DIFFERENTIAL GEOMETRY

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Exercise Sheet 1

(*Note: the reference [AT] in the exercises refers to the textbook by Abate, Tovena, "Geometria Differenziale"*)

Exercise 1. ([AT] Exercise 2.10 p. 116) Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be an atlas of a m-dimensional manifold M, and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}$ an atlas of a n-dimensional manifold N. If we define $\phi_\alpha \times \psi_\beta \colon U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n$ by setting $\phi_\alpha \times \psi_\beta(p, q) = (\phi_\alpha(p), \psi_\beta(q))$, show that $\mathcal{A} \times \mathcal{B} = \{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is an atlas of dimension $m + n$ on $M \times N$.

Exercise 2. ([AT] Exercise 2.13 p. 116) Prove that there does not exist a structure of differentiable manifold on $[0, +\infty)$ that is compatible with the Euclidean topology.

Exercise 3. ([AT] Exercise 2.14 p. 116) Prove that there does not exist an atlas on the n-sphere S^n, compatible with the natural topology of S^n, consisting of only one chart.

Exercise 4. ([AT] Exercise 2.21 p. 117) Let $F \colon \Omega \to \mathbb{R}^m$ be a smooth map defined on an open subset $\Omega \subset \mathbb{R}^n$. Show that the set of critical points $\mathrm{Crit}(F)$ is a closed subset of Ω.

Exercise 5. ([AT] Exercise 2.23 p. 117) Let k be an integer with $0 \leq k \leq \min(m, n)$. Prove that the subset of $M_{m,n}(\mathbb{R})$ of matrices of rank $\geq k$ is open, and hence it has a natural structure of smooth manifold of dimension mn.

Exercise 6. ([AT] Exercise 2.25 p. 117) Let $S(n, \mathbb{R}) \subset M_{n,n}(\mathbb{R})$ be the subspace of symmetric matrices; we can identify $S(n, \mathbb{R})$ with $\mathbb{R}^{n(n+1)/2}$. Let $F \colon M_{n,n}(\mathbb{R}) \to S(n, \mathbb{R})$ be the function defined by $F(X) = X^T X$. Show that

$$dF_X(A) = X^T A + A^T X,$$

for every $A, X \in M_{n,n}(\mathbb{R})$. Using this result, prove that for any $X \in O(n)$ the differential

$$dF_X \colon M_{n,n}(\mathbb{R}) \to S(n, \mathbb{R})$$

is surjective, hence $O(n)$ has a structure of smooth manifold of dimension $n(n-1)/2$.

Exercise 7. ([AT] Exercise 2.29 p. 118) Let $\{U_\alpha\}$ be an open covering of a smooth manifold M, and let N be another manifold. Let us assume that for any α there is a smooth function $F_\alpha \colon U_\alpha \to N$, such that $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$, when $U_\alpha \cap U_\beta \neq \varnothing$. Prove that there exists a unique smooth function $F \colon M \to N$ such that $F|_{U_\alpha} = F_\alpha$, for any α.

Exercise 8. ([AT] Exercise 2.34 p. 118) Prove that the following functions are differentiable:

- For any $n \in \mathbb{Z}$, the map $p_n \colon S^1 \to S^1$ given (in complex notation) by $p_n(z) = z^n$.
- The map $A \colon S^n \to S^n$ given by $A(x) = -x$.

- The function $F\colon S^3 \to S^2$ given by

$$F(z,w) = (2\operatorname{Re}(z\bar{w}), 2\operatorname{Im}(z\bar{w}), |z|^2 - |w|^2),$$

where we identify S^3 with the set $\{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$.

Exercise 9. ([AT] Exercise 2.41 p. 119) Show that $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to S^2.

Exercise 10. Consider the map $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$F(x,y,s,t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0,1) \in \mathbb{R}^2$ is a regular value of F, and that the level set $F^{-1}(0,1)$ is diffeomorphic to S^2.

Exercise 11. ([AT] Exercise 2.60 p. 122) Let M be a smooth manifold and $p \in M$. Let C_p^∞ denote the ring of germs of smooth functions at p. Let us set

$$\mathfrak{m}_p = \{f \in C_p^\infty \mid f(p) = 0\}.$$

Prove that $\mathfrak{m}_p$ is the unique maximal ideal of the ring C_p^∞.
Show that the tangent space T_pM is canonically isomorphic to the dual of $\mathfrak{m}_p/\mathfrak{m}_p^2$, i.e., to the vector space $\operatorname{Hom}_{\mathbb{R}}(\mathfrak{m}_p/\mathfrak{m}_p^2, \mathbb{R})$.

Exercise 12. Let M and N be two smooth manifolds, and let $(p,q) \in M \times N$. Prove that the tangent space $T_{(p,q)}M \times N$ is canonically isomorphic to $T_pM \oplus T_qN$.

Exercise 13. ([AT] Exercise 2.73 p. 123) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by setting

$$F(\phi,\theta) = ((2 + \cos\phi)\cos\theta, (2 + \cos\phi)\sin\theta, \sin\phi).$$

Prove that F is an immersion that induces an embedding of the torus $\mathbb{T}^2$ into $\mathbb{R}^3$.

Exercise 14. ([AT] Exercise 2.74 p. 123) Let $\mathbb{T}^2 = S^1 \times S^1$ be the torus, considered as a subset of $\mathbb{C}^2$. For $\alpha \in \mathbb{R}$ let $\sigma_\alpha\colon \mathbb{R} \to \mathbb{T}^2$ be defined by $\sigma_\alpha(t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$. Prove that:

- if $\alpha \notin \mathbb{Q}$ then σ_α is an injective immersion whose image is dense in $\mathbb{T}^2$, hence it is not an embedding.
- if $\alpha \in \mathbb{Q}$ then σ_α induces an embedding of S^1 into $\mathbb{T}^2$.

Exercise 15. ([AT] Exercise 2.84 p. 125) Let $F\colon S^2 \to \mathbb{R}^4$ be defined by $F(x,y,z) = (x^2 - y^2, xy, xz, yz)$. Prove that F induces an embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

Exercise 16. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.