Random Graphs and Networks - 8° LECTURE

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Goal: consider general random graph models and be able to model scale-free property

→ The simplest way to generalize $G(n,p)$ is to consider edge-connectivities that are independent but not identical: $(p_e)_{e \in E_n}$ so that, $\forall e \in E_n$

$$X_e(G) = \begin{cases} 1 & \text{if } e \in G \\ 0 & \text{if } e \notin G \end{cases} \Rightarrow X_e \sim Be(p_e)$$

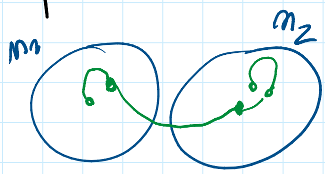
and $(X_e)_{e \in E_n}$ are all independent.

⟶ This leads to the definition of Inhomogeneous RG

Inhomogeneous Random Graph

$\boxed{A.}$ Motivating example: 2-type random graph

- We divide the vertex-set $[n]$ in 2 groups:
  - group of type 1 of size n_1
  - group of type 2 of size n_2



→ We assign a probability connection to an edge (i,j) depending on the type of i and of j. Hence

$$t : [n] \longrightarrow \{1,2\} \qquad \text{and set} \qquad \boxed{p(i,j) = p(t(i), t(j))}$$
$$i \longrightarrow t(i) = \text{type}(i)$$

This is represented by a 2×2 symmetric matrix:

$$\begin{pmatrix} p(1,1) & p(1,2) \\ p(1,2) & p(2,2) \end{pmatrix} \longrightarrow \text{connection with stochastic block model}$$

This model is specified by:

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- The sizes n_1 and n_2 of the two groups, that should increase with n to avoid triviality. Equivalently, we consider a probability density $\mu : \{1,2\} \to [0,1]$ s.t

$$n_1 = \mu(1)\cdot n \qquad \text{and} \qquad n_2 = \mu(2)\cdot n$$

- A 2×2 symmetric matrix providing the edge-connectivities as function of types.

Comment: Inspired by ER random graph, we mainly deal with edges connectivities:

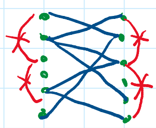
$s,t \in \{1,2\}$ $$p(s,t) = \frac{K(s,t)}{n} \qquad \left(\begin{array}{c}\text{sparse}\\ \text{regime}\end{array}\right)$$

For some matrix $K = (K(s,t))_{s,t\in\{1,2\}}$ symmetric, non-negative (irreducible, bounded).

Examples:

1. $K = \begin{pmatrix} \lambda & \lambda \\ \lambda & \lambda \end{pmatrix} \Longrightarrow G\left(n, \frac{\lambda}{n}\right)$

2. $K = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \Longrightarrow$ Bipartite random graph



type 1 type 2

Consider the degree of a vertex:

- Let $j \in [n]$ s.t. $t(j) = 1$. Then

$$d(j) = \sum_{k\in[n]} \mathbb{1}_{\{k\sim j\}} = \overbrace{\sum_{\substack{k:\\ t(k)=1}}}^{n_1} \underbrace{\mathbb{1}_{\{k\sim j\}}}_{Be(p(1,1))} + \overbrace{\sum_{\substack{k:\\ t(k)=2}}}^{n_2} \underbrace{\mathbb{1}_{\{k\sim j\}}}_{Be(p(1,2))}$$

by independence

$$\ldots \sum_{k\in[n]} \ldots \underbrace{\ldots}_{\substack{t(k)=1 \\ Be(p(1,1))}} \ldots \underbrace{\ldots}_{\substack{t(k)=2 \\ Be(p(1,2))}}$$

by independence of edges

$$\stackrel{d}{=} \mathrm{Bin}\left(\mu(1)\cdot n, \frac{K(1,1)}{n}\right) + \mathrm{Bin}\left(\mu(2) n, \frac{K(1,2)}{n}\right) \quad \text{(indep.)}$$

$$\approx \mathrm{Poi}\left(\mu(1)\cdot K(1,1)\right) + \mathrm{Poi}\left(\mu(2) K(1,2)\right)$$

$\boxed{\text{B.}}$ Definition of IRG

- Let T = set of types and μ a probability measure on T, so that $\mu(t)\cdot n = \#\{\text{vertices of type } t\}$.

