(m,p)$ where $m = \#$ remained vertices (by independence between edges)

2° FACT: When we remove a component that is not giant, then $m = n - o(n)$

$$\Rightarrow G(m,p) \simeq G(n,p) \quad \text{as } n \to \infty$$

3° FACT: When we find and remove a giant component, the remaining graph as distribution. (conjugate parameter of λ)

$$G\Big(\underbrace{n - n\cdot c_\lambda}_{\shortparallel}, \frac{\lambda}{n}\Big) = G\Big(m, \frac{\overbrace{\lambda\cdot \eta_\lambda}^{=:\,\mu_\lambda}}{m}\Big) = G\Big(m, \frac{\mu_\lambda}{m}\Big)$$

$$n(1-c_\lambda) = n\cdot \eta_\lambda =: m \iff n = \frac{m}{\eta_\lambda}$$

$\Rightarrow G(m, \frac{\mu_\lambda}{m})$ is a Erdős–Rényi RG in the sparse regime

$\longrightarrow G(n, \frac{\mu_\lambda}{n})$ is a Erdös-Rényi R.G. in the sparse regime with $\mu_\lambda < 1$ $[\mu_\lambda < 1 < \lambda] \longrightarrow$ sub-critical regime

Thus, by part 1. of Theorem, its maximal component is at most $O(\log n)$. In conclusion:

$$\frac{|\mathcal{C}_2|}{n} \longrightarrow 0 \qquad \left(\begin{array}{l} \text{and in fact that } \exists\, c>0 \text{ s.t.} \\ \mathbb{P}_{np}(|\mathcal{C}_2| \leq c \log n) \longrightarrow 1 \end{array} \right)$$

Comment: This procedure provides the proof of the **Duality principle** for ER random graphs:

Let $\mu_\lambda < 1 < \lambda$ conjugate parameters (of Poisson distributions).

Then: $$G(n, \tfrac{\lambda}{n}) \setminus \mathcal{C}_{max} \stackrel{d}{\approx} G(n \cdot \eta_\lambda, \tfrac{\mu_\lambda}{n}).$$

[notice similarity to the duality principle in BP:

$$BP(Poi(\lambda)) \mid \{T < \infty\} \stackrel{d}{=} BP(Poi(\mu_\lambda))$$]

B. Small-world property of $G(n, \frac{\lambda}{n})$, with $\lambda > 1$

Recall that a sequence $(G_n)_{n \in \mathbb{N}}$ is small-world if for $U_1, U_2 \sim \text{Uniform}[n]$, indip., $\exists\, K > 0$

$$\mathbb{P}\left(\text{dist}_{G_n}(U_1, U_2) \leq K \log n\right) \xrightarrow[n \to \infty]{} 1$$

When we randomize the graph sequence, hence $(G(n, \frac{\lambda}{n}))_{n \in \mathbb{N}}$, When we randomize the graph sequence, hence $\left(G(n, \frac{x}{n})\right)_{n \in \mathbb{N}}$,

$$\overline{\mathbb{P}}\left(dist_{G(n,p)}(U_1, U_2) \leq K \log n \right) \xrightarrow[n \to \infty]{} 1$$

where $\overline{\mathbb{P}} = \mathbb{P}_{n,p} \times P$.

C. Local limit of sequence of graphs

1. Deterministic sequence $(G_n)_{n \in \mathbb{N}}$

a. Consider rooted graphs (G_n, ρ_n), where $\rho_n \in [n]$ is fixed, and let (G, ρ) be a rooted infinite-size graph.

<u>Def</u>: $((G_n, \rho_n))_{n \in \mathbb{N}}$ <u>converges locally</u> to (G, ρ) if

$$\forall r \geq 1 \qquad \lim_{n \to \infty} B_r^{G_n}(\rho_n) \simeq B_r^G(\rho)$$

isomorph $\to$ $\exists\ \phi$ bijection between vertex-sets s.t. $i \sim j \iff \phi(i) \sim \phi(j)$

or equivalently if

$$\boxed{\lim_{n \to \infty} B_r^{G_n}(\rho_n) \simeq H^* \iff B_r^G(\rho) \simeq H^*}$$

$\forall H^* = (H, \rho)$ generic rooted graph.

