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Mixing time for Glauber dynamics beyond $\mathbb{Z}^d$

Doctoral Thesis

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Abstract

This dissertation studies the rate of convergence to equilibrium (mixing time) of the Glauber dynamics for the Ising model, and its connection with the geometrical properties of the underlying graph. The results fall into two main groups.

In the first part we analyze the effect of boundary conditions on the mixing time for the Glauber dynamics on hyperbolic graphs. Specifically, we show that the spectral gap of this dynamics for the Ising model on an n-vertex ball in hyperbolic tiling with $(+)$-boundary is bounded by a constant independent on n at all low temperatures. This implies that the mixing time is $O(n)$, in contrast to the free boundary case where it is not bounded by any fixed polynomial at low temperatures. The influence of the boundary condition on the mixing time has been long time conjectured, but first rigorous results only appear few years ago in the framework of spin systems on homogenous trees. The method introduced in the above mention work, is here readapted to the hyperbolic graph setting where, due to presence of cycles, a non-trivial analysis is required. The proof of our result is given through the application of analytic methods, and is based on the existence of a decay correlation between spins when the $(+)$-boundary condition is regarded. This kind of spatial mixing, which strictly depends on the geometric properties of the graph, is proved by means of a kind of Peierls argument.

In the second part we develop new techniques to study the relaxation time on random graphs. A first result concerns the Glauber dynamics for the Ising model on the binomial random graph $\mathbb{G}(n,p)$, whit $p = c/n$ and c a positive constant. In particular, for all $\beta > 0$, we prove that asymptotically almost surely the relaxation time of the dynamics on $G(n,p)$ grows with the size of the graph at least as $\exp\left(\Omega(\frac{\log n}{\log\log n})\right)$. Here the main step is to provide a suitable subgraph which slows down the dynamics in a significant

way, and then show that this subgraph is asymptotically almost surely contained in $G(n,p)$. When $c>1$, we extend this result to the dynamics on the giant component of $\mathbb{G}(n,p)$.

The second result concerns the Glauber dynamics for the Ising model on the r-regular random graph $G(n,r)$. For every $r\geq 3$ and every $\beta>\beta_0=\beta_0(r)$ (low temperatures), we prove that asymptotically almost surely the relaxation time of the dynamics on $G(n,r)$ is $\exp(\Omega(n))$. The proof of this result is based again on the asymptotically almost surely existence in $\mathbb{G}(n,r)$ of a geometrical property, which combined with the analytic techniques provides the result.

Though there is an increasing number of papers on the equilibrium behavior of spin systems on random graph, there are very few papers regarding the stochastic dynamics on them. The results presented here are a first attack to this kind of hard problems. We hope that they could stimulate further ideas and works on this direction.

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Asymptotic

- $a_n = O(b_n)$ if $\exists\, C$ such that $|a_n| \leq C\, b_n$
- $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$
- $a_n = \Omega(b_n)$ if $\exists\, C$ such that $|a_n| \geq C\, b_n$
- $a_n = \Theta(b_n)$ if $\exists\, C_1, C_2$ such that $C_1\, b_n \leq |a_n| \leq C_2\, b_n$
- $a_n \ll b_n$ if $a_n \geq 0$ and $a_n = o(b_n)$

Probability

- $\mathbb{P}(\cdot)$ probability measure
- $\mathbb{E}[\cdot]$ expectation
- $X \sim Be(p)$ Bernoulli distribution: $\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p$
- $X \sim Bin(n, p)$ binomial distribution: $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- $X \sim Po(c)$ Poisson distribution: $\mathbb{P}(X = k) = \frac{e^{-c} c^k}{k!}$
- i.i.d. independent identically distributed.

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Introduction

Spin systems are a class of models that originated in Statistical Physics, though interest in them has expanded to many other areas, including Probability Theory, Combinatorial Optimization, and Theoretical Computer Science. A *spin system* can be described as follows: let $G = (V, E)$ be a locally finite graph and associate to every vertex $v \in V$ a variable (*spin*) σ_v taking value in a finite space $\mathcal{S}$. A *configuration* of the spin system is an assignment of a spin value to each vertex of G, denoted by $\sigma \in \Omega_V = \mathcal{S}^V$. Let $U = \{U_\Delta\}_{\Delta \subset\subset V}$, where $\Delta \subset\subset V$ means that Δ is a finite subset of V, be a collection of local functions on Ω determining the local interaction between sites. We would like to define, at least formally, the Hamiltonian (or energy) of a configuration $\sigma \in \Omega_V$ by

$$H_V(\sigma) = \sum_{\Delta \subset\subset V} U_\Delta(\sigma) \tag{0.1}$$

and the Gibbs probability measure at inverse temperature $\beta > 0$ by

$$\mu_V(\sigma) = Z_V(\beta)^{-1} \exp\left(-\beta H_V(\sigma)\right), \tag{0.2}$$

where $Z_V(\beta)$ is a normalizing factor.
If G is a finite graph these two formulas are well defined and can be taken as a definition. If G is an infinite graph we can instead consider, for any $\Lambda \subset\subset V$ and any configuration $\eta \in \Omega$, the set of configurations $\sigma \in \Omega$ which agree with η in Λ^c, denoted by Ω_Λ^η. For any $\sigma \in \Omega_\Lambda^\eta$, the contribution to the energy of σ coming from Λ is given by

$$H_\Lambda^\eta(\sigma) = \sum_{\Delta \cap \Lambda \neq \emptyset} U_\Delta(\sigma) \tag{0.3}$$

and the Gibbs probability measure on Ω_Λ^η is defined as

$$\mu_\Lambda^\eta(\sigma) = Z_\Lambda^\eta(\beta)^{-1} \exp\left(-\beta H_\Lambda^\eta(\sigma)\right), \tag{0.4}$$

where $Z^{\eta}_{\Lambda}(\beta)$ is a normalizing factor. The measure μ^{η}_{Λ} is regarded as the equilibrium state of the spin system in the finite region Λ with η boundary condition. The term *free boundary Gibbs measure* is used to indicate that no boundary condition is specified and is obtained from (0.3) and (0.4) substituting the sum over $\Delta \cap \Lambda \neq \emptyset$ with a sum over $\Delta \subseteq \Lambda$.

As an example we can consider the *Ising model*. In this model, a configuration $\sigma = (\sigma_x)_{x\in V}$ consists of an assignment of ± 1 values to each vertex of V and the potential U is such that, for every finite $\Lambda \subset\subset V$ and $\eta \in \Omega$, the Hamiltonian is given by

$$H^{\eta}_{\Lambda}(\sigma) = - \sum_{(x,y)\in E(\overline{\Lambda})} \sigma_x \sigma_y - h \sum_{x\in\Lambda} \sigma_x \, ,$$

where $E(\overline{\Lambda}) \subset E$ denotes the edge with at least an end vertex in Λ and h is a real constant. According to (0.4), the Gibbs measure on Λ with η boundary condition is given by

$$\mu^{\eta}_{\Lambda}(\sigma) \; = \; Z(\beta)^{-1} \exp \big(\beta \sum_{(x,y)\in E(\overline{\Lambda})} \sigma_x \sigma_y + h \sum_{x\in\Lambda} \sigma_x \big) \, .$$

In this case μ_{Λ} assigns higher probability to configurations in which neighboring spins are aligned, as well as to configurations in which many spins agree with the sign of h. This effect increases with β, so that at high temperatures (low β) the spins behave almost independently, while at low temperature (high β) a global order may occur.

We refer to the collection of all μ^{η}_{Λ}, as Λ and η vary, as the specification for the system. It turns out that for any given specification one can define the Gibbs measures on the infinite space Ω through the notion of DLR compatibility:

Definition 0.1. *A probability measure μ on Ω is called a Gibbs measure for the specification $\{\mu^{\eta}_{\Lambda}\}_{\Lambda,\eta}$ if, for every finite region Λ and μ-almost every configuration σ,*

$$\mu(\, \cdot \, | \sigma_{\Lambda^c}) = \mu^{\sigma}_{\Lambda}(\, \cdot \,) \, . \tag{0.5}$$

Let $\mathcal{G}$ be the set of all (infinite volume) Gibbs measures relative to a given specification derived as above. It is well known that $\mathcal{G}$ is a nonempty convex compact set, with extremal points (measures) called pure phases [Ge, GHM, EFS]. If $\mathcal{G}$ has more then one element we say that the spin system, described by the specification giving rise to $\mathcal{G}$, exhibits a *phase coexistence*.

Consider for example the Ising model on the d-dimensional Cartesian lattice $\mathbb{Z}^d$. If $d > 1$, then there exists a critical inverse temperature $\beta = \beta_c$ (depending on the dimension) such that a *phase transition* occurs. For $\beta < \beta_c$ there is a *unique* Gibbs measure independent from the boundary condition and no long-range correlation between spins is present, while for $\beta > \beta_c$ there are at least two distinct Gibbs measures (phase coexistence) corresponding to the $(+)$ or $(-)$ boundary conditions, and correlations are present at arbitrary distances. See, for instance, [Ge, Pre, Sim] for more details.

Graph setting

Recently an increasing effort has been devoted to the study of spin systems on graphs other than the regular lattices. In this area one is interested to study the basic lattice models from statistical mechanics, but with the typical Euclidean lattice replaced by a fairly general graph. The current surge of interest in this subject has various motivations. Firstly, many new phenomena are not present when one only considers Euclidean lattice. This reveals the presence of an interplay between the geometry of the graph and the behavior of statistical mechanics system, and suggests the possibility of setting results in a more general theory. Secondly, non-Euclidean geometry is important, and sometimes more natural, in many applications where processes are modeled on graphs that do not need to be regular in any sense. Thirdly, problems which remain open on Euclidean lattices may be better understood, and perhaps even eventually solved, by placing them in a broader context.

One of the basic assumption in the deterministic setting, is the *transitivity* (or *quasi-transitivity*) of the graph, where we say that a graph $G = (V, E)$ is transitive if the automorphisms group of G acts transitively on the vertices in V. The family of transitive graph is very large and includes, among others, the Euclidean lattice $\mathbb{Z}^d$, the homogeneous trees and the *hyperbolic graphs* (see the description below) . Several interesting results have been derived for spin systems on infinite, connected, *transitive* graphs. See e.g. [BRZ, Io1, SS, Wu2, JS, Ly1, Ly2, Sch, HSS] for the results concerning the Ising and the Potts model, and [BS, BS2, BS3, GN, H, HJL, Jo, La, LS] for percolation and random cluster model.

Notable are the results concerning the existence of a second kind of

phase transition, proved for different statistical mechanics models on *non amenable graphs* (see [Ly1, Ly2, Sch, Io1, Wu2]).

For example, it has been proved that the Ising model on trees has two phase transitions at the critical temperatures $\beta_0 < \beta_1$. The first one, β_0, refers to the uniqueness/non uniqueness phase transition previously described, whereas β_1 refers to the property of extremality of the free boundary Gibbs measure μ^f. If $\beta \leq \beta_1$ then μ^f is extremal in $\mathcal{G}$, while for $\beta > \beta_1$ it is a convex combination of other extremal measures [BRZ, Io1, Io2]. To better appreciate this result, let us remark that this implies that the Gibbs measures corresponding to the $(+)$ and $(-)$ boundary conditions are not the only extremal measures (automorphism invariant), at the contrary to what happens for $\mathbb{Z}^d$ (see [Aiz, Hi] for the case $d = 2$, and [Bod] for $d \geq 3$). The landscape provided by this model is thus broader then the classical Ising model on $\mathbb{Z}^d$, and offer the possibility of new modeling.

In the first part of this dissertation we will consider the Ising model on a family of non-amenable graphs with a cycle periodic structure: the *hyperbolic graphs* (see, e.g., [Mag]). They can be thought as the hyperbolic plane counterpart of tiling in the Euclidean plane $\mathbb{R}^2$; but while in $\mathbb{R}^2$ there are only three possible tiling ($\mathbb{Z}^2$, the triangular lattice and the hexagonal lattice), their number is infinite in the hyperbolic plane. Hyperbolic graphs can be characterized by two integers, both ≥ 3: v, the number of neighbors of each vertex; and s, the number of sides of each face (tile), and thus denoted by $\mathbb{H}(v, s)$. In order for the embedding in the hyperbolic plane to be well defined, the integers v and s have to satisfy the relation $(v-2)(s-2) > 4$. The typical representation of hyperbolic tilings makes use of the Poincaré disc that is in bi-univocal correspondence with the hyperbolic plane (see fig. 0.1).

In analogy with the behavior of the Ising model on trees, the study of this model on hyperbolic graphs led to the characterization of two different phase transitions appearing at inverse temperatures $\beta_c \leq \beta_c'$, where the strict inequality has been proved when v is large enough or $\mathbb{H}(v, s)$ is self-dual [SS, Wu, Wu2]. The first one, β_c, corresponds to the occurrence of a uniqueness/nonuniqueness phase transition, while the critical temperature β_c' refers to a change of the properties of the free boundary condition measure μ^f. If $\beta_c < \beta \leq \beta_c'$ then μ^f is not a convex combination of μ^+ and μ^-, which implies the existence of an automorphism invariant extremal

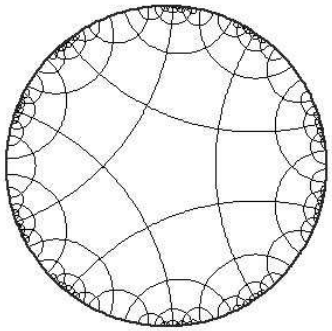

Figure 0.1: The hyperbolic graph $\mathbb{H}(4,5)$ in the Poincaré disc representation.

measure different from μ^+ and μ^-, while if $\beta > \beta_c'$ then μ^f recovers the property $\mu^f = (\mu^+ + \mu^-)/2$. We remark that this behavior distinguishes the hyperbolic graph from the regular tree, where $\mu^f \neq (\mu^+ + \mu^-)/2$ for all $\beta > \beta_0$.

Glauber dynamics

While the classical theory focused on the equilibrium properties of the Gibbs measures, in modern statistical physics the emphasis has shifted towards dynamical question with a computational flavor. The key object here is the *Glauber dynamics*, a (discrete or continuous time) Markov chain on the set of spin configuration Ω_V of a finite graph $G = (V, E)$. For definiteness, we describe the "heat-bath" version of Glauber dynamics: to each vertex $x \in V$ we associate, independently, a Poisson process with mean one. At each arrival of the process at x, the associated spin is refreshed by a random spin drawn from the distribution of σ_x conditional on all the neighboring spins. It is easy to check that the Glauber dynamics is an ergodic, reversible Markov process on Ω_V, with stationary distribution μ_V. We denote by $\mathcal{L}_V$ the *generator* of the dynamics, which is a non positive self-adjoint operator in $\ell^2(\Omega_V, \mu_V)$, and we define the *spectral gap* of the dynamics $c_{gap}(\mathcal{L}_V)$ as the first nonzero eigenvalue of $-\mathcal{L}_V$.

The Glauber dynamics is much studied for two reasons: firstly, it is the basis of Markov chain Monte Carlo algorithms, widely used in computational physics for sampling from the Gibbs distribution; secondly, it is a plausible model for the evolution of the underlying physical system toward

the equilibrium. In both contexts, a central question is to determine the *mixing time*, i.e. the number of steps until the dynamics is close to its stationary measure. Different techniques have been developed to get meaningful bounds on this quantity [Sa, Mar2]. For example, by means of analytic method, it has been proved that the mixing time T_1 of the Glauber dynamics satisfies the following bounds

$$\frac{1}{c_{gap}(\mathcal{L}_V)} \leq T_1 \leq \frac{1}{2c_{gap}(\mathcal{L}_V)}(2 + \log \frac{1}{\mu_V^*}), \tag{0.6}$$

where $\mu_V^* = \min_\sigma \mu_V(\sigma)$. The inverse of the spectral gap is called *relaxation time* and is another a key quantity to understand the relaxation to the equilibrium of the dynamics.

Advances in statistical physics over the past decade have led to the following remarkable characterization of the mixing time on finite n-vertex cubes with free boundary in the 2-dimensional lattice $\mathbb{Z}^2$: when $\beta < \beta_c$ the mixing time is $O(\log n)$, while for $\beta > \beta_c$ it is $\exp(\Omega(\sqrt{n}))$ ([SZ, MO1, MO2, Mar]).

Remark 0.1. *It has been recently proved that any dynamics which updates only finite subsets (e.g. the Glauber dynamics) has mixing time at least* $\Omega(\log n)$*, where* n *is the size of the underlying graph [HS]. A mixing time satisfying this lower bower is thus called* optimal*.*

It turns out that the phase transition in $\mathbb{Z}^2$ has a dynamical manifestation in the form of an explosion from optimal to exponential in the mixing time of the process. The relation between equilibrium and nonequilibrium properties has been investigated for systems on the integer lattice $\mathbb{Z}^d$ [SZ, MO1, MO2, Mar], and more recently some results have been obtained also for non Euclidean graphs [BKMP, MSW]. However, the subject is continuously in progress since a complete understanding of this connection, which would led to a great improvement of many results, is still missing.

Boundary condition and mixing time: first result

One of the most interesting questions left open by the dynamical analysis is the *influence of boundary conditions* on the mixing time. To better appreciate this point, let us go back to the Ising model on $\mathbb{Z}^2$. As remarked below, it is proved that for $\beta < \beta_c$ the mixing time is $O(\log n)$ and correlations decay

exponentially with the distance uniformly in the boundary condition. In the low temperature regime ($\beta > \beta_c$), instead, the mixing time for free boundary condition is $\exp(\Omega(\sqrt{n}))$ and correlations persist over arbitrarily long distances. In contrast to the free boundary case, it has been conjectured that, in the presence of an all-$(+)$ boundary, the mixing time should remain polynomial in n at all temperatures [FH]. This captures the intuition that the only obstacle to rapid mixing for $\beta > \beta_c$ is the long time required for the dynamics to get through the "bottleneck" between the $(+)$-phase and the $(-)$-phase; the presence of the $(+)$-boundary eliminates in some sense the $(-)$-phase and hence the bottleneck. Further support for this intuition comes from the fact that a certain spatial mixing property holds under the all-$(+)$ boundary condition: specifically, it is known that in this case the correlation between the spin of two sites is exponentially small in the distance between them. Even though this conjecture have received a lot of attention in the past decade, obtaining formal results has proved very elusive.

However, recently a new technique to approach this problem has been introduced in [MSW] when the underlying graph is a regular tree. Whereas in the free boundary condition case the mixing time, at low temperature, is polynomial in n with exponent increasing with β [BKMP], in [MSW] it has been proved the following:

Theorem 0.1. *In both of the following situations, the spectral gap and the logarithmic Sobolev constant of the Glauber dynamics on an n-vertex tree are $\Omega(1)$:*

(i) *$\beta < \beta_1$ or $|h| > h_c$, with arbitrary boundary conditions;*

(ii) *$(+)$-boundary condition and arbitrary β, h.*

Together with inequalities 0.6 (and with the analogous, and stronger, bounds concerning the logarithmic Sobolev constant), this implies that the mixing time is optimal also in the low temperature region, providing that the system has $(+)$-boundary. The behavior conjectured for $\mathbb{Z}^d$, is thus proved for this model.

The first part of this dissertation is aimed to analyze the influence of the boundary condition on the dynamics in the hyperbolic graphs case. The similarity between some properties of the trees and hyperbolic graphs, and in particular the non amenability, would indeed suggest the possibility to extend the result stated above. On the other hand the presence of cycles in

$\mathbb{H}(v,s)$ (as in $\mathbb{Z}^d$, $d \geq 1$) which are absent in the tree, makes the analysis more difficult and requires the introduction of new techniques. Before state our result, we recall that in the free boundary condition case the relaxation time of the Glauber dynamics in a n-vertex ball in $\mathbb{H}(v,s)$ is polynomial in n with exponent growing with β [BKMP].
The main result we will prove is the following:

Theorem 0.2. *Let $\mathbb{H}(v,s)$ such that $v>4$ and $s>3$. Then, for all $\beta \gg 1$, the Glauber dynamics on an n-vertex ball in $\mathbb{H}(v,s)$ with (+)-boundary condition has $c_{gap} = \Omega(1)$.*

The presence of $(+)$-boundary condition is thus reflected in the behavior of the relaxation time, which remains $\Theta(1)$ for all high β values instead of growing with n as in the free boundary condition case.

The main ingredient of the proof is a kind of correlation decay between spins which holds under the $(+)$-boundary condition. Indeed, essentially due to the non amenability of the graph, the influence of the $(+)$-boundary condition on a given spin weakens the influence from other regions at arbitrary distance. The deduced notion of spatial mixing is then turned in a temporal mixing condition using analytic and coupling techniques.

Glauber dynamics on Random Graphs: second result

In the second part of this dissertation, we will address statistical mechanics models on *random graphs*. A random graph is a random variable defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is a suitable family of graphs and $\mathbb{P}$ is a probability distribution on Ω.

For example, given a real number $p \in [0,1]$, the *binomial random graph*, denoted by $\mathbb{G}(n,p)$, is defined taking as Ω the set of all graphs on n vertices and setting

$$\mathbb{P}(G) = p^{e_G}(1-p)^{\binom{n}{2}-e_G}\,, \tag{0.7}$$

where e_G stands for the number of edges of G.

Spin models on random graphs have attracted much attention in recent years and a sophisticated theory has been developed for computing the thermodynamic properties of such systems in great generality (see for instance [MP] and references therein). The interest in this subject is at least twofold. On one side one would like to extend to new structures the theory of statistical physics, in order to obtain new features and behaviors which could

explain some real physical systems. On the other side, many combinatorial problems on random structures, in particular optimization and counting problems, can be reformulated in the statistical mechanics setting and then solved, or at least better understood, with the methods of statistical mechanics (see [MM]); we think, for example, to the *random k-sat problem*, that recently has attracted a lot of attention both from computer scientists and from mathematical physicists (see [AP, AR, MMZ]).

In contrast with the huge literature on probabilistic and combinatorial analysis of random graphs, and with the increasing number of papers on the equilibrium behavior of statistical models on random graphs, there are very few papers analyzing the nonequilibrium properties of these models. In particular it doesn't exist, to the best of our knowledge, any rigorous result concerning the relaxation time of the Glauber dynamics. To attack this range of hard problem, new techniques and ideas are then required.

The work in the second part of this thesis is a first tentative of clarifying the behavior of *local dynamics* for Ising model on some random graphs. To this aim, we will combine the standard analytic and probabilistic methods, used to study Glauber dynamics, with the probabilistic and combinatorial techniques coming from the theory of random graph.

Firstly we will consider the Glauber dynamics for the Ising model on the binomial random graph $\mathbb{G}(n,p)$, with $p = c/n$ and $c > 0$ (see, e.g., [JLR, Bo, Sp2]). For this value of the parameter p, it is well known that almost surely in the limit of $n \to \infty$ (a.a.s.), the binomial random graph is disconnected and has a largest component whose size depends on c: if $c < 1$ it has size $O(\log n)$; if $c > 1$ it has size $\Theta(n)$ (giant component). The Gibbs measure on the graph is thus given by the product of Gibbs measures on any component. We will prove the following results

Theorem 0.3. *For every $\beta > 0$, there exists a positive constant c_β such that a.a.s. the spectral gap of the Ising model Glauber dynamics on $\mathbb{G}(n, c/n)$ is less then $\delta_n = \exp\left(-c_\beta \frac{\log n}{\log\log n}\right)$, i.e.*

$$\lim_{n\to\infty} \mathbb{P}(c_{gap}(\mathcal{L}_G) \le \delta_n) = 1 \tag{0.8}$$

Theorem 0.4. *For every $\beta > 0$ and $c > 1$ there exists a positive constant c_β such that a.a.s. the spectral gap of the Ising model Glauber dynamics on the giant component $\mathcal{C}$ of $\mathbb{G}(n, c/n)$ is less then $\delta_n = \exp\left(-c_\beta \frac{\log n}{\log\log n}\right)$, i.e.*

$$\lim_{n\to\infty} \mathbb{P}(c_{gap}(\mathcal{L}_\mathcal{C}) \le \delta_n \,,\, \mathcal{C} \subseteq \mathbb{G}(n, c/n)) = 1 \tag{0.9}$$

From Theorem **0.3** it follows that for every $\beta > 0$ and $c > 0$, a.a.s. the mixing time of the dynamics on $\mathbb{G}(n, c/n)$ grows with n at least as $\exp(-c_\beta \frac{\log n}{\log\log n})$. Moreover, as stated in Theorem **0.4**, if $c > 1$ then a.a.s. there exists a unique giant component $\mathcal{C}$ and the relaxation time of the dynamics on $\mathcal{C}$ grows with n at least as $\exp(-c_\beta \frac{\log n}{\log\log n})$.

The proofs of these results are similar and are both given in two steps. In the first step, for every integer $k \leq n$, we define a suitable subgraph S_k of size k and prove, providing a suitable test function, that the spectral gap of the dynamics on S_k is exponentially small in k. In the second step, using standard techniques like *the first and the second moment methods*, we prove that the probability that S_k is contained in $\mathbb{G}(n, c/n)$ is asymptotically one for any $k \leq \alpha \frac{\log n}{\log\log n}$, with $\alpha < 1/2$. With some extra work we are also able to prove that a.a.s. the underlying subgraph belongs to the giant component of the graph.

Secondly, we consider the *r-regular random graph model* on n vertex, that is the set of all graphs with constant vertex degree equal to r, each one taken with uniform probability [Wo, JLR, Bo]. Here we give a very simple proof that for every $\beta > \beta_1$ (low temperatures) and for almost surely any realization of the random graph, the relaxation time of the dynamics is $\Omega(\exp(n))$, as the size n of the graph tends to infinity. The proof consists again in providing a suitable *test function* to get results from the variational characterization of the spectral gap. Here will be determinant a geometrical property of regular random graph, holding almost surely when n goes to infinity.

Organization

In Chapter 1 we introduce some elements of graph theory and give precise notions of transitive graph and hyperbolic graph. In Chapter 2 we define spin systems, Gibbs measures and Glauber dynamics, and introduce the analytic tools to study the relaxation to equilibrium. A briefly description of results for the Ising model on transitive graph, both from equilibrium and the dynamical point of view, is given in Chapter 4. In Chapter 5 we present and prove our boundary-specific result for hyperbolic graphs. The definition of random graphs and the relative results are all given in Chapter 6.

Chapter 1

Elements of Graph Theory

1.1 Basic definitions and notation

A graph is a pair $G = (V, E)$, where V is an arbitrary set, called the set of vertices, and $E \subseteq V \times V$ is called the set of edges (or bonds) of G. Given an edge $e = (x, y)$, the vertices x and y are called its end-points and e is said to be incident to x and y. If x and y are the end-points of an edge, we call them neighbors or adjacent and write $x \sim y$. Edges are said to be neighbors or adjacent if they share a common vertex.
The *degree* of $x \in V$, $\deg x$, is the number of edges incident with x. A graph is said *locally finite* if the degree of each vertex is finite. Let us define the quantity

$$\Delta(G) \, = \, \sup\{\deg x \, : \, x \in V\} \, ;$$

if $\Delta(G) < \infty$, the graph G is said to be of bounded degree, and $\Delta(G)$ is its maximal degree. A graph of bounded degree is also locally finite whereas one can provide examples of locally finite graphs with $\Delta(G) = \infty$.

A *path* in $G = (V, E)$ is a sequence $(e_1, e_2, \ldots, e_n)$ of distinct edges in E, such that e_j is neighbor to e_{j+1} for each $j = 1, \ldots n$. It can be also identified with the sequence $x_0, x_1, \ldots, x_n$ of vertices in V such that $e_j = (x_{j-1}, x_j)$ for every $j = 1, \ldots n$, and in this case x_0 and x_n are called the *end vertices* of the path. If $x_0 = x_n$ and $n \geq 3$, the path is closed and is called a *cycle*. In both cases we call n the length of the path.
Given two nonempty subsets $S, K \subset V$ such that $S \cap K = \emptyset$, we say that γ is a path from S to K, and write $\gamma : S \mapsto K$, if for some $n \in \mathbb{N}$, $\gamma = (x_0, \ldots, x_n)$ is a path in G with $x_0 \in S$ and $x_n \in K$.

Two vertices are said to belong to the same connected component of the graph, if there is a path which has them as end vertices. With this notion the graph is partitioned into connected components, and it is said connected if it has a single connected component.
The *graph distance* $\mathrm{d}(x, y)$ between two vertices x and y, is defined as the minimal length of paths from x to y.

A graph S, with vertex set $V(S)$ and edge set $E(S)$, is said a *subgraph* of G if $V(S) \subset V$ and $E(S) \subset E$. Given $V' \subset V$, we denote by $E(V')$ the set of all edges in E which have both their end vertices in V' and we call $G(V') = (V', E(V'))$ the *induced subgraph* on V'. Given a subset $K \subset V$, we define the *vertex boundary* of K

$$\partial_V K \,=\, \{x \in V \setminus K \,:\, \exists y \in K \text{ s.t. } x \sim y\}$$

and the *edge boundary* of K

$$\partial_E K \,=\, \{e = (x, y) \in E \text{ s.t. } x \in K\,,\, y \in V \setminus K\}\,.$$

For a subgraph S of G, we will abuse the notation and write $\partial_V(S)$ and $\partial_E(S)$ for the boundaries of $V(S)$.
Its easy to see that for finite $K \subset V$,

$$\frac{|\partial_E K|}{\Delta(G)} \leq |\partial_V K| \leq |\partial_E K| \leq |\partial_V K|\Delta(G)\,, \tag{1.1}$$

where the first and last inequalities become trivial when $\Delta(G) = \infty$.

All the graphs considered in this dissertation will be infinite, connected and of bounded degree.

1.2 Transitive graphs

1.2.1 Definitions and examples

Given a graph $G = (V, E)$, an automorphism of G ia a bijection ϕ of V which preserves the graph structure, i.e. such that $E = \{(\phi(x), \phi(y)) : (x, y) \in E\}$. The set of automorphism of G forms a group denoted $\mathrm{Aut}(G)$. We say that a group $\Gamma \subseteq \mathrm{Aut}(G)$ is *transitive* (or *acts transitively*) if for all $x, y \in V$ there is some $\gamma \in \Gamma$ such that $\gamma x = y$. We say that Γ is *quasi-transitive* if Γ splits V into finitely many orbits, i.e. if there is a finite set $V_0 \subset V$ with the property that each vertex of the graph can be mapped by

Γ into one of the vertices of V_0. We call the graph G transitive (or quasi-transitive) if Aut(G) is transitive (or quasi-transitive). In the sequel we will ignore quasi-transitive graphs, though the extension of results for transitive graphs to quasi-transitive graphs is often reachable.

In the sequel we will present and discuss some subclasses of the huge family of transitive graphs. But before, we want to explain some graph properties in order to identify these subclasses with properties that they may or may not satisfy. Let us give a briefly description of properties we will be mainly concentrated on.

1.2.2 Main properties

Amenability and isoperimetric constants

Given an infinite, locally finite, connected graph $G = (V, E)$, its vertex and edge isoperimetric constants are defined respectively as

$$i_v(G) := \inf \left\{ \frac{\partial_V(K)}{|K|} ; K \subset V \text{ finite} \right\} \tag{1.2}$$

$$i_e(G) := \inf \left\{ \frac{\partial_E(K)}{|K|} ; K \subset V \text{ finite} \right\} . \tag{1.3}$$

From (1.1) we have

$$\frac{i_e(G)}{\Delta(G)} \leq i_v(G) \leq i_e(G) \leq \Delta(G) i_v(G)$$

so that for graphs of bounded degree $i_v(G) > 0$ if and only if $i_e(G) > 0$.

A graph G is *amenable* if its vertex isoperimetric constant (or Cheeger constant) is 0, i.e. if for every $\epsilon > 0$ there is a finite set of vertices K such that $|\partial_V K| < \epsilon |K|$. Otherwise G is *non-amenable*.

A typical examples of (transitive) non-amenable graph is the regular tree $\mathbb{T}^b$. If, for instance, we consider the balls $B(l)$ of radius l around the root r (namely the subtree with depth l), for every l we get that $|\partial_v B(l)|/|B(l)| \leq 1/(b-1)$; one can prove that this is indeed the correct value of the Cheeger constant.

A typical examples of (transitive) amenable graph is the cubic lattice $\mathbb{Z}^d$. Taking in this case the balls of radius l around the origin, we see that the ratio $|\partial_v B(l)|/|B(l)| = O(l^{-1})$ and then goes to 0 for $l \to \infty$.

Number of Ends

Given $K \subset V$, the graph $G \setminus K$ is the graph obtained from the graph G by removing the vertices which belong to K and the edges incident to these vertices. The number of ends of the graph G is

$$\mathcal{E}(G) = \sup_{K \subset V} \{\text{number of infinite connected components of } G \setminus K\}.$$

Any Cartesian product of infinite graphs can easily be seen to have a single end.

It is known that for transitive graphs the number of ends can only be: 1 (e.g., $\mathbb{Z}^d$, $d \geq 2$), 2 (e.g., $\mathbb{Z}$), or ∞ (e.g., $\mathbb{T}_b$, $b \geq 2$). Moreover, when the number of ends is 2, the graph is amenable and when the number of ends is infinity the graph is non-amenable. Graphs with a number of ends equal to 1, can be amenable (as $\mathbb{Z}^d$ with $d \geq 2$) or not (as the hyperbolic graphs, see below).

Planarity

Let Π be the Euclidean or the hyperbolic plane. A graph $G = (V, E)$ is planar if can be embedded in Π satisfying the following restrictions.

(i) Each vertex $x \in V$ is mapped into a point $v_x \in \Pi$.

(ii) Each edge $e = (x, y) \in E$ is mapped into the image $\Gamma_e = \gamma_e([0, 1])$ of a curve $\gamma_e : [0, 1] \mapsto \Pi$, with $\gamma_e(0) = v_x$ and $\gamma_e(1) = v_y$. If $e_1 \neq e_2$, then $\gamma_{e_1}((0, 1)) \cap \gamma_{e_2}((0, 1)) = \emptyset$.

The connected components of $\Pi \setminus (\cup_{e \in E})\Gamma_e$ are called the faces of the embedding. When G is planar, transitive and one end, it is not hard to show that each face is bounded.

Given a planar graph, one can construct the dual multigraph (meaning that more then one edge can connect two vertices and edges may have two identical endpoints) $G^* = (V^*, E^*)$ as follows: to every faces in G is bi-univocally associated a vertex $v \in V^*$; to every $e \in E$ is bi-univocally associated the dual edge e^* connecting the two faces which have Γ_e in their boundary.

1.3 Cayley Graphs

1.3.1 Construction and main subclasses

An important class of transitive graphs is that of the *Cayley graphs* of finitely generated group, defined as follows. Let Γ a finitely generated group and S a finite symmetric generating set for Γ, where, we recall, S is symmetric if $s \in S \Rightarrow s^{-1} \in S$. The (right) Cayley graph $\mathcal{C}(\Gamma, S)$ of Γ is the graph with vertex set $V := \Gamma$ and edge set $E := \{(v, vs);\, v \in \Gamma,\, s \in S\}$. From the symmetry of generators, E is well defined and the obtained graph non-oriented. If the generators'set S is finite, then the corresponding Cayley graph is of bounded degree, with constant vertex degree equal to the number of generators $|S|$.

Cayley graphs of finitely generated groups includes a large number of examples of transitive graphs and most of those which are of great interest.

(i) The basic examples of graph of interest in statistical mechanics and related areas are the cubic lattices $\mathbb{Z}^d$, with $d \geq 1$. The cubic lattice $\mathbb{Z}^d$ is indeed a Cayley graph of the free Abelian group of rank d.

