Random Graphs and Networks - 9° LECTURE

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Recall: IRG is $G(n,K)$ is s.t.

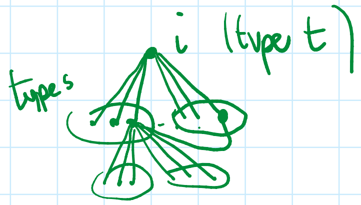
$\mathbb{P}\big(e=(i,j) \text{ is in } G(n,K)\big) = p\big(t(i),t(j)\big)$, indep. $\forall\, i,j\in[n]$, where

$t:[n]\longrightarrow T$ (type set)

- $p(s,t)=\dfrac{K(s,t)}{n}$

Then, $\forall\, i\in[n]$ s.t. $t(i)=t$: $\quad d_{G(n,K)}(i) \overset{d}{\approx} \overbrace{\sum_{s\in T} \mathrm{Poi}(\lambda(t,s))}^{\text{indep}}$

where $\lambda(t,s)=K(t,s)\cdot\mu(s)$

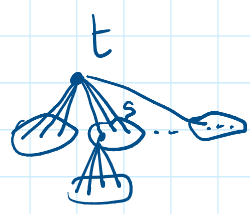


$\longrightarrow$ Multi-type BP with Poisson offspring distribution

- Let $\Lambda=\big(\lambda(s,t)\big)_{s,t\in T}$.

- Assume that $\mathrm{Poi}(\lambda(s,t))$ = law of the number of offsprings of type t generating from a vertex of type s

indipendently for all vertices and types



$\boxed{A.}$ Extinction and survival probabilities

- In the above setting, let $\nu=\sup\limits_{x:\|x\|\le 1}|\Lambda x|$

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(that is ν is the maximal eigenvalue of Λ)

Theorem [survival/extinction probabilities]

1. $\underline{\zeta}_\lambda \neq \underline{0} \iff \nu > 1$. Hence

   · if $\nu \leq 1 \implies$ the BP dies out with prob. 1

   · if $\nu > 1 \implies$ the BP survives with prob. $\zeta_\lambda := \sum_{t \in T} \zeta_\lambda^{(t)}$

2. $\underline{\zeta}_\lambda$ is the largest sol. of $\underline{\zeta}_\lambda = \underline{1} - \underline{G}(\underline{1} - \underline{\zeta}_\lambda)$

   (or equiv., if $\underline{\eta}_\lambda = \underline{1} - \underline{\zeta}_\lambda$ then $\underline{\eta}_\lambda = \underline{G}(\underline{\eta}_\lambda)$)

   and explicitly: $\underline{\zeta}_\lambda = \underline{1} - e^{-\Lambda \underline{\zeta}_\lambda}$.

B. Results

1. Local structure: The IRG $G(n, K)$ converges locally in probability to a multi-type BP such that:

   · the root has type $t \in T$ with prob. $\mu(t)$

   · every vertex of type $t \in T$ has type s-offspring distribution $\mathrm{Poi}(\lambda(t,s))$, indep. for all s, where

   $$\lambda(t,s) = K(t,s) \cdot \mu(s)$$

2. Phase transition: let $\Lambda = (\lambda(s,t))_{s,t \in T}$ s.t

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i: if $\nu = \sup_{x : |x|\le 1} |\Lambda x| > 1$

if $\frac{|\mathcal{C}_{max}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \zeta_\lambda$, $\frac{|\mathcal{C}_2|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$

ii: if $\nu \le 1 \implies \frac{|\mathcal{C}_{max}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$

3. <u>Small-world property</u>: If $\nu > 1$ then, conditionally on the event that $U_1 \longleftrightarrow U_2$, where U_1, U_2 are indep. Uniform $[n]$, it holds

$$\frac{dist_{G(n,K)}(U_1,U_2)}{\log n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\log \nu}$$

$\boxed{C.}$ A sub-family of IRG: Generalized Random Graphs (GRG)

- The connection probabilities $P_{i,j}$, for $i,j \in [n]$ are moderated by <u>vertex weights</u>

<u>Def</u>: Let $\underline{w} = (w_i)_{i\in[n]}$ vertex weight ≥ 0 assigned to every vertex of $[n]$.
Then, $\forall\, i \neq j \in [n]$, independently, set

$$P_{ij} = \frac{w_i \cdot w_j}{\ell_n + w_i \cdot w_j}, \quad \ell_n = \sum_{j\in[n]} w_j$$

The resulting graph is called GRG and denoted by $G(n,\underline{w})$ with law $\mathbb{P}_{n,\underline{w}}$.

with low $\mathbb{P}_{m,\underline{w}}$.

