DIFFERENTIAL GEOMETRY

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Exercise Sheet 2

(*Note: the reference [AT] in the exercises refers to the textbook by Abate, Tovena, "Geometria Differenziale"*)

Exercise 1. ([AT] Exercise 3.6 p. 193) Let $\pi_1 \colon E_1 \to M_1$ and $\pi_2 \colon E_2 \to M_2$ be two vector bundles, and let (L, F) be a morphism of vector bundles

$$\begin{array}{ccc} E_1 & \xrightarrow{L} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow[F]{} & M_2 \end{array}$$

Let us assume that L has constant rank, i.e., that $\dim L(E_P)$ does not depend on $P \in M_1$. Prove that $\operatorname{Ker}(L, F) = \{(P, v) \in E_1 \mid L(v) = 0\} \subset E_1$ is a sub-vector bundle of E_1, and that $\operatorname{Im}(L, F) = L(E_1) \subset E_2$ is a sub-vector bundle of E_2.

Exercise 2. ([AT] Exercise 3.10 p. 193) Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be an atlas on M, and $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ a family of transition functions for a vector bundle E over M. Let us assume that, for every α, we are given a r-tuple of differentiable functions $a_\alpha = (a_\alpha^1, \dots, a_\alpha^r)$ with $a_\alpha^i \in C^\infty(U_\alpha)$, such that on $U_\alpha \cap U_\beta$ the r-tuples a_α and a_β are related by

$$a_\alpha^j = \sum_{h=1}^{r} (g_{\alpha\beta})_h^j a_\beta^h.$$

Show that there exists a unique section σ of E such that the functions a_α^j are the components of σ with respect to a suitable local basis of E on U_α.

Exercise 3. Compute the transition function for TS^2 associated with the two local trivializations determined by stereographic projections.

Exercise 4. ([AT] Exercise 3.14 p. 194) Let $F \colon M \to N$ be a differentiable function, and $\pi \colon E \to N$ a vector bundle of rank r over N. Prove that the space of sections over M of the pull-back bundle F^*E (see definition in the book) is isomorphic to the space of C^∞ functions $\sigma \colon M \to E$ such that $\sigma(P) \in E_{F(P)}$, for every $P \in M$.

Exercise 5. ([AT] Exercise 3.17 p. 194) Let $\tau \in \mathcal{T}_k^h(M)$ be a tensor field of type $\binom{h}{k}$. For $1 \le i \le h$ and $1 \le j \le k$, let $\omega^1, \dots, \omega^i \in A^1(M)$ be 1-forms, and $X_1, \dots, X_j \in \mathcal{T}(M)$ be vector fields. Show that the function

$$P \mapsto \tau_P(\omega_P^1, \dots, \omega_P^i, \cdot, X_1(P), \dots, X_j(P), \cdot)$$

can be naturally interpreted as a tensor field of type $\binom{h-i}{k-j}$.

Exercise 6. Show that there is a smooth vector field on S^2 that vanishes at exactly one point.
[Hint: try using stereographic projection.]

Exercise 7. ([AT] Exercise 3.19 p. 195) For every $z \in S^{2n-1} \subset \mathbb{C}^n$ let $\sigma_z \colon \mathbb{R} \to S^{2n-1}$ be the curve $\sigma_z(t) = e^{it} z$. Prove that by setting $X(z) = \sigma_z'(0)$ we get a nowhere vanishing vector field $X \in \mathcal{T}(S^{2n-1})$.

Exercise 8. ([AT] Exercise 3.25 p. 196) Determine explicitely the flux of the following vector fields on $\mathbb{R}^2$:

- $y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$
- $x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$
- $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$
- $y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

Exercise 9. ([AT] Exercise 3.27 p. 196) Let $X \in \mathcal{T}(M)$ be a vector field, and σ be a maximal integral curve of X.

- Show that if σ is not constant then it is either injective or periodic.
- Prove that if σ is periodic and non constant then there exists a unique positive number T_0 (the period of σ) such that $\sigma(t) = \sigma(t')$ if and only if $t - t' = kT_0$, for some $k \in \mathbb{Z}$.
- Prove that if σ is not constant then it is an immersion, and the image of σ has a natural structure of 1-dimensional variety, diffeomorphic to $\mathbb{R}$ or to S^1.

Exercise 10. Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive, and let $F \colon M \to M$ be the map $F(x, y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y, where

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial x}.$$

Exercise 11. For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

- $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}, \qquad Y = \frac{\partial}{\partial y}$
- $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$
- $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad Y = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}.$

Exercise 12. Let M be a smooth manifold and $X, Y \in \mathcal{T}(M)$ be smooth vector fields on M. Show that, for any $Z \in \mathcal{T}(M)$, we have

$$\mathcal{L}_X\mathcal{L}_Y Z - \mathcal{L}_Y\mathcal{L}_X Z = \mathcal{L}_{[X,Y]}Z.$$