(ii) Another important class of examples is that of the homogeneous or regular trees, i.e. trees which are transitive and in particular have constant vertex degree Δ. We will denote by $\mathbb{T}_b$ the homogeneous tree of degree $\Delta = b + 1$ (the index b stays for the branching number of the tree). If Δ is even, then $\mathbb{T}_b$ is a Cayley graph of the free group with $\Delta/2$ free generators. When Δ is odd, $\mathbb{T}_b$ is a Cayley graph of the group with $b/2$ free generators and one generator which is identical to its inverse.

(iii) The third class we want to recall is that of Cayley graphs of discrete groups of isometries of the hyperbolic spaces H^d, with $d \geq 2$ (Fuchsian groups in the $d = 2$ case). The vertices of such graphs can be thought as the center of a tile in a tesselation of H^d, with edges connecting vertices corresponding to tiles whose boundary intersect in a $d-1$-dimensional surface. Such examples may be seen as "crystals in a non-Euclidean space"; their Euclidean counterparts are Cayley graphs of discrete groups of isometries of Euclidean space, including $\mathbb{Z}^d$ and the triangular and hexagonal lattices. We will deepen the case $d = 2$, which includes the main objects we will work on: hyperbolic graphs.

1.3.2 Hyperbolic Graphs

Let $\mathbb{H}^2$ be the hyperbolic plane and denote by $\mathcal{I}(\mathbb{H}^2)$ the group of isometries of $\mathbb{H}^2$. Consider a finitely generated co-compact subgroup G of $\mathcal{I}(\mathbb{H}^2)$, namely such that there is a compact subset K of $\mathcal{I}(\mathbb{H}^2)$ with $GK = \mathcal{I}(\mathbb{H}^2)$. If S is a finite set of generators for G, then the Cayley graph $\mathcal{C}(G, S)$ is an *hyperbolic graph*. We can embed $\mathcal{C}(G, S)$ in $\mathbb{H}^2$ as follows.
Choose a convex finite-sided geodesic polygon $A \in \mathbb{H}^2$ which is a fundamental domain for G acting on $\mathbb{H}^2$. According to the Poincaré's theorem on fundamental polygons, the set of isometries S which identify the sides of A is a set of generator for G. Thus fix a point $0 \in \text{Int}A$. The vertices of $\mathcal{C}(G, S)$ are the points of $g0$, with $g \in G$, and an edge between $g0$ and $g'0$ is drawn whenever exists $s \in S$ such that $g' = gs$.

The embedding of an hyperbolic graph in $\mathbb{H}^2$, gives out a tessellation of the hyperbolic plane with regular tiles, and indeed the hyperbolic graphs are also known as *hyperbolic tiling*. In particular, they can be characterized by the (constant) vertex degree v and the (constant) number of sides s in each face (tile), and thus denoted by $\mathcal{H}(v, s)$. In order for the embedding in $\mathbb{H}^2$ to be well defined, the integers v and s have to satisfy the relation $(v-2)(s-2) > 4$. The typical representation of hyperbolic tilings make use of the Poincaré disc D^2 that is in bi-univocal correspondence with $\mathbb{H}^2$. See figure 3.1.

It turns out that the hyperbolic graphs are planar. In particular, the dual graph of a given hyperbolic tiling $\mathcal{H}(v, s)$ is an hyperbolic tiling too, and it is easy to check that $[\mathcal{H}(v, s)]^* = \mathcal{H}(s, v)$. See Fig(3.1).
Moreover, the hyperbolic graphs belong to the class of non-amenable one ended transitive graphs; as we will see in chapter 3, this class is related with two interesting conjectures concerning the behavior of some statistical mechanics models.
As shown in [HJL], the edge isoperimetric constant of $\mathbb{H}(v, s)$ is explicitly given by

$$i_e(\mathcal{H}(v, s)) = (v-2)\sqrt{1 - \frac{4}{(v-2)(s-2)}}\,.$$

For a detailed analysis of the hyperbolic graph we refer, e.g., to [Mag, RNO].

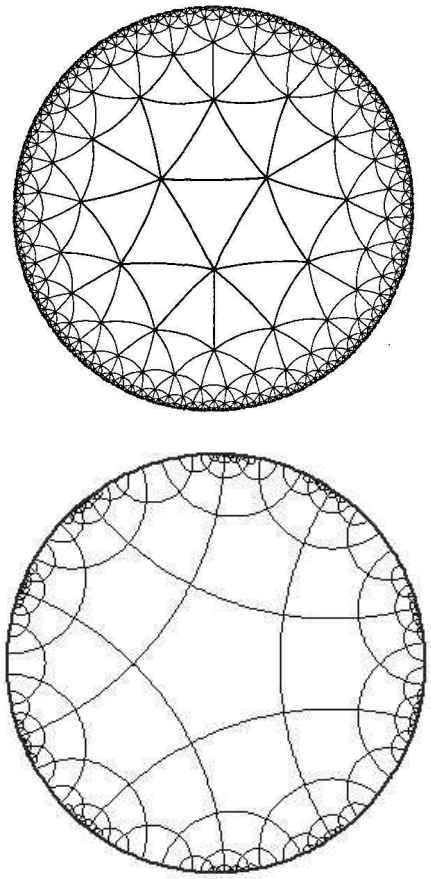

Figure 1.1: Two examples of hyperbolic tiling in the Poincaré disc representation : $\mathbb{H}(3,7)$ on the left and $\mathbb{H}(4,5)$ on the right.

Chapter 2

Spin Systems

2.1 Preliminaries

A spin system is a collection of particles, each one associated to a spin variable taking value in a finite space $\mathcal{S}$, referred to as the *spin space*. Here we will consider systems whose particles are sited on the vertex set of a locally finite graph $G = (V, E)$, with V a countable set.

We use the following terminology and notation. To every vertex $x \in V$ is associated a spin variable $\sigma_x \in \mathcal{S}$ and the state of the total system is specified by a *configuration* $\sigma \in \Omega = \mathcal{S}^V$, that is an assignment of spins on V. We endow Ω with the σ-algebra $\mathcal{F}$ generated by the set of projections $\{\pi_x\}_{x \in V_m}$ from Ω to $\mathcal{S}$, where $\pi_x : \sigma \mapsto \sigma_x$. For $\Lambda \subset V$, we also use the notation σ_Λ for the configuration in $\Omega_\Lambda := \mathcal{S}^{|\Lambda|}$ and denote by $\mathcal{F}_\Lambda$ the σ-algebra generated by π_x , $x \in \Lambda$. We say that σ agrees with η on Λ if $\sigma_\Lambda = \eta_\Lambda$ and let $\Omega^\eta_\Lambda = \{\sigma \in \Omega | \sigma_{\Lambda^c} = \sigma_{\Lambda^c}\}$ denote the set of configurations that agree with η outside Λ. We write $f \in \mathcal{F}_\Lambda$ to indicate that f is $\mathcal{F}_\Lambda$-measurable.

The global behavior of the system depends from the interaction between particles. If there aren't interactions, then every particle behaves independently and the analysis of the system is elementary, since the measure describing the system is just a product measure. While, if a simple interaction is introduced in the system, the global behavior of the system changes deeply and non trivial phenomena (like phase transitions) appear.
Let $\mathbb{F}$ be the set of all nonempty finite subsets on V; the general assumptions on the interactions are the following.

Definition 2.2. *A finite range, automorphism-invariant potential (or interac-*

tion) $J = \{J_\Lambda\}_{\Lambda\in\mathbb{F}}$ *is a real function of non empty finite subsets of* V *with the following properties*

1. $J_A = J_{\phi(A)}$ *for all* A *finite subset of* V *and all* $\phi \in$ *Aut*(G)
2. *There exists* $r > 0$ *such that* $J_A = 0$ *if diam*$(A) > r$. r *is called the range of the interaction.*
3. $\sum_{A\ni 0} |J_A| < \infty$

Given an interaction J with these properties and for every finite subset $\Lambda \in V$, we define the Hamiltonian $H_\Lambda(\sigma) : \Omega \mapsto \mathbb{R}$ by

$$H_\Lambda(\sigma) \;=\; -\sum_{A\cap\Lambda\neq\emptyset} J_A \prod_{x\in A} \sigma_x, \tag{2.1}$$

that can be considered as the contribution to the energy of σ coming from Λ. Though we dropped any subscript, H_Λ clearly depends from the choice of the potential J. Let $\eta \in \Omega$ specify a boundary condition. The *finite region Gibbs measure* on Λ conditioned on η is defined as:

$$\mu^\eta_\Lambda(\sigma) := \begin{cases} (Z^\eta_\Lambda)^{-1} \exp[-\beta H^\eta_\Lambda(\sigma)] & \text{if } \sigma \in \Omega^\eta_\Lambda; \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where Z^η_Λ is the appropriate normalizing factor and β is a non-negative constant often interpreted as the inverse temperature (and sometimes included in the interaction J). We will also refer to the Gibbs measure on Λ with *free boundary conditions*, denoted by μ^f_Λ, meaning the measure resulting from the above definition but considering Λ as disconnected from the rest of the graph, i.e. taking the sum in (2.1) only over subsets $A \subseteq \Lambda$.
Given a measurable bounded function f on Ω, $\mu^\eta_\Lambda(f)$ is the average of f w.r.t. μ^η_Λ and $\mu_\Lambda(f)$ denotes the function $\sigma \mapsto \mu^\sigma_\Lambda(f)$. Clearly $\mu_\Lambda(f) \in \mathcal{F}_\Lambda$.

Spin systems with nearest neighbor interactions are probably the most studied models. Their are defined by associating to every edge $e = (x, y) \in E$ a symmetric pair potential $J_{(x,y)} : \mathcal{S} \times \mathcal{S} \mapsto \mathbb{R} \cup \{\infty\}$, and to every $x \in V$ a self potential $J_x : \mathcal{S} \mapsto \mathbb{R} \cup \{\infty\}$. Notice that for these models, the distribution (2.2), of the particle configurations on Λ, depends from the boundary conditions only through $\partial_V\Lambda$.

The most famous example of spin system with nearest neighbor interactions is certainly the Ising model. For this model the spin space is $\mathcal{S} = \{\pm 1\}$, the pair potential is constant equal to 1 for every edge and the self potential

is constant equal to $h \in \mathbb{R}$ for every vertex, where h represents the external magnetic field. For every $\eta \in \Omega$ b.c. and $\sigma \in \Omega_\Lambda^\eta$, he Hamiltonian of the Ising model is

$$H_\Lambda(\sigma) = \sum_{(xy)\cap\Lambda\neq\emptyset} \sigma_x\sigma_y + h\sum_{x\in\Lambda}\sigma_x\,. \tag{2.3}$$

The relative Gibbs measure distribution μ_Λ^η thus assigns higher weight to configurations in which many neighboring spins are aligned with one another, as well as to configurations in which many spins agree with the sign of h. This two effects increase with β and $|h|$ respectively. In particular, at high temperatures (low β) the spins behave almost independently, while at low temperature large connected regions of equal spins tend to form. The Ising model will be discussed in details in the following chapter, where different graph structure will be considered.

2.2 Gibbs measures

2.2.1 Construction by DLR equations

It is immediate from the definition, that any finite region Gibbs measure satisfies the so called *DLR compatibility conditions*. Namely, for every $\eta \in \Omega$ and every $\sigma \in \Omega_\Lambda^\eta$, it holds

$$\mu_\Lambda^\eta(\,\cdot\,|\sigma_{A^c}) = \mu_A^\sigma(\,\cdot\,) \qquad \forall A \subseteq \Lambda \in \mathbb{F}\,. \tag{2.4}$$

Equivalently, given an event $X \in \mathcal{F}$ and with the notation $\mu_\Lambda(X) \equiv \mu_\Lambda(\mathbb{1}_X)$, where $\mathbb{1}_X$ is the characteristic function on X, it holds

$$\mu_\Lambda^\eta(\mu_A(X)) = \mu_\Lambda^\eta(X) \quad \forall X \in \mathcal{F} \quad \forall A \subseteq \Lambda \in \mathbb{F}\,.$$

We say that μ_Λ^η is stationary under μ_A.
Let us refer to the collection of all finite region Gibbs measures μ_Λ^η as Λ and η vary, as the specification μ. Given a specification μ, namely after specified the configuration space and the potential J that give rise to μ, one can extend the notion of DLR compatibility to measures on the infinite space that are compatible with all finite distributions.

Definition 2.3. *A probability measure ν on $(\Omega, \mathcal{F})$ is called a Gibbs measure for the specification μ if, for every finite region Λ and ν-almost every configuration σ,*

$$\nu(\,\cdot\,|\sigma_{\Lambda^c}) = \mu_\Lambda^\sigma(\,\cdot\,)\,, \tag{2.5}$$

or equivalently if $\nu(\mu_\Lambda(X)) = \nu(X)$ *for every* $X \in \mathcal{F}$ *and* $\Lambda \in \mathbb{F}$.

The physical intuition for a Gibbs measure is that it describes a "macroscopic equilibrium" in which all parts of the system are in equilibrium with their boundaries.

2.2.2 Uniqueness and mixing in space

It is well known that, for any specification μ derived as above, at least one Gibbs measure exists. However, several Gibbs measures for a given specification may coexist (see, e.g., [Ge, GHM, EFS] for details and more on Gibbs measures).

The question of whether the Gibbs measure is unique or not is central in statistical physics, because it corresponds to whether one or more macroscopic equilibria are possible for the given system. Denoting by $\mathcal{G}$ the set of all Gibbs measures relative to a given specification μ, we say that the spin system described by μ exhibits a *phase coexistence* if $\mathcal{G}$ contains more then one element.
Since $\mathcal{G}$ is proved to be a convex compact set, the convex combination of limits of finite volume Gibbs measures along sequences of regions and boundary conditions is a Gibbs measure. Thus, the question of whether the Gibbs measure is unique, can be translated to that of whether there is an asymptotic independence between the configuration on a finite region and a distant boundary condition. To formalize this concept, let us introduce the following notation.
Given two probability measures μ_1, μ_2 on a finite set Y, their *total variation distance* is defined as:

$$\|\mu_1 - \mu_2\| := \frac{1}{2}\sum_{y\in Y} |\mu_1(y) - \mu_2(y)| = \sup_{X\subset Y} |\mu_1(X) - \mu_2(X)| . \qquad (2.6)$$

Given $\Delta \subset \Lambda \in \mathbb{F}$ and the Gibbs measure μ_Λ on Ω_Λ, let $\mu_{\Lambda,\Delta}$ denotes the projection of the measure μ_Λ on Ω_Δ, i.e.

$$\mu_{\Lambda,\Delta}(\sigma) = \sum_{\eta:\, \eta_\Delta = \sigma_\Delta} \mu_\Lambda(\eta) .$$

The sufficient and necessary condition for the Gibbs measure relative to a specification μ to be unique is the following.

Proposition 2.1. *A specification μ admits a unique Gibbs measure if and only if, for every finite region Δ, there exists an infinite sequence of regions $\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_l \subset \ldots$ such that $\bigcup_{l>0} \Lambda_l = V$, and for any two configurations η and σ,*

$$\|\mu^{\eta}_{\Lambda_l,\Delta} - \mu^{\sigma}_{\Lambda_l,\Delta}\| \underset{l\to\infty}{\longrightarrow} 0$$

From proposition **2.1** one can observe that the uniqueness of the Gibbs measure corresponds to a particular form of asymptotic independence in the equilibrium state between configurations on two distant regions. We call this behavior *mixing in space*. Following this idea, we can now define two different notions of spatial mixing.

Definition 2.4. *We say that the Gibbs measure μ_Λ has weak spatial mixing in Λ if there exist constants C and m such that, for every subset $\Delta \subset \Lambda$ and every pair of boundary conditions η and σ,*

$$\|\mu^{\eta}_{\Lambda,\,\Delta} - \mu^{\sigma}_{\Lambda,\,\Delta}\| \leq C|\Lambda| e^{-md(\Delta,\,\partial_V \Lambda)}\,. \tag{2.7}$$

Definition 2.5. *We say that the Gibbs measure μ_Λ has strong spatial mixing in Λ if there exist constants C and m such that, for every subset $\Delta \subset \Lambda$, every $S \subseteq \partial_V \Lambda$ and every pair of boundary conditions η and σ that differ only on S,*

$$\|\mu^{\eta}_{\Lambda,\,\Delta} - \mu^{\sigma}_{\Lambda,\,\Delta}\| \leq C|\Delta| e^{-md(\Delta,\,S)}\,. \tag{2.8}$$

From definitions above, one can verify that the conditions of weak and strong mixing both imply the existence of a unique infinite volume Gibbs measure with exponentially decaying truncated correlations functions. The exponential decay inside Λ grows with the distance from the boundary $\partial_V \Lambda$, in the weak mixing case, or with the distance from the support of the perturbation, in the strong mixing case. Both notions are very useful tools when analyzing the mixing properties of the Glauber dynamics on graph spin system.

2.3 Glauber dynamics on a bounded degree graph

In the previous section we described the equilibrium state of a spin system on an infinite, locally finite graph $G = (V, E)$, and defined some properties which the state may or may not satisfy. In this section, given a finite graph $G = (V, E)$ and a spin system with configuration space $\Omega = \mathcal{S}^V$ and

equilibrium measure μ, we will define a special class of dynamical process on Ω known as *Glauber dynamics*. This is a Markov process with a unique invariant reversible measure corresponding to the Gibbs measure μ, which is often used in algorithmic simulation for sampling from the Gibbs measure. Beyond its relevance in computational problem, Glauber dynamics suggests a model for how the system converges to the equilibrium state, and thus enables to analyze nonequilibrium phenomena. As we will point out at the end of this section, the interest in this subject come also from its connection with the equilibrium properties of the system; the first of this dissertation is indeed devoted to these kind problem.
Here we define the dynamics together with its main mixing properties. Then we introduce some analytic tools to analyze the mixing time of the process.

2.3.1 Construction and existence

Before defining the Glauber dynamics on finite graph, let us recall some basic facts about reversible Markov process on finite spaces (see [Lig, Sa]).

Given a finite set X, an irreducible continuous-time Markov chain $(X_t)_{t\geq 0}$ on X, reversible with respect to the positive probability measure π, is constructed as follows. One first defines the generator $\mathcal{L}$ of the process, namely a self-adjoint operator on $\ell^2(X,\,\pi)$ with kernel $\mathcal{L}(x,y)$ whose matrix elements satisfy:

(i) $\sum_{y\in X}\mathcal{L}(x,y) \,=\, 0$ for any $x\in X$;

(ii) $\mathcal{L}(x,y)\geq 0$ for any $x\neq y\,\in X$;

(iii) $\pi(x)\mathcal{L}(x,y)=\pi(y)\mathcal{L}(y,x)$ for any $x,y\,\in X$ (detailed balanced condition);

(iv) for any pair of $x\neq y\,\in X$ there exists $n\in\mathbb{N}$ such that $(\mathcal{L}^n)(x,y)>0$.

The *Markov semigroup* associated to $\mathcal{L}$ is given by $P_t = e^{t\mathcal{L}}$ and has kernel $P_t(x,y) = e^{t\mathcal{L}(x,y)}$. For every $x\in X$, $P_t(x,\,\cdot\,)$ represents the distribution at time t of the Markov process $(X_t)_{t\geq 0}$ starting at x. Notice that

$$\mathcal{L}(x,y) = \frac{d}{dt}P_t(x,y)|_{t=0}\,,$$

which justifies the name of *jump rate from x to y* for the matrix element $\mathcal{L}(x,y)$, with $x\neq y$.

From the detailed balanced condition (iii) we get that π must be the unique invariant measure for the process, i.e. $\pi(P_t f) = \pi(f)$ for any f. From the Perron-Forensics Theorem (see, for instance [Sa]) then it holds that the distribution of the Markov process converges to its reversible measure, namely $\lim_{t\to\infty} \|P_t f - \pi(f)\|_\infty = 0$.

Now, let $G = (V, E)$ be an infinite locally finite graph, $\mathcal{S}$ be a finite space and J be a finite-range, automorphism invariant potential on $\Omega = \mathcal{S}^V$. This defines a spin system on G with finite volume Gibbs measure μ_Λ^τ as in (2.2). The *Glauber dynamics* on finite subset $\Lambda \subset V$ with boundary condition τ, is a continuous time Markov chain $(\sigma_t)_{t\geq 0}$ on Ω_Λ^τ, with Markov generator $\mathcal{L}$ given by

$$(\mathcal{L}f)(\sigma) \;=\; \sum_{\substack{x\in\Lambda \\ a\in\mathcal{S}}} c_x(\sigma, a)[f(\sigma^{x,a}) - f(\sigma)] \tag{2.9}$$

where $\sigma^{x,a}$ is the configuration obtained by setting the spin at x equal to the value a and the quantities $c_x(\sigma, a)$ are the jump (or transition) rates.
The general assumption on the transition rates, beyond hypothesis (i)-(iv) above, are

(1) *Finite range interactions.* If $\sigma_y = \eta_y$ for all y such that $d(x, y) \leq r$, then $c_x(\sigma, a) = c_x(\eta, a)$ for all $a \in \mathcal{S}$.

(2) *Positivity and boundedness.* There exist positive real numbers c_m and c_M such that

$$0 < c_m \leq \inf_{x,a,\sigma} c_x(\sigma, a) \qquad \sup_{x,a,\sigma} c_x(\sigma, a) \leq c_M$$

(3) *Automorphism invariance.* If, for some $\phi \in \mathrm{Aut}(G)$, $\sigma_y = \eta_{\phi(y)}$ for all $y \in V$, then $c_x(\sigma, a) = c_{\phi(x)}(\eta, a)$ for all $x \in \Lambda$ and $a \in \mathcal{S}$.

In the sequel we will focus for simplicity on a specific choice of the flip rates known as the *heat-bath* dynamics:

$$c_x(\sigma, a) \;=\; \mu_x^\sigma(a)\,. \tag{2.10}$$

It is easy to check that the heat bath Glauber dynamics is ergodic and satisfies the detailed balance condition w.r.t. the Gibbs measure μ_Λ^τ, i.e. it holds

$$\exp[-H_x(\sigma)]c_x(\sigma, a) \;=\; \exp[-H_x(\sigma^{x,a})]c_x(\sigma^{x,a}, \sigma_x)\,.$$

Thus, from the Perron-Forensics Theorem, the distribution of the Markov process $(\sigma_t)_{t\geq 0}$ converges to the stationary distribution μ_Λ^τ.

When $\mathcal{S} = \{\pm 1\}$ (Ising spin space), for all $a \in \mathcal{S}$ the difference $f(\sigma^{v,a}) - f(\sigma)$ is non zero only when $a = -\sigma_x$, namely when the new configuration $\sigma^{x,a}$ is obtained from σ by flipping the spin at the site x. We denote such a configuration by σ^x and we abbreviate the flip rates $c_x(\sigma, -\sigma_x) = c_x(\sigma)$; the generator formula then reduces to the following

$$(\mathcal{L}f)(\sigma) \;=\; \sum_{x\in V_m} c_x(\sigma)\nabla_x f(\sigma)\,, \tag{2.11}$$

where we also introduced the symbol $\nabla_x f(\sigma) = [f(\sigma^x) - f(\sigma)]$.

2.3.2 Relaxation to equilibrium: coercive bounds

The crucial problem in several applications, like combinatorial computation and statistical mechanics, concerns the time needed, for a reversible Markov process $(\mathcal{L},\, \pi)$, to be arbitrary close to its stationary measure. The standard proofs of the Perron-Forensics theorem, do not provide any information (or very lousy one) concerning this problem. Thus new technique are required.

Here we will present the main analytic tools to get bounds on the rate of convergence to equilibrium of Markov process as defined in the previous paragraph. We will serve of these methods to prove the main results of this dissertation.

Let us first consider the general Markov process $(\mathcal{L},\, \pi)$ on the finite space X, satisfying the conditions (i)-(iv) settled at the beginning of the previous paragraph. Given a real function f on X, we introduce the variance of f with respect to π

$$\mathrm{Var}_\pi(f) \;=\; \pi(f^2) - \pi(f)^2$$

and the *Dirichlet form* of f associated with the process $(\mathcal{L},\, \pi)$, that thanks to the reversibility condition can be written as

$$\mathcal{D}(f) \;= \frac{1}{2}\sum_{x,y} \pi(x)\mathcal{L}(x,y)[f(x) - f(y)]^2\,.$$

Then one can easily prove the equality

$$\frac{d}{dt}\mathrm{Var}_\pi(P_t f) \;=\; -2\mathcal{D}(P_t f)\,, \tag{2.12}$$

that suggests the use of functional coercive inequalities, like *the Poincaré inequality* or the *logarithmic Sobolev inequalities*, to obtain bounds on the long time behavior of the semigroup P_t. Let as formalize this idea.

The first quantity of interest is the *spectral gap*, defined as follows. Recall that, since the generator $\mathcal{L}$ is a nonpositive self-adjoint operator on $\ell^2(X, \pi)$, its spectrum consists of discrete eigenvalues of finite multiplicity that can be arranged as $0 > -\lambda_1 \geq -\lambda_2 \geq \ldots, \geq \lambda_{n-1}$, $|X| = n$, with $\lambda_i > 0$ for all $i > 0$.

Definition 2.6. *The spectral gap of the Markov process $(\mathcal{L}, \pi)$, denoted c_{gap}, is the absolute value of the first nonzero eigenvalue of $\mathcal{L}$, namely $c_{gap} = \lambda_1$. It satisfies*

$$c_{gap} \;=\; \inf\left\{\frac{\mathcal{D}(f)}{Var_\pi(f)}; Var_\pi(f) \neq 0\right\}. \tag{2.13}$$

Equivalently c_{gap}^{-1} is the best constant c in the *Poincaré inequality*

$$\mathrm{Var}_\pi(f) \;\leq\; c\mathcal{D}(f)\;\forall f \in l^2(X,\,\pi),$$

that together (2.12) and the fact $P_0(f) = f$, implies

$$\mathrm{Var}_\pi(P_t f) \;\leq\; e^{-2c_{gap}\,t}\mathrm{Var}_\pi(f)\,. \tag{2.14}$$

From last inequality, we notice that the spectral gap "gives a measure" of the exponential decay of the variance, and indeed the *relaxation time*, T_0, is defined as the inverse of the spectral gap.
Moreover, let h_t^x denote the density of the distribution at time t of the process starting at x w.r.t. π, i.e. $h_t^x(y) = \frac{P_t(x,y)}{\pi(y)}$. For $1 \leq p \leq \infty$ and a function $f \in \ell^p(X,\,\pi)$, let $\|f\|_p$ denote the ℓ^p norm of f and define the time of convergence

$$T_p \;= \min\left\{t > 0 \;:\; \sup_x \|h_t^x - 1\|_p \leq e^{-1}\right\}, \tag{2.15}$$

that for $p = 1$ is called *mixing time*. Then it holds the following result (see [Sa]):

Theorem 2.5. *Let $(\mathcal{L}, \pi)$ be a continuous-time, reversible Markov chain on a finite set X with spectral gap $c_{gap} > 0$. Denoting $\pi^* = \min_x \pi(x)$, it holds that*

$$c_{gap}^{-1} \leq T_p \leq c_{gap}^{-1}\left(2 + \log\frac{1}{\pi^*}\right), \quad \forall\, 1 \leq p \leq 2$$

$$c_{gap}^{-1} \leq T_p \leq c_{gap}^{-1}\left(1 + \log\frac{1}{\pi^*}\right), \quad \forall\, 2 < p \leq \infty$$

The bounds provided by this theorem can be proved to be optimal in some cases; but in other cases, the presence of the term $\log \frac{1}{\pi^*}$, which can be very large, worsen the tightness of the bounds. In these cases, can be useful to introduce another quantity, related to a new coercive inequality for the process $(\mathcal{L}, \pi)$, which improves the result of theorem **2.5**.
Given a function $f \geq 0$, let us recall that the *entropy* of f w.r.t. π is given by

$$\text{Ent}(f) \;=\; \mu(f \log f) - \mu(f) \log \mu(f)\,,$$

and observe that $\text{Ent}(f) \geq 0$ from the Jensen inequality.

Definition 2.7. *The logarithmic Sobolev constant of the Markov process $(\mathcal{L}, \pi)$, denoted by c_{sob}, is defined by*

$$c_{sob} \;=\; \inf_{f \geq 0} \frac{\mathcal{D}(\sqrt{f})}{\mathit{Ent}(f)} \tag{2.16}$$

In particular c_{sob} is the best constant c in the logarithmic Sobolev inequality

$$\text{Ent}(f) \;\leq\; c\,\mathcal{D}(\sqrt{f})\,, \quad \forall\, f \geq 0\,.$$

The main interest for the logarithmic Sobolev constant c_{sob} comes from its relation with hypercontractivity properties of the Markov semigroup P_t, that we briefly recall (see [Gro]).

Definition 2.8. *Given a strictly increasing function $q : \mathbb{R}^+ \mapsto [q(0), \infty]$, we say that the Markov semigroup P_t is hypercontractive with contraction function q if for any function f and any $t \geq 0$*

$$\|P_t f\|_{q(t)} \leq \|f\|_{q(0)}$$

We now list a set of results (see [Sa])

Theorem 2.6. *Let c_{sob} the logarithmic Sobolev constant of the Markov process $(\mathcal{L}, \pi)$. Then:*

(i) *P_t is hypercontractive with contraction function $q(t) \;=\; 1 + e^{\frac{4\,t}{c_{sob}}}$.*

(ii) *$\mathit{Ent}_\pi(P_t f) \leq e^{\frac{4\,t}{c_{sob}}} \mathit{Ent}_\pi(f)$ for any $f \geq 0$.*

(iii) *The relaxation time is bounded as follows:*

$$c_{gap}^{-1} \leq T_p \leq \frac{c_{sob}^{-1}}{4} \left(4 + \log_+ \log \frac{1}{\pi^*}\right)\,, \quad \forall\, 1 \leq p \leq 2$$

$$\frac{c_{sob}^{-1}}{2} \leq T_p \leq \frac{c_{sob}^{-1}}{2} \left(3 + \log_+ \log \frac{1}{\pi^*}\right)\,, \quad \forall\, 2 < p \leq \infty$$

where $\log_+ t = \max\{0, \log t\}$.

Statement (ii) is the entropy counterpart of inequality (2.12), from which we can interpret the logarithmic Sobolev constant as a measure of the decay of the entropy. Statement (iv) provides the announced bounds on the relaxation time.

The results stated above, clearly apply to Glauber dynamics $(\mathcal{L}, \mu_\Lambda^\tau)$, which is a reversible and ergodic Markov process on the finite space Ω_Λ^τ. When the spin system under consideration is the Ising model, defined by means of the Hamiltonian in (2.3) (see also chapter 3), and the dynamics is specified by the generator $\mathcal{L}$ in (2.11) with heat-bath jump rates (given in (2.10)), then the *Dirichlet form* of f takes the following equivalent expression

$$\mathcal{D}(f) \,=\, \frac{1}{2}\sum_{x\in\Lambda}\mu_\Lambda^\tau\left(c_x[\nabla_x f]^2\right) \,=\, \sum_{x\in\Lambda}\mu_\Lambda^\tau(\mathrm{Var}_x(f))\,. \tag{2.17}$$

In the case of heat-bath dynamics on a graph G with n vertices and maximum vertex degree $\Delta < \infty$, the main bounds on the mixing time, obtained through c_{gap} and c_{sob}, are the following:

$$\begin{array}{l} c_{gap}^{-1} \,\le\, T_1 \,\le\, c_{gap}^{-1} \,\times\, C_1 n \\ c_{gap}^{-1} \,\le\, T_1 \,\le\, c_{sob}^{-1} \,\times\, C_2 \log n \end{array} \tag{2.18}$$

where C_1 and C_2 are constants depending on β and Δ. We refer to [Mar, Mar2, Sa] for a wider discussion on the relation between c_{gap}, c_{sob} and the relaxation time for Markov chains, and to [GZ, St, var] for analysis and results regarding the logarithmic Sobolev inequalities.

2.3.3 Mixing time and phase transitions

Advances in statistical physics over the last past decade have led to the following remarkable characterization of the mixing time on finite n-vertex cubes with free boundary in the 2-dimensional lattice $\mathbb{Z}^2$. Let β_c denote the critical value marking the separation between the uniqueness phase ($|\mathcal{G}| = 1$) and the coexistence phase ($|\mathcal{G}| > 1$). It is proved (see, for instance [Mar]) that for all $\beta < \beta_c$ the mixing time T_1 is $O(\log n)$, while for all $\beta > \beta_c$, $T_1 = \exp(\Omega(\sqrt{n}))$. Thus the phase transition at β_c, that is a static and spatial phenomenon, has a dramatic manifestation in the form of an explosion from optimal to exponential in the mixing time for the dynamical process. This result is perhaps the most convincing example of an intimate connection

between phase transitions and mixing time. In particular, it manifests the correspondence between the mixing in space, related to the uniqueness of the Gibbs measure as remarked after proposition **2.1**, and the mixing in time. This relation is still under investigation, though notable results for spin system on the integer lattice $\mathbb{Z}^d$ has been derived [SZ, MO1, MO2, Mar1, Wei]. We recall, without proving, some of these results.

We first introduce a notion of temporal mixing for Glauber dynamics $(\mathcal{L}, \mu_\Lambda^\tau)$, with Λ a finite region of V. Assume that $|\Lambda| = n$.

Definition 2.9. *We say that the continuous time dynamics $(\mathcal{L}, \mu_\Lambda^\tau)$ has optimal temporal mixing if there exist constants C and $\alpha > 0$ such that, for any $t > 0$ and all $\sigma \in \Omega_\Lambda^\tau$,*

$$\|P_t(\sigma, \cdot) - \mu_\Lambda^\tau\| \leq Cn \exp(-\alpha t)\,, \tag{2.19}$$

where P_t is the Markov semigroup at time t associated with the dynamics for μ_Λ^τ.

Notice that a simple inversion reveals that optimal temporal mixing is equivalent to a mixing time $T_1 = O(\log n)$. Indeed it has been recently proved (see [HS]) that the mixing time of a dynamics which updates only finite size regions (e.g., the Glauber dynamics) is at least $\Omega(\log n)$, where n is the size of the underlying graph.
The main connections between spatial and temporal mixing for spin systems on the lattice $\mathbb{Z}^d$ are captured in the following results (see [SZ, Ces, MO1, MO2, Wei] and also [Mar] for a nice review on the subject):

Theorem 2.7. *If the Glauber dynamics $(\mathcal{L}, \mu_\Lambda)$, with $\Lambda \in \mathbb{Z}^d$, has optimal temporal mixing for some boundary condition τ then the system has weak spatial mixing. If in addiction the optimal temporal mixing holds uniformly in the boundary condition, then the system has strong spatial mixing.*

Theorem 2.8. *If a system $(\Omega_\Lambda, \mu_\Lambda)$ has strong spatial mixing then the Glauber dynamics has optimal temporal mixing uniformly in the boundary condition.*

As a consequence of Theorems **2.7** and **2.8**, we get that if the mixing time of the Glauber dynamics on the lattices is optimal, i.e. $T_1 = \Omega(\log n)$, then the system is in the uniqueness phase region; viceversa, if the system is in the uniqueness phase region then the mixing time of the Glauber dynamics is $T_1 = \Omega(\log n)$. In particular, for these systems, the occurrence of

a "uniqueness/non-uniqueness" phase transition has a dynamical manifestation as an increment of the mixing time from optimal (order $\log n$) to non optimal.

Remark 2.2. *The statement of Theorems **2.7** and **2.8** can be extended to any graph of subexponential growth rather than just $\mathbb{Z}^d$, as remarked in [Wei]. However, subexponential growth of the underlying graph is required for the theorem to hold. Explicit counterexamples are provided by regular trees, for which it has been proved the there exists a region of the phase space where optimal temporal mixing holds but the Gibbs measure is not unique, and in particular weak spatial mixing does not hold [MSW].*

In the following section we will discuss the Ising model on transitive graphs, and for some particular system we will give precise statements on the behavior of the mixing time. We will also provide other examples of the connection between mixing time and phase transition phenomena.