Comments:

- We may assume that $w_i > 0$ (otherwise i is isolated)
- The weights $\underline{w} = (w_i)_{i \in \mathbb{N}}$ may be deterministic or random: e.g. taken to be iid r.v. $(W_i)_{i \in \mathbb{N}} \longrightarrow$ double source of randomness.
- Notice that if $w_i = c \; \forall i \in \mathbb{N}$, then we obtain ER random graph (Specifically $\sim G(m, \frac{c}{m+c})$)
- There are different choices of $P_{ij} = \phi(\underline{w})$ [Chung-Lu model, Norros-Reittu model]
- Let $T = [0,1]$ set of types and assign to each vertex $i \in [n]$ the type $t_m(i) = \frac{i}{m} \in T$, $\forall m \in \mathbb{N}$

Then set $K\left(\frac{i}{m}, \frac{j}{m}\right) = w_i \cdot w_j \cdot \frac{m}{l_m}$

and $$P_{ij} = \frac{K\left(\frac{i}{m}, \frac{j}{m}\right)}{m + K\left(\frac{i}{m}, \frac{j}{m}\right)}$$

- For given $i \in [n]$:

$$\boxed{\mathbb{E}_{m,\underline{w}}\left(d_G(i)\right)} = \mathbb{E}_{m,w}\left(\sum_{j \in [n]} \mathbb{1}_{\{i \sim j\}}\right)$$

$$= \sum_{j \in [n]} \frac{w_i \cdot w_j}{\underbrace{l_m + w_i \cdot w_j}_{\approx l_m}} \simeq \frac{w_i}{l_m} \overbrace{\sum_{j \in [n]} w_j}^{= l_m} = \boxed{w_i}$$

Assumptions on weights

Let $W_n := w_U$, where $U \sim \text{Uniform}[n]$

a. $\exists\, W$ (r.v.) s.t. $\boxed{W_n \xrightarrow{d} W}$

Equiv: $P(W_n \le x) = \underbrace{\frac{1}{n}\sum_{j\in[n]} \mathbb{1}_{\{w_j \le x\}}}_{\text{empirical weight distribution}} \longrightarrow P(W \le x)$ $\quad \forall\, x \in \mathbb{R}$

(as or convergence in probability)

↳ If $\underline{w} = (W_i)_{i\in\mathbb{N}}$ iid r.v., it is immediately verified by LLN with $W \overset{d}{=} W_i$

b. $\boxed{E(W_n) \xrightarrow[n\to\infty]{} E(W)} < \infty$

↳ if $\underline{w} = (w_i)_{i\in\mathbb{N}}$ iid r.v., then $E(W_n) = h(\underline{w})$ is a r.v., and the convergence in b is replaced by

$$E(W_n) \xrightarrow[n\to\infty]{P} E(W)$$

Remark: under this assumption we can deduce that

$$E_{n,\underline{w}}\big[|E(G)|\big] \approx E(W)\cdot\frac{n}{2} \quad \to \text{sparse regime}$$

Theorem 1: Under conditions a. and b.:

1. $d_{G(n,\underline{w})}(U) \xrightarrow[n\to\infty]{d} D$, when D has mixed Poisson dist.:

$$P(D=k) = E\left[\frac{W^k}{k!}e^{-W}\right],$$ where W is as in a., b.

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(H) $\quad \mathbb{P}(W \geq x) \simeq c \cdot x^{-(\tau-1)}$ as $x \to \infty$

$$\Longrightarrow \underline{\mathbb{P}(D \geq x)} \simeq \mathbb{P}(W \geq x) \simeq \underline{c \cdot x^{-(\tau-1)}}$$ [scale-free property]

2. If $\nu := \dfrac{\mathbb{E}(W^2)}{\mathbb{E}(W)} > 1$ (also ∞), then in $G(n, \underline{w})$

$$\frac{|\mathcal{C}_{max}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \zeta_{\underline{W}}, \quad \frac{|\mathcal{C}_2|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$$ [high connectivity]

$\updownarrow$

Phase transition

while if $\nu \leq 1 \Longrightarrow \dfrac{|\mathcal{C}_{max}|}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$

3. If $\nu > 1$ then, conditionally on the event that $U_1 \longleftrightarrow U_2$, where U_1, U_2 indep. Uniform $[n]$,

$$\frac{dist_{G(n,\underline{w})}(U_1, U_2)}{\lg n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\lg \nu}$$ [small world property]

__Remark:__ If $\nu = \infty$ (which happens if $\mathbb{E}(W^2) = \infty$), result 3. reads as $\dfrac{dist_{G(n,\underline{w})}(U_1, U_2)}{\lg n} \xrightarrow{\mathbb{P}} 0$

This suggests an ultra-small world behavior, that indeed can be proved under hypothesis (H) with $\tau \in (2,3)$ so that

$$\frac{dist_{G(n,\underline{w})}(U_1, U_2)}{\lg \lg n} \xrightarrow{\mathbb{P}} \frac{2}{|\lg(\tau-2)|}$$

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