Hyp: to simplify the discussion we assume $|T| < \infty$, though in general it may be $|T| = \infty$ (countable or not)

- Let $K = (K(s,t))_{s,t\in T}$ a symmetric, non-negative matrix (irreducible, bounded), called kernel, s.t. $\forall\, i,j \in [n]$

$$p(i,j) = p(t(i), t(j)) = \frac{K(t(i), t(j))}{n}$$

where $t : [n] \to T$.

The resulting graph is denoted $G(n,K)$, and it has law denoted by $P_{n,K}$.

$\boxed{\text{C.}}$ Vertex-degree sequence of $G(n,K)$

- Let $j \in [n]$ s.t. $t(j) = t \in T$

$$d(j) = \sum_{i\in[n]} \mathbb{1}_{\{i\sim j\}} = \sum_{s\in T} \overbrace{\sum_{\substack{i\in[n]: \\ t(i)=s}}}^{\mu(s)\cdot n} \overbrace{\mathbb{1}_{\{i\sim j\}}}^{Be\left(\frac{K(s,t)}{n}\right)}$$

stays for $d_{G(n,K)}(j)$

$$\stackrel{d}{=} \sum_s \mathrm{Bin}\left(n\cdot\mu(s), \frac{K(s,t)}{n}\right) \approx \mathrm{Poi}\Big(\underbrace{\sum \mu(s) K(s,t)}_{\mu(t,s)}\Big) = \mathrm{Poi}\left(\lambda(t)\right)$$

$t(i)=s$ $\quad \mu(t,s)$

stays for $d_{G(n,K)}(j)$

$$\stackrel{d}{=} \sum_{s\in T} \underbrace{\mathrm{Bin}\left(n\cdot\mu(s), \frac{K(s,t)}{n}\right)}_{\approx \mathrm{Poi}(\mu(s)K(s,t))} \approx \mathrm{Poi}\left(\underbrace{\sum_{s\in T}\mu(s)K(s,t)}_{\underbrace{K\cdot\mu(t)}_{\stackrel{!!}{=}\lambda(t)}}\right) = \mathrm{Poi}(\lambda(t))$$

- Let $U \sim \mathrm{Uniform}[n]$ and consider $d(U)$

$$P(d(U)=k) = \underbrace{\frac{1}{n}\sum_{i\in[n]} \mathbb{1}(d(i)=k)}_{\text{empirical distr. of a vertex degree}} = \sum_{t\in T}\sum_{\substack{i\in[n]:\\ t(i)=t}} \frac{\mathbb{1}\{d(i)=k\}}{n}$$

Averaging w.r.t. $\mathbb{P}_{n,K}$, we get:

$$\mathbb{P}_{n,K}(d(U)=k) = \sum_{t\in T}\sum_{\substack{i\in[n]\\ t(i)=t}} \frac{\mathbb{P}_{n,K}(d(i)=k)}{n}$$

$$= \sum_{t\in T}\overbrace{\sum_{\substack{i\in[n]:\\ t(i)=t}}}^{\mu(t)\cdot n} \frac{1}{n}\frac{\lambda(t)^k}{k!}e^{-\lambda(t)} =$$

$$= \sum_{t\in T} \mu(t)\cdot\frac{\lambda(t)^k}{k!}e^{-\lambda(t)}$$

called mixed Poisson distribution

In short: $d_{G(n,K)}(U) \stackrel{d}{\approx} \mathrm{Poi}(Z)$ s.t. Z is random:

$$\mathbb{P}(Z=\lambda(t)) = \mu(t)$$

<u>Consequence</u>:

<u>1° FACT</u>: It is possible to consider kernel K s.t.

$$\mathbb{P}_{n,K}(d(U)>k) \approx \mathbb{P}(Z>k)$$

where $\mathbb{P}(Z=\lambda(t)) = \mu(t)$ and $\lambda = K\cdot\mu$

<u>2° FACT</u>: It is possible to prepare λ (and hence K) so that

$$\mathbb{P}(Z>k) \sim c\cdot k^{-(\tau-1)} \quad \text{as } k\to\infty$$

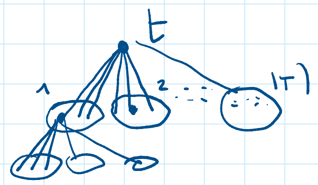
that $\mathbb{P}(Z > k) \sim c \cdot k^{-(\tau-1)}$ as $k \to \infty$

$\longrightarrow \quad \mathbb{P}_{m,k}(d(U) > k) \sim c \cdot k^{-(\tau-1)} \longrightarrow$ scale-free behavior

D. Multi-type branching processes

- Assign to each individual a type $t \in T$
- Provide a branching rule that depends on the type (of parent and child)

1. Start with 1 individual of given type $t \in T$
2. Let $\underline{j} = (j_1, \dots, j_{|T|}) \in \mathbb{N}_0^{|T|}$ and consider the joint offspring density



$$p_{\underline{j}}^{(t)} = \mathbb{P}\left(\begin{array}{l}\text{a vertex of type } t \text{ gives rise} \\ \text{to offspring } \underline{j}: j_1 \text{ of type } 1, \dots \\ \dots, j_{|T|} \text{ of type } |T|\end{array}\right), \forall t \in T, \forall \underline{j} \in \mathbb{N}_0^{|T|}$$

Assume that each individual of type t, indep. of the others, produce offsprings of given type accordingly to $p_{\underline{j}}^{(t)}$.

We will focus on the case in which:

$$p_{\underline{j}}^{(t)} = \prod_{s \in T} p_{j_s}^{(t,s)}, \quad \text{where } \left(p_{j_s}^{(t,s)}\right)_{j_s \in \mathbb{N}_0} \leftrightarrow \mathrm{Poi}(\lambda(t,s))$$

Notation: $\forall s, t \in T$

- $Z_{m,s}^{(t)} :=$ # vertices in generation m of type s and parent t
- $\underline{Z}_m^{(t)} = \left(Z_{m,s}^{(t)}\right)_{s \in T}$
- $\zeta^{(t)} := \mathbb{P}\left(\underline{Z}_m^{(t)} \neq \underline{0}, \forall m \in \mathbb{N}\right)$

- $\mathcal{G}^{(t)} := \mathbb{P}\left(\underline{Z}_m^{(t)} \neq \underline{0}, \ \forall m \in \mathbb{N} \right)$

and let $\underline{\mathcal{G}} = \left(\mathcal{G}^{(t)} \right)_{t \in T}$

- $G^{(t)} : [0,1]^{|T|} \longrightarrow \mathbb{R}$ s.t

$$G^{(t)}(\underline{q}) = \sum_{\underline{j} \in \mathbb{N}_0^{|T|}} p_{\underline{j}}^{(t)} \cdot \prod_{\ell \in T} q_\ell^{j_\ell} = \underline{q}^{\underline{j}}$$

Notice that if $\left(p_{j_s}^{(t,s)} \right)_{j_s \in \mathbb{N}_0} \leftrightarrow \underline{Poi(\lambda(t,s))}$, then

$$G^{(t)}(\underline{q}) = \mathbb{E}\left(\prod_{s \in T} q_s^{Z_{1,s}^{(t)}} \right) = e^{\sum_{s \in T} \lambda(t,s)(q_s - 1)}$$

$$= e^{\Lambda(\underline{q} - \underline{1})(t)}$$

where $\Lambda = \left(\lambda(s,t) \right)_{s,t \in T}$.