Chapter 3

Ising Model on Transitive Graphs

3.1 Introduction

The Ising model is an extremely simplified description of ferromagnetism. This is a phenomena that happens, for example, in some metal, when a finite fraction of the spins of the atoms become spontaneously polarized in some direction, giving rise to a macroscopic field. This happens, however, only when the temperature is lower then a characteristic temperature (Curie Temperature T_c), whereas above T_c, the spins are oriented at random producing no magnetic field. As T_c is approached from both sides, the specific heat of the metal approaches infinity and there is phase transition.

The Ising model was introduced by Ising in 1925 following a suggestion of his thesis adviser Lenz. Ising solved the problem in $\mathbb{Z}$, where non phase transitions occurs, and conjectured the same behavior to hold in any dimensions. Contrasting the Ising conjecture, in 1936 Peierls stated the existence of a phase transition for both $\mathbb{Z}^2$ and $\mathbb{Z}^3$, though the Peierls' method was made rigorous only twenty years later by Griffiths and Dobrushin. In 1944 Onsager solved analytically the model in two dimension ($\mathbb{Z}^2$), in the absence of an external magnetic field, and confirmed the existence of a phase transition. The three dimensional model is still not solved exactly.

A lot of works followed this result and the existence of a phase transition for the Ising model on $\mathbb{Z}^d$, for all $d > 1$, was established. At the same time the study of the Ising model on graphs other than the Euclidean lat-

tices started to be analyzed (see, e.g., [Ge, Pre]). Particularly important in this context is the existence of new phenomena which are not present when only consider Euclidean lattice. This reveals the presence of an interplay between the geometry of the graph and the behavior of statistical mechanics system, and suggests the possibility of setting results in a more general theory. The connection between the geometric properties of the graph with the behavior of the model, has motivated many recent works, both concerning the equilibrium properties (see e.g. [BRZ, Io1, Io2, SS, Wu, Wu2, JS, Ly1, Ly2, Sch, HSS]) and the dynamical properties (see, e.g, [BKMP, MSW]).

In this chapter we will present the main results concerning both the equilibrium and dynamical behavior of the Ising model on the three main classes of transitive graphs: the integer lattices $\mathbb{Z}^d$, the regular trees $\mathbb{T}^b$ and the hyperbolic graphs $\mathbb{H}(v,s)$.

3.2 Ising model and phase transition

The Ising model on given infinite, locally finite graph $G = (V, E)$, is defined as follows. To every vertex $x \in V$ is assigned a spin variable $\sigma_x \in \mathcal{S} = \{\pm 1\}$ so that the global system is described by configurations $\sigma \in \Omega = \{\pm 1\}^{|V|}$. The potential is given by nearest neighbor interactions with constant strength J (positive in the ferromagnetic case) and by a constant self potential equal to a parameter $h \in \mathbb{R}$, which represents the external magnetic field. In the following we will assume J to be equal 1.
For every finite subset $\Lambda \subset V$ and $\sigma \in \Omega_\Lambda$, the Hamiltonian of the Ising model (with free boundary condition) is

$$H_\Lambda(\sigma) = \sum_{(xy)\in E(\Lambda)} \sigma_x\sigma_y + h\sum_{x\in\Lambda}\sigma_x \tag{3.1}$$

and the correspondent finite volume Gibbs measure at inverse temperature β, is given by (see 2.2)

$$\mu_\Lambda(\sigma) = (Z_\Lambda)^{-1}\exp[-\beta H_\Lambda(\sigma)] \qquad \forall\,\sigma\in\Omega_\Lambda. \tag{3.2}$$

Notice that the measure μ_Λ, which describes the equilibrium state of the system, assigns higher weight to configurations in which many neighboring spins are aligned with one another, as well as to configurations in which many spins agree with the sign of h. This two effects increase with β and

$|h|$ respectively. In particular, at high temperatures (low β) the spins behave almost independently, while at low temperature large connected regions of equal spins tend to form.

The normalization constant $Z_\Lambda = Z_\Lambda(\beta, h)$, called *partition function*, is of importance because of its connections with the basic thermodynamic quantities. Indeed it is quite natural to define the *internal energy per particle* as the expected value of H_Λ divided by the volume, i.e

$$e_\Lambda(\beta, h) := \frac{1}{|\Lambda|}\mu_\Lambda(H) = -\frac{d}{d\beta}(Z_\Lambda(\beta, h))$$

and thus the *free energy per particle*

$$f_\Lambda(\beta, h) := -\frac{1}{\beta|\Lambda|}\log Z_\Lambda(\beta, h)$$

and the *magnetization*

$$m_\Lambda(\beta, h) := \frac{1}{|\Lambda|}\sum_x \mu(\sigma_x) = -\frac{\partial}{\partial h} f_\Lambda(\beta, h)\,.$$

Since the finite volume Gibbs measure is continuous in β and h, and it is invariant under a global spin flip on Λ, the limit for $h \downarrow 0$ of $m_\Lambda(\beta, h)$ is trivially 0. Instead, if we consider a increasing sequence of subsets $(\Lambda_\ell)_{\ell\geq 0}$ such that $\bigcup_{\ell\geq 0}\Lambda_\ell = V$ and we perform the limit

$$m_0(\beta) = \lim_{h\downarrow 0}\lim_{\ell\to\infty} m_{\Lambda_\ell}(\beta, h) = -\lim_{h\downarrow 0}\frac{\partial}{\partial h}(\lim_{\ell\to\infty} f_{\Lambda_\ell}(\beta, h)) \tag{3.3}$$

a different result can occur.
First of all, for the Ising potential it has been proved that there exists a function $f(\beta, h)$, concave in h, such that

$$f(\beta, h) = \lim_{\ell\to\infty} f_{\Lambda_\ell}(\beta, h)\,.$$

Let us define $D^\pm f(\beta) := \lim_{h\downarrow 0}\frac{f(\beta,\pm h)-f(\beta,0)}{\pm h}$; from the concavity of f we deduce that the derivatives $D^\pm f(\beta)$ exist and moreover, due to the symmetry $h \to -h$, it holds that $D^+ f(\beta) = -D^- f(\beta)$.
If $f \in C^1$ then $D^+ f(\beta) = D^- f(\beta) = 0$, $m_0(\beta) = 0$ and nothing happens; but if there exists a range of β such that f is discontinuous in $h = 0$, then $m_0(\beta) \neq 0$ for all these values of β and we say that a *first order phase transition* occurs.

It is proved that a first order phase transition corresponds to have more than one (automorphism invariant) infinite volume Gibbs measure. In particular, if there exists a unique infinite volume Gibbs measure μ, then from the definition of m_0 we get $m_0(\beta) = \mu(\sigma_0) = 0$, otherwise, as μ varies among different (automorphism invariant) infinite volume limit,

$$m_0(\beta) = \mu(\sigma_0) \in [-D^+ f(\beta), -D^- f(\beta)]\,. \tag{3.4}$$

More precisely, let $\mathcal{G}$ be the set of infinite volume Gibbs measures relative to the Ising potential on G at fixed temperature β. We denote by $\mu^+_{\Lambda_\ell}$ and $\mu^-_{\Lambda_\ell}$ the Gibbs measures over Ω_{Λ_ℓ} with $(+)$-b.c (all $(+)$-spins on Λ^c_ℓ) and $(-)$-b.c. (all $(-)$-spins on Λ^c_ℓ) respectively. The measures $\mu^\pm := \lim_{\Lambda_\ell \uparrow V} \mu^\pm_{\Lambda_\ell}$ satisfy the following properties:
1) $\mu^\pm$ are automorphism invariant measures in $\mathcal{G}$.
2) $\mu^- \leq \mu \leq \mu^+$ for any $\mu \in \mathcal{G}$.
3) $\mu^\pm$ are extremal point of $\mathcal{G}$.
4) $|\mathcal{G}| > 1$ if and only $\mu^+ \neq \mu^-$.
5) $|\mathcal{G}| > 1$ if and only $\mu^+(\sigma_0) \neq \mu^-(\sigma_0)$.
6) $\mu^+(\sigma_0) + \mu^-(\sigma_0) = 0$.
From this properties and (3.4), it holds that $\mu^+(\sigma_0) = -D^- f(\beta)$ and $\mu^-(\sigma_0) = -D^+ f(\beta)$. Thus, a first order phase transition occurs if and only if there exist more then one Gibbs measure.

As we will show in the next section, providing explicit examples, the structure of the Gibbs measure set $\mathcal{G}$ is related to other kind of phase transition phenomena.

3.3 Ising model on lattices: equilibrium and dynamical properties

As remarked in the introduction, the Ising model on the lattice $\mathbb{Z}^d$ was the first non trivial example of a phase transition, that can be solved exactly in $\mathbb{Z}$ and $\mathbb{Z}^2$.
Here we want to give a brief overview of the main equilibrium and dynamical properties of the model, making distinction between the one dimensional case, the two dimensional case and the d-dimensional case, with $d \geq 3$.

3.3.1 One dimensional case

The one-dimensional can be easily solved with the method of transfer matrix (see, e.g, [Sim]). In particular, given a subset $\Lambda \in \mathbb{Z}$ with labeled vertices from 1 to n, the partition function Z_Λ can be expressed as

$$Z_\Lambda = \sum_{\sigma \in \Omega_\Lambda} \prod_{i=1}^{n-1} V(\sigma_i, \sigma_{i+1}) V(\sigma_n, \sigma_1)$$

where $V(\sigma_i, \sigma_j) = \exp\left(\beta(\sigma_i\sigma_j) + \frac{\beta}{2}h(\sigma_i\sigma_j)\right)$. Then, given the real matrix V with entrances $V(\sigma_i, \sigma_j)$ defined as

$$V = \begin{pmatrix} V(++) & V(+,-) \\ V(-,+) & V(-,-) \end{pmatrix} = \begin{pmatrix} e^{\beta(1+h)} & e^{-\beta} \\ e^{-\beta} & e^{\beta(1-h)} \end{pmatrix},$$

we get that $Z_\Lambda = \mathrm{Tr}V^n$. The matrix V is symmetric and with strictly positive elements; thus, by the Perron-Forensics theorem, there exist two eigenvalues λ_1 and λ_2 such that $\lambda_1 > 0$ and $\lambda_1 > \lambda_2$. Then the partition function is given by $Z_\Lambda = \lambda_1^n + \lambda_2^n = \lambda_1^n(1 + (\frac{\lambda_2}{\lambda_1})^n)$, with $\frac{\lambda_1}{\lambda_2} < 1$, and it holds

$$f(\beta, h) = -\frac{1}{\beta} \lim_{n\to\infty} \frac{\log Z_\Lambda}{n} = -\frac{1}{\beta} \log \lambda_1 \,. \tag{3.5}$$

Computing λ_1 one thus obtain the explicit expression for the free energy per particle and all the thermodynamic quantities which can be derived from the partition function. The main consequences we want to stress are the following

1. the free energy is a real analytic function of β and h, and thus the system doesn't show phase transition.

2. the correlation between two spin σ_i and σ_j decays exponentially in the distance $|i-j|$ uniformly in the boundary condition, i.e. $\mu_\Lambda^\tau(\sigma_i\sigma_j) \leq ce^{-m|i-j|}$ with $c, m > 0$ and $\forall\, \tau \in \Omega$.

For all temperature, the one dimensional Ising model is thus described by a unique Gibbs measure μ. Moreover, any finite Gibbs measure μ_Λ has exponential decay of correlations and verifies the strong mixing condition (2.8) uniformly in the boundary condition. From Theorem **2.8** we conclude that, for all temperature $\beta > 0$, the Glauber dynamics on a n-vertex subset of $\mathbb{Z}$ has optimal temporal mixing $T_1 = \Omega(\log n)$.

3.3.2 Two dimensional case

The exact solution of two dimensional model given by Onsager, for $h = 0$, relays again on the method of transfer matrix, but in this case the proof is rather difficult, essentially due the complexity in diagonalizing the transfer matrix. We refer the reader to the books [Sim, **?**], where the proof is discussed in details.
The important consequence of this result, is the existence of a first order phase transition which occurs at $h = 0$ and inverse temperature $\beta_c > 0$. We stress that a proof of this fact can be obtained, independently from the analytic solution, with the Peierls-method, introduced by Peierls and later made rigorous by Griffiths and Dobrushin. The Peirels method also applies to the case $\mathbb{Z}^d$ with $d \geq 3$.
For high temperature $\beta < \beta_c$ or $h \neq 0$, the system is thus in the uniqueness phase region, whereas for $\beta > \beta_c$ and $h = 0$ the system is in the multiple phase region. In particular, as described in the previous section, the set of Gibbs measures $\mathcal{G}$ at least contains the two extremal measures μ^+ and μ^- .

Afterwards the problem of the existence of Gibbs states, different from μ^+, μ^- and their convex combinations, has been considered and solved independently in [Aiz, Hi] . The result is that in $\mathbb{Z}^2$ there is no non-translation-invariant Gibbs state, and μ^+ and μ^- are the only extremal measures.

Let us now consider the Glauber dynamics for the Ising model on an n-vertex squares $\Lambda \in \mathbb{Z}^2$. We recall the following results:

1. For all $\beta < \beta_c$ or $h \neq 0$ (uniqueness phase region) the measure μ_Λ satisfies the strong mixing condition (2.8) uniformly in the boundary condition [MOS]. From Theorem **2.8** then it holds that the mixing time of the Glauber dynamics on Λ, with Λ finite square of $\mathbb{Z}^2$, is $T_1 = \log n$.

Remark 3.3. *If $\beta < \beta_c$ and for any finite subset $V \in \mathbb{Z}^2$, it holds that the measure μ_V satisfies the weak mixing condition (2.7) (see [Hi, MO1]). The authors of [MOS] then show that weak mixing for all finite $V \in \mathbb{Z}^2$ implies the strong mixing for all finite squares $\Lambda \in \mathbb{Z}^2$, and the result follows.*

2. For all $\beta \geq \beta_c$ and $h = 0$ (phase coexistence region), the spectral gap of the Glauber dynamics on Λ decays exponentially with $\sqrt{n}$, i.e. there exists $c > 0$ such that $c_{gap} = \Theta(exp{-}(c\sqrt{n}))$ [CGMS, Mar1]. From

Theorem **2.5** then it holds that the mixing time of the Glauber dynamics on Λ, with Λ finite square of $\mathbb{Z}^2$, is $T_1 = \Theta(\exp(c'\sqrt{n}))$.

The occurrence of a phase transition in $\mathbb{Z}^2$ is thus associated with a dramatic increase of the mixing time, from logarithmic to exponential, of the associated dynamics. The intrinsic motivation for this behavior is the presence of a "bottleneck" in the phase space, given by the set of configurations of zero magnetization. Indeed, let us consider the dynamics started from all minuses. Since the measure μ_Λ with free boundary condition has two maxima, corresponding to the two configurations identically equal to $(+)$ or $(-)$, the dynamics, in order to relax to equilibrium, necessarily crosses the bottleneck. Thus, since the Gibbs measures gives to the latter a weight of the order of a negative exponential of the surface of Λ, i.e. $\sqrt{n}$, the result follows.

The same motivations also suggests that if the bottleneck is removed, for example taking all $(+)$ spin on the boundary of Λ, then the relaxation time must be much shorter. In particular it has been conjectured the following [FH, BM]:

Conjecture 3.1. *In the phase coexistence region, the Glauber dynamics for the Ising model on an n-vertex square of $\mathbb{Z}^2$ with $(+)$-boundary condition has mixing time $T_1 = n \log n$*

Though this conjecture seems very natural, proving results in this direction has proved very elusive and, until now, the only available bounds are upper bounds on the spectral gap and the logarithmic Sobolev constant. More precisely, has been proved (see [BM]) that in the same situation of Conjecture 3.1 it holds: $c_{gap} \leq 1/\sqrt{n}$ and $c_{sob} \leq 1/n$.
This result essentially improves over all the previous existing bounds.

3.3.3 High dimension lattice case

The Ising model on $\mathbb{Z}^d$, with $d \geq 3$, is similar in many aspects with the two-dimensional case, and many results described in the previous paragraph also apply to this setting. Let us first discuss the equilibrium properties.
As just remarked, with the Peierls' method one can prove the existence of a first order phase transition which occur at $h = 0$ and $\beta_c = \beta_c(d) > 0$. Thus the system is in the uniqueness phase for all $\beta < \beta_c$ or $h \neq 0$, and is in the

coexistence pase for all $\beta > \beta_c$ and $h = 0$.
Whereas in the two dimensional case there are only two extremal measures, μ^+ and μ^-, in higher dimensions Dobrushin [Do] proved that for low temperature there exist non-translation invariant measures, which of course are not combinations of μ^+ and μ^-. These state arise from some suitable mixed boundary conditions which create a rigid interface separating the system into two regions (see also [DS]).
The problem of whether there exist translation invariant extremal measures, other then μ^+ and μ^-, has been completely solved only recently [Bod] and it has been proved, as was widely expected, that μ^+ and μ^- are the only translation invariant extremal measures.

Consider now the Glauber dynamics on an n-cubes $\Lambda \in \mathbb{Z}^d$, with $d \geq 3$. We recall the following results.

1. For all $\beta < \beta_c$ or for $\beta \gg 1$ and $h \neq 0$ the measure μ_Λ satisfies the strong mixing condition (2.8) uniformly in the boundary condition [MOS]. From Theorem **2.8** then it holds that the mixing time of the Glauber dynamics on Λ, with Λ finite square of $\mathbb{Z}^2$, is $T_1 = \log n$.

Remark 3.4. *Notice that whereas in the two dimensional case it is proved that optimal mixing time holds inside all the uniqueness region, in higher dimensions this statement excludes the region of $h \neq 0$ and β near β_c. The reason of this difference is that, in high dimension, the weak mixing condition has been proved only for $\beta < \beta_c$ or for $\beta \gg 1$ and $h \neq 0$ (see [Hi, MO1]). Thus the result, which is obtained by proving that the weak mixing for all finite $V \in \mathbb{Z}^d$ implies strong mixing for all finite cubes $\Lambda \in \mathbb{Z}^d$ [MOS], only apply in this region. However there is no result concerning weak spatial mixing in the region of $h \neq 0$ and β near β_c.*

2. For all $\beta \geq \beta_c$ and $h = 0$, the spectral gap of the Glauber dynamics on Λ decays exponentially with $n^{\frac{d-1}{d}}$, i.e. there exists $c > 0$ such that $c_{gap} = \exp\left(-O(n^{\frac{d-1}{d}})\right)$ [CGMS, Mar1]. From Theorem **2.5** then it holds that the mixing time of the Glauber dynamics on Λ, with Λ finite cubes of $\mathbb{Z}^d$, is $T_1 = \exp(\Omega(n^{\frac{d-1}{d}}))$.

The behavior of the dynamics on $\mathbb{Z}^d$ with $d \geq 3$ is thus qualitatively the same as for Z^2. In particular, also in this general case, it is expected the existence of a the dependence of the dynamics from the boundary condition [FH, BM]:

Conjecture 3.2. *In the phase coexistence region, the Glauber dynamics for the Ising model on an n-vertex cube of $\mathbb{Z}^d$ with $(+)$-boundary condition has mixing time $T_1 = n^{\frac{d-1}{d}} \log n$*

In [BM], in the same situation of Conjecture 3.1, it is proved that for all $d \geq 2$ it holds the following bound: $c_{sob} \leq n^{-\frac{2}{d}}$.

3.4 Ising model on regular trees

Regular tree graphs were the first non Euclidean graphs to be considered as support for statistical mechanics models. (In this context, the tree is sometimes referred to as the "Bethe approximation" of the lattice). Firstly because they include, as a particular case, the one-dimensional integer lattice. Secondly because they are simpler in many respect then $\mathbb{Z}^d$, due to the lack of cycles.
In spite of their simplicity, they exhibits two phase transitions: a first order phase transition with a non trivial dependence from the magnetic field; a new phase transition, related to the structure of the Gibbs measure set, which does not appear on the lattice[Ge, Pre, BRZ, Io1, Io2, Ly2]. Moreover, as we will show in section 3.4.2, the dynamical behavior of model, and its relation with the equilibrium properties, need a completely different description respect to the lattice case [BKMP, MSW].

3.4.1 Equilibrium properties

We recall that, for every $b \geq 2$, $\mathbb{T}^b$ denotes the infinite regular tree with vertex degree equal to $b+1$. Let us first consider the case with zero magnetic field.
This model is known to have two critical temperatures β_0 and β_1. The first one, $\beta_0 = \frac{1}{2} \log \frac{b+1}{b-1}$, marks the line between uniqueness and non-uniqueness of the Gibbs measure and thus corresponds to a first order phase transition. In particular, if $\beta \leq \beta_0$ then the system is in the uniqueness region, whereas for $\beta > \beta_0$ is in the coexistence region [Ge, Pre].
However, in contrast to the model on $\mathbb{Z}^d$, there is now a second critical point $\beta_1 = \frac{1}{2} \log \frac{\sqrt{b}+1}{\sqrt{b}-1}$ [BRZ, Io1, MSW], which delimits the region where the free boundary condition measure μ^f is extremal. More precisely, for all β such that $\beta_0 < \beta \leq \beta_1$ then μ^f is an extremal Gibbs measure, whereas

if $\beta > \beta_1$ then μ^f is a convex combination of other extremal measures. In other word, in the intermediate region $\beta_0 < \beta \leq \beta_1$ the $(+)$ and $(-)$ boundary conditions exert long-range influence but "typical" boundaries do not, while in the low temperature region $\beta > \beta_1$ long-range influence occurs even for typical boundaries. We refer this phenomenon as an *extremality phase transition*.

Let us now examine what happens when an external magnetic fields h is added to the system. It turns out that for all $\beta > \beta_0$, there is a critical value $h_c = h_c(\beta) > 0$ such that the Gibbs measure is not unique if $|h| \leq h_c$, and is unique when $|h| > h_c$. The landscape of the phase diagram can be summarized as follows (see fig):

- For $\beta \leq \beta_0$ or for all $|h| > h_c$ there is a unique Gibbs measure.
- For $\beta_0 < \beta \leq \beta_1$ and $|h| \leq h_c$ there are infinite Gibbs measures and μ^f is an extremal state.
- For $\beta > \beta_1$ and $|h| \leq h_c$ there are infinite Gibbs measure and μ^f is not an extremal state.

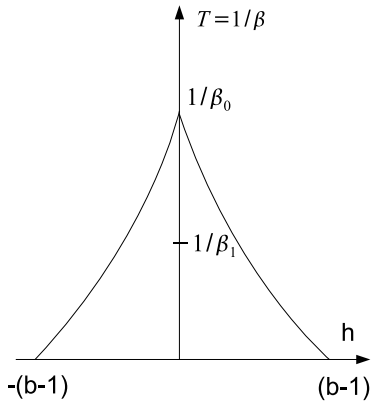

Figure 3.1: Phase diagram of $\mathbb{T}^b$. The Gibbs measure is unique above the curve

3.4.2 Dynamical properties

Recently, the analysis of the Glauber dynamics on regular trees yields interesting results, which clarify the dynamical behavior of such a model.

The first remarkable result, obtained in [BKMP], shows that the relaxation time of the dynamics on finite trees, with zero external field, is $\Omega(1)$ for all $\beta < \beta_1$. Moreover, at the critical point $\beta = \beta_1$ the same relaxation time is bounded above by $c \log n$, and as soon as $\beta > \beta_1$ it becomes smaller than an $n^{\alpha(\beta)}$, with $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$.

Part of this result, is obtained generalizing an argument for $\mathbb{Z}^d$ (see [Mar, JeS]) to graphs with finite vertex degree. This argument provides a general bound on the relaxation time depending from some graph parameters. Afterwards, the graph structure is analyzed in order to find suitable bounds on these parameters. More precisely, the authors of [BKMP] prove the following Proposition.

Definition 3.10. *The* cut-width $\xi(G)$ *of a finite graph G is the smallest integer such that there exist a labeling $\{v_1, v_2, \dots, v_n\}$ of the vertices such that for all $1 \leq k \leq n$, the number of edges from $\{v_1, v_2, \dots, v_k\}$ to $\{v_{k+1}, \dots, v_n\}$ is at most $\xi(G)$.*

Proposition 3.2. *Let G a finite graph with n vertices and maximal vertex degree Δ. The relaxation time of the Glauber dynamics for the Ising model on G is at most $ne^{(4\xi(G)+2\Delta)\beta}$.*

The polynomial bound on the relaxation time then follows by proving, with a rather simple argument, that the cut-width of the finite tree is bounded by a quantity proportional to the logarithm of its size.

From this result one can deduce the main differences between the dynamical behavior on regular trees and on lattices. Firstly, whereas in the lattice case, at low temperature, the relaxation time for a box of size n decays exponentially in $n^{\frac{d-1}{d}}$, in the tree case the relaxation time never grows more than polynomially in the size. Secondly, whereas in the lattice case the jump in the behavior of the relaxation time, as a function of the size, reflects the first order phase transition, in the tree case the same behavior appears at the critical point β_1, and thus reflect the extremality phase transition.

The second important result, obtained in [MSW], concerns the Glauber dynamics on trees with $(+)$-boundary condition. It is proved the following:

Theorem 3.9. *In both of the following situations, the spectral gap and the logarithmic Sobolev constant of the Glauber dynamics on an n-vertex tree are $\Omega(1)$:*

(i) $\beta < \beta_1$ *or* $|h| > h_c$*, with arbitrary boundary conditions;*

(ii) $(+)$*-boundary condition and arbitrary* β*,* h*.*

Notice first that together Theorems **2.5** and **2.6**, this result implies that the mixing time is $T_1 = \Omega(\log n)$ in all the situations described by Theorem **3.9**. Of particular interest is the result when $\beta > \beta_1$, $h = 0$ and the boundary condition is $(+)$. In this case, changing the $(+)$-boundary condition with a free boundary condition, the relaxation time jumps from $\Theta(1)$ to $O(n^{\alpha(\beta)})$, with $\alpha(\beta)$ increasing in β. This behavior shows with evidence that on the regular tree there is a strong dependence of the mixing time from the boundary conditions. Notice again, that on $\mathbb{Z}^d$ not much is known about the spectral gap when $\beta > \beta_c$, $h = 0$ and the boundary condition is $(+)$, and the only available bounds are upper bounds.

3.5 Ising model on hyperbolic graphs

3.5.1 Equilibrium properties

The study on the Ising model on hyperbolic graphs, when the magnetic field is zero, led to the characterization of two different phase transitions appearing at inverse temperatures $\beta_c \leq \beta_c'$ [SS, Wu, Wu2]. The first one, β_c, corresponds to the occurrence of a first order phase transition; thus if $\beta \leq \beta_c$ there is a unique Gibbs measure, whereas if $\beta > \beta_c$ there are infinite Gibbs measures (at least all the convex combinations between μ^+ and μ^-). The critical temperature β_c' is defined as follows:

$$\beta_c' = \inf\{\beta \geq \beta_c : \mu^f = (\mu^+ + \mu^-)/2\}.$$

Thus, the phase transition at β_c' refers to a change of the properties of the free boundary condition measure μ^f: for $\beta_c < \beta \leq \beta_c'$ then μ^f is not a convex combination of μ^+ and μ^-, but if $\beta > \beta_c'$ then μ^f recover the property $\mu^f = (\mu^+ + \mu^-)/2$.
The most interesting scenario clearly appears when the strict inequality $\beta_c < \beta_c'$ holds, namely when there exists a nonempty interval between β_c and β_c'. In this case, for all $\beta \in (\beta_c, \beta_c']$, it holds that $\mu^f \neq (\mu^+ + \mu^-)/2$ which implies the existence of a translation invariant Gibbs measure different from μ^+ and μ^-.

Remark 3.5. *Notice that this behavior does not appear in the Ising model on lattices and on regular trees. On $\mathbb{Z}^d$ the only translation invariant Gibbs states are μ^+ and μ^-. On $\mathbb{T}^b$, for all $\beta > \beta_c$, the measure μ^f never satisfies the property to be a convex combination of μ^+ and μ^-.*

Although it is believed that the strict inequality $\beta_c < \beta_c'$ holds for any hyperbolic graph, at the moment it has only been proved for a subclass satisfying some conditions on v and s (see [Wu2] for details).
For that concerns the dependence of the first order phase transition from the magnetic fields, we can recall the following result which holds for all non amenable graphs [JS].

Theorem 3.10. *Consider the Ising model on G, with G an infinite non amenable graph with finite degree. Then there exists $h > 0$ and $\beta \in [0, \infty)$ such that G exhibits a phase transition.*

This result clearly applies to the hyperbolic graphs and thus show, as in the tree case, the existence of a first order phase transition for $h \neq 0$.

Other open questions about the Gibbs measures for the hyperbolic graphs are the following: is μ^f extremal when $\beta_c < \beta < \beta_c'$? are μ^+ and μ^- the only extremal measures for $\beta > \beta_c'$?
We remark that Series and Sinai in [SS] gave a possible answer to this last question, proving that for low temperatures there exists uncountably many mutually singular Gibbs states which they conjectured to be extremal. The measures they constructed are similar to the non-translational invariant extremal measures for the Ising model on $\mathbb{Z}^d$, $d \geq 3$, constructed by Dobrushin ([Do, Do2]). But in contrast of what happens on $\mathbb{Z}^d$, they are an uncountable set.

Summarizing, the known landscape of the phase diagram for the Ising model on $\mathbb{H}(v, s)$ is the following [Wu, Wu2, SS, JS]:

- For $\beta \leq \beta_c$ or for all $|h| > h_c$ there is a unique Gibbs measure;
- For $\beta_c < \beta < \beta_c'$ and $|h| \leq h_c$ there are infinite Gibbs states and $\mu^f \neq (\mu^+ + \mu^-)/2$, i.e. μ^+ and μ^- are not the only extremal measures;
- For $\beta > \beta_c'$ and $|h| \leq h_c$ there are infinite Gibbs states and $\mu^f = (\mu^+ + \mu^-)/2$;
- For $\beta \gg \beta_c'$ there exist uncountably many mutually singular, and presumably extremal, Gibbs states.

Connection with the bond percolation

Unfortunately sharp estimates of the two critical temperature for the Ising model on hyperbolic graphs still do not exist. Nevertheless, some bounds on β_c and β_c' are available thanks to the connection, through the random cluster model (or FK model), between Ising model and percolation. Independent bond (and site) percolation on hyperbolic graphs have been indeed widely studied and many useful results have been proved, see for examples [BS2, BS3, GN, Jo, La, LS].

The independent bond percolation on a graph $G = (V, E)$ is the random subgraph with vertices V, and where each edge is in the percolation subgraph with probability p, independently. The main focus in this setting concerns the existence of one or more infinite clusters depending from the probability $p \in [0, 1]$ to keep each edge in the percolation subgraph. For any p, the number of infinite cluster is almost surely constant taking values 0, 1, or ∞ [BS, NS]. A basic result about percolation on transitive graph G is that there are two critical points, $0 < p_c(G) \leq p_u(G) \leq 1$, which define the boundaries of three distinct phases described as follows.

1. if $p < p_c(G)$ then almost surely there is no infinite cluster;

2. if $p_c(G) < p < p_u(G)$ then almost surely there are infinitely many infinite clusters ;

3. if $p > p_u(G)$ then almost surely there is exactly one infinite cluster.

In analogy with the results listed above for the Ising model, the existence of two distinct phase transitions for percolation strictly depends from the graph structure. For example, it has been proved (see [BK]) that on amenable transitive graphs $p_c(G) = p_u(G)$; moreover on all infinite transitive graphs with two ends, which are amenable (see section 1.2.2), $p_c(G) = p_u(G) = 1$. On the other hand, there are many examples of non amenable graphs for which $p_c(G) < p_u(G)$, such as non amenable transitive graphs with infinity ends. For this class of graphs, including $\mathbb{T}^b$, it has been proved that $p_c(G) < p_u(G) = 1$, meaning that the uniqueness of the infinite cluster only occurs at the trivial value $p = 1$.
The landscape for transitive graphs with one end is not really understood and there are two important conjectures concerning them.

Conjecture 3.3. *Suppose that G is an infinite, locally finite, connected, transitive graph. Then $p_c(G) < p_u(G)$ if and only if G is non-amenable.*

Conjecture 3.4. *Suppose that G is an infinite, locally finite, connected transitive graph. Then $p_u(G) < 1$ if and only if G has one end.*

As remarked, the first conjecture, due to Benjamini and Schramm [BS], has been proved for many non-amenable graphs. Benjamini and Schramm themselves, proved in [BS2] that $0 < p_c(G) < p_u(G) < 1$ for any planar transitive graph with one end, including hyperbolic graphs.

The proof of the existence of the second phase transition for the Ising model on hyperbolic graphs ([Wu2]) exploits this last relation and the connection between Ising model and percolation, to obtain the bounds

$$\ln(\frac{1}{1-p_c}) \leq \beta_c \leq \ln(\frac{1+p_c}{1-p_c}) \quad ; \quad \ln(\frac{1}{1-p_u}) \leq \beta_c' \leq \ln(\frac{1+p_u}{1-p_u})$$

(β_c and p_c are the critical points referred to hyperbolic graphs). Using some estimates of p_c and p_u, involving some graph properties like isoperimetric constants and spectral radius, one then get the following bounds

$$\ln(\frac{i_e}{i_e+1}) \leq \beta_c \leq \ln\left(\frac{i_e+2}{i_e}\right) \quad ; \quad \beta_c' \geq \ln\left(\frac{\sqrt{v-i_e^2}}{\sqrt{v-i_e^2}-1}\right),$$

where i_e denotes the isoperimetric edge constant (see definition (1.2).

3.5.2 Dynamical properties

From the dynamical point of view, the Ising model on hyperbolic graphs has been recently investigated in the work of Peres [BKMP]. In analogy with the results for regular trees, it is proved that, for every temperature, the relaxation time of the Glauber dynamics on an n-vertex ball in the hyperbolic graph is polynomial in n.

The proof is based again on Proposition **3.2**. But in order to apply this result, a bound on the cut-width of the graph is required. The following result, proved in [BKMP], provide such a bound for a family of graphs which includes hyperbolic tiling.
Given an infinite graph G, let G_r denote the ball of radius r around a fixed vertex o, and let $|G_r| = n_r$.

Proposition 3.3. *For every $c > 0$ and $\Delta < \infty$, there exists a constant $C = C(c, \Delta)$ such that if G is an infinite planar graph with*

- *vertex isoperimetric constant at least* c
- *maximum degree bounded by* Δ *and*
- *for every* r *no cycle from* G_r *separates two vertices of* $G \setminus G_r$*,*

then $\xi(G_r) \leq C \log n_r$ *for all* r*.*

Combining Propositions **3.2** and **3.3**, we get that for all $\beta > 0$ the Glauber dynamics for the Ising models on balls in the hyperbolic tiling has relaxation time bounded as $T_0 \leq cn^{\alpha(\beta)}$, with $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$.
The differences of the low temperature behavior of the dynamics between hyperbolic graphs and integer lattices, shows once again the tight connection between the graph properties and dynamics.

The dependence of the dynamics from the boundary condition in the hyperbolic case will be investigated in the next chapter, where we will state and prove our main result.