<u>Comment</u>: This definition is restrictive in the following sense:

* It strongly depends on $(\rho_n)_{n \in \mathbb{N}}$
  $\longrightarrow$ We take instead a r.v. $U \sim \text{Uniform}[n]$
* We want the freedom to randomize the limiting graph (G, ρ)

b. **Def**: $(G_n)_{n\in\mathbb{N}}$ converges weakly locally to (G,ρ) if

$$\begin{cases}\forall r\geq 1 \\ \forall H^*\end{cases} : \mathbb{P}\left(B_r^{G_n}(U)\simeq H^*\right) \xrightarrow[n\to\infty]{} \tilde{\mathbb{P}}\left(B_r^{G}(\rho)\simeq H^*\right)$$

$$\| $$

$$\frac{1}{n}\sum_{j\in[n]} \mathbb{1}\{B_r^{G_n}(j)\simeq H^*\}$$

empirical distribution of r-Balls in G_n

2. Local convergence of random graphs: $(G_n)_{n\in\mathbb{N}}$

Def: $(G_n)_{n\in\mathbb{N}}$ converges local in probability to (G,ρ) if

$$\begin{cases}\forall r\geq 1 \\ \forall H^*\end{cases} \mathbb{P}\left(B_r^{G_n}(U)\simeq H^*\right) \xrightarrow[n\to\infty]{\mathbb{P}} \tilde{\mathbb{P}}\left(B_r^{G}(\rho)\simeq H^*\right)$$

(G_n random; $=: Y_n$)

Remark: Since $(Y_n)_{n\in\mathbb{N}}$ is uniformly bounded in $n\in\mathbb{N}$ ($\in[0,1]$), it also converges in average w.r.t. $\mathbb{E}_{n,p}$.

Theorem 1: $G(n,\frac{\lambda}{n})$ with $\lambda>1$ converges locally in probability to $BP(Poi(\lambda))$.

Theorem 2: $G(n,\frac{\lambda}{n})$ with $\lambda>1$. Conditionally on the event $\{U_1\longleftrightarrow U_2\}$, for $U_1,U_2\sim \text{Uniform}[n]$, it holds:

$$\frac{dist_{G(n,p)}(U_1,U_2)}{\log n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\log\lambda}$$

Proof idea:

$$\bar{\mathbb{P}}\left(dist_{G(n,p)}(U_1,U_2)\leq k\right) = \mathbb{E}\left(\mathbb{1}_{\{dist_{G(n,p)}(U_1,U_2)\leq k\}}\right)$$

averaging over

averaging over U_2

$$= \frac{1}{n} \mathbb{E}\left(\sum_{j \in [n]} \mathbb{1}\{ dist_{G(n,p)}(U_1, j) \leq k \} \right)$$

$$= \frac{1}{n} \mathbb{E}\left(\left| B_k^{G(n,p)}(U_1) \right| \right) \approx \frac{1}{n} \sum_{j=0}^{k} \tilde{\mathbb{E}}(Z_j) = \frac{1}{n} \sum_{j=0}^{k} \lambda^j$$

by theorem 1, $B_k^{G(n,p)}(U_1) \approx B_k^T(1)$

where $T \sim BP(Poi(\lambda))$ identified by $(Z_j)_{j \in \mathbb{N}_0}$ (# individuals in generation j)

$$= \frac{1}{n} \frac{\lambda^{k+1} - 1}{(\lambda - 1)}$$

$k = \log n \cdot \frac{1}{\log \lambda}$

$$= \frac{1}{n} e^{k \log \lambda + ct}$$

$\longrightarrow 1$ if $k = \log n \cdot \frac{1}{\log \lambda}$ + small corrections

$\longrightarrow 0$ if $k < \log n \cdot \frac{1}{\log \lambda}$

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