Sheet 1, Exercise 15

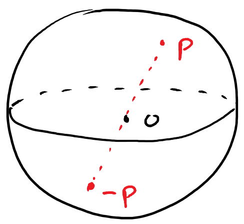
Let $F: S^2 \longrightarrow \mathbb{R}^4$ be defined by

$$F(x,y,z) = (x^2 - y^2,\ xy,\ xz,\ yz)$$

Prove that F induces an embedding of $P^2(\mathbb{R})$ into $\mathbb{R}^4$.

Proof On the sphere S^2 let us define an equivalence relation by requiring that any two antipodal points be equivalent:

$$P = (x,y,z) \sim -P = (-x,-y,-z), \quad \forall P \in S^2$$



We know that $S^2/\sim \ \cong P^2(\mathbb{R})$

Moreover, the projection

$$\pi : S^2 \longrightarrow S^2/\sim \ \cong P^2(\mathbb{R})$$

is a 2-sheeted covering (hence, in particular, a local diffeomorphism) [Esempio 2.2.18, page 79]

Let's see if $F: S^2 \longrightarrow \mathbb{R}^4$ is injective.

Let $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1) \in S^2$ be such that $F(P_0) = F(P_1)$. This means that

$$\begin{cases} x_0^2 - y_0^2 = x_1^2 - y_1^2 \\ x_0 y_0 = x_1 y_1 \\ x_0 z_0 = x_1 z_1 \\ y_0 z_0 = y_1 z_1 \end{cases} \qquad \text{and} \quad \begin{array}{l} x_0^2 + y_0^2 + z_0^2 = 1 \\ x_1^2 + y_1^2 + z_1^2 = 1 \end{array}$$

A case-by-case analysis shows that the only solutions to the previous equations are:

$$P_0 = (x_0, y_0, z_0) = P_1 = (x_1, y_1, z_1)$$

or

$$P_0 = (x_0, y_0, z_0) = -P_1 = (-x_1, -y_1, -z_1)$$

This means that $F : S^2 \to \mathbb{R}^4$ is <u>not</u> injective, but it factors through an <u>injective</u> map

$$\bar{F} : S^2/\sim \; \cong \mathbb{P}^2(\mathbb{R}) \longrightarrow \mathbb{R}^4$$

In other words, there is a commutative diagram

$$\begin{tikzcd} S^2 \arrow[rr, "F"] \arrow[dr, "\pi"'] & & \mathbb{R}^4 \\ & S^2/\sim \; \cong \mathbb{P}^2(\mathbb{R}) \arrow[ur, "\bar{F}"'] & \end{tikzcd}$$

with <u>$\bar{F}$ injective</u>.

Now we must prove that the differential of $\bar{F}$ is injective, at every point of $\mathbb{P}^2(\mathbb{R})$.

Since $S^2 \xrightarrow{\pi} \mathbb{P}^2(\mathbb{R})$ is a covering (hence a local diffeomorphism), to prove that $d\bar{F}$ is injective is equivalent to proving that $dF_P : T_P S^2 \longrightarrow T_{F(P)}\mathbb{R}^4 \cong \mathbb{R}^4$ is injective, $\forall P \in S^2$.

If $P = (x, y, z)$, the Jacobian matrix of F at P is

$$JF_P = \begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$$

We have : $T_p S^2 = \{ (a,b,c) \in \mathbb{R}^3 \mid ax + by + cz = 0 \}$

hence, if $v = (a, b, c) \in T_p S^2$, the equation $dF_p(v) = 0$ is equivalent to :

$$\begin{cases} 2ax - 2by = 0 \\ ay + bx = 0 \\ az + cx = 0 \\ bz + cy = 0 \\ ax + by + cz = 0 \end{cases} \quad \text{and } x^2 + y^2 + z^2 = 1$$

A direct computation shows that the unique solution of this system is $v = (a, b, c) = (0, 0, 0)$, $\forall P = (x, y, z) \in S^2$.

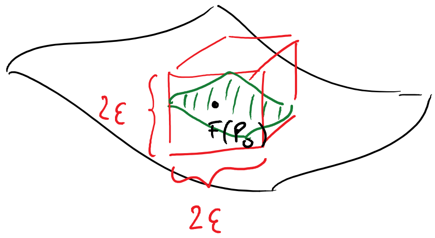
This means that $\ker(dF_p) = \{0\}$, $\forall P \in S^2$, hence dF_p is injective, $\forall P \in S^2$.

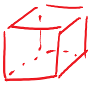
To prove that $\overline{F} : \mathbb{P}^2(\mathbb{R}) \longrightarrow \mathbb{R}^4$ is an embedding it only remains to show that $\overline{F}$ is a homeomorphism on its image. This means that we must prove that $\mathbb{P}^2(\mathbb{R}) \cong S^2/\sim$, with the quotient topology coming from the usual topology of S^2, is homeomorphic to $\overline{F}(\mathbb{P}^2(\mathbb{R})) = F(S^2)$, with the topology induced from the usual topology of $\mathbb{R}^4$.