Chapter 4

Fast mixing inside the pure phase: hyperbolic graph case

The goal of this work will be to prove, through spectral properties, fast relaxation time for the Glauber dynamics of the Ising model on hyperbolic graphs with $(+)$-boundary conditions at low temperatures.
This approach first appeared in [MSW], where different spin systems defined on a finite regular tree are considered and the mixing time for the Glauber dynamics with fixed b.c. on them is analyzed. Whereas standard techniques do not distinguish between b.c., then giving uniform bounds on the mixing time, this new approach allows to perform analysis sensitive to b.c.. This technique is particularly important when the temperature slows down the critical one ($\beta > \beta_c$) and the system reaches the multiple phase region. In this case we say that a bottleneck in the phase space appears so that any uniform estimates of the mixing time give "drastic" bounds. Fixing for example (+)-b.c., one would like to select the μ^+ state between the pure phases in order to avoid such a bottleneck. This could change deeply the behavior of the mixing time and it is indeed the case for the Glauber dynamics on a regular tree of size n, with (+)-b.c., for which the mixing time in the phase coexistence region remains of order $\log n$, as proved in [MSW].
We point out that the influence of the boundary conditions on the mixing time for systems in the multiple phase region, is one of the most interesting and difficult question left open by the dynamical analysis of systems lying on $\mathbb{Z}^d$. Though uniform in the b.c. bounds on the mixing time are of exponential order in $n^{1-1/d}$ (see [Th, Mar]), it has been long time conjectured that,

in the presence of (+)-b.c., the relaxation process is driven by the mean-curvature motion of interfaces separating droplets of the $(-)$-phase inside the $(+)$-phase. This would suggest that the mixing time should be polynomial in n, most likely $n^{2/d}\log n$ [BM, FH]. In particular it has been argued that the spectral gap for the Glauber dynamics of the Ising model in the pure phases at high enough dimension should be $\Omega(1)$. These conjectures are supported by the recently proved results obtained on $\mathbb{Z}^2$ showing that, with $(+)$-b.c., the spectral gap in a square with n sites shrinks to zero at least as fast as $1/\sqrt{n}$ [BM]. Unfortunately not much is known about $\mathbb{Z}^d$ for $d > 2$ in the multiple phase region.

The strong dependence from boundary conditions, proved for regular trees, does not seem to be a peculiar behavior of these graphs but rather to be related to some graph properties that trees satisfy.
Among them we point attention on *non-amenability*.

4.1 Notation and statement of results

Let us consider the infinite hyperbolic graph $\mathbb{H}(v,s)$, with V denoting the vertex set and E the edge set. The set V admits a natural notion of distance as the length of the shortest path between two vertices. For every $x, y \in V$ we denote their distance by $d(x,y)$.
We fix a vertex $o \in V$ (origin or root) and for any integer k we denote by $B_k = (V_k, E_k) \in \mathbb{H}(v,s)$ the ball centered in o and with radius k, namely the finite subgraph induced on $V_k = \{x \in V : d(o,x) \leq k\}$. We will fix an integer m and let $B \equiv B_m$ denote the corresponding ball. Let us give a few useful definitions:

(i) For each $i = 0, \dots, m$ we define the i-th *level* L_i as

$$L_i = \{x \in V \,:\, d(x,o) = i\} = \{x \in V_i \,:\, \exists y \in V_i^c \text{ s.t. } x \sim y\}\,,$$

so that $B_m = \sum_{i=0}^m L_i$ and for all $i > 0$ $L_i = \partial_V B_{i-1}$;

(ii) For each $i = 0, \dots, m+1$ we define the subset $F_i \subset B$

$$F_i := \{v \in \bigcup_{k=i}^{m} L_k\} = \{v \in B_{i-1}^c\}\,,$$

so that $\{F_i\}_{i=0}^{m+1}$ is a decreasing sequence of subsets such that

$$V_m = F_0 \supset F_1 \supset \ldots \supset F_{m+1} = \emptyset\,;$$

(iii) For each $i = 0, \dots, m+1$ we define the σ-algebra $\mathcal{F}_i$ generated by the functions π_x for $x \in F_i^c = B_{i-1}$.

We now consider the Ising model on B at inverse temperature β, zero external field and with $(+)$-boundary condition on $\partial_V B$.
The system is specified by configuration $\sigma \in \Omega^+ := \{\pm 1\}^{B \cup \partial_V B}$ which agree with the $(+)$-configuration on $\partial_V B$. We endow Ω^+ with the σ-algebra $\mathcal{F}$ generated by the set of projections $\{\pi_x\}_{x \in V_{m+1}}$ from Ω^+ to $\{\pm 1\}$, where $\pi_x : \sigma \mapsto \sigma_x$. For any $A \subset V_m$ we also consider the set of configurations in $\Omega_A := \{\pm 1\}^A$ with σ-algebra $\mathcal{F}_A$ generated by π_x , $x \in A$; we write $f \in \mathcal{F}_A$ to indicate that f is $\mathcal{F}_A$-measurable.
For every $\eta \in \Omega^+$ and any subset $A \subset V_m$, we denote by Ω_A^η the set of configurations $\sigma \in \Omega$ that agree with η on $\partial_V A$ and by μ_A^η the Gibbs distribution over Ω^+ conditioned on the configuration outside A being η. Namely to every $\sigma \in \Omega_A^\eta$ is assigned probability

$$\mu_A^\eta(\sigma) \,=\, \frac{1}{Z(\beta)} \exp(\,\beta \sum_{(xy) \in E(\overline{A})} \sigma_x \sigma_y)\,,$$

where $Z(\beta)$ is a normalizing constant and the sum runs over every couples of nearest neighbors in the induced subgraph of $\overline{A} = A \cup \partial_V A$, otherwise $\mu_A^\eta(\sigma) = 0$. When $A = V_m$, $\mu_{V_m}^\eta$ is simply the Gibbs distribution on B with boundary condition $(+)$, and we shall use the shorthand μ for $\mu_{V_m}^+$.

Moreover, we will use the short notation $\mu_i^\eta := \mu_{F_i}^\eta = \mu(\cdot\,|\eta \in \mathcal{F}_i)$ for the Gibbs distribution on F_i and boundary condition $\eta \in \Omega^+$, and analogously we denote by Var_i^η the variance w.r.t. μ_i^η.
We recall that if $f : \Omega^+ \to \mathbb{R}$ is a measurable function, the expectation of f w.r.t. the measure μ_i^η is given by

$$\mu_i^\eta(f) = \sum_{\sigma \in \Omega} \mu_i^\eta(\sigma) f(\sigma)\,.$$

We usually think of it as a function of η, that is $\mu_i(f)(\eta) = \mu_i^\eta(f)$; in particular $\mu_i(f) \in \mathcal{F}_i$.
Throughout the discussion c will denote a constant which is independent from $|B| = n$ but may depend on the parameters (v, s) of the hyperbolic graph and on β. The particular value of c may change from line to line as the discussion progresses.

Given $f \in L^2(\Omega^+, \mathcal{F}, \mu)$ and an inverse temperature $\beta \gg 1$ (multiple phases region), we will look for a Poincaré inequality for the Gibbs measure

μ of the kind

$$\mathrm{Var}(f) \leq c\,\mathcal{D}(f)\,,$$

with $\mathcal{D}(f)$ the Dirichlet function defined through rates (2.10) (heat-bath dynamics) and $c = c(\beta, B)$ a constant. From definition (2.13) of spectral gap and inequalities (2.18), we thus get bounds on the mixing time of the dynamics; the smaller constant c is the better the bound.

The aim of our analysis is to prove that inside the $(+)$-pure phase, the measure μ satisfies a Poincaré inequality with constant c independent from the size of the system. In this case the spectral gap remains bounded from 0 uniformly in the system size, implying the an upper bound on the mixing of the dynamics of order n. We state the following:

Theorem 4.11. *Let $\mathbb{H}(v, s)$ such that $v > 4$ and $s > 3$. Then, for all $\beta \gg 1$, the Glauber dynamics on an n-vertex ball $B \in \mathbb{H}(v, s)$ with (+)-boundary condition has $c_{gap} = \Omega(1)$ and then it holds:*

$$T_0 = \Theta(1) \quad \textit{and} \quad T_1 \leq cn \tag{4.1}$$

Remark 4.6. *To better appreciate the result, let us remark that for typical boundary conditions the only known bound on the relaxation time for balls in an hyperbolic graph is polynomial in n, and more precisely it holds that $T_0 \leq cn^{\alpha(\beta)}$, with exponent $\alpha(\beta) \uparrow \infty$ as $\beta \to \infty$ (see proposition **(3.2)** and **(3.3)**). Thus, the presence of the $(+)$-boundary conditions gives rise to an abrupt jump of the relaxation time from polynomial in n to non increasing with n. This behavior provides a convincing example of the influence of boundary conditions on the spectral gap.*

4.2 Sketch of the Proof

Let us first recall the following decomposition property of the variance

$$\mathrm{Var}_C^\eta(f) \;=\; \mu_C^\eta[\mathrm{Var}_D(f)] + \mathrm{Var}_C^\eta[\mu_D(f)] \;\;, \quad \text{for } D \subseteq C \subseteq V_m\,. \tag{4.2}$$

Applying recursively (4.2) to subsets $F_0 \supset F_1 \supset \ldots \supset F_{m+1}$, and with equalities $\mu_i(\mu_{i+1}(f)) = \mu_i(f)$ and $\mu_{m+1}(f) = f$, we obtain

$$\begin{aligned} \mathrm{Var}(f) &= \mu[\mathrm{Var}_m(f)] + \mathrm{Var}[\mu_m(f)] \\ &= \mu[\mathrm{Var}_m(\mu_{m+1}(f))] + \mu[\mathrm{Var}_{m-1}(\mu_m(f))] + \mathrm{Var}[\mu_{m-1}(\mu_m(f))] \\ &= \vdots \\ &= \sum_{i=0}^{m} \mu[\mathrm{Var}_i(\mu_{i+1}(f))]\,. \end{aligned} \tag{4.3}$$

To simplify notation we define $g_i := \mu_i(f)$, for all $i = 0, \ldots, m+1$; clearly $g_i \in \mathcal{F}_i$. Inserting g_i in (4.3) we then have

$$\mathrm{Var}(f) = \sum_{i=0}^{m} \mu[\mathrm{Var}_i(g_{i+1})]\,. \tag{4.4}$$

There are two key steps that we will perform in order to obtain a Poincaré inequality for μ, with constant independent of the size of the system:

1. proving that $\forall\, \tau \in \Omega^+$, it holds the following Poincaré inequality with constant uniformly bounded in the size of L_i:

$$\mathrm{Var}_i^\tau(g_{i+1}) \leq c \sum_{x \in L_i} \mu_i^\tau(\mathrm{Var}_x(g_{i+1})) \tag{4.5}$$

2. relating the local variance of $g_i = \mu_i(f)$ to the local variance of f in order to get an inequality of the kind

$$\sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_x(g_{i+1})) \leq c\mathcal{D}(f) + \epsilon \sum_{i=0}^{m} \sum_{x \in L_i} \mu(\mathrm{Var}_x(g_{i+1})) \tag{4.6}$$

with ϵ a small quantity for $\beta \gg 1$.

The main ingredient that enables us to prove these two steps, is a kind of correlation decay between spins on a fixed level L_i under the measure μ_i^τ with $\tau \in \Omega^+$. The proof of this property is based on the fact that the influence of the $(+)$-b.c. on a given spin is stronger then the influence of a nearest spin, also in the worst case (all (-) neighboring spins). In particular, we will reformulate the problem as the existence of suitable negative connected components. By means of a kind of Peierls argument, together with some geometrical properties of the graph, we will obtain a precise bound on the probability of this event and thus the decay correlation property will follows. The first part on the proof of Theorem **4.11** is aimed to explain this argument, whereas in the second part we will come back to prove the two steps listed above.

4.3 Main tool: correlation decay inside the pure phase

In this section we will analyze the existence of a decay of correlation between spins sited on a level L_i, under the measure μ_i^τ, with τ some boundary condition in Ω^+.
Essentially due to the non-amenability of hyperbolic graphs, we will prove that the influence of the positive boundary on a fixed vertex is "with high probability" stronger then the influence of vertices lying on the same and previous level. We will give a precise mathematical definition of this event and introduce new objects in order to estimate its probability. We will prove that correlation between spins on a level decay with their "distance on the level" and that this phenomenon increases with β.

4.3.1 Notation and statement of the result

Let us define a linear order on the levels L_i as follows: let T_B be a shortest path spanning tree of B, namely such that for every $x \in V_m$ the path from o to x in T_B is a shortest path in B. Clearly the i-th level of T_B is equal to the level L_i of B; we thus choose, for any $i = 0, \dots, m$, a vertex $x_0^i \in L_i$ and order in counterclockwise sense all the vertices in L_i along T_B. This order clearly depends on the choice of the x_0^i, but it does not affect the next computations.
Notice that taking $x_{m+1}^i = x_0^i$, then for all $i = 0, \dots, m$ the vertices x_i and x_{i+1} belongs to the same tile of B. We will call a pair of vertices in the same level and with this property *level-neighboring vertices*.

Let us introduce tho following notion of distance on L_i:

Definition 4.11 (L_i-distance). *Given $n, m \in \{1, \cdots, h\}$ such that $n \geq m$ and $h = |L_i|$, the L_i-distance between x_n^i and x_m^i in L_i is defined as*

$$d_i(x_n^i, x_m^i) = \min\{|\{k : n < k \leq m\}|\,,\, |\{k : k < n \text{ or } k \geq m\}|\}\,.$$

Remark 4.7. *Let us remark that $d_i(x_n^i, x_m^i)$ is just the minimal number of jumps between L_i-neighboring vertices from x_n^i to x_m^i. Notice also that the definition of L_i-distance doesn't depend from the choice of the ordering on L_i. In general, for $x, y \in L_i$, we have $d_i(x, y) \neq d(x, y)$, where $d(\cdot, \cdot)$ is the usual graph distance.*

Let us fix a vertex $x \subset L_i$ and introduce the set $K_x := \{x\} \cup F_{i+1}$; as usually we denote by $\mu^{\sigma}_{K_x}$ the Gibbs measure on K_x with $\sigma \in \Omega^+$ boundary conditions. With this notation we can state the following:

Proposition 4.4. *Let $B \in \mathbb{H}(v,s)$ such that $v > 4$. Then there exist two positive constants c_1 and c_2 dependent from the parameters of the hyperbolic graph such that, for every $\beta > \beta_0 = \frac{c_2}{c_1}$, every configuration $\sigma \in \Omega^+$ and every couples of distinct vertices $x, y \in L_i$, with $i \in \{1 \dots m\}$, it holds*

$$|\mu^{\sigma}_{K_x}(\sigma_x = +) - \mu^{\sigma^y}_{K_x}(\sigma_x = +)| \leq c e^{-\beta' d_i(x,y)}\,, \tag{4.7}$$

with $\beta' := c_1\beta - c_2 > 0$.

The proof of proposition **4.4** is organized as follows.
Through a geometric argument, we first express the probability of discrepancy in a site, appearing in (4.7), as the probability to have suitable negative subsets. This last event will be then analyzed by mean of a kind of Peierls argument.

4.3.2 Negative spin components inside the (+) pure phase

Let us recall the setting of proposition **4.4** just to fix some notation.
We consider two vertices $x, y \in L_i$ such that $d_i(x,y) = \ell$ with $\ell \geq 1$, and a configuration $\sigma \in \Omega^+$. The quantity under analysis is the correlation between site x and y w.r.t the Gibbs measure conditioned on the configuration outside $K_x = F_{i+1} \cup \{x\}$ being $\sigma \in \Omega^+$, i.e.

$$|\mu^{\sigma}_{K_x}(\sigma_x = +) - \mu^{\sigma^y}_{K_x}(\sigma_x = +)|$$

Now let $\sigma^{y,+}$ be the configuration that agree with σ in all sites but y and has a $(+)$-spin on y, define analogously $\sigma^{y,-}$ and denote by $\mu^{y,+}_{K_x}$ and $\mu^{y,-}_{K_x}$ the measures conditioned respectively to $\sigma^{y,+}$ and $\sigma^{y,-}$.
With this notation and from the obvious fact that the $\{\sigma_x = +\}$ is an increasing event, we thus focus our analysis to correlation

$$\mu^{y,+}_{K_x}(\sigma_x = +) - \mu^{y,-}_{K_x}(\sigma_x = +)\,. \tag{4.8}$$

Before defining new objects, we want to clarify the main idea that is beyond this proof. Since the measure $\mu^{\sigma}_{K_x}$ fixes the configuration on all the sites in K_x^c (i.e. on $B_i \setminus \{x\} \ni y$), the vertex y can communicate with x only

through paths going from x to y and crossing vertices in K_x. However, the effect of this communication can be very small respect to the information coming to x from the $(+)$-boundary. In particular if every path starting from y crosses a $(+)$-spins before arriving to x, then the communication between them is interrupted. Let us formalize this assertion.

Negative components and correlation decay

For every vertex $z \in L_i$, with $i = 0 \dots, m$, let N_z denote the set of nearest neighbors of z belonging to the level L_{i+1}.
Then, let us denote by $\mathcal{C}$ the set of subsets $C \subset K_x$ such that $C \cup \{y\}$ is a connected subset in $K_x \cup \{y\}$. We call an element $C \in \mathcal{C}$ a *component* of y and notice that $\emptyset \in \mathcal{C}$.
Again, for any configuration $\sigma \in \Omega^+$, let $C^{(\sigma)}$ denotes the maximal negative component of y admitted by σ, i.e.

$$C^{(\sigma)} \in \mathcal{C} \quad \text{s.t.} \begin{cases} \sigma_z = - & \forall\, z \in C^{(\sigma)} \\ \sigma_z = + & \forall\, z \in (\partial_V(C^{(\sigma)}) \cap K_x) \end{cases},$$

where the case $C^{(\sigma)} = \emptyset$ corresponds to have $\sigma_z = +$ for all $z \in N_y$ with $\partial_V(\emptyset) := N_y$. We define the event $A := \{\sigma \in \Omega^+ : C^{(\sigma)} \cap N_x = \emptyset\}$ and write it as

$$A = \bigsqcup_{C \in \mathcal{C}_\emptyset} \{\sigma \in \Omega^+ : C^{(\sigma)} = C\},$$

where we used the notation $\mathcal{C}_\emptyset := \{C \in \mathcal{C} \text{ s.t. } C \cap N_x = \emptyset\}$.
It holds the following computation

$$\begin{aligned} \mu^{y,-}_{K_x}(\sigma_x = + \,|\, A) &= \sum_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\sigma_x = +, C^{(\sigma)} = C \,|\, A) \\ &= \frac{\sum_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\sigma_x = +, C^{(\sigma)} = C\,)}{\sum_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\,C^{(\sigma)} = C\,)} \\ &= \frac{\sum_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\sigma_x = + \,|\, C^{(\sigma)} = C\,)\mu^{y,-}_{K_x}(\,C^{(\sigma)} = C\,)}{\sum_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\,C^{(\sigma)} = C\,)} \\ &\geq \min_{C \in \mathcal{C}_\emptyset} \mu^{y,-}_{K_x}(\sigma_x = + \,|\, C^{(\sigma)} = C). \end{aligned} \tag{4.9}$$

By definition of $C^{(\sigma)}$, we can observe that $\mu^{y,-}_{K_x}(\,\cdot \,|\, C^{(\sigma)} = C)$, with $C \in \mathcal{C}_\emptyset$, is a measure over the set $K_x \setminus (C \cup \partial_V C) \subseteq K_x \setminus N_y$ conditioned to have $\sigma^{y,-}$ b.c. over K_x^c, $(+)$-b.c. over $\partial_V C \cap K_x$ and $(-)$-b.c. over C. The presence

of (+)-b.c. over $\partial_V C \cap K_x$ makes the events in $\mathcal{F}_{K_x \setminus (C \cup \partial_V C)}$ independent from the $(-)$-b.c. over C, and then it holds the stochastic domination

$$\begin{aligned} \mu^{y,-}_{K_x}(\,\cdot\,|\,C^{(\sigma)} = C) &= \mu^{y,-}_{K_x}(\,\cdot\,|\sigma_z = +\,,\, z \in C \cup \partial_V C) \\ &\geq \mu^{y,+}_{K_x}(\,\cdot\,|\sigma_z = +\,,\, z \in C \cup \partial_V C) \\ &\geq \mu^{y,+}_{K_x}(\,\cdot\,)\,. \end{aligned}$$

Being $\{\sigma_x = +\}$ an increasing event, from (4.9) we get

$$\mu^{y,-}_{K_x}(\sigma_x = +\,|\,A) \geq \mu^{y,+}_{K_x}(\sigma_x = +)\,,$$

which with the obvious fact that $\mu^{y,-}_{K_x}(\sigma_x = +) \geq \mu^{y,-}_{K_x}(\sigma_x = +\,|\,A)\,\mu^{y,-}_{K_x}(A)$, implies

$$\mu^{y,+}_{K_x}(\sigma_x = +) - \mu^{y,-}_{K_x}(\sigma_x = +) \leq \mu^{y,-}_{K_x}(A^c)\,. \tag{4.10}$$

This means that the correlation between x and y in the r.h.s of (4.10), can be estimated by the probability that exists a maximal negative component of y intersecting N_x. We will argue that this event is very unlikely due to the presence of $(+)$-b.c. on the boundary of B. At this point of the proof the geometrical properties of the hyperbolic graph, and in particular the non-amenability, become crucial. In the next section we will clarify this point, and prove, by means of a kind of Peierls argument, the exponential decay in the distance $d_i(x, y)$ of (4.10).

4.3.3 Influence of the boundary on the correlation decay

We first observe that A^c, the existence of a negative component of y intersecting N_x, is a decreasing event. Then, denoting by $\mu^-_{K_x}$ the measure with σ-b.c., where $\sigma \in \Omega^+$ is such that $\sigma_z = -$ for all $z \in K_x^c = B_i \setminus \{x\}$, by monotonicity we get

$$\mu^{y,-}_{K_x}(A^c) \leq \mu^-_{K_x}(A^c)\,.$$

Since the decay of the probability $\mu^-_{K_x}(A^c)$ implies the decay in the general case with $\sigma^{y,-}$-b.c., we will just focus on this quantity.

Let us denote by $\mathcal{C}_{\neq\emptyset}$ the set of components of y with nonempty intersection with N_x, and for every $m \in \mathbb{N}$, let $\mathcal{C}_m$ be the set of components in $C_{\neq\emptyset}$ with m vertices, i.e

$$\mathcal{C}_m := \{C \in \mathcal{C}_{\neq\emptyset} \text{ s.t. } |C| = m\} \qquad \mathcal{C}_{\neq\emptyset} := \bigcup_{m>0} \mathcal{C}_m.$$

Notice that a component of y containing a vertex in N_x has at least cardinality ℓ, since $d_i(x,y) = \ell$. Thus A^c can be expressed by mean of disjoint events as

$$A^c = \bigcup_{m\geq \ell} \bigcup_{C\in\mathcal{C}_m} \{\sigma \in \Omega^+ : C^{(\sigma)} = C\},$$

and we get

$$\mu^-_{K_x}(A^c) \leq \sum_{m\geq\ell} \sum_{C\in\mathcal{C}_m} \mu^-_{K_x}(\{\sigma \sim C\}) \tag{4.11}$$

where $\{\sigma \sim C\}$ denotes the set of configurations $\sigma \in \Omega^-_{K_x}$ s.t. $C(\sigma) = C$.

Let us recall the following Lemma due to Kesten (see [Kes]).

Lemma 4.1. *Let G an infinite graph with maximum degree Δ and let $\mathcal{K}_m$ be the set of connected sets with m vertices containing a fixed vertex v. Then $|\mathcal{K}_m| \leq (e(\Delta+1))^m$.*

Since to every component $C \in \mathcal{C}_m$ is uniquely associated the connected component $C \cup \{y\}$ with $m+1$ vertices, it holds

$$|\mathcal{C}_m| \leq |\{C \text{ is a connected component : } y \in C,\, |C| = m+1\}| \leq e^{c_2(m+1)}, \tag{4.12}$$

where in the last inequality we just applied Lemma 4.1.
If we show that for each $C \in \mathcal{C}_{\neq\emptyset}$

$$\mu^-_{K_x}(\{\sigma \sim C\}) \leq e^{-c_1\,\beta|C|}, \quad c_1 > 0 \tag{4.13}$$

then by inserting this bound into (4.11) and using inequality 4.12, we get

$$\mu^-_{K_x}(A^c) \leq |N_x| e^{c_2} \sum_{m\geq\ell} e^{c_2 m} e^{-c_1\beta m} \leq c e^{-\beta'\ell}$$

where the second inequality holds for $\beta' = c_1\beta - c_2 > 0$, i.e. $\beta > c_2/c_1$.

To prove (4.13), we first apply a like Peierls argument that runs as follows (see also [JS]) .
First, for a given $C \in \mathcal{C}_{\neq\emptyset}$, we consider the edge boundary $\partial_E C$ and define

$$\partial_+ C := \{e = (z,w) \in \partial_E C : z,\, w \in K_x\}$$

$$\partial_- C := \{e = (z,w) \in \partial_E C : z \text{ or } w \in K_x^c\}$$

with $\partial_E C = \partial_+ C + \partial_- C$.
The meaning of this notation can be understood if we consider a configuration $\sigma \in \Omega^-_{K_x}$ such that $C(\sigma) = C$. In this case σ has $(-)$-spins on both the

end-vertices of every edge in $\partial_- C$ and a $(+)$-spin in one end-vertex of every edge in $\partial_+ C$.

Then, for every $\sigma \in \Omega^-_{K_x}$ such that $C^{(\sigma)} = C$, let $\sigma^* \in \Omega^-_{K_x}$ denote the configuration obtained by a global spin flip of σ on the subset C, and observe that the map $\sigma \to \sigma^*$ is injective. This flipping changes the Hamiltonian contribute of the interactions just along the edges in $\partial_E C$ and in particular σ^* loses the positive contribute of the edges in $\partial_+ C$ and gains the contribute of the edges in $\partial_- C$. We get

$$H^-_{K_x}(\sigma^*) \;=\; H^-_{K_x}(\sigma) - 2(|\partial_+ C| - |\partial_- C|)\,. \tag{4.14}$$

and hence

$$\begin{aligned} \mu^-_{K_x}(\{\sigma \sim C\}) \;&=\; \sum_{\{\sigma\sim C\}} \frac{e^{-\beta H^-_{K_x}(\sigma)}}{Z^-_{K_x}} \\ &\leq\; \frac{\sum_{\{\sigma\sim C\}} e^{-\beta H^-_{K_x}(\sigma)}}{\sum_{\{\sigma\sim C\}} e^{-\beta H^-_{K_x}(\sigma^*)}} \\ &\leq\; e^{-2\beta(|\partial_+ C| - |\partial_- C|)}\,, \end{aligned} \tag{4.15}$$

where in the first inequality we reduced the partition function to a summation on $\{\sigma \sim C\}$ and in the second inequality we applied (4.14).

The quantity under analysis is now the difference $|\partial_+ C| - |\partial_- C|$, that we would like to relate to the size of C over which the sum in (4.11) runs. This key step is essentially based on the geometrical properties of the graph that we will stress in that follows.

Geometrical properties

Before proceeding our analysis, we state the following Lemma concerning the growth properties of the nearest neighborhood of a vertex in $B \subset \mathbb{H}(v,s)$. This result does not apply to the case $s = 3$ (triangular tilings) which is indeed excluded from the result stated in Theorem **4.11** . From now on we restrict attention to the case $s > 3$.

Lemma 4.2 (link-property). *For any hyperbolic graph $\mathbb{H}(v,s)$, with $s > 3$, and any vertex $x \in \mathbb{H}(v,s)$ at distance $i \geq 0$ from a reference point $o \in \mathbb{H}(v,s)$, the number of neighbors of x at distance $i+1$ from o is at least $v-2$.*

Remark 4.8. *We recall that the hyperbolic relation* $(v-2)(s-2) > 4$ *imposes that* $v > 2$ *for all* s *, and then the number of neighbors of* x *in the next level is always positive.*

Proof. Being v the vertex degree of the graph, Lemma 4.2 can be equivalently stated, by saying that each vertex $x \in L_i$, with respect a given root o, is linked to the vertices in the same or previous level by at most 2 edges. Indeed, as can be directly verified from a figure (see 3.1, case $s > 3$), only three situations can appear regarding these edges (see fig. 4.1):

1. x is linked with two ancestors and none vertex on the same level;
2. x is linked with one ancestor and one vertex on the same level;
3. x is linked with one ancestor.

The exclusion of the other possibilities comes from the planarity of the graph together with the requirement $s > 3$. □

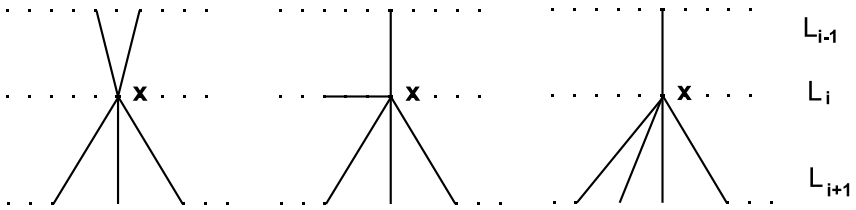

Figure 4.1: The three possible connections between a site on a level and the neighbors

Now let us come back to the analysis of the edge boundaries of a component $C \in \mathcal{C}_{\neq\emptyset}$; the main step to complete the proof is to show the following claim

Claim 4.1. *Assume that the vertex degree of the hyperbolic graph is* $v > 4$*. Then, for every* $C \in \mathcal{C}_{\neq\emptyset}$ *as defined above, it holds*

$$|\partial_+ C| \geq (1+\delta)|\partial_- C|\,, \tag{4.16}$$

where $\delta = \delta(v)$ *is a positive constant equal to* $\frac{v-4}{2}$ *for all* $s > 3$*.*

To verify Claim 4.1 let us consider the subset $S := C \cap \partial_V K_x^c$ corresponding to the set of end-vertices (in C) of the edges in $\partial_- C$.

For every vertex $z \in L_i$, with $i = 0 \dots, m$, we also introduce the notation P_z for the set of nearest neighbors of z belonging to $L_{i-1} \cup L_i$ (i.e. to the same or previous level of z).
Due to the shape of K_x, S is a subset of $\{x\} \cup L_{i+1} \subset L_i \cup L_{i+1}$ whereas $K_x^c \subset B_i$; every edge in $\partial_- C$ can then be written as $e = (z, w)$ with $z \in S$ and $w \in P_z$.

From this observation and using the link property, we get

$$\begin{aligned} |\partial_- C| &= \sum_{z \in S} |\{w \in P_z \,, w \notin C\}| \\ &= \sum_{z \in S} |\{w \in P_z\}| - \sum_{z \in S} |\{w \in P_z \,, w \in C\}| \\ &\leq 2|S| - \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \,, \end{aligned}$$

which corresponds to inequality

$$|S| \geq \frac{1}{2} \left(|\partial_- C| + \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \right) . \tag{4.17}$$

To prove Claim 4.1, we want to find a suitable relation between $\partial_+ C$ and S. To this aim, let us consider the increasing sequence of subsets of C defined as follows

$$C_0 = S \quad \text{and} \quad C_j = C_{j-1} \cup \{z \in C \cap L_{i+j}\} \;\; \forall j \geq 1 \,,$$

and notice that for some finite integer k, depending on C, $C_k \equiv C$.
For all $j \geq 0$, let $|\partial_+ C_j| := \{e = (z, w) \in \partial_E(C_j) \,:\, z, w \in K_x\}$. Then it holds:

1. $|\partial_+ C_0| = |\partial_+ S| \geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}|$,
   due to the trivial fact that $\{w \sim z \,, w \notin S \cup K_x^c\} \supset \{w \in N_z \,, w \notin S\}$;

2. $|\partial_+ C_j| \geq |\partial_+ C_{j-1}| + (v-4)\,|\{z \in C \cap L_{i+j}\}| \quad \forall j \geq 1$,
   that comes from the definition of $\{C_j\}_{j \geq 0}$ together with the link property of Lemma 4.2. Indeed, by construction, $C_j \subseteq B_{i+j}$ for all $j \geq 1$ and $C_0 \subset B_{i+1}$. Then, for every $z \in C \cap L_{i+j}$ and $j \geq 1$, there are at most $|P_z|$ edges between z and C_{j-1}, i.e. edges in $\partial_+ C_{j-1} \setminus \partial_+ C_j$, and at least $|N_z|$ edges between z and $K_x \setminus C_j$, i.e. edges in $\partial_+ C_j \setminus \partial_+ C_{j-1}$. Thanks to the link property, it holds $|P_z| \leq 2$ and $|N_z| \geq v - 2$, and the inequality follows.

From these last inequalities and being $v > 4$ by hypothesis, we get

$$\partial_+ C \geq \partial_+ C_0 \geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}| \,. \tag{4.18}$$

Remark 4.9. *Since $S \subset \{x\} \cup L_{i+1}$, x is the only vertex in K_x which can satisfies both the conditions $x \in S$ and $N_x \cap S \neq \emptyset$ and then it holds that $|\{w \in N_z \,, w \in S\}| = 0$ for all $z \in S$ different from x. However, we prefer to use this more general notation in order to better understand the possible extension of this computation to components which satisfy different conditions from those ones we required. In particular, it easy to verify that all the above construction continues to holds if we consider, instead of K_x, a subset $U = B_{i-1} \cup V$ with $V \in L_{i+1}$, $y \in U$ and $x \in U^c$. This can be useful, for example, to compute the correlation $\mu_U^{y,+}(\sigma_x = +) - \mu_U^{y,-}(\sigma_x = +)$ or simply the probability $\mu_U^-(\sigma_x = -)$. In this last case, since the event $\{\sigma_x = -\}$ corresponds to the existence of a negative connected component passing through x, we can write (as in 4.11 and 4.15)*

$$\mu_U^-(\sigma_x = -) \leq \sum_{m \geq 1} \sum_{C \in \mathcal{C}_m} \mu_U^-(\{\sigma \sim C\}) \leq \sum_{m \geq 1} \sum_{C \in \mathcal{C}_m} e^{-2\beta(|\partial_+ C| - |\partial_- C|)} \,,$$

where $\mathcal{C}_m$ will denote the set of m-vertex connected component in U^c passing through x. Thus again the computation reduces to analyze $|\partial_+ C| - |\partial_- C|$.

From (4.17)-(4.18) and again by the link property, it holds

$$\begin{aligned}
\partial_+ C &\geq \sum_{z \in S} |\{w \in N_z \,, w \notin S\}| \\
&= \sum_{z \in S} (v - |\{w \in P_z \,\}|) - \sum_{z \in S} |\{w \in N_z \,, w \in S\}| \\
&\geq (v-2)|S| - \sum_{z \in S} |\{w \in N_z \,, w \in S\}| \\
&= (v-2)|S| - \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \\
&\geq \frac{(v-2)}{2} |\partial_- C| + \frac{(v-4)}{2} \sum_{z \in S} |\{w \in P_z \,, w \in S\}| \\
&\geq (1 + \frac{(v-4)}{2}) |\partial_- C| \,,
\end{aligned} \tag{4.19}$$

which corresponds to the statement of Claim 4.1.

From Claim 4.1 and trivial computations, we obtain

$$\begin{cases} |\partial_+ C| - |\partial_- C| \geq \frac{\delta}{1+\delta} |\partial_+ C| \\ |\partial_+ C| + |\partial_- C| \leq \frac{2+\delta}{1+\delta} |\partial_+ C| \end{cases} \implies |\partial_+ C| - |\partial_- C| \geq \frac{\delta}{2+\delta} |\partial_E C| \,.$$

Due to the non-amenability of the hyperbolic graphs, we can now use the isoperimetric inequality $\partial_E C \geq i_e |C|$, to obtain

$$|\partial_+ C| - |\partial_- C| \geq \frac{\delta}{1+\delta} i_e |C| ,$$

that inserted in (4.15) give us the required inequality (4.13), namely

$$\mu^-_{K_x}(\{\sigma \sim C\}) \leq e^{-2i_e \frac{\delta}{2+\delta} \beta |C|} = e^{-c\beta|C|} .$$

We want to stress that the existence of a positive isoperimetric constant (i.e.the non-amenability of the graph) is really fundamental for this kind of argument. Indeed, as shown in (4.13), it allowed to contrast the entropic term counting the number of components of fixed size and growing exponentially with that size. This conclude the proof of Proposition (**4.4**). □

4.4 First step: Poincaré inequality for a marginal Gibbs measure

In this section we will prove that at low temperatures and for any n-vertex ball in the hyperbolic graph $\mathbb{H}(s, v)$, with $v > 4$, the marginal measure on a level satisfies a Poincaré inequality with constant independent from the level size. We first state the result in its main generality and then, by means of the *coupling technique* together Proposition **4.4**, we prove the statement.

