DIFFERENTIAL GEOMETRY

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Exercise Sheet 1

(Note: the reference [AT] in the exercises refers to the textbook by Abate, Tovena, “Geometria Differenziale”)

Exercise 1. ([AT] Exercise 2.10 p. 116) Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be an atlas of an $m$-dimensional manifold $M$, and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ an atlas of an $n$-dimensional manifold $N$. If we define $\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n$ by setting $\phi_\alpha \times \psi_\beta(p, q) = (\phi_\alpha(p), \psi_\beta(q))$, show that $\mathcal{A} \times \mathcal{B} = \{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is an atlas of dimension $m + n$ on $M \times N$.

Exercise 2. ([AT] Exercise 2.13 p. 116) Prove that there does not exist a structure of differentiable manifold on $[0, +\infty)$ that is compatible with the Euclidean topology.

Exercise 3. ([AT] Exercise 2.14 p. 116) Prove that there does not exist an atlas on the $n$-sphere $S^n$, compatible with the natural topology of $S^n$, consisting of only one chart.

Exercise 4. ([AT] Exercise 2.21 p. 117) Let $F : \Omega \to \mathbb{R}^m$ be a smooth map defined on an open subset $\Omega \subset \mathbb{R}^n$. Show that the set of critical points $\text{Crit}(F)$ is a closed subset of $\Omega$.

Exercise 5. ([AT] Exercise 2.23 p. 117) Let $k$ be an integer with $0 \leq k \leq \min(m, n)$. Prove that the subset of $M_{m,n}(\mathbb{R})$ of matrices of rank $\geq k$ is open, and hence it has a natural structure of smooth manifold of dimension $mn$.

Exercise 6. ([AT] Exercise 2.25 p. 117) Let $S(n, \mathbb{R}) \subset M_{n,n}(\mathbb{R})$ be the subspace of symmetric matrices; we can identify $S(n, \mathbb{R})$ with $\mathbb{R}^{n(n+1)/2}$. Let $F : M_{n,n}(\mathbb{R}) \to S(n, \mathbb{R})$ be the function defined by $F(X) = X^TX$. Show that

$$dF_X(A) = X^TA + A^TX,$$

for every $A, X \in M_{n,n}(\mathbb{R})$. Using this result, prove that for any $X \in O(n)$ the differential

$$dF_X : M_{n,n}(\mathbb{R}) \to S(n, \mathbb{R})$$

is surjective, hence $O(n)$ has a structure of smooth manifold of dimension $n(n-1)/2$.

Exercise 7. ([AT] Exercise 2.29 p. 118) Let $\{U_\alpha\}$ be an open covering of a smooth manifold $M$, and let $N$ be another manifold. Let us assume that for any $\alpha$ there is a smooth function $F_\alpha : U_\alpha \to N$, such that $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$, when $U_\alpha \cap U_\beta \neq \emptyset$. Prove that there exists a unique smooth function $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$, for any $\alpha$.

Exercise 8. ([AT] Exercise 2.34 p. 118) Prove that the following functions are differentiable:

- For any $n \in \mathbb{Z}$, the map $p_n : S^1 \to S^1$ given (in complex notation) by $p_n(z) = z^n$.
- The map $A : S^n \to S^n$ given by $A(x) = -x$. 
• The function $F : S^3 \to S^2$ given by

$$F(z, w) = (2 \text{Re}(z\bar{w}), 2 \text{Im}(z\bar{w}), |z|^2 - |w|^2),$$

where we identify $S^3$ with the set $\{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$.

**Exercise 9.** ([AT] Exercise 2.41 p. 119) Show that $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to $S^2$.

**Exercise 10.** Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1) \in \mathbb{R}^2$ is a regular value of $F$, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $S^2$.

**Exercise 11.** ([AT] Exercise 2.60 p. 122) Let $M$ be a smooth manifold and $p \in M$. Let $C^\infty_p$ denote the ring of germs of smooth functions at $p$. Let us set

$$m_p = \{ f \in C^\infty_p \mid f(p) = 0 \}.$$

Prove that $m_p$ is the unique maximal ideal of the ring $C^\infty_p$. Show that the tangent space $T_p M$ is canonically isomorphic to the dual of $m_p/m^2_p$, i.e., to the vector space $\text{Hom}_\mathbb{R}(m_p/m^2_p, \mathbb{R})$.

**Exercise 12.** Let $M$ and $N$ be two smooth manifolds, and let $(p, q) \in M \times N$. Prove that the tangent space $T_{(p,q)} M \times N$ is canonically isomorphic to $T_p M \oplus T_q N$.

**Exercise 13.** ([AT] Exercise 2.73 p. 123) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by setting

$$F(\phi, \theta) = ((2 + \cos \phi) \cos \theta, (2 + \cos \phi) \sin \theta, \sin \phi).$$

Prove that $F$ is an immersion that induces an embedding of the torus $\mathbb{T}^2$ into $\mathbb{R}^3$.

**Exercise 14.** ([AT] Exercise 2.74 p. 123) Let $\mathbb{T}^2 = S^1 \times S^1$ be the torus, considered as a subset of $\mathbb{C}^2$. For $\alpha \in \mathbb{R}$ let $\sigma_\alpha : \mathbb{R} \to \mathbb{T}^2$ be defined by $\sigma_\alpha(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$. Prove that:

- if $\alpha \notin \mathbb{Q}$ then $\sigma_\alpha$ is an injective immersion whose image is dense in $\mathbb{T}^2$, hence it is not an embedding.
- if $\alpha \in \mathbb{Q}$ then $\sigma_\alpha$ induces an embedding of $S^1$ into $\mathbb{T}^2$.

**Exercise 15.** ([AT] Exercise 2.84 p. 125) Let $F : S^2 \to \mathbb{R}^4$ be defined by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Prove that $F$ induces an embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

**Exercise 16.** Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of $F$ are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.