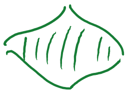
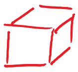
So, let $P_0 = (x_0, y_0, z_0) \in S^2$, and

$F(P_0) = (x_0^2 - y_0^2, x_0 y_0, x_0 z_0, y_0 z_0) \in F(S^2) \subset \mathbb{R}^4$

Let us consider a small open neighborhood of $F(P_0)$ in $F(S^2)$, obtained by intersecting a small (hyper)cube (of size 2ε) centered at $F(P_0)$ with $F(S^2)$



 = small hypercube $\subset \mathbb{R}^4$

 = intersection of  with $F(S^2)$

This means that we are looking at points

$F(P) = (x^2 - y^2, \ xy, \ xz, \ yz) \subset F(S^2)$ that satisfy

$$(*) \begin{cases} |(x^2-y^2) - (x_0^2 - y_0^2)| < \varepsilon \\ |xy - x_0 y_0| < \varepsilon \\ |xz - x_0 z_0| < \varepsilon \\ |yz - y_0 z_0| < \varepsilon \end{cases}$$

Let us consider the case $x_0 > 0$, $y_0 > 0$, $z_0 > 0$

(other cases are treated in a similar way).

$$|xy - x_0 y_0| < \varepsilon \iff \left| \frac{x}{x_0} \frac{y}{y_0} - 1 \right| < \frac{\varepsilon}{x_0 y_0} = \varepsilon_1$$

In a similar way, we get

$$\left| \frac{x}{x_0} \frac{z}{z_0} - 1 \right| < \varepsilon_2 \quad ; \quad \left| \frac{y}{y_0} \frac{z}{z_0} - 1 \right| < \varepsilon_3$$

To simplify the notation, let us write

$$a = \frac{x}{x_0}, \quad b = \frac{y}{y_0}, \quad c = \frac{z}{z_0}$$

Hence we have:

$$|ab-1| < \varepsilon_1 \quad ; \quad |ac-1| < \varepsilon_2 \quad ; \quad |bc-1| < \varepsilon_3$$

I will write $ab \approx 1$ (approximately $= 1$) instead of $|ab-1| < \varepsilon$ (this is just to simplify the exposition). Then we have:

$$\begin{array}{ccccc} ab \approx 1 & ; & ac \approx 1 & ; & bc \approx 1 \\ \Downarrow & & \Downarrow & & \Downarrow \\ b \approx \frac{1}{a} & & c \approx \frac{1}{a} & \Rightarrow & a^2 \approx 1 \end{array}$$

Similarly, we find $b^2 \approx 1$, $c^2 \approx 1$

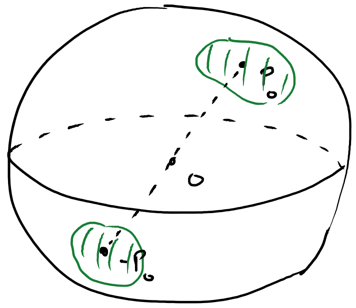
This means that

$$\left(\frac{x}{x_0}\right)^2 \approx 1 \iff x^2 \approx x_0^2$$

$$\left(\frac{y}{y_0}\right)^2 \approx 1 \iff y^2 \approx y_0^2$$

$$\left(\frac{z}{z_0}\right)^2 \approx 1 \iff z^2 \approx z_0^2$$

A closer inspection of the inequalities (*) reveals that the solutions of (*) are given by points $P = (x, y, z)$ such that either $P = (x, y, z) \approx (x_0, y_0, z_0)$ or $P = (x, y, z) \approx (-x_0, -y_0, -z_0)$. This means that, on the sphere S^2, we obtain a small open neighborhood of $P_0 = (x_0, y_0, z_0)$ together with a small open neighborhood of its antipodal point $-P_0 = (-x_0, -y_0, -z_0)$.



But this is precisely what is needed to get a small open neighborhood of the image of P_0 in $\mathbb{P}^2(\mathbb{R}) = S^2/\sim$, for the quotient topology.

This proves that, in the map $\bar{F} : \mathbb{P}^2(\mathbb{R}) \longrightarrow \mathbb{R}^4$, open subsets of the image of $\bar{F}$ for the induced topology from $\mathbb{R}^4$, correspond one-to-one with open subsets of $\mathbb{P}^2(\mathbb{R})$ for the quotient topology coming from the identification $\mathbb{P}^2(\mathbb{R}) = S^2/\sim$.
Hence we conclude that $\bar{F}$ is a homeomorphism onto its image.
Finally, this means precisely that $\bar{F}$ is an embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

Remarks.