4.4.1 Main result

Definition 4.12 (Interval). *A subset $S \subseteq L_i$ is an interval if its vertices can be ordered as $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$, with $d_i(x_{i_j}, x_{i_{j+1}}) = 1$ for all $j = 1, \cdots, k-1$.*

Let us fix an interval S and introduce the notation ν^τ_S for the marginal measure on S of the Gibbs measure conditioned to the configuration in $B_i \backslash S$ is τ, i.e.

$$\nu^\tau_S(\sigma) = \sum_{\eta : \eta_S = \sigma_S} \mu(\eta \,|\, \tau \in \mathcal{F}_{B_i \backslash S}) .$$

We denote by $\mathrm{Var}_{\nu^\tau_S}$ the variance w.r.t. ν^τ_S and then state the following:

Theorem 4.12. *For all $\beta \gg 1$ and for every interval $S \subseteq L_i$, $\tau \in \Omega^+$ and $f \in L^2(\Omega, \mathcal{F}_S, \nu^\tau_S)$, the measure ν^τ_S satisfies the Poincaré inequality*

$$Var_{\nu^\tau_S}(f) \leq c \sum_{x \in S} \nu^\tau_S(Var_x(f)) . \tag{4.20}$$

with $c = c(\beta) = 1 + O(e^{-c\beta})$ *uniformly in the size of* S.

Before proceeding with the proof, we want to point out that this result includes, as a particular case, inequality (4.5) which is the key object of the first step of the proof of theorem **4.11**. Indeed, taking $S = L_i$ and $f = g_{i+1}$ and observing that $\mu(\,\cdot\,|\mathcal{F}_{B_i\setminus S}) \equiv \mu_i$ and $\mu_i^\tau(g_{i+1}) \equiv \nu_{L_i}^\tau(g_{i+1})$, we can apply theorem **4.12**, to obtain exactly the Poincaré inequality

$$\mathrm{Var}_i^\tau(g_{i+1}) \le c \sum_{x\in L_i} \mu_i^\tau(\mathrm{Var}_x(g_{i+1}))\,.$$

4.4.2 Proof of the Poincaré inequality for marginal measures

The proof of theorem **4.12** is based on a well known and useful technique to bound from above the mixing time of Markov processes known as "*coupling*", introduced for the first time in this setting by Aldous [Al] and subsequently refined to the "*path coupling*" [BD, LRS]. See also [Lin] for a wider discussion on the coupling method, and [Je] for its applications in combinatorial problems.
We first define the Glauber dynamics on S with reversible measure ν_S. Then we construct a coupled process between two dynamics on S with different initial configurations. Using Proposition **4.4** together with some properties of the coupling, we will obtain a bound on the spectral gap of the dynamics on S, and then a Poincaré inequality for ν_S.

Glauber dynamics on levels: the coupled process

First of all let us fix some ideas about Glauber dynamics on an interval $S \in L_i$ with reversible measure ν_S^τ and in particular let us explicit the jump rates corresponding to heath bath dynamics. Given a configuration $\sigma \in \Omega_S^\tau$, a vertex $x \in S$ and a spin value $a \in \mathcal{S} = \{\pm 1\}$, from the definition (2.10) of jump rates we get

$$\begin{aligned} c_x(\sigma, a) &= \nu_S^\tau(\sigma_x = a \,|\, \sigma \in \mathcal{F}_{S\setminus x}) \\ &= \mu(\sigma_x = a \,|\, \sigma \in \mathcal{F}_{S\setminus x}, \tau \in \mathcal{F}_{B_i\setminus S}) \\ &= \mu_{K_x}^\sigma(\sigma_x = a)\,, \end{aligned} \tag{4.21}$$

where in the second line we used the fact that $\{\sigma_x = a\} \in \mathcal{F}_S$ and in the third line we introduced the notation $K_x = \{x\} \cup F_{i+1}$ as in the previous section. The generator of this dynamics will be denoted by $\mathcal{L}_S$.

Let us remark that the flip rate $c_x(\sigma, a) = \mu^{\sigma}_{K_x}(a)$ depends from whole spin configuration on S, rather then only from nearest neighbors of x, due to the presence of paths in $S \cup F_{i+1}$ connecting x to $S\backslash\{x\}$. To analyze the dynamics we will indeed exploit the correlations decay between vertices on L_i stated in Proposition **4.4**.

Let $(\sigma_t, \eta_t)_{t\geq 0}$ be the coupled process on $\Omega_S \times \Omega_S$ with initial configurations (σ, η) defined as follows.
Using the graphical construction of the Glauber dynamics [Sch, Mar], we can assume that a Poisson clock with rate $c_M = \sup_{x,\sigma,a} c_x(\sigma, a)$ is associated to every site $x \in S$. We assume independence as x varies in S and we denote by $\{t_{x,n}\}_{n\in\mathbb{N}}$ the successive arrivals after time $t = 0$ of the Poisson clock in x. Both Markov processes evolve at the same time $t \in \{t_{x,n}\}_{\substack{n\in\mathbb{N}\\ x\in S}}$ and what remains to specify are the coupling jump rates $\tilde{c}_x((\sigma, a), (\eta, b))$, with $a, b \in \mathcal{S}$, to go from (σ, η) to $(\sigma^{x,a}, \eta^{x,b})$.
Since the jump rate $c_x(\sigma, \cdot\,)$ just corresponds to the measure $\mu^{\sigma}_{K_x}(\,\cdot\,)$ on the spin configuration in x , we can define the coupling jump rate $\tilde{c}_x((\sigma, \cdot), (\eta, \cdot\,)$ as the optimal coupling (see [Lin]) between the measures $\mu^{\sigma}_{K_x}(\,\cdot\,)$ and $\mu^{\eta}_{K_x}(\,\cdot\,)$. Explicitly the coupling process will run as follows. Suppose that $t = t_{x,n}$ for some $x \in S$ and $n \in \mathbb{N}$, and that the process immediately before t was in the state (σ, η); then the coupled jumps are defined as follows

- $\tilde{c}_x((\sigma, +), (\eta, +)) = \min\{\mu^{\sigma}_{K_x}(\sigma_x = +)\,;\, \mu^{\eta}_{K_x}(\sigma_x = +)\}$;
- $\tilde{c}_x((\sigma, -), (\eta, -)) = 1 - \max\{\mu^{\sigma}_{K_x}(\sigma_x = +)\,;\, \mu^{\eta}_{K_x}(\sigma_x = +)\}$;
- $\tilde{c}_x((\sigma, +), (\eta, -)) = \max\{0\,;\, \mu^{\sigma}_{K_x}(\sigma_x = +) - \mu^{\eta}_{K_x}(\sigma_x = +)\}$:
- $\tilde{c}_x((\sigma, -), (\eta, +)) = \max\{0\,;\, \mu^{\eta}_{K_x}(\sigma_x = +) - \mu^{\sigma}_{K_x}(\sigma_x = +)\}$.

We denote by $\widetilde{\mathcal{L}}$ the generator specified by the above jump rates, and by $\widetilde{P}_t$ the correspondent Markov semigroup (see Section 2.3.1). Notice that this coupling is monotone and then, taking two initial configurations (σ, η) such that $\sigma \geq \eta$, we get $\sigma_t \geq \eta_t$ for all future time t.

Now, let us consider the subset $H \subset \Omega_S \times \Omega_S$ given by all couples of configurations which differ by a single spin flip in some vertex of S. One can easily verify that the graph $(\Omega_S \times \Omega_S, H)$ is connected and that the induced graph distance $D(\sigma, \tau)$ between configurations $(\sigma, \tau) \in \Omega_S \times \Omega_S$

just corresponds to their Hamming distance.
Let also denote by $\mathbb{E}_{\sigma,\eta}[D(\sigma_t,\eta_t)] \equiv \mathbb{E}[D(\sigma_t,\eta_t)|(\sigma,\eta)]$ the average distance at time t between two coupled configurations of a process starting at (σ,η). Taking $\beta \gg 1$, we claim the following

Claim 4.2. *For all $\beta \gg 1$ there exists a positive constant $\alpha \equiv \alpha(\beta)$ such that, for every $(\sigma,\eta) \in H$ initial configurations, the coupling process $(\sigma_t,\eta_t)_{t\geq 0}$ verifies the inequality*

$$\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma_t,\eta_t)]\,|_{t=0} \leq -\,\alpha\,. \tag{4.22}$$

Proof of Claim 4.2. Let us first explicit the derivative of the average distance as follows

$$\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma_t,\eta_t)]\,|_{t=0} = \frac{d}{dt}\left(\widetilde{P}_t D\right)(\sigma,\eta)\,|_{t=0} = (\widetilde{\mathcal{L}}\,D)(\sigma,\eta)$$

$$= \sum_{x\in S}\sum_{a,b\in\{\pm 1\}} \tilde{c}_x((\sigma,a)(\eta,b))[D(\sigma^{x,a},\eta^{x,b}) - D(\sigma,\eta)]\,, \tag{4.23}$$

where the sum over $a,b \in \{\pm 1\}$ can be restricted by monotonicity to the set $a \leq b$ or $a \geq b$ depending if $\sigma \leq \eta$ or $\sigma \geq \eta$, respectively.

From the definition of the coupling jump rates, the probability of disagreement in x after one update in x of (σ,η), is given by

$$P^x_{dis}(\sigma,\eta) = |\mu^{\sigma}_{K_x}(\sigma_x = +) - \mu^{\eta}_{K_x}(\sigma_x = +)|\,.$$

In particular, since by hypothesis there exists a vertex $y \in S$ such that $\eta = \sigma^y$, from Proposition **4.4** it holds that for all $\beta' = c_1\beta - c_2 > 0$

$$P^x_{dis}(\sigma,\sigma^y) = |\mu^{\sigma}_{K_x}(\sigma_x = +) - \mu^{\sigma^y}_{K_x}(\sigma_x = +)| \leq c e^{-\beta' d_i(x,y)}\,.$$

The distance between coupled configurations after one updates can then be computed as follows.

(i) If $x = y$ then $P^x_{dis}(\sigma,\sigma^x) = 0$, which means the disagreement at the site x is removed after the update in x with certainty so that the new configurations coincide;

(ii) If $x \neq y$ then

- **(a)** with probability $1 - P^x_{dis}(\sigma,\sigma^y)$ the updated configurations have the same spin at x, and thus their distance is equal to $D(\sigma,\eta)$;

(b) with probability $P^x_{dis}(\sigma,\sigma^y)(x)$ the updated configurations have different spin at x, and thus their distance is equal to $D(\sigma,\eta)+1$.

From these observations and continuing from (4.23), we get

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}_{\sigma,\eta}[D(\sigma_t,\eta_t)]\,|_{t=0} &= -1+\sum_{\substack{x\in S\\ x\neq y}} P^x_{dis}(\sigma,\eta)\\ &\le -1+c\sum_{\ell\ge 1} e^{-\beta'\ell}\\ &\le -(1-ce^{-\beta'})\,; \end{aligned}\tag{4.24}$$

taking $\alpha=(1-ce^{-\beta'})$ and for $\beta\gg 1$, Claim 4.2 follows. □

The result of Claim 4.2 can be easily extended to arbitrary initial configurations $(\sigma,\eta)\in\Omega_S\times\Omega_S$, using the *path coupling* technique introduced in [BD]). More explicitly, one can consider a path γ in $(\Omega_s\times\Omega_s,H)$ from σ to η, namely $\gamma=\{\sigma=\sigma^0,\sigma^1,\cdots,\sigma^k=\eta\}$ with $(\sigma^i,\sigma^{i+1})\in H$ for all $i=0,\cdots,k$, and then interpolate the coupling between neighboring configurations. Essentially due to the triangular inequality on the metric in $(\Omega_s\times\Omega_s,H)$, one get

$$\mathbb{E}_{\sigma,\eta}[d(\sigma_t,\eta_t)]\le\sum_{j=1}^{S}\mathbb{E}_{\sigma^{j-1}\sigma^j}[d(\sigma_t^{j-1},\sigma_t^j)].\tag{4.25}$$

Since by construction $(\sigma_t^{j-1},\sigma_t^j)\in H$ for all $j\ge 1$ and all $t>0$, we can apply Claim 4.2 to every terms in the r.h.s. of 4.25 and then obtain

$$\frac{d}{dt}\,\mathbb{E}[D(\sigma_t,\eta_t)\,|\,(\sigma,\eta)]\,|_{t=0}\le-\alpha D(\sigma,\eta)\,.\tag{4.26}$$

By means of the Markov property, for every time $t\ge 0$ and any coupled configurations $(\sigma_t,\eta_t)\in H$ we get

$$\begin{aligned}\frac{d}{dt}\,E_{\sigma,\eta}[D(\sigma_t,\eta_t)] &= \frac{d}{ds}\mathbb{E}_{\sigma,\eta}[D(\sigma_{t+s},\eta_{t+s})]\,|_{s=0}\\ &= \frac{d}{ds}\mathbb{E}_{\sigma,\eta}[\mathbb{E}[D(\sigma_{t+s},\eta_{t+s})|\sigma_t,\eta_t]]\,|_{s=0}\\ &= \mathbb{E}_{\sigma,\eta}\left[\frac{d}{ds}\,E[D(\sigma_{t+s},\eta_{t+s})\,|\,(\sigma_t,\eta_t)]\,|_{s=0}\right]\\ &\le \mathbb{E}_{\sigma,\eta}[-\alpha D(\sigma_t,\eta_t)]\\ &= -\alpha\,\mathbb{E}_{\sigma,\eta}(D(\sigma_t,\eta_t))\,. \end{aligned}\tag{4.27}$$

It follows that $E_{\sigma,\eta}[D(\sigma_t,\eta_t)] \leq e^{-\alpha\, t} D(\sigma,\eta)$ for every $(\sigma,\eta) \in \Omega_S \times \Omega_S$ and then we get

$$\mathbb{P}(\sigma_t \neq \eta_t) = \mathbb{E}_{\sigma,\eta}(\mathbb{1}(\sigma_t \neq \eta_t)) \leq \mathbb{E}_{\sigma,\eta}(D(\sigma_t,\eta_t)) \leq e^{-\alpha\, t} D(\sigma,\eta)\,. \quad (4.28)$$

To bound the spectral gap c_{gap} of the dynamics on S, we then consider an eigenfunction f of $\mathcal{L}_S$ with eigenvalue $-c_{gap}$, so that

$$\mathbb{E}_\sigma f(\sigma_t) = e^{t\mathcal{L}_S} f(\sigma) \;=\; e^{-c_{gap}\, t} f(\sigma)\,.$$

Since the identity function has eigenvalue 0 and is therefore orthogonal to f, we have $\nu_S^\tau(f) = 0$ and $\nu_S^\tau(\mathbb{E}_\eta f(\eta_t)) = 0$, where ν_S^τ is the invariant measure for $\mathcal{L}_S$. From these considerations and inequality (4.28), it holds

$$\begin{aligned} e^{t\mathcal{L}_S} f(\sigma) &= \mathbb{E}_\sigma f(\sigma_t) - \nu_S^\tau(\mathbb{E}_\eta f(\eta_t)) \\ &= \sum_\eta \nu_S^\tau(\eta)[\, E_\sigma f(\sigma_t) - E_\eta f(\eta_t)\,] \\ &\leq 2\|f\|_\infty \sup_{\sigma,\eta} \mathbb{P}(\sigma_t \neq \eta_t) \\ &\leq 2\|f\|_\infty |S| e^{-\alpha\, t}\,. \end{aligned} \quad (4.29)$$

Since the above inequality holds for all t, we finally obtain that $c_{gap} \geq \alpha$ independently from the size of S, which implies the Poincaré inequality 4.20 with constant $c = \alpha^{-1} = 1 + O(e^{-c\beta})$ and concludes the proof of Theorem **4.12**. □

As previously remarked, applying Theorem **4.12** for $S = L_i$ and $f = g_{i+1}$, we obtain the desired Poincaré inequality for the marginal Gibbs measure on the level L_i, i.e.

$$\mathrm{Var}_i^\tau(g_{i+1}) \leq c \sum_{x \in L_i} \mu_i^\tau(\mathrm{Var}_x(g_{i+1}))\,.$$

This conclude the first step toward the proof of Theorem **4.11**.

4.5 Second step: Poincaré inequality for the Gibbs measure

With the previous analysis we obtained a Poincaré inequality for the marginal measures on levels. By means of this result, the formula (4.3)

becomes

$$\mathrm{Var}(f) \;\le\; c \sum_{i=0}^{m} \sum_{x \in L_i} \mu \left[\mu_i(\mathrm{Var}_x(g_{i+1})) \right] . \tag{4.30}$$

Using the same notation as in [MSW], let us denote the sum in the r.h.s. of (4.30) by $\mathrm{P}_{\mathrm{var}}(f)$.
The aim of the following analysis is to analyze $\mathrm{P}_{\mathrm{var}}$ in order to find an inequality of the kind $\mathrm{P}_{\mathrm{var}}(f) \le D(f) + \varepsilon \mathrm{P}_{\mathrm{var}}(f)$, with $\varepsilon = \varepsilon(\beta) < 1$ independent from the size of the system. This would imply that

$$\mathrm{Var}(f) \le c \cdot \mathrm{P}_{\mathrm{var}}(f) \le \frac{c}{1-\varepsilon} \mathcal{D}(f)$$

and the proof of Theorem (**4.11**) would follow.

As a first step, we want to extract the local variance of f from the local variance of g_{i+1}, in order to reconstruct from (4.30) the Dirichlet form of f. For $x \in L_i$ and $\tau \in \Omega^+$, let $p(\tau) = \mu_x^\tau(\sigma_x = +)$ and $q(\tau) = \mu_x^\tau(\sigma_x = -)$, and then consider the quantity

$$\mu_i^\eta \left(\mathrm{Var}_x(g_{i+1}) \right) \;=\; \sum_\tau \mu_i^\eta(\tau) p(\tau) q(\tau) \left(\nabla_x \, g_{i+1}(\tau) \right)^2 \qquad x \in L_i \, , \eta \in \Omega^+$$

Using the martingale property $g_{i+1} = \mu_{i+1}(g_{i+2})$, the local variance $\mathrm{Var}_x(g_{i+1})$ can be split in two terms stressing the dependence from x of g_{i+2} and of the conditioned measure μ_{i+1}. Let us formalize this idea.
For a given configuration $\tau \in \Omega^+$ we introduce the symbols

$$\tau^+ := \begin{cases} \tau_y^+ = \tau_y & \text{if } y \neq x \\ \tau_y^+ = + & \text{if } y = x \end{cases} \qquad \tau^- := \begin{cases} \tau_y^- = \tau_y & \text{if } y \neq x \\ \tau_y^- = - & \text{if } y = x \end{cases}$$

and then define the density

$$h_x(\sigma) := \frac{\mu_{i+1}^{\tau^+}(\sigma)}{\mu_{i+1}^{\tau^-}(\sigma)} \, , \qquad \text{with} \qquad \mu_{i+1}^{\tau^-}(h_x) = 1$$

Whit this notation, it holds the following computation

$$\begin{aligned} \mu_i^\eta(\mathrm{Var}_x(g_{i+1})) \;&=\; \sum_\tau \mu_i^\eta(\tau) p(\tau) q(\tau) \left[\nabla_x \, \mu_{i+1}(g_{i+2})(\tau) \right]^2 \\ &=\; \sum_\tau \mu_i^\eta(\tau) p(\tau) q(\tau) \left[\mu_{i+1}^{\tau^-}(g_{i+2}) - \mu_{i+1}^{\tau^+}(g_{i+2}) \right]^2 \\ &=\; \sum_\tau \mu_i^\eta(\tau) p(\tau) q(\tau) \left[\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2}) - \mu_{i+1}^{\tau^-}(h_x, g_{i+2}) \right]^2 \\ &\le\; 2 \sum_\tau \mu_i^\eta(\tau) p(\tau) q(\tau) \left[\left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2}) \right)^2 + \left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}) \right)^2 \right] . \end{aligned} \tag{4.31}$$

Now it is simple to verify that $\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2}) = \mu_{i+1}^{\tau^+}(\nabla_x f)$.
To understand this fact it is enough to observe that the dependence from x of $g_{i+2} = \mu_{i+2}(f)$ come only from f, since the b.c. on B_{i+1} are fixed equal τ^+. Substituting $\mu_{i+1}^{\tau^+}(\nabla_x f)$ and applying the Jensen inequality, the first term of (4.31) can be bounded as

$$\sum_{\tau} \mu_i^{\eta}(\tau) p(\tau) q(\tau) \left(\mu_{i+1}^{\tau^+}(\nabla_x g_{i+2})\right)^2 \leq \mu_i^{\eta}\left(q(\tau)(\,\nabla_x f\,)^2\right) \leq \mu_i^{\eta}\mathrm{Var}_x(f)\,. \tag{4.32}$$

Substituting this term in the r.h.s. of (4.31) and then summing both sides over $x \in L_i$ and $i \in \{0, \ldots, m\}$, we obtain

$$\mathrm{P}\text{var}(f) \;\leq\; 2\,\mathcal{D}(f) + 2\sum_{i=0}^{m-1}\sum_{x\in L_i} \mu\left[\sum_{\tau} \mu_i(\tau) p(\tau) q(\tau) \left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2})\right)^2\right], \tag{4.33}$$

where we excluded the value m in the summation over i, since in that case $g_{m+2} = f$ is constant w.r.t. $\mu_{m+1}^{\tau^-}$ and thus $\mu_{m+1}^{\tau^-}(h_x, g_{m+2}) \equiv 0$.
The more involved analysis of $\mu_{i+1}^{\tau^-}(h_x, g_{i+2})$ will be discuss in the next section.

4.5.1 Recursive Argument

Notation: Recall that for every $x \in L_i$, N_x denotes the set of nearest neighbors of x in the level L_{i+1}. Given $x \in L_i$ and $\ell \in \mathbb{N}$, let us define the following objects:

(i) The ℓ-neighborhood of N_x in L_{i+1}

$$N_{x,\ell} := \{y \in L_{i+1} \,:\, d_{i+1}(y, N_z) \leq \ell\}\,;$$

(ii) The σ-algebra generated by the spin configuration in the vertices of $B_{i+1} \setminus N_{x,\ell}$

$$\mathcal{F}_{x,\ell} \;:=\; \sigma\left(\sigma_y \,:\, y \in B_{i+1} \setminus N_{x,\ell}\right)\,;$$

(iii) The measure conditioned on the configuration in $B_{i+1} \setminus N_{x,\ell}$

$$\mu_{x,\ell}(\,\cdot\,) := \mu\left(\cdot\,|\mathcal{F}_{x,\ell}\right).$$

We remark that $N_{x,0} = N_x$ and that there exists some $\ell_0 \leq |L_{i+1}|$ such that, for all integers $\ell \geq \ell_0$, it holds $N_{x,\ell} = L_{i+1}$ and then $\mu_{x,\ell} = \mu_i$.
We also remark that $\mathcal{F}_{x,0} \supset \mathcal{F}_{x,1} \supset \ldots \supset \mathcal{F}_{x,\ell_0} \equiv \mathcal{F}_{i+1}$, namely the family

of σ-algebra $\{\mathcal{F}_{x,\ell}\}_{\ell=0,1,\dots,\ell_0}$ is a filtration. In particular for any function $f \in L^1(\Omega, \mathcal{F}_{i+1}, \mu)$, the set of variables $\{\mu_{x,\ell}(f)\}_{\ell\in\mathbb{N}}$ is a Martingale.

Let us come back to our proof and recall the following property of the covariance (analogous to property (4.2) for the variance)

$$\mu_C^\eta(f,g) \;=\; \mu_C^\eta(\mu_D(f,g)) + \mu_C^\eta(\mu_D(f),\mu_D(f)) \;\;, \quad \text{for } D\subseteq C\subseteq B \;. \tag{4.34}$$

Since the support of μ_{i+1} strictly contains the support of $\mu_{x,0}$, the property (4.34) can be applied to the square covariance $(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}))^2$ appearing in (4.33) in order to get

$$(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}))^2 \;\le\; 2(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x, g_{i+2})))^2 + 2(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2})))^2 \,. \tag{4.35}$$

Through the Schwartz inequality, the first term in (4.35) can be bound as

$$(\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x, g_{i+2})))^2 \le \mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(h_x)) \cdot \mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(g_{i+2}))\,. \tag{4.36}$$

Applying standard argument of probability, the second term can be rearranged and bounded as follows:

$$\begin{aligned}
[\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2}))]^2 &= [\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x) - \mu_{i+1}(h_x), g_{i+2})]^2 \\
&= \left[\mu_{i+1}^{\tau^-}\left(\sum_{\ell=1}^{\ell_0} \mu_{x,\ell-1}(h_x) - \mu_{x,\ell}(h_x), g_{i+2}\right)\right]^2 \\
&\le \sum_{\ell=1}^{\ell_0} \ell^2 \left[\mu_{i+1}^{\tau^-}(\mu_{x,\ell-1}(h_x) - \mu_{x,\ell}(h_x), g_{i+2})\right]^2 \\
&= \sum_{\ell=1}^{\ell_0} \ell^2 \left[\mu_{i+1}^{\tau^-}\left(\mu_{x,\ell}(\mu_{x,\ell-1}(h_x), g_{i+2})\right)\right]^2 \;, \qquad (4.37)
\end{aligned}$$

where in the second line, due to the fact that $\mu_{x,\ell_0} = \mu_i$ for some ℓ_0, we substituted $\mu_{x,0}(h_x) - \mu_{i+1}(h_x)$ by the telescopic sum $\sum_{\ell=1}^{\ell_0}(\mu_{x,\ell-1}(h_x) - \mu_{x,\ell}(h_x))$. Applying the Cauchy-Schwartz inequality to the last term in 4.37, we get

$$\begin{aligned}
&[\mu_{i+1}^{\tau^-}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2}))]^2 \le \\
&\qquad \le \sum_{\ell=1}^{\ell_0} \ell^2\, \mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x))\right) \cdot \mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right) \qquad (4.38)
\end{aligned}$$

From inequalities (4.36) and (4.38), there are basically three quantity to analyze

(i) $\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right)$, for all $\ell = 0, 1, \ldots, \ell_0$;

(ii) $\mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,0}(h_x))$;

(iii) $\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x))\right)$, for all $\ell = 1, \ldots, \ell_0$,

We now proceed to estimate separately these three terms; at the end we will come back to Eqs. (4.36,4.38) and finally to (4.33).

First term: Poincaré inequality for the marginal measure on $N_{x,\ell}$
Let us consider the variance $\mathrm{Var}_{x,\ell}(g_{i+2})$ appearing in (i).
From definition the function g_{i+2} depends from the spin configuration on B_{i+1}, but under the measure $\mu_{x,\ell}^{\eta}$ it only depends from $N_{x,\ell}$ and then it holds

$$\mu_{x,\ell}^{\eta}(g_{i+2}) = \mu_{x,\ell|_{N_{x,\ell}}}^{\eta}(g_{i+2})\,.$$

Since $\tau^- \in \Omega^+$, for every configuration $\eta \in \Omega_{i+1}^{\tau^-}$ we can apply the Poincaré inequality stated in Theorem (**4.12**) to $\mathrm{Var}_{x,\ell}^{\eta}(g_{i+2})$ and then obtain

$$\mu_{i+1}^{\tau^-}\left(\mathrm{Var}_{x,\ell}(g_{i+2})\right) \;\le\; c \sum_{y \in N_{x,\ell}} \mu_{i+1}^{\tau^-}(\mathrm{Var}_y(g_{i+2})), \tag{4.39}$$

with $c = 1 + O(e^{c\beta})$ independent from the size of system.

Second term: computation of the variance of h_x
We first recall the definition $h_x(\sigma) := \frac{\mu_{i+1}^{\tau^+}(\sigma)}{\mu_{i+1}^{\tau^-}(\sigma)}$, from which we can deduce that h_x is a variable with mean one w.r.t. $\mu_{i+1}^{\tau^-}$ and only dependent from the vertices $y \in N_x$. In particular it can be expressed as

$$h_x(\sigma) \;=\; \frac{\exp(2\beta \sum_{y\in N_x} \sigma_y)}{\mu_{i+1}^{\tau^-}(\,\exp(2\beta \sum_{y\in N_x} \sigma_y)\,)} = \frac{\exp(2\beta \sum_{y\in N_x} (\sigma_y - 1))}{\mu_{i+1}^{\tau^-}(\,\exp(2\beta \sum_{y\in N_x} (\sigma_y - 1))\,)}\,,$$

where in the second equality we introduced a constant in the exponent in order to get the next computations easier.

Let us consider the (mean) variance $\mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,\ell}(h_x))$ with $\ell \ge 0$; through the DLR equations and the Jensen inequality we get

$$\mu_{i+1}^{\tau^-}(\mathrm{Var}_{x,\ell}(h_x)) \le \mu_{i+1}^{\tau^-}(h_x^2) - (\mu_{i+1}^{\tau^-}(h_x))^2 = \mu_{i+1}^{\tau^-}(h_x^2) - 1\,.$$

Using the above expression of h_x, $\mu^{\tau^-}_{i+1}(h_x^2)$ can be bounded as follows

$$\begin{aligned}\mu^{\tau^-}_{i+1}(h_x^2) &= \frac{\mu^{\tau^-}_{i+1}(\exp(4\beta\sum_{y\in N_x}(\sigma_y-1)))}{[\mu^{\tau^-}_{i+1}(\exp(2\beta\sum_{y\in N_x}(\sigma_y-1)))]^2}\\ &\le 1/\exp(4\beta\sum_{y\in N_x}\mu^{\tau^-}_{i+1}(\sigma_y-1))\\ &\le \exp(8\beta\, v\,\mu^{\tau^-}_{i+1}(\sigma_y=-))\,, \end{aligned} \tag{4.40}$$

where in the second line we use that $\sigma_y - 1 \le 0$ and the Jensen inequality to bound the numerator and the denominator respectively, and in the third line we bound the cardinality of N_x by v, the vertex degree of B.

All the computation then reduce to the probability $\mu^{\tau^-}_{i+1}(\sigma_y = -)$. Denoting by μ^-_{i+1} the measure conditioned to have all minus spins in B_i and plus spins in $\partial_V B$, by monotonicity it holds

$$\mu^{\tau^-}_{i+1}(\sigma_y=-) \le \mu^-_{i+1}(\sigma_y=-)\,.$$

The event $\{\sigma \in \Omega^-_{F_{i+1}} : \sigma_y = -\}$ corresponds to the set of configurations $\sigma \in \Omega^-_{F_{i+1}}$ such that there exists a negative connected component $C^{(\sigma)} \in F_{i+1}$ with $y \in C^{(\sigma)}$. Then, by the same argument developed in the previous section (see also Remark 4.9), it holds

$$\mu^-_{i+1}(\sigma_y=-) \le \sum_{m\ge 1}\sum_{\substack{C\in\mathcal{C}_m\\ y\in C}} \mu^-_{i+1}(\{\sigma\sim C\}) \le ce^{-\beta'}\,,$$

with $\beta' = c_1\beta - c_2$ as in Proposition **4.4**.
Inserting this result in (4.40) and with the previous computation, we get that for $\beta \gg 1$ and $\ell \ge 0$

$$\mu^{\tau^-}_{i+1}(\mathrm{Var}_{x,\ell}(h_x)) \le \exp(c\,\beta\, e^{-\beta'}) - 1 \sim c\,\beta e^{-\beta'} =: c_\beta\,. \tag{4.41}$$

We then keep in mind this result and proceed analyzing the last term.

Third term: the variance of $\mu_{x,\ell-1}(h_x)$
We now consider the variance $\mathrm{Var}^\eta_{x,\ell}(\mu_{x,\ell-1}(h_x))$ with $\eta \in \Omega^+$ and $\ell \ge 1$. Applying the Poincaré inequality stated in Theorem **4.12**, we obtain

$$\begin{aligned}\mathrm{Var}^\eta_{x,\ell}(\mu_{x,\ell-1}(h_x)) &\le \sum_{z\in N_{x,\ell}} \mu^\eta_{x,\ell}(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x)))\\ &= \sum_{z\in N_{x,\ell}\setminus N_{x,\ell-1}} \mu^\eta_{x,\ell}(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x)))\,, \end{aligned} \tag{4.42}$$

where the last inequality is due to the fact that the function $\mu_{x,\ell-1}(h_x)$ doesn't depend from the spin configuration on $N_{x,\ell-1}$.
Notice that the set $N_{x,\ell} \setminus N_{x,\ell-1}$ only contains the two vertices (eventually coincident) at distance ℓ from N_x.
For any configuration $\zeta \in \Omega^{\eta}_{x,\ell}$, let us denote by ζ^+ and ζ^- the configurations that agree with ζ in all sites but z and have respectively a $(+)$-spin and a $(-)$-spin on z; the summand in 4.42 can then be trivially bounded as

$$\mu^{\eta}_{x,\ell}(\mathrm{Var}_z(\mu_{x,\ell-1}(h_x))) \leq \frac{1}{2} \sup_{\zeta \in \Omega^{\eta}_{x,\ell}} ((\mu^{\zeta^+}_{x,\ell-1}(h_x) - \mu^{\zeta^-}_{x,\ell-1}(h_x))^2.$$

Due to the stochastic domination $\mu^{\zeta^+}_{x,\ell-1} \geq \mu^{\zeta^-}_{x,\ell-1}$, and to the fact that h_x is an increasing function it holds

$$\mu^{\zeta^+}_{x,\ell-1}(h_x) \geq \mu^{\zeta^-}_{x,\ell-1}(h_x)\,.$$

Moreover, there exists a coupling $\nu(\sigma,\sigma')$, with marginal measure $\mu^{\zeta^+}_{x,\ell-1}$ and $\mu^{\zeta^-}_{x,\ell-1}$, assigning probability one to the set $\{(\sigma,\sigma') \in \Omega_{N_{x,\ell-1}} \times \Omega_{N_{x,\ell-1}} : \sigma \geq \sigma'\}$. Then it holds

$$\begin{aligned}
\mu^{\zeta^+}_{x,\ell-1}(h_x) - \mu^{\zeta^-}_{x,\ell-1}(h_x) &= \sum_{\sigma,\sigma'} \nu(\sigma,\sigma') \left(h_x(\sigma) - h_x(\sigma')\right) \\
&\leq 2\|h_x\|_\infty \, \nu(\sigma_y \neq \sigma'_y\,, y \in N_x) \\
&\leq 2\,v\|h_x\|_\infty \, \left(\nu(\sigma_z = +) - \nu(\sigma'_z = +)\right) \\
&= 2\,v\|h_x\|_\infty \, (\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_z = +)), \qquad (4.43)
\end{aligned}$$

where we used the fact that the function h_x only depends from $y \in N_x$.

The quantity $\|h_x\|_\infty$ can be easily bounded using the same procedure as in (4.41). Indeed, for all $\sigma \in \Omega^+$, it holds

$$\begin{aligned}
h_x(\sigma) &= \frac{\exp(2\beta \sum_{y \in N_x}(\sigma_y - 1))}{\mu^{\tau^-}_{i+1}(\exp(2\beta \sum_{y \in N_x}(\sigma_y - 1)))} \\
&\leq 1/\exp(2\beta \sum_{y \in N_x} \mu^{\tau^-}_{i+1}(\sigma_y - 1)) \\
&\leq \exp(4\beta\, v\, \mu^{\tau^-}_{i+1}(\sigma_y = -)) \\
&\leq \exp(c\beta e^{-\beta'}) =: k_\beta\,, \qquad (4.44)
\end{aligned}$$

which implies that $\|h_x\|_\infty \leq k_\beta$. To bound the probability of disagreement appearing in (4.43), we again refer to Proposition **4.4** and to its proof (see

also Remark 4.9). Denoting by B the event that there exists a negative component of z with nonempty intersection with N_y, it easy to verify, with the same computations as in (4.9) and (4.10), that $\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_z = +) \leq \mu^{\zeta^-}_{x,\ell-1}(B)$. Since $d_i(z,y) \geq d_i(z,N_x) = \ell$, all these components have at least cardinality ℓ and thus, performing the same computation as in Section 4.3.3, we get

$$\mu^{\zeta^+}_{x,\ell-1}(\sigma_y = +) - \mu^{\zeta^-}_{x,\ell-1}(\sigma_y = +) \leq c\, e^{-\beta'\ell}\,. \tag{4.45}$$

Putting together formulas (4.42)-(4.45), we finally obtain

$$\mathrm{Var}_{x,\ell}(\mu_{x,\ell-1}(h_x)) \;\leq\; k'_\beta e^{-2\beta'\,\ell} \tag{4.46}$$

with $k'_\beta = c\,k_\beta^2 = c(1 + O(e^{-c\beta})\,.$

Conclusion

Let us come back to inequalities (4.36) and (4.38). Applying the bounds (4.39),(4.41) and (4.46), we get respectively

- $(\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x, g_{i+2})))^2 \leq c_\beta \sum_{y\in N_x} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))$;
- $[\mu^{\tau^-}_{i+1}(\mu_{x,0}(h_x), \mu_{x,0}(g_{i+2}))]^2 \leq k'_\beta \sum_{\ell=1}^{\ell_0} \ell^2 e^{-2\beta'\,\ell} \sum_{y\in N_{x,\ell+1}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))$,

where we included in c_β and k'_β all constants non depending from β.
For all $\beta \gg 1$, there exists a constant $\varepsilon_\beta = O(e^{-c\beta})$ such that $c_\beta \leq \varepsilon_\beta$ and $k'_\beta \ell^2 e^{-\beta'\ell} \leq \varepsilon_\beta$; summing the two terms above and rearranging the summation we thus obtain

$$\left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2 \leq \varepsilon_\beta \sum_{\ell=0}^{\ell_0} e^{-\beta'\ell} \sum_{y\in N_{x,\ell}} \mu^{\tau^-}_{i+1}(\mathrm{Var}_y(g_{i+2}))\,.$$

Inserting this result in the second term of formula (4.33) and rearranging the summation, we get

$$\begin{aligned}
\sum_{i=0}^{m-1} \sum_{x\in L_i} \mu \left[\sum_\tau \mu_i(\tau) p(\tau) q(\tau) \left(\mu^{\tau^-}_{i+1}(h_x, g_{i+2})\right)^2 \right] \leq \\
&\leq \varepsilon_\beta \sum_{i=0}^{m-1} \sum_{x\in L_i} \sum_{\ell=0}^{\ell_0} \sum_{y\in N_{x,\ell}} e^{-\beta'\ell} \mu(\mathrm{Var}_y(g_{i+2})) \\
&\leq \varepsilon_\beta \sum_{i=0}^{m-1} \sum_{y\in L_{i+1}} \mu(\mathrm{Var}_y(g_{i+2})) \sum_{\ell=0}^{\ell_0} e^{-\beta'\ell} n(\ell)\,,
\end{aligned} \tag{4.47}$$

where in the last line we introduced the functor $n(\ell)$ which bounds the number of vertices x such that a fixed vertex y belongs to N_x. Since $n(\ell)$ growth linearly with ℓ, the product $e^{-\beta\ell}n(\ell)$ decays exponentially with ℓ for all $\beta \gg 1$. Thus the sum over $\ell \in \{0, \dots, \ell_0\}$ can be bounded by a finite constant c which will be included in the factor ε_β in front of the summations. Continuing from (4.47), we get

$$\begin{aligned} \sum_{i=0}^{m-1} \sum_{x\in L_i} \mu \left[\sum_{\tau} \mu_i(\tau) p(\tau) q(\tau) \left(\mu_{i+1}^{\tau^-}(h_x, g_{i+2}) \right)^2 \right] &\leq \varepsilon_\beta \sum_{i=1}^{m} \sum_{y\in L_i} \mu(\mathrm{Var}_y(g_{i+1})) \\ &\leq \varepsilon_\beta \, \mathrm{Pvar}(f) \,. \end{aligned} \tag{4.48}$$

Inserting this result in (4.33), and noticing that $\varepsilon_\beta < 1$ for β large enough, we obtain

$$\mathrm{Pvar}(f) \leq \mathcal{D}(f) + \varepsilon_\beta \mathrm{Pvar}(f) \Longrightarrow \mathrm{Pvar}(f) \leq \frac{1}{1-\varepsilon_\beta} \mathcal{D}(f) \,,$$

and from inequality (4.30) we finally get

$$\mathrm{Var}(f) \leq c \mathrm{Pvar}(f) \leq c' \mathcal{D}(f) \,,$$

that is the desired Poincaré inequality with $c' = c\,/(1-\varepsilon_\beta) = 1 + O(e^{-c\beta})$, independent from the size of the system. This conclude the proof of Theorem (**4.11**). □

Chapter 5

Stochastic Ising Model on Random Graphs

5.1 Introduction

Random graph is an active area of research which combines probability theory and graph theory. The subject was started in 1959-1961 by Erdös and Rényi, see [ER1, ER2, ER3, ER4]. At first, the study of random graphs was used to prove deterministic properties of the graphs. For example, if we can show that a random graph has with positive probability a certain property, then a graph with this property must exist. The method of proving deterministic statements used probabilistic arguments is called the *probabilistic method* and Erdös was one of the first to use it in the paper [Er], where it was shown the existence of a graph with a specific Ramsey property. One can find a good explanation of this method in the work devoted to it *The Probabilistic Method* [AS], whereas standard references on random graphs are [Bo, JLR].

The initial work of Erdös and Rényi has incited a great amount of work on the field. Over the forty years that have passed since then, the theory has developed into an independent and fast-growing branch of discrete mathematics with applications to theoretical computer science, communications networks, natural and social science and to discrete mathematics itself.

Spin models on random graphs have attracted much attention in recent years and a sophisticated theory has been developed for computing the thermodynamic properties of such systems in great generality (see for instance

[MP] and references therein). The interest in this subject is at least twofold. From a side one would like to extend to new structures the theory of statistical physics, in order to obtain new features and behaviors which could explain some real physical systems.

On the other side, many combinatorial problems on random (and non random) structures, in particular optimization and counting problems, can be reformulated in the statistical mechanics setting and then solved, or at least better understood, with the methods of statistical mechanics (see [MM]). Between the most interesting combinatorial problems with random structure, it is the case to mention the "number partitioning problem" (see [BCMN] for recent result, and reference therein) , and the "random k-sat problem ", that recently has attracted a lot of attention both from computer scientists and from mathematical physicists (see [AP, AR, MMZ]).

We also point out that random graphs give rise to a large number of threshold phenomena which are evocative of phase transition in statistical physics, and that one would like to reexamine in a new perspective provided by statistical models [Mo].

In contrast with the huge literature on probabilistic and combinatorial analysis of random graphs, and with the increasing number of papers on the equilibrium behavior of statistical models on random graph, there are very few papers analyzing the dynamical evolution of these models. In particular it doesn't exist, to the best of our knowledge, any result concerning the mixing time of dynamics but only general, not proved beliefs. Our work is devoted to obtain some simple results on the behavior of local dynamics for the Ising model on random graphs. We will consider Erdös-Rényi (or binomial) random graphs and random r-regular graphs, and we will prove some lower bounds on the relaxation time for Glauber dynamics.

5.2 Preliminaries

Generally speaking, a random graph is a random variable defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is a suitable family of graphs and $\mathbb{P}$ is a probability distribution on Ω. Usually there is also a parameter involved which measures the size of the graph which typically tends to infinity.

The model introduced by Erdös can be described as choosing a graph at random, with uniform probabilities, from the set of all $2^{\binom{n}{2}}$ graphs on n

vertices. The probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ is then defined taking Ω equal to the set of all graphs on n vertices, and assigning to every $G \in \Omega$ probability $\mathbb{P}(G) = 2^{-\binom{n}{2}}$.
It can also be viewed as the product of $\binom{n}{2}$ binary spaces, each one associated to an edge of the complete graph on n vertices (denoted by K_n).
This simple model gave rise to two basic models of random graphs, namely the binomial random graph, or Erdös-Rényi random graph, and the uniform random graph.

Given a real number $p \in [0, 1]$, the *binomial random graph*, denoted by $\mathbb{G}(n, p)$, is defined letting Ω be the set of all graphs on n vertices and setting

$$\mathbb{P}(G) = p^{e_G}(1-p)^{\binom{n}{2}-e_G},$$

where e_G stands for the number of edges of G. For $p = 1/2$ this corresponds to the model introduced by Erdös [Er]. The binomial model is a special case of a reliability network. In this general model, Ω is the family of all spanning subgraphs of a given graph F, and $\mathbb{P}(G) = p^{e_G}(1-p)^{e_F - e_G}$. Taking F equal to K_n (the complete graph on n vertices) we thus reobtain the model $\mathbb{G}(n, p)$.

Uniform random graphs are defined by taking the uniform distribution over a family of graph Ω. Clearly there are many kinds of uniform random graphs, depending from the choice of Ω. The model introduced by Erdös [Er] belongs here too, with Ω being the family of all graphs on n vertices. Between the most popular uniform model we recall the following.

- $\mathbb{G}(n, M)$, with M an integer in $[0, \binom{n}{2}]$, is the uniform random graph with Ω being the family of all graphs on n vertices and with M edges. For every $G \in \Omega$, we thus have $\mathbb{P}(G) = \binom{\binom{n}{2}}{M}^{-1}$. $\mathbb{G}(n, M)$ is closely related to the Erdös-Rényi model. Indeed, providing $p\binom{n}{2}$ is close to M in a suitable way, one can prove the asymptotically equivalence between the two models.

- $\mathbb{G}(n, r)$, with an r integer such that nr is an even number, is the uniform random r-regular graph, where Ω is the family of all graphs on n vertices of equal degree r.

- random trees are uniform random graph with Ω being the family of all n^{n-2} trees on n labeled vertices.

The most interesting discovery made by Erdös and Rényi when investigating random graphs, is the *phenomenon of thresholds*.
Consider for example a random graph on n vertices identified with $(\Omega, \mathfrak{F}, \mathbb{P})$, and let Γ denote the family of all 2-element subsets of $\{1, \dots, n\}$ (or equivalently the set of edges in the complete graph K_n). For any given graph theoretic property A, let identify A with the family $\mathcal{Q}$ of subsets of 2^Γ, correspondent to a family of graphs on n vertices, satisfying A.

Definition 5.13. *A property A of a random graph $(\Omega, \mathfrak{F}, \mathbb{P})$, is called increasing monotone if $S \in \mathcal{Q}$ and $S \subseteq P$ imply that $P \in \mathcal{Q}$. A is called decreasing monotone if its complement in 2^Γ is increasing.*

Examples of increasing properties are "containing a triangle" and "be connected", whereas examples of decreasing properties are "having at least one isolated vertex" and "having at most k-edges". The property "having exactly k isolated vertices" is not monotone.
The probability that a random graph, say $\mathbb{G}(n,p)$, has a property A, depends from the graph parameters (n and p in this case) and it is denoted by $\mathbb{P}(A \in \mathbb{G}(n,p))$. Taking p as a function n, one can thus consider the limiting probability of A. Let us first introduce the following notation: given two sequences of numbers a_n and b_n depending on a parameter $n \to \infty$, we will write

- $a_n = O(b_n)$ if $\exists C$ such that $|a_n| \leq C\, b_n$
- $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$
- $a_n = \Omega(b_n)$ if $\exists C$ such that $|a_n| \geq C\, b_n$
- $a_n = \Theta(b_n)$ if $\exists C_1, C_2$ such that $C_1\, b_n \leq |a_n| \leq C_2\, b_n$
- $a_n \ll b_n$ if $a_n \geq 0$ and $a_n = o(b_n)$

Definition 5.14. *If A is an increasing monotone property and if there exists a sequence $\hat{p} = \hat{p}(n)$ such that*

$$\mathbb{P}(\mathbb{G}(n,p) \in A) \to \begin{cases} 0 & \text{if } p \ll \hat{p} \\ 1 & \text{if } p \gg \hat{p} \end{cases},$$

then $\hat{p}$ is called a threshold.

The threshold for a decreasing monotone property is defined as the threshold of their complements. Notice that the threshold is not uniquely determined, but only within constant factors; nevertheless, it is customary to talk about the threshold.
It is proved that every monotone property has a threshold (see for example [JLR]), whereas non-monotone properties, for which one can give a "local definition" of threshold as above, may have no threshold or countably many thresholds (see [Sp1]). Classic examples of thresholds refer to the property of containing a given graph as subgraph (e.g. isolated vertices, trees, cycles).

The presence of thresholds allows one to get an asymptotic characterization of the random graph, as p (or other involved parameters) varies as a function of n. In the next sections we will consider binomial and r-regular random graphs and discuss the probability of existence of some properties. Aside from having a better understanding of the shape of these random graphs, this will enables to recall the main techniques to analyze random graphs.

5.3 Binomial Random Graphs

Let us consider the binomial (or Erdös-Rényi) random graph $\mathbb{G}(n,p)$. We recall that it is the probability space of graphs on n vertices such that, for each $G \in \Omega$,

$$\mathbb{P}(G) \;=\; p^{e_G}(1-p)^{\binom{n}{2}-e_G}\,,$$

where $e_G = |E(G)|$ stands for the number of edges of G. It can also be viewed as a result of $\binom{n}{2}$ independent coin flippings, one for each pair of vertices, with probability of success (i.e. drawing an edge) equal to p.

Definition 5.15. *Given an event A_n, describing a property of a random structure depending on a parameter n, we say that A_n holds asymptotically almost surely if*

$$\lim_{n\to\infty} \mathbb{P}(A_n) = 1 \qquad (\textit{ write } A_n \textit{ holds a.a.s})\,.$$

Remark 5.10. *In many publications on random graph the phrase "almost surely" or a.s. is used, thought its normally meaning in probability is different. To not create confusion, we prefer to adopt this notation, which was introduced for the first time in [SU].*

As just remarked, for any monotone property $\mathcal{P}$ of $\mathbb{G}(n,p)$ there exists a threshold $\tilde{p} = \tilde{p}(n)$ such that, for any $p \gg \tilde{p}$, $\mathcal{P}$ holds a.a.s.. Depending on the p value, taken as a function of n, $\mathbb{G}(n,p)$ can then be characterized by properties holding with hight probability, and usually analyzed in the asymptotic limit $n \to \infty$. Here we briefly discuss the asymptotic behavior of certain properties of $\mathbb{G}(n,p)$ as p varies in $(0,1)$ as a function of n. See for details [Bo], [JLR] and [Sp2].

5.3.1 Thresholds for small subgraph containment

One of the first problems studied in the theory of random graphs was that of the existence in $\mathbb{G}(n,M)$ of at least a copy of a given graph H ([ER2]). Since the graph H is fixed and the random graph grows with $n \to \infty$, copies of H are called small subgraphs, as opposed to subgraphs which grow with n, like a Hamiltonian cycle. The problem was solved in full generality in [Bo], though a simpler proof was given later in [RV]. Here we present the result and just explain the idea beyond the proof. We refer to [Bo, RV, JLR] for a discussion on the problem and for the details of the proofs.

Consider a graph $H \subseteq K_n$, where K_n is the complete graph on n vertices, and denote by v_H the number of vertices of H and by e_H the number of edges. We search for the asymptotic probability that $\mathbb{G}(n,p)$ contains a copy of the subgraph H, when v_H and e_H are finite integers.

Let X_H be the (random) number of copies of H that can be found in the binomial random graph $\mathbb{G}(n,p)$. For each copy H' of H in K_n, define the indicator $\mathbb{1}_{H'} = \mathbb{1}[H' \subseteq \mathbb{G}(n,p)]$ so that $X_H = \sum_{H' \sim H} \mathbb{1}_{H'}$.
The number of copies of H in K_n is given by the function

$$f(n,H) = \binom{n}{v_H} (v_H)!/\mathrm{aut}(H) = \Theta(n^{v_H}) \,, \tag{5.1}$$

where $\mathrm{aut}(H)$ denotes the number of automorphisms of H. For every copy H', the probability that H' is a subgraph in $\mathbb{G}(n,p)$ is equal to p^{e_H}. Then

$$\mathbb{E}(X_H) \;=\; f(n,p)p^{e_H} = \Theta(n^{v_H}p^{e_H}) \to \begin{cases} 0 & \text{if } p \ll n^{-v_H/e_H} \\ \infty & \text{if } p \gg n^{-v_H/e_H} \end{cases} .$$

By the first moment method (an instance of the Markov inequality for integer valued random variables) we get

$$\mathbb{P}(X_H > 0) \le \mathbb{E}(X_H) = o(1) \qquad \text{if } p \ll n^{-v_H/e_H} \,.$$

To prove that $p = n^{-v_H/e_H}$ is a threshold for the property that H is a subgraph in $\mathbb{G}(n,p)$, it remains to verify that $\mathbb{P}(X_H > 0) = 1 - o(1)$ for $p \gg n^{-v_H/e_H}$. But this turns out to be false as can be shown by explicit examples. The correct threshold for this property is indeed the following. Let us define

$$m(H) := \max\left\{\frac{e_K}{v_K} : K \subseteq H,\, v_K > 0\right\};$$

it holds the following

Theorem 5.13. *For any subgraph $H \in K_n$ with at least one edge,*

$$\lim_{n\to\infty} \mathbb{P}(H \subseteq \mathbb{G}(n,p)) = \begin{cases} 0 & \text{if } p \ll n^{-1/m(H)} \\ 1 & \text{if } p \gg n^{-1/m(H)} \end{cases} . \tag{5.2}$$

The proof of Theorem **5.13** relays on the so called second moment method. This is an instance of the Chebyshev'inequality holding for random variables with positive average. Through this inequality, one obtains

$$\mathbb{P}(H \not\subseteq \mathbb{G}(n,p)) = \mathbb{P}(X_H = 0) \leq \frac{\mathrm{Var}(X_H)}{(\mathbb{E}(X_H))^2},$$

and thus the probability under investigation con be computed comparing the variance $\mathrm{Var}(X_H)$ with $(\mathbb{E}(X_H))^2$.

From theorem **5.13** one can deduce some first useful consequences. First of all we observe that any cycle C of whatever length, has a threshold in $p = n^{-1}$, since $m(C) = 1$. Thus, for any sequence $p = p(n)$ such that $p(n) \ll n^{-1}$, any cycle or subgraph containing a cycle has probability $o(1)$ to be contained in $\mathbb{G}(n,p)$. In particular, for such values of p, only trees are admissible subgraphs and, more strictly, for any p such that $n^{-\frac{k}{k-1}} \ll p \ll n^{-\frac{k+1}{k}}$, the random graph $\mathbb{G}(n,p)$ is a forest of trees with at most k vertices.

For many subgraphs, this result can be refined, by considering the behavior at the threshold.
Let H be a graph with v_H vertices and e_H edges and denote by $\rho(H) = v_H/e_H$ its density. H is called *strictly balanced* if every proper subgraph H' has $\rho(H') < \rho(H)$ (then $\rho(H) = m(H)$). It holds the following

Theorem 5.14. *Let H a strictly balanced graph with v_H vertices and e_H edges and denote by X_H the number of copies of H in $\mathbb{G}(n,p)$. For $c > 0$ and $p = cn^{-v_H/e_H}$ it holds*

$$X_H \xrightarrow{d} \mathit{Poiss}(\lambda) \quad , \quad \lambda = \frac{c^{v_H}}{\mathit{aut}(H)} \tag{5.3}$$

Cycles are examples of strictly balanced graphs. If C_l denotes a cycle of finite length l and $p = c/n$, from Theorem **5.14** it follows

$$\mathbb{P}(C_l \subseteq \mathbb{G}(n,p)) = \mathbb{P}(X_{C_l} > 0) \longrightarrow 1 - e^{-\lambda}\,.$$

Any cycle of arbitrary finite length has thus asymptotic positive probability to be contained in $\mathbb{G}(n,p)$.
Of course, the problem of subgraph containment is much more developed and here we only gave some basic results that will be helpful in that follows.

5.3.2 Giant component and connectivity

As we argue by the former analysis, for any $p = o(n^{-1})$, $\mathbb{G}(n,p)$ is asymptotically a forest of trees with finite cardinality.
When $p = c/n$, with $c > 0$, the landscape changes and a largest component, with size increasing with n, appears. Moreover, depending on the value of the constant c, the random graph could contain a *giant component*, namely a subgraph with size comparable to n. Again, the existence of a phase transition which appears as a "sudden" jump of the size of the largest component, has been studied for the first time in the fundamental paper of Erdös and Rényi [ER2]. A great improvement of these results has been obtained twenty years later by Bollobás in [Bo2].
Here we present, without prove, the main result due to Erdös and Rényi.

Theorem 5.15. *Let $p = \frac{c}{n}$, where $c > 0$ is a constant.*

(i) *If $c < 1$, then a.a.s. the largest component of $\mathbb{G}(n,p)$ has at most $\frac{3}{(1-c)^2}\log n$ vertices.*

(ii) *Let $c > 1$ and let $\rho = \rho(c) \in (0,1)$ such that $\rho + e^{c\rho} = 1$, or in other terms let $1-\rho$ be the extinction probability of a branching process with offsprings distribution $Poiss(c)$. Then a.a.s. $\mathbb{G}(n,p)$ contains a giant component of $(1+o(1))\rho\, n$ vertices whereas the size of the other components is at most $\frac{16c}{(c-1)^2}\log n$.*

The proof of this theorem (see for instance [LRS]) is made analyzing the size of a connected component through a fixed vertex v. The construction of this component, usually denoted by $C(v)$, can be realized defining a suitable branching process. This explains why in the second part of the theorem appears the constant ρ.

The value $p = n^{-1}$ is thus a threshold for the existence of the giant component; nevertheless, for this p, the random graph $\mathbb{G}(n,p)$ is a.a.s. not connected. Indeed, it is proved that the existence of isolated graphs with finite cardinality (that is a monotone decreasing property), has a threshold for $p \gg n^{-1}$.

More precisely, one can prove that the existence of an isolated vertex in $\mathbb{G}(n,p)$ has a threshold for $p = \frac{\log n + c + o(1)}{n}$. At the threshold, the variable counting the number of isolated vertices in $\mathbb{G}(n,p)$ is asymptotically distributed as Poiss(e^{-c}).
On the other hand, for this value of p, the probability that exists an isolated graph with finite size $k > 1$ is a.a.s. 0. Thus the threshold for the existence of isolated vertices and the connectivity is the same. For more details on the problem of connectivity see [Bo, Sp2].

5.4 Random Regular Graphs

An *r-regular* graph is a graph with degree equal to r at each vertex. If the number of vertices are n, then the number of edges is $rn/2$, so that rn has to be even.

The *random r-regular graph* $\mathbb{G}(n,r)$ is defined as the probability space of all r-regular graphs on n vertices, taken with uniform probability. The case $r = 0$ and $r = 1$ are trivial; the first corresponds to en empty graph whereas the second is the set of perfect matchings, each one taken with probability $1/(n-1)!!$. Indeed $(n-1)!!$ is the number of perfect matchings on n vertices, provided that n be even.

Though the definition of random regular graphs is conceptually simple, it is not so easy to use it; indeed there is no simple formula for the total number of r-regular graphs on n vertices, and thus also the uniform probability is unknown. Most work on random regular graphs, both theoretical and practical, is based on a parallel model, whose construction has two different versions due to Bender and Canfield [BC] and Bollobàs [Bo1].

In the Bollobàs version, it is called the *configurational model* and is defined as follows. Let W the product space between a set V with n elements and a set R with r elements. Then W has rn elements, called half-edges, that will denoted as the couple (v,x), $v \in V = \{1,\dots,n\}$ and $r \in R = \{1,\dots,r\}$.

A *configuration* is a perfect matching of W, i.e. a partition of W into $rn/2$ pairs. These pair are called edges. The key point is that the natural projection π of W onto V, ignoring the second coordinate, maps each configuration F to a multigraph $\pi(F)$ on V with constant degree r. We recall that a multigraph is a graph in which loops or multiple edges are admitted, unlike of the so called simple graphs; of course the set of r-regular multigraphs contains the smaller set of r-regular (simple) graphs.

Taking a random configuration F with uniform probability, namely $\mathbb{P}(F) = 1/(rn-1)!!$, and projecting on V, we thus obtained an r-regular multigraph $\mathbb{G}^*(n,r)$. Unfortunately the distribution of $\mathbb{G}^*(n,r)$ is not uniform, because multigraphs with a different number of loops and multiple edges arise from a different number of configurations. Instead, if we take the projection $\pi(F)$ of a random configuration and condition on it being a simple graph, we obtain a random r-regular graph on V with uniform probability, namely $\mathbb{G}(n,r)$. This is because every r-regular graph on V is the projection of exactly $r!^n$ configurations.

Many properties of $\mathbb{G}^*(n,r)$ can be analyzed through combinatorics on the configuration model and then it is somehow simpler working with it then with $\mathbb{G}(n,r)$. On the other hand it is proved that any property that holds a.a.s. for $\mathbb{G}^*(n,r)$, holds a.a.s. for $\mathbb{G}(n,r)$ too (see Theorem **5.17** below). This provides an easy technique of investigation, and indeed many results concerning $\mathbb{G}(n,r)$ were obtained in this way. We give a briefly description of the main properties of $\mathbb{G}(n,r)$, in order to get a better understanding of its shape. A wider discussion on the subject can be found in the survey article [Wo] and in [JLR].

5.4.1 Small Cycles: Poisson paradigm

The term "Poisson paradigm" was introduced for the first time in [AS] to describe the fact that the sum of many nearly independent rare events has Poisson distribution. In [Bo1, Wo1] it has been proved that the number of short cycles in random regular graph with small degree has asymptotically Poisson distribution, and thus a kind of Poisson paradigm has been shown. The method for proving this result is an asymptotic version of the fact that a Poisson random variable is determined by its moments. The main argument of the proof is developed on the random r-regular multigraph $\mathbb{G}^*(n,r)$ and exploits the properties of the correspondent configuration model. The result

for $\mathbb{G}(n,r)$ is then obtained as a consequence of the result for $\mathbb{G}^*(n,r)$.

Given an r-regular (multi)graph G, we let $Z_k = Z_k(G)$ denote the number of cycle of length k in G; for multigraphs we let $k \geq 1$, but for simple graphs we let $k \geq 3$. Taking G to be the random regular graph $\mathbb{G}(n,r)$ or $\mathbb{G}^*(n,r)$, clearly Z_k becomes a random variable. Then it holds the following (see [Bo1, Wo1, Wo2]). By joint convergence of an infinite number of variables we mean joint convergence of every finite subset.

Theorem 5.16. *Let $\lambda_k = \frac{1}{2k}(r-1)^k$ and let $Z_{k,\infty} \overset{d}{\sim} P_o(\lambda_k)$ be independent Poisson distributed random variables for all $k \geq 1$. Then the random variables $Z_k(\mathbb{G}^*(n,r))$ converge in distribution to $Z_{k\infty}$, $Z_k(\mathbb{G}^*(n,r)) \xrightarrow{d} Z_{k\infty}$ as $n \to \infty$, jointly for all k.*

Notice now that a realization G of $\mathbb{G}^*(n,r)$ is simple if and only if $Z_1(G) = Z_2(G) = 0$, since in this way we exclude possible loops or multiple edges, and that $\mathbb{G}^*(n,r)$ conditioned on $Z_1 = Z_2 = 0$ yields $\mathbb{G}(n,r)$. The next result follows.

Corollary 5.1. *Let λ_k and Z_k be as in Theorem **5.16**. Then the random variables $Z_k(\mathbb{G}(n,r))$ converge in distribution to $Z_{k\infty}$ as $n \to \infty$, jointly for all $k \geq 3$.*

Using this result, much of the local properties of the r-regular graph can be computed. For example, it is not difficult to prove that, for a given vertex $v \in V$ and $k < c_r \log n$, with c_r a positive constant depending on the degree r, the probability that there exists a cycle of length at most k passing through v is asymptotically 0. In particular, the structure of $\mathbb{G}(n,r)$ inside a ball centered at a given vertex of radius less then $c_r \log n$, is a.a.s a tree.

Another consequence of Theorem **5.16**, is that one can compute the asymptotic probability that $\mathbb{G}^*(n,r)$ is simple. This result, together the fact that the number of r-regular graph is equal to the number of r-regular multigraph times the probability $\mathbb{P}(\mathbb{G}^*(n,r)$ is simple$)$, enables us to obtain an asymptotic formula for the number of labeled regular graph.

Corollary 5.2. *If $n \to \infty$, then*

$$\mathbb{P}(\mathbb{G}^*(n,r) \textit{ is simple}) \to e^{-(r^2-1)/4} > 0\,.$$

Corollary 5.3. *The number L_n of labeled r-regular graph on n vertices satisfies, as $n \to \infty$ for fixed r,*

$$L_n \;=\; \sqrt{2}e^{-(r^2-1)/4}(r^{r/2}e^{-r/2}/r!)^n n^{rn/2}(1+o(1))$$

From Corollary 5.2, it holds that the asymptotic probability $\mathbb{P}(\mathbb{G}^*(n,r)$ is simple$)$ is positive. With this information, it easy to prove the following result.

Theorem 5.17. *Any property that holds a.a.s for $\mathbb{G}^*(n,r)$ holds a.a.s for $\mathbb{G}(n,r)$ too.*

The converse clearly doesn't hold, as the trivial example of containing a loop shows. Theorem **5.17** thus provides a tight connection between the properties in $\mathbb{G}^*(n,r)$ and in $\mathbb{G}(n,r)$.

5.4.2 Isoperimetric constant

Another important property that we want to point out, concern the edge isoperimetric constant of $G(n,r)$. For any finite graph $G=(V,E)$, we recall (see (1.2)) that

$$i_e(G) \;=\; \min_K \frac{|\partial_E K|}{|K|},$$

where the minimum is taken over all subsets $K \in V$ with $|K| \leq |V|/2$.
Many regular graphs tend to have small isoperimetric constant. For example, by folding an n vertex line on a circle, we get a 2-regular graph with isoperimetric constant about $2/n$.
Also the existence, for every n and r, of a regular graph with positive isoperimetric constant is a non trivial problem. A useful approach in this context, comes from the probabilistic method ([AS]).
The existence of regular graph with positive isoperimetric constant, for large n and $r \geq 3$, is consequence of the following theorem due to Bollobás [Bo3].

Theorem 5.18. *Let $r \geq 3$ and $\eta \in (0,1)$ be such that*

$$2^{4/r} < (1-\eta)^{(1-\eta)}(1+\eta)^{(1+\eta)}\,.$$

Then a.a.s. $\mathbb{G}(n,r)$ has isoperimetric constant at least $(1-\eta)r/2$, i.e.

$$\mathbb{P}(i_e(G) \geq (1-\eta)r/2) \underset{n\to\infty}{\longrightarrow} 1 \tag{5.4}$$

We refer to [Bo3] for the proof of the theorem. We only mention that the proof is based on the correspondence between $\mathbb{G}(n,r)$ and the configurational model, and on combinatorics.

5.5 Analysis of Glauber dynamics

In this section we will consider the Glauber dynamics for the Ising model on a random graphs with n vertices; we will concentrate on the two random graph models previously defined. The aim of our analysis is to shed some light on the behavior of the relaxation time for the dynamics on random graphs.

5.5.1 Notation

Let $\mathcal{G}$ be a general random graph on n vertices and with finite mean vertex degree, correspondent to the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.
For any graph realization G of $\mathcal{G}$, the Ising model is defined as for deterministic graphs. Let us just fix some notation.
Let V denote the n-vertex set of $\mathcal{G}$. To every vertex $x \in V$ we associate a spin variable $\sigma_x \in \{\pm 1\}$, and denote by $\Omega_\sigma := \{\pm 1\}^n$ the space of spin configurations and by $\mathcal{F}$ the σ-algebra generated by the projections of Ω_σ onto the single spin space.
Given $G \in \Omega$, we consider the Ising model on G specified by the Gibbs measure μ_G, which assign probability

$$\mu_G(\sigma) = \frac{1}{Z_G(\beta)} \exp\big(\beta \sum_{(x,y)\in E_G} \sigma_x \sigma_y\big)$$

to every $\sigma \in \Omega_\sigma$. Clearly, the measure μ_G, and all the quantities derived from that measure, are random variables depending on the realization G of $\mathcal{G}$. To denote this dependence from the random graph, we will add the subscript G to these quantities.
In the following section, we will analyze the heat-bath Glauber dynamics on G, with Markov generator denoted by $\mathcal{L}_G$ (see section 3.5.2), when G is a realization of $\mathbb{G}(n,p)$ and of $\mathbb{G}(n,r)$. In particular we will consider the problem of determining, with high probability, the behavior of the relaxation time of the dynamics.

5.5.2 Slow mixing on binomial random graphs at all temperatures: first result

Let us consider $\mathbb{G}(n,p)$ with $p = \frac{c}{n}$ and c a finite positive constant, in order to get a finite mean vertex degree. For this value of p, we recall that

$\mathbb{G}(n, c/n)$ can be (asymptotically) described as follows (see section 5.3).

- a.a.s $\mathbb{G}(n, c/n)$ is not connected;
- if $c < 1$, a.a.s. the largest component of $\mathbb{G}(n, c/n)$ has at most $\frac{3}{(1-c)^2} \log n$ vertices;
  if $c > 1$, a.a.s. $\mathbb{G}(n, c/n)$ contains a giant component of $(1 + o(1))\rho n$ vertices;
- a.a.s the only finite subgraphs in $\mathbb{G}(n, c/n)$ are trees and cycles. Moreover the number of cycles of given length, say k, is a.a.s a Poisson variable with mean $c^k/2k$.

Consider now the Glauber dynamics for the Ising model on $\mathbb{G}(n, c/n)$. From the analysis of the subgraph containment (see Section 5.3.1), one can easily deduce that, given a fixed vertex $x \in V$, the $\log n$-neighborhood of x is a.a.s. a tree. As just recalled in the previous chapters, the Ising model Glauber dynamics on trees has spectral gap uniformly bounded by a constant for all temperature bigger then the critical one, which depends on the maximal degree of the tree. Roughly, this would led to deduce that there is a finite a range of temperatures such that the dynamics on $\mathbb{G}(n, c/n)$ shows a similar behavior. However, the random graph could contain trees with maximal degree increasing with n, and so having a critical temperature that goes to infinity with n, where the dynamics slows down in a significant way for all finite temperature. Since the spectral gap takes in account of the worst case that can appears in the graph, this would imply that the relaxation time is increasing with n for all realizations G of the random graph containing these subgraphs.
To analyze the dynamics on $\mathbb{G}(n, c/n)$, we will follow this approach: first we will identify a subgraph on which the dynamics slows down for all $\beta > 0$, and then we will prove that, with positive probability, $\mathbb{G}(n, c/n)$ contains such a subgraph.