We can parametrize S^2 as follows:

$$\begin{cases} x = \sin\vartheta \cos\varphi \\ y = \sin\vartheta \sin\varphi \\ z = \cos\vartheta \end{cases} \qquad \begin{array}{l} 0 \le \varphi \le 2\pi \\ 0 \le \vartheta \le \pi \end{array}$$

Hence we can parametrize $F(S^2) = \bar{F}(\mathbb{P}^2(\mathbb{R})) \subset \mathbb{R}^4$ as follows:

$$(x^1, x^2, x^3, x^4) = F(x, y, z) = \big(\sin^2\vartheta \cos^2\varphi - \sin^2\vartheta \sin^2\varphi,$$

$\sin^2\vartheta \sin\varphi \cos\varphi,$

$\sin\vartheta \cos\vartheta \cos\varphi,$

$\sin\vartheta \cos\vartheta \sin\varphi \ )$

You can now "visualize" the image of $\mathbb{P}^2(\mathbb{R}) \subset \mathbb{R}^4$ by projecting from $\mathbb{R}^4$ to $\mathbb{R}^3$ (just forget one of the 4 coordinates). For instance, we can use the projection

$$p : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$
$$(x^1, x^2, x^3, x^4) \longmapsto (x^2, x^3, x^4)$$

Then $p(\bar{F}(\mathbb{P}^2(\mathbb{R}))) \subset \mathbb{R}^3$ is parametrized by:

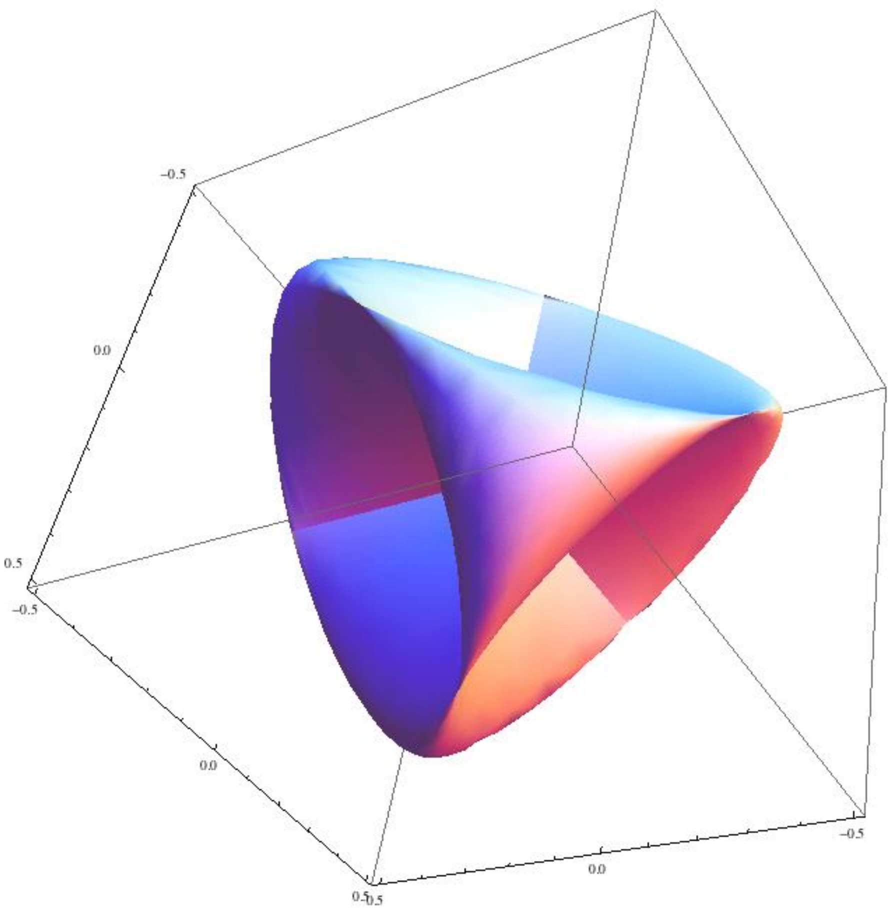
$$(x^2, x^3, x^4) = (\sin^2\vartheta \sin\varphi \cos\varphi, \sin\vartheta \cos\vartheta \cos\varphi, \sin\vartheta \cos\vartheta \sin\varphi)$$

Of course, different points $P_1 \neq P_2$ of $\bar{F}(\mathbb{P}^2(\mathbb{R}))$ that differ only in their first coordinate x^1, will be projected down to the same point $p(P_1) = p(P_2)$ of $\mathbb{R}^3$. This means that the surface $p(\bar{F}(\mathbb{P}^2(\mathbb{R}))) \subset \mathbb{R}^3$ will have some apparent self-intersections, that are not present in the "true" $\bar{F}(\mathbb{P}^2(\mathbb{R})) \subset \mathbb{R}^4$ (the self-intersections are an artifact of the projection from $\mathbb{R}^4$ to $\mathbb{R}^3$).

You can visualize the surface $p(\bar{F}(\mathbb{P}^2(\mathbb{R}))) \subset \mathbb{R}^3$ with a software like "Mathematica" (for instance) by typing:

```
ParametricPlot3D[ { Sin[u]^2 * Sin[v] * Cos[v], Sin[u] * Cos[u] * Cos[v], Sin[u] * Cos[u] * Sin[v] },
   {u, 0, Pi}, {v, 0, 2*Pi}, Mesh -> None ]
```

The result is the following:



This is known as the "Steiner surface" or the "Roman surface".