Remark 5.11. *Notice that this kind of approach is conceptually similar to the one used to analyze the Glauber dynamics for the random Ising model on a finite volume $\Lambda \in \mathbb{Z}^d$ in the so-called Griffhits phase (see, e.g. [CMM, Fr, Mar]). The presence of peculiar subgraphs in $\mathbb{G}(n, c/n)$, on which the dynamics slows down for all $\beta > 0$, plays the same role of the presence of "rare bad cluster" for the random Ising model in Λ, on which the dynamics slows*

down due to the presence of strong interactions between particles.
There is also a kind of similarity with argument used in [H2], where the author consider the zero-temperature Glauber dynamics on $\mathbb{G}(n,p)$ and $\mathbb{G}(n,M)$ and analyze the problem of whether the dynamics reaches the global minimum energy for some value of $p = \Theta(n^{-1})$. Also in this case, the proof is based on the observation that there is a subgraph which imposes a certain global property to the dynamics. Then the proof is reduced to the analysis of the probability that such a subgraph is contained in the random graph.

Let us formalize these ideas. With the notation introduced above, we state the following:

Theorem 5.19. *For every $\beta > 0$, there exists a positive constant c_β such that a.a.s. the spectral gap of the Ising model Glauber dynamics on $\mathbb{G}(n, c/n)$ is less then $\delta_n = \exp\left(-c_\beta \frac{\log n}{\log\log n}\right)$, i.e.*

$$\lim_{n\to\infty} \mathbb{P}(c_{gap}(\mu_G) \leq \delta_n) = 1 \tag{5.5}$$

Proof. We will proceed in the following two steps.

(i) for every integer $k \leq n$, we define a suitable graph S_k and prove that if G is a graph realization of $\mathbb{G}(n,p)$ containing S_k, then the spectral gap of the dynamics on G is exponentially small in k.

(ii) if $k \leq \alpha \frac{\log n}{\log\log n}$, with $\alpha < 1$, we prove that the probability that S_k is contained in $\mathbb{G}(n, c/n)$ is asymptotically one.

The statement will easily follow.
Proof of (i). Let us consider the graph S_k given by $k+1$ vertices with a distinguished root-vertex linked to all the other k vertices and no more edges; we refer to it as k-star. Suppose that S_k belongs to a realization G of the random graph $\mathbb{G}(n,p)$ as an isolated subgraph, and let's analyze the Glauber dynamics on S_k.

The way to bound from above the spectral gap, generally consists in providing a test function f on the spin configuration space having a very slow decaying. More precisely, from the definition of spectral gap (see 2.13), for any nonconstant function $f \in L^2(\Omega, \mathcal{F}, \mu_G)$

$$c_{gap}(\mu_G) \leq \frac{\mathcal{D}_G(f)}{\mathrm{Var}_G(f)}\,. \tag{5.6}$$

Thus, for any configuration $\sigma \in \Omega_\sigma$, let $m_k(\sigma)$ denote the magnetization of S_k, i.e. $m_k(\sigma) = \sum_{x \in V(S_k)} \sigma_x$, and take as a test function the following indicator function

$$\mathbb{1}_k(\sigma) := \mathbb{1}_{\{m_k > 0\}}(\sigma) = \begin{cases} 1 & \text{if } m_k(\sigma) > 0 \\ 0 & \text{if } m_k(\sigma) \leq 0 \end{cases} . \tag{5.7}$$

Without loss of generality, we can restrict the analysis to even k values. Since in this case $m_k(\sigma) \neq 0$ for all σ, by symmetry it holds that $\mathrm{Var}_G(\mathbb{1}_k) = 1/4$ so that the spectral gap is bounded as $c_{gap}(\mathcal{L}_G) \leq 4\,\mathcal{D}_G(\mathbb{1}_k)$. We thus concentrate on the Dirichlet form

$$\mathcal{D}_G(\mathbb{1}_k) := \frac{1}{2} \sum_{x \in V(S_k)} \mu_G\left(c_x \left[\nabla_x \mathbb{1}_k\right]^2\right) .$$

We first observe that $\nabla_x \mathbb{1}_k(\sigma) \neq 0$ only if σ is such that $|m_k(\sigma)| = 1$, namely for configurations σ having $(k+2)/2$ spins with the same value on the vertices of the subgraph S_k. Thus, for every configuration σ of this kind, $[\nabla_x \mathbb{1}_k(\sigma)]^2 = 1$ only if x is one of the $(k+2)/2$ vertices in S_k with spin of the same sign of $m_k(\sigma)$; then we get

$$\mathcal{D}_G(\mathbb{1}_k) \leq \frac{k+2}{4}\, \mu_G(|m_k| = 1)\,. \tag{5.8}$$

Let us state the following:

Lemma 5.3. *For every $\beta > 0$ and k high enough, there exists a positive constant α_β such that*

$$\mu_G(|m_k| = 1) \leq e^{-\alpha_\beta k} \tag{5.9}$$

Proof. Let us denote by r the only vertex in S_k with degree equal to k, namely the center of the k-star, and label from 1 to k all the other vertices. Using the notation $\mu_G^{r,+}(\,\cdot\,) = \mu_G(\,\cdot\,|\sigma_r = +)$ (and analogously for $\sigma_r = -$) and observing that $\mu_G^{r,+}(|m_k| = 1) = \mu_G^{r,-}(|m_k| = 1)$ by symmetry, we get

$$\begin{aligned} \mu_G(|m_k| = 1) &= \frac{1}{2}\,\mu_G^{r,+}(|m_k| = 1) + \frac{1}{2}\,\mu_G^{r,-}(|m_k| = 1) \\ &= \mu_G^{r,+}(|m_k| = 1) \\ &\leq \mu_G^{r,+}\Big(\sum_{i=1}^{k} \sigma_i \leq 0\Big)\,. \end{aligned} \tag{5.10}$$

Since S_k is an isolated subgraph, all variables σ_i, with $i \in \{1, \dots, k\}$, are independent w.r.t $\mu_G^{r,+}$, with probabilities

$$\begin{cases} \mu_G^{r,+}(\sigma_i = +) = \frac{e^\beta}{e^\beta + e^{-\beta}} = p_\beta \\ \mu_G^{r,+}(\sigma_i = -) = \frac{e^{-\beta}}{e^\beta + e^{-\beta}} = 1 - p_\beta \end{cases}$$

and with mean and variance

$$\mu_G^{r,+}(\sigma_i) = 2p_\beta - 1 \quad ; \quad \mathrm{Var}_G^{r,+}(\sigma_i) = 4p_\beta(1 - p_\beta)\,.$$

In particular, if $(X_i)_{i=1}^k$ are independent Bernoulli random variables with $\mathbb{P}$-mean p_β, then $\sigma_i \overset{d}{\sim} 2X_i - 1$ for all $i \in \{1, \dots, k\}$ and the sum $\sum_{i=1}^k \sigma_i$ is distributed as $2Z - k$, with $Z = \sum_{i=1}^k X_i$ a random binomial variable $\mathrm{Bi}(k, p_\beta)$. Using the Chernoff's inequality for binomial variables, we get

$$\mu_G^{r,+}(\sum_{i=1}^k \sigma_i \le 0\,) = \mathbb{P}(\, Z \le \frac{k}{2}\,) \;\le\; \exp\left(-\frac{(p_\beta - 1/2)^2}{2p_\beta}\, k\right)$$

and then inequality (5.9) follows taking $\alpha(\beta) := \frac{(p_\beta - 1/2)^2}{2p_\beta}$ and noticing that $\alpha(\beta)$ is positive and increasing in β for all $\beta > 0$. □

From inequality (5.8)and Lemma 5.3, we get $\mathcal{D}(\mathbb{1}_k) \le \frac{(k+2)}{4}\, e^{-\alpha_\beta\, k}$. Thus, for all $\beta > 0$ there exists an integer k_0 and a positive constant c_β such that, for all $k \ge k_0$, it holds

$$c_{gap}(\mu_G) \le (k+2)\, e^{-\alpha_\beta k} \le e^{-c_\beta k}\,. \tag{5.11}$$

This concludes the proof of step (i), in which we proved that for every $\beta > 0$ there exists an integer k_0 and a positive constant c_β such that, for all $k \ge k_0$, if G contains an isolated S_k, then $c_{gap}(\mu_G) \le e^{-c_\beta k}$. Notice that this implies

$$\mathbb{P}(c_{gap}(\mu_G) \le e^{-c_\beta k}) \ge \mathbb{P}(\text{ isolated } S_k \subseteq G)\,.$$

Proof of (ii). Here we want to compute the asymptotic probability that $\mathbb{G}(n, c/n)$ contains an isolated S_k. We thus consider the random variable X_k counting the number of copies of S_k in $\mathbb{G}(n, c/n)$; clearly it holds $\mathbb{P}(\text{ isolated } S_k \subseteq \mathbb{G}(n, c/n)) = \mathbb{P}(X_k > 0)\,.$

As a first step we can use the first moment method (see subsection 5.3.1) to rule out the range of k such that $\mathbb{P}(X_k > 0) \le \mathbb{E}(X_k) \underset{n\to\infty}{\longrightarrow} 0$. To compute

$\mathbb{E}(X_k)$, we first express X_k as sum of the following indicator functions. For every $v \in V$, define the event ξ_v that a k-star rooted in v is contained in $\mathbb{G}(n, c/n)$, i.e.

$$\xi_v := \{\mathbb{G}(n, c/n) \supseteq S_k \text{ with root in } v\}\,.$$

Denoting by $\mathbb{1}(\xi_v)$ the indicator function of ξ_v, we thus get that $X_k = \sum_{v\in V} \mathbb{1}(\xi_v)$ and then, by linearity of expectation, $\mathbb{E}(X_k) = \sum_{v\in V} \mathbb{P}(\xi_v)$. By simple combinatorics, one obtains the formula

$$\mathbb{P}(\xi_v) = \binom{n-1}{k} p^k (1-p)^{(k+1)(n-k-1)+\binom{k+1}{2}-k}\ ,\ \forall v \in V\,.$$

Substituting the value $p = c/n$ and applying the Stirling formula for large n and $k \equiv k(n) = o(n)$ but such that $k(n) \uparrow \infty$ as $n \to \infty$, we get

$$\mathbb{P}(\xi_v) = e^{-k\log k}(1 + o(1))\,. \tag{5.12}$$

The asymptotic behavior of $\mathbb{E}(X_k)$ clearly depends from the choice of $k(n)$; taking $k = \alpha\frac{\log n}{\log\log n}$, with α a real positive number, we get

$$\mathbb{E}(X_k) = n\exp\left(-k\log k\right)(1 + o(1)) = \Omega(n^{1-\alpha})\,.$$

If $k = \alpha\frac{\log n}{\log\log n}$ with $\alpha \geq 1$, and in general for all $k \gg \frac{\log n}{\log\log n}$, $\mathbb{E}(X_k)$ tends asymptotically to 0. By the first moment method this implies that a.a.s. $X_k = 0$, namely the existence of an isolated k- star has a vanishing asymptotic probability.

If $k = \alpha\frac{\log n}{\log\log n}$ with $\alpha < 1$, and in general for all k increasing with n but such that $k \ll \frac{\log n}{\log\log n}$, the expectation of X_k becomes large with n. Clearly this is not enough to assure that $\mathbb{P}(X_k > 0) \to 1$. We will thus exploit the second moment method and search for k values such that $\mathrm{Var}(X_k) = o(\mathbb{E}(X_k)^2)$, in order to get

$$\mathbb{P}(X_k = 0) \leq \frac{\mathrm{Var}(X_k)}{\mathbb{E}(X_k)^2} = o(1)\,,$$

which implies that a.a.s $\mathbb{G}(n, c/n)$ contains an isolated S_k.

In order to bound the variance we first recall the following decomposition property of the variance (see for instance [Sa]).

Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, \dots, n$ be n probability spaces and let $(\Omega, \mathcal{F}, \mu)$ be the associated product space, with $\mu = \otimes_i \mu_i$. Then

$$\mathrm{Var}_\mu(f) \leq \sum_{i=1}^{n} \mu(\mathrm{Var}_{\mu_i}(f)) \,. \tag{5.13}$$

Notice that the measure $\mathbb{P}$, defined on the probability space $\mathbb{G}(n,p)$, is the product measure of independent Bernoulli variables with mean p associated to the $n(n-1)/2$ edges of K_n, namely

$$\mathbb{P} = \otimes_{e \in K_n} \mathrm{Be}(p) \,.$$

Let us introduce the following notation: given a function f on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathrm{Var}_e(f)$ denote the variance of f with respect to the Bernoulli(p) measure associated to e, and define the gradient ∇_e as

$$\nabla_e f\,(G) := f(G^e) - f(\xi_v)(G) \,,$$

where G^e denotes the graph obtained from G by switching the value of the Bernoulli variable $\eta_e \in \{0,1\}$ associated to e, or, in other word, by removing or adding the edge e, depending respectively if $e \in E(G)$ or $e \notin E(G)$.

Applying the decomposition property (5.13) to $\mathrm{Var}(X_k)$, we obtain

$$\begin{aligned} \mathrm{Var}(X_k) &\leq \sum_{e \in E(K_n)} \mathbb{E}\,(\mathrm{Var}_e(X_k)) \\ &= p(1-p) \sum_{e \in E(K_n)} \mathbb{E}(\,\nabla_e X_k\,)^2 \\ &= p(1-p) \sum_{\substack{e=(x,y):\\ x<y \in V}} \mathbb{E}(\sum_{v \in V} \nabla_e \mathbb{1}(\xi_v)\,)^2 \,. \end{aligned} \tag{5.14}$$

Let us analyze the quantity in the r.h.s. of (5.14).
For any given edge $e = (x,y)$, the vertex v (the root of a k-star) could correspond to an end-vertex of e, namely $v = x$ or $v = y$, or it could be $v \notin \{x,y\}$. We thus introduce the characteristic functions $\chi_{v=x}$, $\chi_{v=y}$ and $\chi_{v \notin \{x,y\}}$ in the sum over $v \in V$ in (5.14), and obtain

$$\begin{aligned} \mathbb{E}\,(\sum_{v \in V} \nabla_e \mathbb{1}(\xi_v)\,)^2 &= \mathbb{E}\,(\sum_{v \in V} \nabla_e \mathbb{1}(\xi_v)(\chi_{v=x} + \chi_{v=y} + \chi_{v \notin \{x,y\}})\,)^2 \\ &\leq 2\mathbb{E}\,(\nabla_e \mathbb{1}(\xi_x))^2 + \mathbb{E}\,(\sum_{v \notin \{x,y\}} \nabla_e \mathbb{1}(\xi_v)\,)^2 \quad , \end{aligned} \tag{5.15}$$

where in the last line, due to the fact that e is an unoriented edge, we identified the quantities $\nabla_e \mathbb{1}(\xi_x)$ and $\nabla_e \mathbb{1}(\xi_y)$. We analyze separately the two terms appearing in (5.15).

First term: Notice that $\nabla_e \mathbb{1}(\xi_x)(G) = \mathbb{1}(\xi_x)(G^e) - \mathbb{1}(\xi_x)(G) \neq 0$ if and only if $\mathbb{1}(\xi_x)(G^e) = 1$ or $\mathbb{1}(\xi_x)(G) = 1$. Since each one of these occurrences excludes the other and in both cases $|\nabla_e \mathbb{1}(\xi_x)(G)| = 1$, it holds

$$\begin{aligned}\mathbb{E}\left(\nabla_e \mathbb{1}(\xi_x)\right)^2 &= \sum_{G\in\Omega} \mathbb{P}(G)(\mathbb{1}(\xi_x)(G) + \mathbb{1}(\xi_x)(G^e)) \\ &= \mathbb{P}(\xi_x) + \sum_{G\in\Omega} \mathbb{P}(G)(\mathbb{1}(\xi_x)(G^e)) \\ &\leq e^{-k\log k}(1+o(1)) + \sum_{G\in\Omega} \mathbb{P}(G)(\mathbb{1}(\xi_x)(G^e))\,. \quad (5.16)\end{aligned}$$

We now observe that if $\mathbb{1}(\xi_x)(G^e) = 1$, namely if G^e contains an isolated k-star with root in x, one of the following two properties must be satisfied in G

$$P_1 = \left\{ \begin{array}{l} \exists \text{ a subgraph } T_k \text{ such that } T_k \setminus \{e\} \text{ is an isolated } k\text{-star rooted at } x \end{array} \right\}$$

$$P_2 = \left\{ \begin{array}{l} \exists \text{ an isolated } (k-1)\text{-star rooted at } x \text{ and } y \text{ is an isolated vertex} \end{array} \right\}$$

and thus we can write

$$\sum_{G\in\Omega} \mathbb{P}(G)(\mathbb{1}(\xi_x)(G^e)) \leq \sum_{G\in\Omega} \mathbb{P}(G)(\mathbb{1}(P_1)(G) + \mathbb{1}(P_2)(G)) = \mathbb{P}(P_1) + \mathbb{P}(P_2)\,. \quad (5.17)$$

Since the subgraph T_k is constructed from an isolated k-star rooted at x adding the edge e, we get

$$\mathbb{P}(P_1) = \frac{p}{1-p}\,\mathbb{P}(\xi_x) \leq \frac{1}{n}\,e^{-k\log k}(1+o(1))\,.$$

To compute the probability of P_2 we use simple combinatorics and apply the Stirling formula for large n and $k \equiv k(n) = o(n)$, such that $k(n) \uparrow \infty$ as $n \to \infty$, as for the computation of $\mathbb{P}(\xi_v)$ in (5.29); we get

$$\begin{aligned}\mathbb{P}(P_2) &= \binom{n-2}{k-1} p^{k-1}(1-p)^{(k+1)(n-k-1)+\binom{k+1}{2}-k+1} \\ &= e^{-k\log k}(1+o(1))\end{aligned}$$

Inserting these result in (5.16), we finally obtain

$$\mathbb{E}\left(\nabla_e \mathbb{1}(\xi_x)\right)^2 \leq e^{-k\log k}(1+o(1))\,. \quad (5.18)$$

Second term: We first observe that for a given edge $e = (x, y)$ and for any vertex $v \notin \{x, y\}$, $\nabla_e \mathbb{1}(\xi_v)(G) \neq 0$ only if at least one of the two end-vertices of e is connected to v in G, namely if $\eta_{(x,v)} = 1$ or $\eta_{(y,v)} = 1$. Otherwise the structure of the 2-neighborhood of v remains unchanged by switching the value of η_e.

Denoting by $\mathbb{1}_{x \mapsto v}$ and by $\mathbb{1}_{y \mapsto v}$ the indicator functions respectively of the events $\{\eta_{(x,v)} = 1\}$ and $\{\eta_{(y,v)} = 1\}$, it holds

$$\begin{aligned}
\mathbb{E}\,[\sum_{v \notin \{x,y\}} \nabla_e \mathbb{1}(\xi_v)\,]^2 \;&\leq\; E\,[\sum_{v \notin \{x,y\}} (\mathbb{1}_{x \mapsto v} + \mathbb{1}_{y \mapsto v}) \nabla_e \mathbb{1}(\xi_v)\,]^2 \\
&\leq \sum_G \mathbb{P}(G)[\sum_{v \notin \{x,y\}} (\mathbb{1}_{x \mapsto v}(G) + \mathbb{1}_{y \mapsto v}(G))(\mathbb{1}(\xi_v)(G) + \mathbb{1}(\xi_v)(G^e))\,]^2 \\
&= \sum_G \mathbb{P}(G)[\sum_{v \notin \{x,y\}} (\mathbb{1}(\xi_v^x)(G) + \mathbb{1}(\xi_v^y)(G) + \mathbb{1}(\xi_v^x)(G^e) + \mathbb{1}(\xi_v^y)(G^e))\,]^2 \quad (5.19)
\end{aligned}$$

where in the second line we used the bound $\nabla_e \mathbb{1}(\xi_v)(G) \leq \mathbb{1}(\xi_v)(G) + \mathbb{1}(\xi_v)(G^e)$, and in the last line we introduced the symbol ξ_v^x for the event "exists an isolated k-star rooted at v and with a leaf in x", in order to have $\mathbb{1}(\xi_v^x)(G) = \mathbb{1}_{x \mapsto v}(G)\mathbb{1}(\xi_v)(G)$.
Computing the square of the sum in (5.19) and observing that, from the definition of isolated k-star, it holds

$$\begin{cases}
\mathbb{1}(\xi_v^x)(G)\mathbb{1}(\xi_w^x)(G) \;=\; \mathbb{1}(\xi_v^y)(G)\mathbb{1}(\xi_w^y)(G) \equiv 0\,, & \forall\, v \neq w \\
\mathbb{1}(\xi_v^x)(G^e)\mathbb{1}(\xi_w^x)(G^e) \;=\; \mathbb{1}(\xi_v^y)(G^e)\mathbb{1}(\xi_w^y)(G^e) \equiv 0\,, & \forall\, v \neq w \\
\mathbb{1}(\xi_v^x)(G)\mathbb{1}(\xi_w^x)(G^e) \;=\; \mathbb{1}(\xi_v^y)(G)\mathbb{1}(\xi_w^y)(G^e) \equiv 0\,, & \forall\, v, w \\
\mathbb{1}(\xi_v^x)(G)\mathbb{1}(\xi_w^y)(G^e) \equiv 0\,, & \forall\, v, w
\end{cases}$$

we can continue from (5.19) and obtain

$$\begin{aligned}
&\leq\; 2 \sum_{v \notin \{x,y\}} \sum_G \mathbb{P}(G)[(\mathbb{1}(\xi_v^x)(G) + \mathbb{1}(\xi_v^x)(G^e))] + \qquad (5.20) \\
&+\; 2 \sum_{v,w \notin \{x,y\}} \sum_G \mathbb{P}(G)[\mathbb{1}(\xi_v^x)\mathbb{1}(\xi_w^y)(G) + \mathbb{1}(\xi_v^x)\mathbb{1}(\xi_w^y)(G^e))]\,.
\end{aligned}$$

Proceeding as for the computation of $\mathbb{P}(\xi_v)$ in (5.29), for large n and $k = k(n) = o(n)$ such that $k(n) \uparrow \infty$ as $n \to \infty$, we get

$$\begin{aligned}
\mathbb{P}(\xi_v^x) \;&=\; \sum_G \mathbb{P}(G)\mathbb{1}(\xi_v^x)(G) \;=\; \frac{1}{n}\, e^{-k \log k}(1 + o(1)) \\
\mathbb{P}(\xi_v^x, \xi_w^y) \;&=\; \sum_G \mathbb{P}(G)\mathbb{1}(\xi_v^x)\mathbb{1}(\xi_w^y)(G) \;=\; \frac{1}{n^2}\, e^{-2k \log k}(1 + o(1))
\end{aligned}$$

Then, noticing that $\mathbb{1}(\xi_v^x)(G^e) = 1$ if and only if there is a subgraph T_k in G such that $T_k \setminus \{e\}$ is an isolated k-star rooted at v and with a leaf in x, and that $\mathbb{1}(\xi_v^x)(\xi_w^y)(G^e) = 1$ if and only if there is a subgraph $\widetilde{T}_k$ in G such that $\widetilde{T}_k \setminus \{e\}$ is given by two isolated k-stars rooted at v and w, and with a leaf in x and y, respectively, it holds

$$\begin{aligned}\sum_G \mathbb{P}(G)\mathbb{1}(\xi_v^x)(G^e) &= \frac{p}{1-p}\,\mathbb{P}(\xi_v^x) \,\leq\, \frac{1}{n^2}\,e^{-k\log k}(1+o(1)) \\ \sum_G \mathbb{P}(G)\mathbb{1}(\xi_v^x)\mathbb{1}(\xi_w^y)(G^e) &= \frac{p}{1-p}\,\mathbb{P}(\xi_v^x,\xi_w^y) \,\leq\, \frac{1}{n^3}\,e^{-2k\log k}(1+o(1))\,.\end{aligned}$$

Inserting the above expressions in (5.20) and summing over the vertex set, we finally obtain the bound

$$E\,[\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(\xi_v)\,]^2 \,\leq\, e^{-k\log k}(1+o(1))\,. \tag{5.21}$$

Substituting the expressions (5.18) and (5.21) in (5.15), and then inserting the result in (5.14), we obtain

$$\mathrm{Var}(X_k) \,\leq\, p\,(1-p)\,n^2\,e^{-k\log k}(1+o(1)) = n\,e^{-k\log k}(1+o(1))\;.$$

If $k = \alpha\frac{\log n}{\log\log n}$ with $\alpha < 1$, or in general for all k increasing with n but such that $k \ll \frac{\log n}{\log\log n}$, and from the second moment method, it holds

$$\mathbb{P}(X_k = 0) \,\leq\, \mathrm{Var}(X_k)/\mathbb{E}(X_k)^2 \leq (n\,e^{-k\log k})^{-1}(1+o(1)) \underset{n\to\infty}{\longrightarrow} 0\,,$$

completing the proof of step (ii).

The statement of Theorem **5.19** will now easily follows.
Indeed, in step (i) we proved that for all $\beta > 0$ there exists an integer k_0 and a positive constant c_β such that, for all $k \geq k_0$,

$$\mathbb{P}(c_{gap}(\mu_G) \leq e^{-c_\beta k}) \geq \mathbb{P}(\text{ isolated } S_k \subseteq \mathbb{G}(n, c/n))\,,$$

whereas in step (ii) we proved that for all integers k increasing with n and such that $k < \frac{\log n}{\log\log n}$, it holds

$$\mathbb{P}(\text{ isolated } S_k \subseteq \mathbb{G}(n, c/n)) \underset{n\to\infty}{\longrightarrow} 1\,.$$

Taking $k = \alpha\frac{\log n}{\log\log n}$ and $\delta_n = \exp\left(-\alpha c_\beta \frac{\log n}{\log\log n}\right)$, with $\alpha \in (0,1)$, we obtain

$$\mathbb{P}(c_{gap}(\mu_G) < \delta_n) \geq \mathbb{P}(S_k \subseteq \mathbb{G}(n,p)) \underset{n\to\infty}{\longrightarrow} 1,$$

which completes the proof of Theorem **5.19**. □

5.5.3 Slow mixing on binomial random graphs at all temperatures: second result

In the previous section we proved that the random graph $\mathbb{G}(n, c/n)$ contains a.a.s. an isolated subgraph, with size increasing with n, where the dynamics slows down in a significant way. Let us now recall that if $c > 1$, then a.a.s. $\mathbb{G}(n, c/n)$ has a giant component of size $(1 + o(1))\rho n$, with $\rho \in (0, 1)$ uniquely determined by the equation $\rho + e^{-c\,\rho} = 1$. At this point one can ask what happens if one restricts the dynamics to the giant component of $\mathbb{G}(n, c/n)$: will the relaxation time grow with n for any $\beta > 0$?
This section is aimed to answer this question.

From now on, we will suppose that $c > 1$ and denote by $\{\mathcal{C} \subseteq \mathbb{G}(n, c/n)\}$ the a.a.s. event that there exists a unique giant component $\mathcal{C}$ in $\mathbb{G}(n, c/n)$. We thus state the following:

Theorem 5.20. *For every $\beta > 0$ there exists a positive constant c_β such that a.a.s. the spectral gap of the Glauber dynamics on the giant component $\mathcal{C}$ of $\mathbb{G}(n, c/n)$ is less then $\delta_n = \exp\left(-c_\beta \frac{\log n}{\log\log n}\right)$, i.e.*

$$\lim_{n\to\infty} \mathbb{P}(c_{gap}(\mu_{\mathcal{C}}) \le \delta_n \,;\, \mathcal{C} \subseteq \mathbb{G}(n, c/n)) = 1 \tag{5.22}$$

Proof. We will proceed in the following two steps.

(i) for every $k \le n$, we define a suitable graph $\tilde{S}_k$ and prove that if G is a graph realization of $\mathbb{G}(n, c/n)$ containing $\tilde{S}_k$, then the spectral gap of the dynamics on G is exponentially small in k for all β.

(ii) if $k \le \alpha \frac{\log n}{\log\log n}$, with $0 < \alpha < \frac{1}{2}$, we prove that the probability that $\tilde{S}_k$ is contained in the giant component $\mathcal{C}$ of $\mathbb{G}(n, c/n)$ is asymptotically 1.

The statement will easily follow.

Proof of (i). Let us fix a finite integer ℓ and consider the graph $\widetilde{S}_k \equiv \widetilde{S}_k(\ell)$ defined as follows:
a root-vertex r is linked to k vertices indexed by $i \in \{1, \dots, k\}$, and each of them is the end vertex of a segment of line with ℓ vertices, denoted by Z^i_ℓ. In other words, $\widetilde{S}_k$ is a non-regular tree with ℓ levels such that the root has k children whereas all the other vertices have one child. Notice that $|\widetilde{S}_k| = \ell\, k + 1$, and that the graph S_k defined in the previous section is a subgraph of $\widetilde{S}_k$.

Suppose that $\widetilde{S}_k$ belongs to a realization G of the random graph $\mathbb{G}(n,p)$ as an induced subgraph, but in such a way that only the vertices on the leafs of $\widetilde{S}_k$, namely the vertices in the ℓ-th level denoted by L_ℓ, can be linked to vertices outside $\widetilde{S}_k$. In this case we will say that $\widetilde{S}_k$ is an *almost isolated (a.i)* subgraph of G. Let's analyze the Glauber dynamics on $\widetilde{S}_k$.

To bound from above the spectral gap, we use the same test function as in the proof of Theorem **5.19**. Thus, let

$$m_k(\sigma) = \sum_{x \in V(S_k)} \sigma_x = \sigma_r + \sum_{i=1}^{k} \sigma_i$$

be the magnetization of the subgraph S_k in $\widetilde{S}_k$, and consider the indicator function $\mathbb{1}_k = \mathbb{1}_{\{m_k > 0\}}$. Without loss of generality, we can restrict the analysis to even k values. Since in this case $m_k(\sigma) \neq 0$ for all σ, by symmetry it holds that $\mathrm{Var}_G(\mathbb{1}_k) = 1/4$ so that the spectral gap is bounded as $c_{gap}(\mu_G) \leq 4\,\mathcal{D}_G(\mathbb{1}_k)$. We thus concentrate on the Dirichlet form

$$\mathcal{D}_G(\mathbb{1}_k) := \frac{1}{2} \sum_{x \in V} \mu_G\left(c_x \left[\nabla_x \mathbb{1}_k\right]^2\right) .$$

We first observe that $\nabla_x \mathbb{1}_k(\sigma) \neq 0$ only for configurations σ such that $|m_k(\sigma)| = 1$, i.e. for configurations σ having $(k+2)/2$ spins with the same value on the vertices of the subgraph S_k. In particular, for every configuration σ of this kind, $[\nabla_x \mathbb{1}_k(\sigma)]^2 = 1$ only if x is one of the $(k+2)/2$ vertices in S_k with spin of the same sign of $m_k(\sigma)$; then we get

$$\mathcal{D}_G(\mathbb{1}_k) \;\leq\; \frac{k+2}{4}\, \mu_G(|m_k| = 1)\,. \tag{5.23}$$

Let us state the following

Lemma 5.4. *For every $\beta > 0$, there exists a finite integer $\ell_0 = \ell_0(\beta)$ and a positive constant α_β, such that for every $\ell \geq \ell_0$*

$$\mu_G(|m_k| = 1) \leq e^{-\alpha_\beta k} \tag{5.24}$$

Proof. Let us first consider the measure μ_G conditioned to the spin at the root r. Using the notation $\mu_G^{r,+}(\,\cdot\,) = \mu_G(\,\cdot\,|\sigma_r = +)$ (and analogously for $\sigma_r = -$), we observe that $\mu_G^{r,+}(|m_k| = 1) = \mu_G^{r,-}(|m_k| = 1)$ by symmetry,

and then we get

$$\begin{aligned}\mu_G(|m_k|=1) &= \frac{1}{2}\,\mu_G^{r,+}(|m_k|=1)+\frac{1}{2}\,\mu_G^{r,-}(|m_k|=1)\\ &= \mu_G^{r,+}(|m_k|=1)\\ &\le \mu_G^{r,+}(\sum_{i=1}^{k}\sigma_k\le 0). \end{aligned} \tag{5.25}$$

In this case, at the contrary of what happens when we consider the isolated subgraph S_k, the spins on the first level vertices remain correlated under the measure $\mu_G^{r,+}$, due to the possible presence of paths in $V\setminus\{r\}$ between them. We thus introduce a second conditioning on the spin configuration outside $\widetilde{S}_k\setminus L_\ell$. Given a configuration $\tau\in\Omega_\sigma$, we use the notation

$$\mu_k^{\tau;\,r,+}(\,\cdot\,) \;:=\; \mu_G^{r,+}(\,\cdot\,|\,\tau\in\mathcal{F}_{(\widetilde{S}_k\setminus L_\ell)^c})$$

and then write

$$\mu_G^{r,+}(\sum_{i=1}^{k}\sigma_i\le 0\,) \;=\; \sum_{\tau}\mu_G^{r,+}(\tau)(\mu_k^{\tau;\,r,+}(\sum_{1}^{k}\sigma_i\le 0)) \tag{5.26}$$

Since, by hypothesis, all paths from $\widetilde{S}_k$ to $\widetilde{S}_k^c$ intersect L_ℓ, all variables σ_i become independent w.r.t $\mu_k^{\tau;\,r,+}$, with probabilities

$$\begin{cases}\mu_k^{\tau;\,r,+}(\sigma_i=+) =: p_i\\ \mu_k^{\tau;\,r,+}(\sigma_i=-) =: 1-p_i\end{cases}$$

and mean $\mu_k^{\tau;\,r,+}(\sigma_i)=2p_i-1$, where p_i also depends from β and τ.

Notice that for all $i\in\{1,\dots,k\}$, p_i corresponds to the probability to have a $(+)$-spin on the end vertex x_i of Z_ℓ^i, given a $(+)$-spin on its neighbor and τ-b.c. on the other end vertex. Since the Z_ℓ^i are one dimensional systems, for any τ-b.c. this probability can be explicitly computed using the method of transfer matrix (see, e.g., [Sim]). In particular one can verify that for every $\beta>0$, the influence of the τ-b.c. on the spin in x_i has a fast decay in the distance between them, which is equal to $\ell-1$. If we take ℓ big enough, σ_i becomes independent from the τ-b.c., and thus the $(+)$-spin at its neighbor makes the probability of $\{\sigma_i=+\}$ bigger then the probability of $\{\sigma_i=-\}$. Moreover this effect increases with β. It follows that for all $\beta>0$, there exists a finite integer $\ell_0=\ell_0(\beta)$ and a number $\xi(\beta)>0$ such that for all $\ell\ge\ell_0$

$$\mu_k^{\tau;\,r,+}(\sigma_i=+)=p_i\ge\frac{1}{2}+\xi(\beta)\,,\qquad \forall i\in\{1,\dots,k\}\ \text{and}\ \forall\tau\in\Omega\,.$$

If $(X_i)_{i=1}^k$ are independent Bernoulli random variables with $\mathbb{P}$-mean p_i, then $\sigma_i \overset{d}{\sim} 2X_i - 1$ for all $i \in \{1, \dots, k\}$ and the sum $\sum_{i=1}^k \sigma_i$ is distributed as $2Z - k$, with $Z = \sum_{i=1}^k X_i$. Using the Chernoff's inequality for the sum of independent Bernoulli variables $Be(p_i)$, we get

$$\begin{aligned} \mu_k^{\tau;0,+}(\sum_1^k \sigma_i \le 0) &= \mathbb{P}(\, Z \le \frac{k}{2}\,) \\ &\le \exp\left(-\frac{(\sum_{i=1}^k p_i - k/2)^2}{2\sum_{i=1}^k p_i}\right) \\ &\le \exp\left(-\frac{\xi_\beta^2}{2}\, k\right). \end{aligned} \tag{5.27}$$

Inequality (5.24) follows taking $\alpha_\beta := \frac{\xi_\beta^2}{2}$, which is positive for all $\beta > 0$. □

From inequality (5.23)and Lemma 5.4, we get $\mathcal{D}(\mathbb{I}_k) \le (k+2)\, e^{-\alpha_\beta\, k}$. Thus, for all $\beta > 0$ there exists an integer k_0 and a positive constant c_β such that, for all $k \ge k_0$, it holds

$$c_{gap}(\mu_G) \le (k+2)\, e^{-\alpha_\beta k} \le e^{-c_\beta k}\,. \tag{5.28}$$

This concludes the proof of step (i), in which we proved that for every $\beta > 0$ there exist two finite integers k_0 and ℓ_0 and a positive constant c_β such that, for all $k \ge k_0$ and $\ell \ge \ell_0$, if G contains an a.i. $\widetilde{S}_k$ then $c_{gap}(\mu_G) \le e^{-c_\beta k}$. Moreover, since μ_G is the product measure over the components of G, denoting by $\mathcal{C}(\widetilde{S}_k)$ a connected component containing $\widetilde{S}_k$ it holds $c_{gap}(\mu_{\mathcal{C}(\widetilde{S}_k)}) \le e^{-c_\beta k}$. Notice that this implies

$$\mathbb{P}(c_{gap}(\mu_{\mathcal{C}}) \le e^{-c_\beta k}\,;\, \mathcal{C} \subseteq \mathbb{G}(n, c/n)) \ge \mathbb{P}(\text{ a.i } \widetilde{S}_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n))\,,$$

where the event in the r.h.s. of the above inequality corresponds to the existence of an a.i. $\widetilde{S}_k$ in the unique giant component $\mathcal{C}$ of $\mathbb{G}(n, c/n)$.

Proof of (ii). Let us first consider the property that an a.i. $\widetilde{S}_k$ is contained in $\mathbb{G}(n, c/n)$. Proceeding as in the proof of Theorem **5.19**, we introduce the random variable $\widetilde{X}_k$ counting the number of copies of $\widetilde{S}_k$ in $\mathbb{G}(n, c/n)$ and such that

$$\mathbb{P}(\text{ a.i. } \widetilde{S}_k \subseteq \mathbb{G}(n, c/n)) = \mathbb{P}(\widetilde{X}_k > 0)\,.$$

To find the values of k such that $\mathbb{P}(\widetilde{X}_k > 0) \underset{n\to\infty}{\longrightarrow} 1$, we use again the first and second moment methods and then proceed by computing the mean and

the variance of $\widetilde{X}_k$.
For every $v \in V$, let us first define the event $\widetilde{\xi}_v$ that an a.i. $\widetilde{S}_k$ rooted at v is contained in $\mathbb{G}(n, c/n)$, i.e.

$$\widetilde{\xi}_v \;:=\; \{\text{ a.i. } \widetilde{S}_k \subseteq \mathbb{G}(n,c/n),\ \widetilde{S}_k \text{ rooted at } v\}\,.$$

Denoting by $\mathbb{1}(\widetilde{\xi}_v)$ the indicator function of $\widetilde{\xi}_v$, we thus get that $\widetilde{X}_k = \sum_{v\in V}\mathbb{1}(\widetilde{\xi}_v)$ and then, by linearity of expectation, $\mathbb{E}(\widetilde{X}_k) = \sum_{v\in V}\mathbb{P}(\widetilde{\xi}_v)$.
The probability $\mathbb{P}(\widetilde{\xi}_v)$ can be expressed with some combinatorics as

$$\mathbb{P}(\widetilde{\xi}_v) = \binom{n-1}{\ell k}\frac{(\ell k)!}{k!}p^{\ell k}(1-p)^{[((\ell-1)k+1)(n-(\ell-1)k-1)+\binom{\ell k+1}{2}-\ell k]}\ ,\ \forall v\in V\,.$$

Remark 5.12. *Notice that in this case the number of automorphisms of $\widetilde{S}_k$ is $k!$, which corresponds to the number of possible orderings of the k lines Z^i_ℓ connected to the root.*

For finite ℓ and $k \equiv k(n) = o(n)$ such that $k(n) \uparrow \infty$ as $n \to \infty$, substituting the value $p = c/n$ and applying the Stirling formula for large n , we obtain

$$\mathbb{P}(\widetilde{\xi}_v) = e^{-k\log k}(1+o(1))\,. \tag{5.29}$$

Taking $k = \alpha\frac{\log n}{\log\log n}$, with α a real positive number, we get

$$\mathbb{E}(\widetilde{X}_k) \;=\; n\exp\left(-k\log k\right)(1+o(1)) = \Omega(n^{1-\alpha})\,. \tag{5.30}$$

Thus if $k < \frac{\log n}{\log\log n}$, and in general for all $k \ll \frac{\log n}{\log\log n}$, the expectation of $\widetilde{X}_k$ tends asymptotically to infinite. Anyway this is not enough to assure that $\mathbb{P}(\widetilde{X}_k > 0) \to 1$, but we need to compute $\mathrm{Var}(\widetilde{X}_k)$ in order to apply the second moment method.

Using the same notation introduced in the proof of Theorem **5.20** and applying the decomposition property (5.13) to $\mathrm{Var}(\widetilde{X}_k)$, we obtain the analogous of formula (5.14), i.e.

$$\mathrm{Var}(\widetilde{X}_k) \;\le\; p(1-p)\sum_{\substack{e=(x,y):\\ x<y\in V}}\mathbb{E}\,(\,\sum_{v\in V}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,)^2\,. \tag{5.31}$$

For any given edge $e = (x, y)$ we introduce the characteristic functions

$\chi_{v=x}$, $\chi_{v=y}$ and $\chi_{v\notin\{x,y\}}$ in the sum over $v \in V$ in (5.31), and obtain

$$\begin{aligned}\mathbb{E}\,(\sum_{v\in V}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,)^2 &= \mathbb{E}\,(\sum_{v\in V}\nabla_e\mathbb{1}(\widetilde{\xi}_v)(\chi_{v=x}+\chi_{v=y}+\chi_{v\notin\{x,y\}})\,)^2\\ &\le 2\mathbb{E}\,(\nabla_e\mathbb{1}(\widetilde{\xi}_x))^2+\mathbb{E}\,(\sum_{v\notin\{x,y\}}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,)^2\\ &\le 2+\mathbb{E}\,(\sum_{v\notin\{x,y\}}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,)^2\,, \qquad (5.32)\end{aligned}$$

where in the second line, due to the fact that e is an unoriented edge, we identified the quantities $\nabla_e\mathbb{1}(\widetilde{\xi}_x)$ and $\nabla_e\mathbb{1}(\widetilde{\xi}_y)$, and in the last line we use the bound $(\nabla_e\mathbb{1}(\widetilde{\xi}_x))^2 \le 1$.

Remark 5.13. *Due to the more difficult structure of $\widetilde{S}_k$ with respect to S_k, at this point we cannot perform the same computations as in the proof of Theorem **5.19** (see Eqs.(5.19)-(5.21)). In that follows, we thus simplify the analysis bounding in a suitable way the quantities that are more difficult to analyze, as just made in (5.32). With this approximation, we probably lose a factor of order $e^{-k\log k}$ in front of every terms. Nevertheless, the effect of this loss is a worsening (increase) of the value of the constant δ_n in the statement of Theorem **5.20** that doesn't modify the validity of the theorem.*

From (5.32), it remains to analyze the quantity $\mathbb{E}\,(\sum_{v\notin\{x,y\}}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,)^2$. We first observe that for a given edge $e=(x,y)$ and for any vertex $v\notin\{x,y\}$, $\nabla_e\mathbb{1}(\xi_v)(G)\neq 0$ only if there exists a path γ in G from at least one of the two end-vertices of e and v, whose length is at most $\ell-1$. Otherwise the structure of the ℓ-neighborhood of v remains unchanged by switching the value of η_e. Notice also that since $v\notin\{x,y\}$, the length of this path is at least 1.
Denoting by $\mathbb{1}_{\gamma:\,x\mapsto v}$ and by $\mathbb{1}_{\gamma:\,y\mapsto v}$ the indicator functions of the events $\{\exists\gamma\in G:\gamma:x\mapsto v\,,\,|\gamma|\le\ell-1\}$ and $\{\exists\gamma\in G:\gamma:x\mapsto v\,,\,|\gamma|\le\ell-1\}$, respectively, it holds

$$\begin{aligned}\mathbb{E}\,[\sum_{v\notin\{x,y\}}\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,]^2 &\le \mathbb{E}\,[\sum_{v\notin\{x,y\}}(\mathbb{1}_{\gamma:\,x\mapsto v}+\mathbb{1}_{\gamma:\,y\mapsto v})\nabla_e\mathbb{1}(\widetilde{\xi}_v)\,]^2\\ &\le \mathbb{E}\,[\sum_{v\notin\{x,y\}}(\mathbb{1}_{\gamma:\,x\mapsto v}+\mathbb{1}_{\gamma:\,y\mapsto v})\,]^2\\ &= 2\sum_{v\notin\{x,y\}}\mathbb{E}(\mathbb{1}_{\gamma:\,x\mapsto v})+\sum_{\substack{v,w\notin\{x,y\}\\ v\neq w}}\mathbb{E}(\mathbb{1}_{\gamma:\,x\mapsto v}\mathbb{1}_{\gamma:\,y\mapsto v}) \qquad (5.33)\end{aligned}$$

where in the second line we used the obvious bound $\nabla_e \mathbb{1}(\widetilde{\xi}_v)(G) \leq 1$.

The quantity $E(\mathbb{1}_{\gamma:\, x \mapsto v})$ corresponds to the probability that there exists a path in G, between x and v, with length at most $\ell - 1$. With some combinatorics and applying the Stirling formula for large n, we get

$$E(\mathbb{1}_{\gamma:\, x \mapsto v}) = \sum_{j=1}^{\ell-1} \binom{n-2}{j-1} p^j = n^{-1}[\sum_{j=1}^{\ell-1} \frac{c^j}{(j-1)!}(1+o(1))\,] = \frac{1}{n}(c_0 + o(1))\,, \tag{5.34}$$

where c_0 is a finite constant. We assume from now that the particular value of c_0 may change from line to line as the discussion progresses.

The event $\mathbb{1}_{\gamma:\, x \mapsto v}\mathbb{1}_{\gamma:\, y \mapsto v}$ can be view as the disjoint union of the following two events:

1. there exists a path $\gamma : x \mapsto v$ of length at most $\ell - 1$ and at least 2, such that $y \in \gamma$;

2. there exists a path $\gamma : x \mapsto v$ of length at most $\ell - 1$ such that $y \notin \gamma$, and for some $z \in \gamma$, a path $\gamma' : z \mapsto y$ such that the length of γ' is at most $\ell - 1 - d_\gamma(z, x) \leq \ell - 1$.

Then it holds the following computation

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\gamma:\, x \mapsto v}\mathbb{1}_{\gamma:\, y \mapsto v}) &\leq \sum_{j=2}^{\ell-1} \binom{n-3}{j-2} p^j + \sum_{j=1}^{\ell-1} \binom{n-3}{j-1} p^j (\ell-1) \sum_{i=1}^{\ell-1} \binom{n-j-2}{i-1} p^i \\
&= n^{-2}(1+o(1))[\sum_{j=2}^{\ell-1} \frac{c^j}{(j-2)!} + (\ell-1)\sum_{j,i=1}^{\ell-1} \frac{c^{j+i}}{(j-1)!(i-1)!}\,] \\
&= \frac{1}{n^2}(c_0 + o(1))\,,
\end{aligned} \tag{5.35}$$

where we first expressed by combinatorics the probability of the two disjoint events described above, and then we applied the Stirling formula for large n and finite ℓ. Inserting the result of these computations in (5.33) and summing over the vertex sets, we get

$$\mathbb{E}\,[\sum_{v \notin \{x,y\}} \nabla_e \mathbb{1}(\widetilde{\xi}_v)\,]^2 \leq c_0 + o(1) \tag{5.36}$$

that together (5.31) and (5.32) yields

$$\mathrm{Var}(\widetilde{X}_k) \leq p(1-p)n^2\,(c_0 + o(1)) = n\,(c_0 + o(1))\,. \tag{5.37}$$

Taking $k = \alpha \frac{\log n}{\log\log n}$ with $\alpha < \frac{1}{2}$, or in general for all k increasing with n but such that $k \ll \frac{1}{2}\frac{\log n}{\log\log n}$, and applying the second moment method, from (5.30) and (5.37) we get

$$\mathbb{P}(\widetilde{X}_k = 0) \;\le\; \mathrm{Var}(\widetilde{X}_k)/\mathbb{E}(\widetilde{X}_k)^2 \le (n\, e^{-2k\log k})^{-1}(c_0 + o(1)) \underset{n\to\infty}{\longrightarrow} 0\,,$$

which prove that $\mathbb{P}(\text{ a.i. } \widetilde{S}_k \subseteq \mathbb{G}(n, c/n)) \underset{n\to\infty}{\longrightarrow} 1$.

To complete the proof of step (ii), we now have to strengthen the statement above, and compute the asymptotic probability of the event

$$\{\text{ a.i. } \widetilde{S}_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n)\}\,.$$

For $A \in \mathbb{N}$, let us introduce the symbol $A \propto n$ if there exists a positive real number ε such that $A = \varepsilon n$, and notice that the event "there exists an a.i. $\widetilde{S}_k$ contained in the unique giant component $\mathcal{C}$" is asymptotically equivalent to the event $\{\text{ a.i. } \widetilde{S}_k \subseteq \mathbb{G}(n, c/n)\,,\ |\mathcal{C}(\widetilde{S}_k)| \propto n\}$. Then we define

$$\zeta_v := \{\text{ a.i. } \widetilde{S}_k \subseteq \mathbb{G}(n, c/n),\ \widetilde{S}_k \text{ rooted at } v\,,\ |\mathcal{C}(\widetilde{S}_k)| \propto n\}$$

and let Y_k denote the random variable counting the number of a.i. $\widetilde{S}_k$ such that $\mathcal{C}(\widetilde{S}_k) \propto n$, i.e. $Y_k := \sum_{v\in V} \mathbb{1}(\zeta_v)$. Since, by definition,

$$\mathbb{P}(\text{ a.i. } \widetilde{S}_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n)) \;=\; \mathbb{P}(Y_k > 0) \;=\; 1 - \mathbb{P}(Y_k = 0)\,,$$

if we prove that $\mathbb{P}(Y_k = 0) \underset{n\to\infty}{\longrightarrow} 0$, the proof of step (ii) will follows.

We first consider the event ζ_v and notice that by definition it holds that $\mathbb{1}(\zeta_v) = \mathbb{1}(\widetilde{\xi}_v)\mathbb{1}(|\mathcal{C}(v)| \propto n)$ where $\mathcal{C}(v)$ denotes the connected component of v. Moreover, since for any given vertex $v \in V$ there exists at most one almost isolated $\widetilde{S}_k$ rooted at v, we can express the event $\widetilde{\xi}_k$ as the union of disjoint events as follows.
We label the vertices of $\widetilde{S}_k$ from the first to the ℓ-th level, in such a way that the vertices in the first level are labeled from 1 to k, the vertices in the second level are labeled from $k+1$ to $2k$, and so on until the vertices in the ℓ-th level, which are labeled from $(\ell-1)k+1$ to ℓk. For any $\bar{x} = (x_1, \dots, x_{\ell k}) \in V^{\ell k}$, we denote by $\{\widetilde{S}_k = (v, \bar{x})\}$ the event that $\mathbb{G}(n, c/n)$ contains an almost isolated $\widetilde{S}_k$ with vertex set specified by $(v, x_1, \dots, x_{\ell k})$. We can write $\widetilde{\xi}_v = \bigsqcup_{\bar{x}\in V^{\ell k}} \{\widetilde{S}_k = (v, \bar{x})\}$ and then

$$\mathbb{1}(\zeta_v) = \mathbb{1}(\widetilde{\xi}_v)\mathbb{1}(|\mathcal{C}(v)| \propto n) = \sum_{\bar{x}\in V^{\ell\, k}} \mathbb{1}(\widetilde{S}_k = (v,\bar{x}))\mathbb{1}(|\mathcal{C}(v)| \propto n)\,.$$

With this notation, the probability of ζ_v can be express in the more convenient form

$$\begin{aligned}
\mathbb{P}(\zeta_v) &= \sum_{\bar{x}} \mathbb{P}(\widetilde{S}_k = (v,\bar{x})\,,\ |\mathcal{C}(v)| \propto n) \\
&= \sum_{\bar{x}} \mathbb{P}(\widetilde{S}_k = (v,\bar{x})\,,\ |\mathcal{C}(x_{\ell k})| \propto n) \\
&= \sum_{\bar{x}} \mathbb{P}(|\mathcal{C}(x_{\ell k})| \propto n|\ \widetilde{S}_k = (v,\bar{x}))\mathbb{P}(\widetilde{S}_k = (v,\bar{x})) \qquad (5.38)
\end{aligned}$$

Since $x_{\ell k}$ is vertex on the ℓ-th level, the conditioning on the existence of $\widetilde{S}_k = (v,\bar{x})$ does not affect the Bernoulli random variables associated to edges from $x_{\ell k}$ to $V \setminus \widetilde{S}_k$. Thus, for any given $\widetilde{S}_k = (v,\bar{x})$, we consider the random graph $\mathbb{G}(n-\ell k, c/n)$ obtained from $\mathbb{G}(n,c/n)$ cutting all the vertices in $\widetilde{S}_k$ but $x_{\ell k}$, and denoting by $\mathbb{P}_-$ its probability measure, we get

$$\mathbb{P}(|\mathcal{C}(x_{\ell k})| \propto n|\ \widetilde{S}_k = (v,\bar{x})) \geq \mathbb{P}_-(|\mathcal{C}(x_{\ell k})| \propto n-\ell k)\,.$$

For all $k \ll n$, the probability in the r.h.s. of the last inequality corresponds asymptotically to the probability that a given vertex belongs to the giant component of $\mathbb{G}(n-\ell k, c/n)$.
We now observe that for any $c_0 > 1$ the probability that a given vertex in $\mathbb{G}(n-\ell k, c_0/n-\ell k)$ belongs to the giant component is equal to $\rho(1+o(1))$. Taking $c_0 = c-(c-1)/2$, where $c > 1$ by hypothesis, then $c_0 > 1$ and for large n it holds that $\frac{c_0}{n-\ell k} = \frac{c}{n-\ell k}\,(1-\frac{c-1}{2c}) \leq \frac{c}{n-\ell k}\,(1-\frac{\ell k}{n}) = \frac{c}{n}$. In particular, the probability that a vertex belong to the giant component in $\mathbb{G}(n-\ell k, c/n)$ is bigger then the same probability in $\mathbb{G}(n-\ell k, c_0/(n-\ell k))$, and thus it holds

$$\mathbb{P}(|\mathcal{C}(x_{\ell k})| \propto n\,|\,\widetilde{S}_k = (v,\bar{x})) \geq \rho(1+o(1))\,.$$

Inserting this bound in (5.38) we can conclude the computation and obtain

$$\mathbb{P}(\zeta_v) \ \geq\ \rho(1+o(1))\mathbb{P}(\widetilde{\xi}_v) \qquad (5.39)$$

The first immediate consequence of(5.39) is that $\mathbb{E}(Y_k) \geq \rho(1+o(1))\mathbb{E}(\widetilde{X}_k)$ and in particular, if $k = \alpha\frac{\log n}{\log\log n}$ with $\alpha < 1$, it holds

$$\mathbb{E}(Y_k) \ =\ \Omega(n^{1-\alpha}) \underset{n\to\infty}{\longrightarrow} \infty\,. \qquad (5.40)$$

To apply the second moment method we now have to compute $\mathrm{Var}(Y_k)$. With the same notation introduced in the proof of Theorem **5.20** and applying the decomposition property (5.13) to $\mathrm{Var}(Y_k)$, we obtain the formula

$$\mathrm{Var}(Y_k) \leq p(1-p) \sum_{\substack{e=(x,y):\\ x<y\in V}} \mathbb{E}\,(\sum_{v\in V} \nabla_e \mathbb{1}(\zeta_v)\,)^2\,. \tag{5.41}$$

For any given edge $e=(x,y)$ we introduce the characteristic functions $\chi_{v=x}$, $\chi_{v=y}$ and $\chi_{v\notin\{x,y\}}$ in the sum over $v\in V$ in (5.41), and obtain

$$\begin{aligned}
\mathbb{E}\,(\sum_{v\in V} \nabla_e \mathbb{1}(\zeta_v)\,)^2 &= \mathbb{E}\,(\sum_{v\in V} \nabla_e \mathbb{1}(\zeta_v)(\chi_{v=x}+\chi_{v=y}+\chi_{v\notin\{x,y\}})\,)^2 \\
&\leq 2\mathbb{E}\,(\nabla_e \mathbb{1}(\zeta_x))^2 + \mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(\zeta_v)\,)^2 \\
&\leq 2 + \mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(\zeta_v)\,)^2\,,
\end{aligned} \tag{5.42}$$

where in the second line, due to the fact that e is an unoriented edge, we identified the quantities $\nabla_e \mathbb{1}(\zeta_x)$ and $\nabla_e \mathbb{1}(\zeta_y)$, and in the last line we use the bound $(\nabla_e \mathbb{1}(\zeta_x))^2 \leq 1$. Writing the gradient $\nabla_e \mathbb{1}(\zeta_v)(G)$ as

$$\begin{aligned}
\nabla_e \mathbb{1}(\zeta_v)(G) &= [\nabla_e \mathbb{1}(\widetilde{\xi}_v)(G)]\mathbb{1}(|\mathcal{C}(v)| \propto n)(G^e) + [\nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)(G)]\mathbb{1}(\widetilde{\xi}_v)(G) \\
&\leq \nabla_e \mathbb{1}(\widetilde{\xi}_v)(G) + \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)(G)
\end{aligned}$$

and inserting the result in the last line of 5.42, we get

$$\begin{aligned}
\mathbb{E}\,(\sum_{v\in V} \nabla_e \mathbb{1}(\zeta_v)\,)^2 &\leq 2 + \mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(\widetilde{\xi}_v) + \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)\,)^2 \\
&\leq 2 + 2\mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(\widetilde{\xi}_v))^2 + 2\mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)\,)^2 \\
&\leq c_0 + o(1) + 2\mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)\,)^2\,,
\end{aligned} \tag{5.43}$$

where in the last line we use the bound (5.36).

Now we observe that the gradient $\nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)(G) \neq 0$ only if the connected component $C(v)$ has size proportional to n in G or in G^e, but it has size $o(n)$ in the graph $G\setminus\{e\}$. In other words, cutting e then v is disconnected from the component of size proportional to n both in G and G^e. Since $e=(x,y)$, this event can be described as the existence in $G\setminus\{y\}$ of a component $|\mathcal{C}(v)| = o(n)$ such that $\mathcal{C}(v) \ni x$ or as the existence in $G\setminus\{x\}$ of a component $|\mathcal{C}(v)| = o(n)$ such that $\mathcal{C}(v) \ni y$. By the symmetry

between x and y we can just consider the first case and then, denoting by $\mathbb{P}^-$ and $\mathbb{E}^-$ respectively the probability and the mean on the random graph $\mathbb{G}(n-1, c/n)$, obtained from $\mathbb{G}(n, c/n)$ cutting y, we get

$$\mathbb{E}\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)\,)^2 = \mathbb{E}^-(\sum_{v\neq x} \mathbb{1}(x\in\mathcal{C}(v)\,;\,|\mathcal{C}(v)| = o(n)))^2\,.$$

Moreover, since a.a.s $\mathbb{G}(n-1, c/n)$ has a giant component of size proportional to n whereas the size of the other components is at most of order $\log n$, we can write $\mathbb{1}(x \in \mathcal{C}(v)\,;\,|\mathcal{C}(v)| = o(n))$ as the indicator function $\mathbb{1}(\gamma_n : v \mapsto x)$ of the event that there exists a path γ from v to x of length at least 1 and at most $\log n$. Then we get

$$\begin{aligned} E\,(\sum_{v\notin\{x,y\}} \nabla_e \mathbb{1}(|\mathcal{C}(v)| \propto n)\,)^2 &\le \mathbb{E}^-(\sum_{v\neq x} \mathbb{1}(\gamma_n : v\mapsto x))^2 \\ &\le \sum_{v\neq x} \mathbb{P}^-(\gamma_n : v\mapsto x) + \sum_{\substack{v,w\neq x\\ v\neq w}} \mathbb{P}^-(\gamma_n : v\mapsto x\,;\,\gamma_n : w\mapsto x)\,. \end{aligned} \tag{5.44}$$

To compute these two probabilities, we perform essentially the same computations as in (5.34) and (5.35). In particular we get

$$\mathbb{P}^-(\gamma_n : v\mapsto x) = \sum_{j=1}^{\log n} \binom{n-3}{j-1} p^j = n^{-1}[\sum_{j=1}^{\log n} \frac{c^j}{(j-1)!}(1+o(1))\,] = \frac{1}{n}(c_0+o(1))\,, \tag{5.45}$$

$$\begin{aligned} &\mathbb{P}^-(\gamma_n :\, v\,\mapsto x\,,\,\gamma_n :\, w\,\mapsto x) \le \\ &\quad\le \sum_{j=2}^{\log n} \binom{n-4}{j-2} p^j + \sum_{j=1}^{\log n} \binom{n-4}{j-1} p^j \log n \sum_{i=1}^{\log n} \binom{n-j-3}{i-1} p^i \\ &\quad= n^{-2}[\sum_{j=2}^{\log n} \frac{c^j}{(j-2)!}(1+o(1)) + \log n \sum_{j,i=1}^{\log n} \frac{c^{j+i}}{(j-1)!(i-1)!}(1+o(1)\,] \\ &\quad= \frac{\log n}{n^2}(c_0+o(1))\,. \end{aligned} \tag{5.46}$$

Collecting formulas from (5.43) to (5.46) we get that $\mathbb{E}\,(\sum_{v\in V} \nabla_e \mathbb{1}(\zeta_v)\,)^2 \le \log n\,(c_0+o(1))$, which inserted in (5.41) yields

$$\mathrm{Var}(Y_k) \le n\log n\,(c_0+o(1))\,.$$

Together with the bound in (5.40) we finally obtain that for all $k = \alpha\frac{\log n}{\log\log n}$ and $\alpha < 1/2$ it holds

$$\mathbb{P}(Y_k = 0) \leq \frac{n \log n \, (c_0 + o(1))}{\rho^2 n^2 e^{-2k \log k}} = O(n^{2\alpha - 1}) \log n \underset{n \to \infty}{\longrightarrow} 0$$

which implies that $\mathbb{P}(\text{ a.i. } \widetilde{S}_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n)) = \mathbb{P}(Y_k > 0) \underset{n \to \infty}{\longrightarrow} 1$, and then concludes the proof of step (ii).

The statement of Theorem **5.20** will now easily follows.
Indeed, in step (i) we proved that for all $\beta > 0$ there exist two finite integers k_0 and ℓ_0 and a positive constant c_β such that, for all $k \geq k_0$ and $\ell \geq \ell_0$,

$$\mathbb{P}(c_{gap}(\mu_{\mathcal{C}(\widetilde{S}_k)}) \leq e^{-c_\beta k}) \geq \mathbb{P}(\text{ a.i } \widetilde{S}_k \subseteq \mathbb{G}(n, c/n)) \,,$$

while in step (ii) we proved that for all finite integer ℓ and all integers k increasing in n and such that $k < \frac{1}{2} \frac{\log n}{\log \log n}$, it holds

$$\mathbb{P}(\text{ a.i. } \widetilde{S}_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n)) \underset{n \to \infty}{\longrightarrow} 1 \,.$$

Taking $k = \alpha \frac{\log n}{\log \log n}$ and $\delta_n = \exp(-\alpha c_\beta \frac{\log n}{\log \log n})$, with $\alpha \in (0, \frac{1}{2})$, we get

$$\mathbb{P}(c_{gap}(\mu_{\mathcal{C}}) < \delta_n \,; \mathcal{C} \subseteq \mathbb{G}(n, c/n)) \geq \mathbb{P}(\text{ a.i. } S_k \subseteq \mathcal{C} \subseteq \mathbb{G}(n, c/n)) \underset{n \to \infty}{\longrightarrow} 1,$$

which completes the proof of Theorem **5.20**. □

From Theorem **5.20** we can conclude that a.a.s the relaxation time of the Ising Glauber dynamics on the giant component of $\mathbb{G}(n, c/n)$ is increasing in n for all $\beta > 0$.

5.5.4 Slow mixing on regular random graphs at low temperatures

Here we consider the Glauber dynamics for the Ising model on the random r-regular graph $\mathbb{G}(n, r)$, with $r \geq 3$ and rn an even number.
Let $\eta \in (0, 1)$ such that $2^{4/r} < (1 - \eta)^{(1-\eta)}(1 + \eta)^{(1+\eta)}$ and recall (see section 5.4.2) that a.a.s. $\mathbb{G}(n, r)$ has isoperimetric constant $i_e \geq (1 - \eta)r/2$.
Introducing the symbol $\alpha = (1 - \eta)r/2$, we state the following:

Theorem 5.21. *For every $\beta > \log 2/\alpha$, there exists a constant c_β such that a.a.s. the spectral gap of the Ising model Glauber dynamics on $\mathbb{G}(n, r)$ is less then $\delta_n = \exp(-c_\beta n)$, i.e.*

$$\lim_{n \to \infty} \mathbb{P}(c_{gap}(\mu_G) < \delta_n) = 1 \tag{5.47}$$

Remark 5.14. *To better appreciate this result, one should keep in mind that the relaxation time of the Glauber dynamics on deterministic almost regular graphs, like the lattice $\mathbb{Z}^d$ and the regular tree $\mathbb{T}^b$, grows less then exponentially in the size of the graph. Indeed, as described in Chapter 3, the relaxation time on n-vertex squares of $\mathbb{Z}^d$ is at most exponential in $n^{\frac{d-1}{d}}$, and even, on n-vertex balls in the regular trees $\mathbb{T}^b$ or in the hyperbolic graphs $\mathbb{H}(v,s)$, it grows at most polynomially in n. Thus, the dynamics on random regular graphs shows a different behavior with respect to the correspondent deterministic case.*

Proof. To prove the statement, we want to exploit the variational definition of spectral gap, and then provide a suitable test function with a very slow relaxation. Let us consider the magnetization of a configuration $\sigma \in \Omega_\sigma$, defined by $m(\sigma) = \sum_{x\in V} \sigma_x$, and then introduce the following characteristic function

$$\mathbb{1}_{\{m>0\}}(\sigma) \;=\; \begin{cases} 1 & \text{if } m(\sigma) > 0 \\ 0 & \text{if } m(\sigma) \le 0 \end{cases} \;.$$

Without loss of generality, we can restrict the analysis to odd n values. Since in this case $m_k(\sigma) \neq 0$ for all σ, by symmetry it holds that $\mathrm{Var}_G(\mathbb{1}_k) = 1/4$ so that the spectral gap is bounded as $c_{gap}(\mu_G) \le 4\,\mathcal{D}_G(\mathbb{1}_k)$. We thus concentrate on the Dirichlet form

$$\mathcal{D}_G(\mathbb{1}_{\{m>0\}}) := \frac{1}{2}\sum_{x\in V} \mu_G\left(c_x\left[\nabla_x \mathbb{1}_{\{m>0\}}\right]^2\right) \;.$$

We first observe that $\nabla_x \mathbb{1}_{\{m>0\}}(\sigma) \neq 0$ only if σ is such that $|m(\sigma)| = 1$, namely for configurations σ having $(n+2)/2$ spins with the same value. Moreover, for every configuration σ of this kind, $[\nabla_x \mathbb{1}_{\{m>0\}}(\sigma)]^2 = 1$ only if x is one of the $(n+2)/2$ vertices with spin of the same sign of $m(\sigma)$; then we get

$$\mathcal{D}_G(\mathbb{1}_{\{m>0\}}) \;\le\; \frac{n+2}{4}\;\mu_G(|m|=1) \;\le\; \frac{n+2}{2}\;\mu_G(m=1)\,. \tag{5.48}$$

Let us analyze the probability $\mu_G(m=1)$.
For every configuration σ such that $m(\sigma) = 1$, we define the subsets

$$V^+(\sigma) := \{x \in V \text{ s.t. } \sigma_x = +\} \quad \text{and} \quad V^-(\sigma) = \{x \in V \text{ s.t. } \sigma_x = -\}\,,$$

and we observe that, from the condition $m(\sigma) = 1$, it holds that $|V^+(\sigma)| = (n+1)/2$ and $|V^-(\sigma)| = (n-1)/2$. Thus, any configuration with magnetization equal to 1, is univocally correspondent to a partition of the vertex

set V in two subsets A and B of size $(n+1)/2$ and $(n-1)/2$, respectively. Moreover, if σ is a configuration such that $V^+(\sigma) = A$ and $V^-(\sigma) = B$, and we denote by $E(A)$ and $E(B)$ the edge sets in the induced subgraphs of A and B respectively, and by $E(A,B)$ the set of edges between A and B, we can write the Hamiltonian of σ as

$$H_G(\sigma) \;=\; -(\,|E(A)| + |E(B)|\,) + |E(A,B)|\,.$$

Denoting by $\mathcal{P}$ the set of partitions of V in two subsets (A,B) such that $|A| = \frac{n+1}{2}$ and $|B| = \frac{n-1}{2}$, it holds the following computation

$$\begin{aligned}
\mu_G(m=0) &= \sum_{\sigma\,:m(\sigma)=0} \frac{\exp[-\beta H_G(\sigma)]}{Z_G(\beta)} \\
&= \sum_{(A,B)\in\mathcal{P}} \frac{\exp(\,\beta(|E(A)| + |E(B)| - |E(A,B)|)\,)}{Z_G(\beta)} \\
&\le \sum_{(A,B)\in\mathcal{P}} \frac{\exp(\,\beta(|E(A)| + |E(B)| - |E(A,B)|)\,)}{\exp(\,\beta(|E(A)| + |E(B)| + |E(A,B)|)\,)} \\
&= \sum_{(A,B)\in\mathcal{P}} \exp(-2\beta|E(A,B)|)\,. \qquad (5.49)
\end{aligned}$$

We now observe that $|E(A,B)| = |\partial_E A|$, the edge boundary of A. Thus, if G has isoperimetric constant $i_e(G) \ge \delta$, with δ a positive real number, it holds the bounds

$$|E(A,B)| \ge \delta|A| = \delta\frac{n+1}{2}$$

and then, continuing from (5.49), we get

$$\begin{aligned}
\mu_G(m=0) &\le \sum_{(A,B)\in\mathcal{P}} e^{-\beta\,\delta(n+1)} \\
&= \binom{n}{\frac{n+1}{2}} e^{-\beta\,\delta(n+1)} \\
&\le 2^n e^{-\beta\,\delta(n+1)} \\
&= e^{-(\beta\,\delta - \log 2)n}\,, \qquad (5.50)
\end{aligned}$$

where in the third line we approximated the binomial factor using the Stirling formula for large n.
Inserting this bound in 5.48, we get that for all G such that $i_e(G) \ge \delta$ and for all $\beta > \frac{\log 2}{\delta}$, there exists a positive constant c_β such that

$$\mathcal{D}_G(\mathbb{1}_{\{m>0\}}) \;\le\; \frac{n+2}{2} e^{-(\beta\delta - \log 2)n}(1+o(1)) \le e^{-c_\beta n},$$

which implies that $c_{gap}(\mu_G) \leq e^{-c_\beta n}$, or in other words that

$$\mathbb{P}(c_{gap}(\mu_G) \leq e^{-c_\beta n}) \geq \mathbb{P}(i_e(G) \geq \delta)\,.$$

Taking $\delta = \alpha$ and recalling that a.a.s. $i_e(G) \geq \alpha$ (see Theorem **5.18**), it follows that for all $\beta > \frac{\log 2}{\alpha}$, there exists a positive constant c_β such that

$$\mathbb{P}(c_{gap}(\mu_G) \leq e^{-c_\beta n}) \geq \mathbb{P}(i_e(G) \geq \alpha) \underset{n\to\infty}{\longrightarrow} 1\,,$$

and thus Theorem **5.21** follows taking $\delta_n = e^{-c_\beta n}$. □

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